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Digital System Analysis - ~~Z domain~~ Poles & zeros

Any LTI ~~system~~ filter, $H(z) = \dfrac{P(z)}{Q(z)} = \dfrac{(a_0 + a_1 z^{-1} + \dots a_M z^{-M})}{1 - (b_1 z^{-1} + b_2 z^{-2} + \dots b_N z^{-N})} z^{R}$

$R = 0$ for causal system

$= A \dfrac{(z-z_1)(z-z_2)\cdots(z-z_M)}{z^P} \bigg/ \dfrac{z^Q}{(z-p_1)(z-p_2)\cdots(z-p_N)}$

For causal system → $P = M$ & $Q = N$

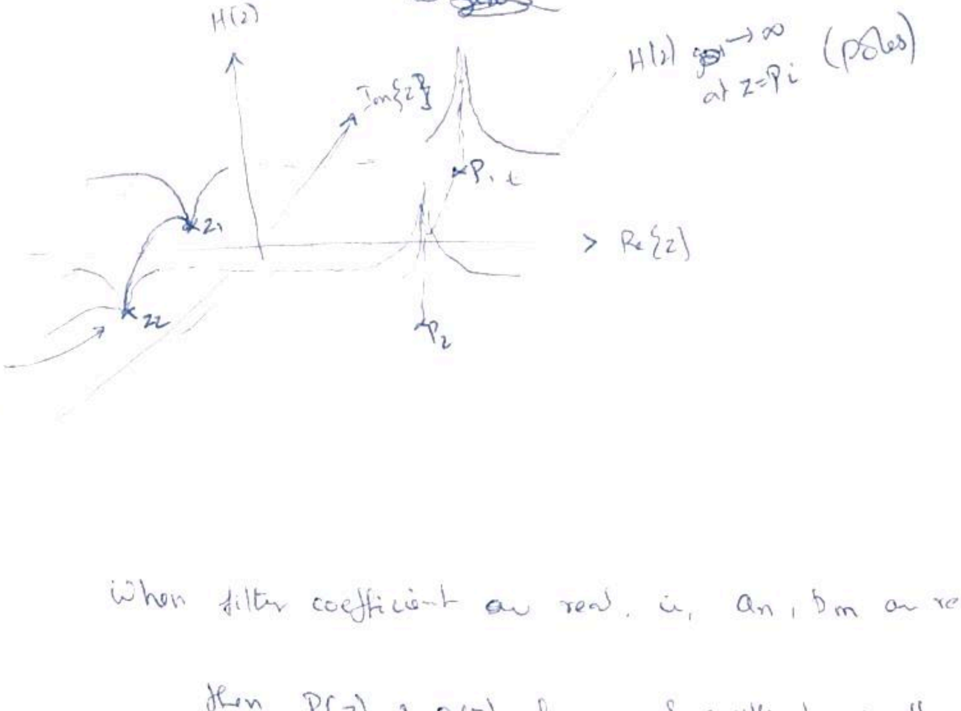
roots of $P(z)$ ← z_i → zeros of ~~f~~filter

roots of $Q(z)$ ← p_i - poles of filter.

~~When the filter has real coefficient~~

~~ie, a_n, b_m → Real $\forall n, m$~~

~~then~~



When filter coefficient are real. ie, a_n, b_m are real $\forall n, m$.

then $P(z)$ & $Q(z)$ have real coefficient & all roots are either real or complex conjugate pairs.

$$\therefore H(z) = A \prod_{k=1}^{M_{real}} \frac{z - z_k}{z} \prod_{k=1}^{M_{conj}} \frac{(z-z_k)(z-z_k^*)}{z^2} \prod_{k=1}^{N_{real}} \frac{z}{(z-p_k)} \prod_{k=1}^{N_{conj}} \frac{z^2}{(z-p_k)(z-p_k^*)}$$

$M = M_{real} + 2M_{conj}$ $\qquad$ $N = N_{real} + 2N_{conj}$.

No. of real roots of $P(z)$ $\qquad$ No. of complex conjugate root pairs of $P(z)$

ie, any LTI filter can be decomposed into of cascade of 1^{st} & 2^{nd} order filter.

$$H_1(z) = \frac{z - z_k}{z} = 1 - z_k z^{-1} \Rightarrow h_1[n] = \begin{cases} 1 & , n=0 \\ -z_k & , n=1 \\ 0 & , \text{Otherwise} \end{cases}$$

← FIR filter
All-Zero filter

$$y[n] = x[n] - z_k x[n-1]$$

← Non-recursive filter.

$$H_2(z) = \frac{z}{z - p_k} = \frac{1}{1 - p_k z^{-1}} = 1 + p_k z^{-1} + p_k^2 z^{-2} + p_k^3 z^{-3} + \cdots$$

$$\Rightarrow h_2[n] = p_k^n u[n]$$

← IIR filter.
All pole Filter

$$y[n] = x[n] - p_k y[n-1]$$

← Recursive filter.

$$H_3(z) = \frac{(z - z_k)(z - z_k^*)}{z^2}$$

$z_k \to$ complex valued, with as $z_k = r_k e^{j\theta_k}$.

$$= \frac{(z - r_k e^{j\theta_k})(z - r_k e^{j\theta_k})}{z_2} = \frac{z^2 - 2z r_k \left(\frac{e^{j\theta_k} + e^{-j\theta_k}}{2}\right) + r_k^2}{2}$$

$$= \frac{z^2 - 2z r_k \cos\theta_k + r_k^2}{z^2}$$

$$= 1 - (2 r_k \cos\theta_k) z^{-1} + r_k^2 z^{-2}$$

← Non-recursive

$$\Rightarrow h_3[n] = \begin{cases} 1 & , n=0 \\ -2r_k \cos\theta_k & , n=1 \\ r_k^2 & , n=0 \\ 0 & , \text{Otherwise} \end{cases}$$

← FIR filter.

All-Zero filter

$$H_4(z) = \frac{z^2}{(z-p_k)(z-p_k^*)} = \frac{z^2}{z^2 - 2z\rho_k\cos\omega_k + \rho_k^2} \quad \text{where } p_k = \rho_k e^{j\omega_k}.$$

$$= \frac{1}{1-(\rho_k e^{j\omega_k})z^{-1}} \cdot \frac{1}{1-(\rho_k e^{-j\omega_k})z^{-1}}$$

All pole filter

$$\Rightarrow h_4[n] = \left((\rho_k e^{j\omega_k})^n u[n]\right) * \left((\rho_k e^{-j\omega_k})^n u[n]\right)$$

$$= \underbrace{\left(\rho_k^n e^{jn\omega_k} u[n]\right)}_{p_1[n]} * \underbrace{\left(\rho_k^n e^{-jn\omega_k} u[n]\right)}_{p_2[n]}$$

$$= \sum_m p_1[m]p_2[n-m] \overset{u[m]u[n-m]}{=} \sum_{m=0}^{n} p_1[m]\, p_2[n-m], \quad n>0.$$

$$= \sum_{m=0}^{n} \rho_k^m e^{jm\omega_k} \rho_k^{n-m} e^{-j(n-m)\omega_k}, \quad n>0.$$

$$= \rho_k^n e^{-jn\omega_k} \sum_{m=0}^{n} e^{2jm\omega_k} \quad u[n]$$

$$= \rho_k^n e^{-jn\omega_k} \frac{1-e^{j2\omega_k(n+1)}}{1-e^{j2\omega_k}} u[n].$$

$$= \rho_k^n \frac{e^{-j(n+1)\omega_k} - e^{j(n+1)\omega_k}}{e^{-j\omega_k} - e^{j\omega_k}} u[n]$$

$$= \rho_k^n \frac{\sin(n+1)\omega_k}{\sin\omega_k} u[n] \quad \longleftarrow \text{IIR filter.}$$

$$= \frac{1}{\sin\omega_k}\left(\underbrace{\rho_k^n \sin(\omega_k(n+1))}_{\text{exponentially growing or decay sinusoid.}}\right) u[n]$$

Scaling factor dependent on ω_k

Stability of LTI filter

FIR filters are always stable

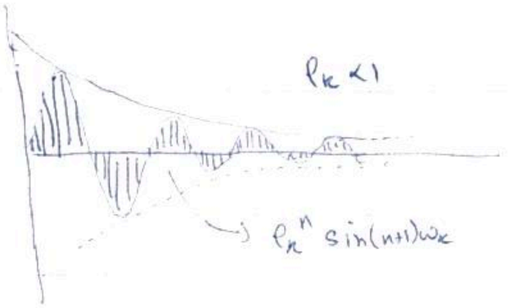
$H_1(z) = 1 - z_k z^{-1}$ & $H_2(z) = \frac{(z-z_k)(z-z_k^*)}{z^2}$ are always stable.

$H_3(z) = \frac{z}{(z-p_k)}$, $h_3[n] = p_k^n u[n]$ $p_k \to$ real

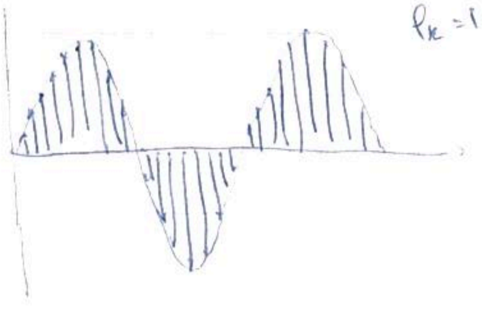
grows exponentially if $|p_k| > 1$ $\to$ Unstable.

decays exponentially if $|p_k| < 1$ $\to$ Stable

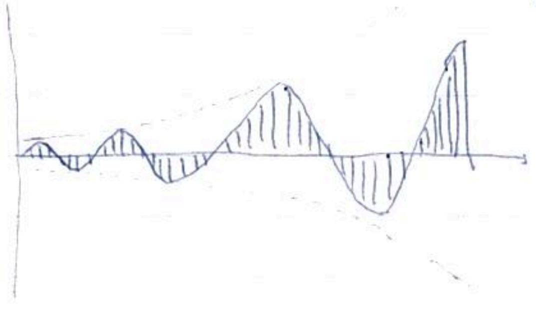
$H_4(z) = \frac{z^2}{(z-p_k)(z-p_k^*)}$; $h_4[n] = \frac{1}{\sin\omega_k} \rho_k^n \sin[(n+1)\omega_k] u[n]$



Stable



Quasi-Stable.



Unstable.

Stability Triangle of 2nd order syst (all-pole)

$$H(z) = \frac{z^2}{z^2 + b_1 z + b_2} = \frac{1}{1 + b_1 z^{-1} + b_2 z^{-2}}$$

$$\text{Poles, } P_1, P_2 = \frac{-b_1}{2} \pm \frac{\sqrt{b_1^2 - 4b_2}}{2}.$$

Syst stable when $|P_1| < 1$ & $|P_2| < 1$

$$\Rightarrow \left| \frac{-b_1}{2} \pm \frac{\sqrt{b_1^2 - 4b_2}}{2} \right| < 1$$

P_1 & P_2 are complex conjugates when $b_1 < 4b_2$.

$$\Rightarrow \left| \frac{-b_1 \pm j\sqrt{4b_2 - b_1^2}}{2} \right| < 1$$

$$\sqrt{\left(\frac{-b_1}{2}\right)^2 + \left(\frac{\sqrt{4b_2 - b_1^2}}{2}\right)^2} < 1.$$

$$\Rightarrow \frac{b_1^2 + (4b_2 - b_1^2)}{4} < 1$$

$$\Rightarrow b_2 < 1$$

If P_1 & P_2 are real, $b_1 > 4b_2$. then

$$-1 < \frac{-b_1}{2} \pm \frac{\sqrt{b_1^2 - 4b_2}}{2} < 1$$

$$\Rightarrow -2 + b_1 < \pm\sqrt{b_1^2 - 4b_2} < 2 + b_1$$

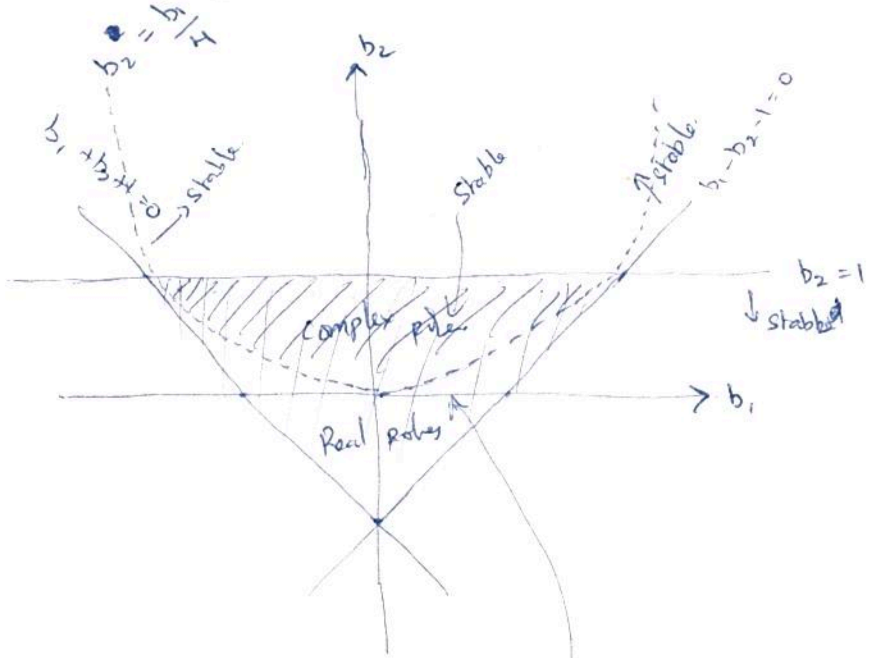
$(-2+b_1) < -\sqrt{b_1^2 - 4b_2}$ and $\sqrt{b_1^2 - 4b_2} < 2 + b_1$

$[-(b_1 - 2)]^2 > b_1^2 - 4b_2$ $\qquad$ $b_1^2 - 4b_2 < (b_1 + 2)^2$

$b_1^2 - 4b_1 + 4 > b_1^2 - 4b_2$ $\qquad$ $b_1^2 - 4b_2 < b_1^2 + 4b_1 + 4$

$b_2 - b_1 - 1 < 0$ $\qquad$ $b_1 + b_2 + 1 > 0.$

(6)



~~2nd System stable~~ 2nd order filter stable if poles lie inside the triangle.

~~Frequency~~

Every LTI system $H(z)$ can be decomposed into a cascade of 1st & 2nd order ~~FIR~~ all zero & ~~IIR~~ all pole filter.

→ If all IIR segments are stable then $H(z)$ is stable

↳ If any one of them is unstable then $H(z)$ is unstable

~~In Afterword~~

Since 1st order all-pole filter is stable if $|p_k| < 1$ & 2nd order all pole filter is stable if $\rho_k < 1$ where $\rho_k = |p_k|$

Any LTI filter is stable if $|p_k| < 1$, $\forall k$

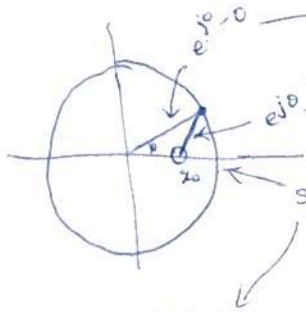
→ All its poles are inside the unit circle on the z-plane.

Magnitude ~~Frequency~~ Response of LTI filter → pole-zero perspective

$$H(z) = \frac{z - z_0}{z}$$

$$\hat{h}(\theta) = \frac{e^{j\theta} - z_0}{e^{j\theta} - 0}$$

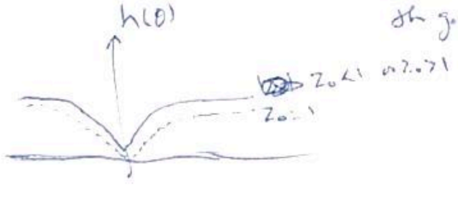
$$|\hat{h}(\theta)| = \frac{|e^{j\theta} - z_0|}{|e^{j\theta} - 0|}$$



Ratio is ~~magn~~ gain at θ, $|\hat{h}(\theta)|$

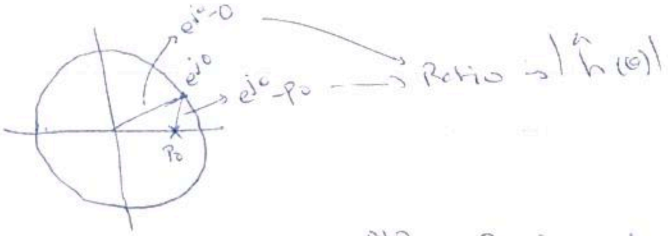
Smallest gain at $\theta = 0$ if $z_0 > 0$, $\theta = \pi$ if $z_0 < 0$

~~If z_0~~ Closer the zero is to the unit circle, the smaller the gain

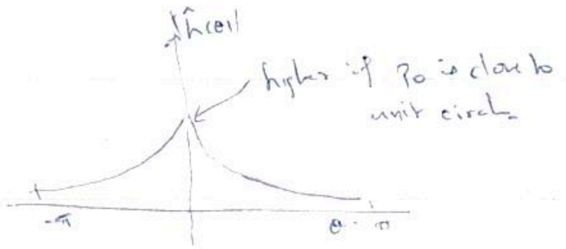


Similarly

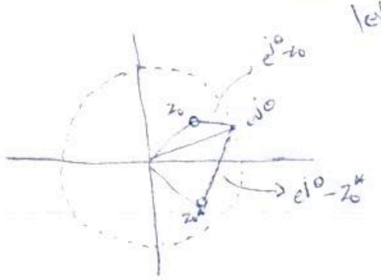
when, $H(z) = \dfrac{z}{z-p_0}$, $|\hat{h}(\theta)| = \dfrac{|e^{j\theta}-0|}{|e^{j\theta}-p_0|}$

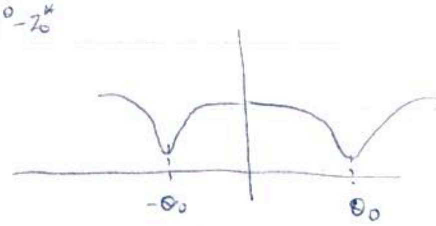


Noting p_0 is real & $|p_0|<1$ for stable syst.



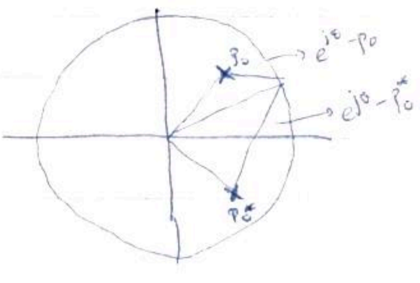
when, $H(z) = \dfrac{(z-z_0)(z-z_0^*)}{z^2}$, $|\hat{h}(\theta)| = \dfrac{|e^{j\theta}-z_0||e^{j\theta}-z_0^*|}{|e^{j\theta}-0|^2}$

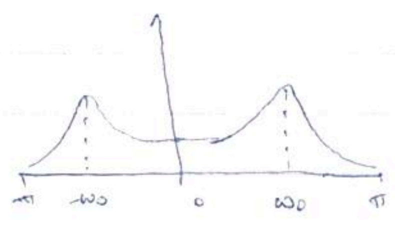




where $z_0 = r_0 e^{j\theta_0}$

2 when $H(z) = \dfrac{z^2}{(z-p_0)(z-p_0^*)}$, $|\hat{h}(\theta)| = \dfrac{|e^{j\theta}-0|^2}{|e^{j\theta}-p_0|\,|e^{j\theta}-p_0^*|}$



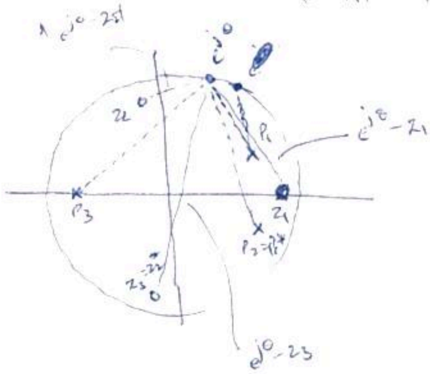


where $p_0 = P_0 e^{j\omega_0}$

In general,

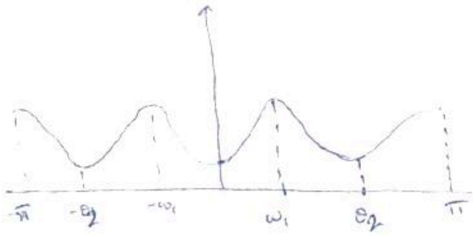
For any LTI filter, $H(z) = \dfrac{A\,(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$.

$$|\hat{h}(\theta)| = A\,\frac{|z-z_1|\,|z-z_2|\cdots|z-z_M|}{|z-p_1|\,|z-p_2|\cdots|z-p_N|}$$



$$|\hat{h}(\theta)| = \frac{|e^{j\theta}-z_1|\,|e^{j\theta}-z_2|\,|e^{j\theta}-z_2^*|}{|e^{j\theta}-p_1|\,|e^{j\theta}-p_1^*|\,|e^{j\theta}-p_3|}\,A$$

Not known from pole zero positions



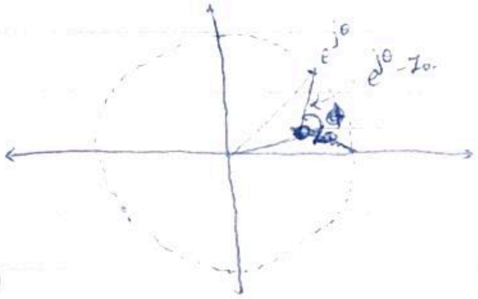
$p_1 = r_1 e^{j\omega_1}$

$z_2 = r_2 e^{j\theta_2}$

Phase Response of LTI filter - Pole-zero Perspective

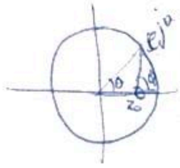
Single zero LTI filter, $H(z) = \frac{z - z_0}{z}$

any complex or real



$$\angle \hat{h}(\theta) = \underbrace{\angle(e^{j\theta} - z_0)}_{\phi} - \underbrace{\angle(e^{j\theta} - 0)}_{\theta}$$

when z_0 is real

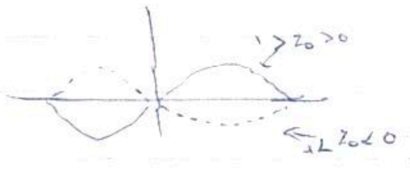


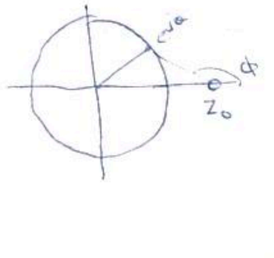
at $\theta = 0$, $\phi = 0$

at $\theta = \pi$, $\phi = \pi$

~~at all other~~ otherwise, $\theta \neq \phi$ in general.

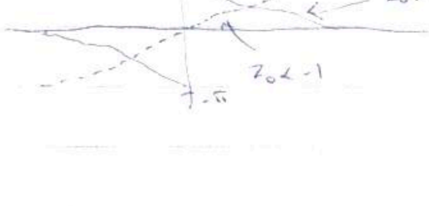
$$\therefore \angle \hat{h}(\theta) = \phi - \theta$$





at $\theta = 0$, $\phi = \pi$

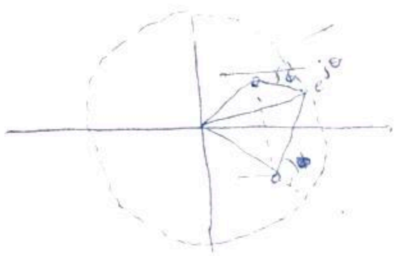
at $\theta = \pi$, $\phi = \pi$



Complex conjugate zeros.

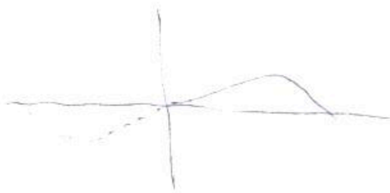
$$H(z) = \frac{(z-z_0)(z-z_0^*)}{z^2}$$

$$\angle \hat{h}(\theta) = \angle e^{j\theta}-z_0 + \angle e^{j\theta}-z_0^* - 2\,\angle e^{j\theta}-0.$$



at $\theta = 0$, $\phi = -\phi^*_{*}$ o.

at $\theta = \pi$, $\phi = -\phi^*_{*}$



In general

for any all-zero LTI filter (FIR filter).

$$H(z) = \frac{(z-z_1)(z-z_2)\cdots(z-z_M)}{z^M}$$

if $|z_i| < 1 \ \forall i$ ⟶ All zeros inside unit circle

~~then phase at $\theta = 0$, $\angle \hat{h}(\theta)$.~~

then $\angle \hat{h}(0) = 0$

$\angle \hat{h}(\pi) = 0$.

Referred to as minimum phase system.

↳ will have minimum group delay.

If $|z_i| > 1 \ \forall i$ ⟶ all zeros outside unit circle ⟶ maximum phase.

Some zeros inside & some outside unit circle ⟶ mixed phase.

Consider any FIR filter

$$H(z) = (z-z_1)(z-z_2)\cdots(z-z_m)$$

Then $H(z^{-1})$ has zeros $z_1^{-1}, z_2^{-1}, \cdots, z_m^{-1}$

$$H(z) \longrightarrow |\hat{h}(\theta)|$$

$$H(z^{-1}) \longrightarrow |\hat{h}(-\theta)| = |\hat{h}^*(\theta)| = |\hat{h}(\theta)|$$

If $H(z)$ is minimum phase then $H(z^{-1})$ is maximum phase.

~~but have~~

Identical magnitude response within a scaling factor.

In general, by replacing a zero of a system by its reciprocal you preserve magnitude response but change phase response.

All-Pass Filter

→ Filter with 'unity' magnitude response, but non-zero phase response.

Trivial all-pass filter $H(z) = z^{-k}$.

'Unity' magnitude response:

$$|\hat{h}(\theta)| = 1, \quad \forall\theta$$

$$\Rightarrow |\hat{h}(\theta)|^2 = 1, \quad \forall\theta$$

$$\Rightarrow \hat{h}(\theta) \cdot \hat{h}^*(\theta) = 1$$

$$\Rightarrow \hat{h}(\theta)\,\hat{h}(-\theta) = 1$$

$$\hat{h}(\theta) \longrightarrow H(z)$$
$$\hat{h}(-\theta) \longrightarrow H(z^{-1}).$$

$$\therefore H(z)\,H(z^{-1}) = 1 \quad \text{for all-pass filter.}$$

$$\text{a: } H(z) = \frac{P(z)}{Q(z)}$$

$$\frac{P(z)}{Q(z)} \cdot \frac{P(z^{-1})}{Q(z^{-1})} = 1$$

$$\Rightarrow P(z) = Q(z^{-1})$$
$$\text{or } Q(z) = P(z^{-1}).$$

~~If H(z)~~

$$\Rightarrow H(z) = 1/H(z^{-1})$$

If p_k is a pole → $1/p_k$ should be a zero.

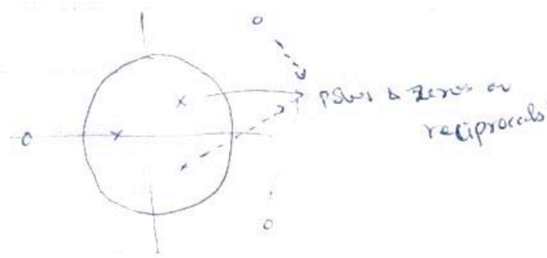
$$\therefore H(z) = \frac{A(z)}{A(z^{-1})}\, z^{-k}$$

$$\text{If } A(z) = \sum_{k=0}^{M} a_k z^{-k}$$

$$A(z^{-1}) = \sum_{k=0}^{M} a_k z^{k} = \sum_{k=0}^{M} a_k z^{-(M-k)} \cdot z^{M}$$

$$= z^{M} \sum_{l=0}^{M} a_{M-l}\, z^{-l}$$

$$\therefore H(z) = \frac{a_0 + a_1 z^{-1} + \dots a_M z^{-M}}{a_M + a_{M-1} z^{-1} + \dots a_0 z^{-M}} \cdot z^{-M} \quad \text{for all-pass filter.}$$

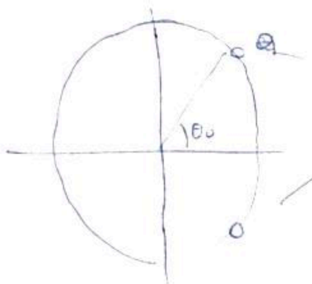


All-Pass Filter

Note: Any LTI filt can be decomposed into combin of all-pass filt & minimum phase filt

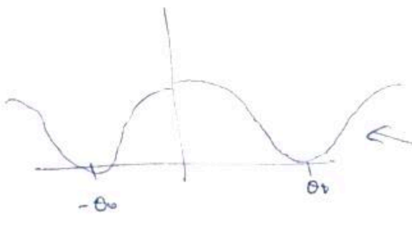
Notch Filter

→ Filter designed to selectively remove one frequency.



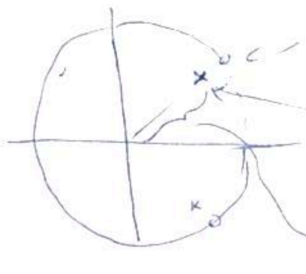
→ Place zeros (complex conjugate) on the unit circle at the frequency to be removed

but this leads to attenuation near θ_0.



Not selective enough
↓
not a notch.

Placing a pole at $\pm\theta_0$, near unit circle leads to high gain at $\pm\theta_0$.



← Place zero on unit circle at θ_0

Place pole near unit circle at θ_0 to get better selectivity.

3dB width.

$$r \approx 1 - \left(\frac{\Delta f}{f_s}\right)\pi$$ ← good for $0.9 \le r < 1$

$$H(z) = K\frac{(z - e^{j\theta_0})(z + e^{j\theta_0})}{(z - re^{j\theta_0})(z + re^{j\theta_0})}$$

Adjust to have unit passband gain.
→ i.e. $|H(0)| = 1$ at $\theta = 0$.

Oscillators → what if pole is on unit circle?

We know, a 2-pole (all-pole) filter.

$$H(z) = \frac{z^2}{(z-p_0)(z-p_0^*)} \quad \text{with } p_0 = \rho_0 e^{j\omega_0}$$

$$= \frac{z^2}{z^2 - 2z\rho_0\cos\omega_0 + \rho_0^2} \quad \& \quad h[n] = \frac{1}{\sin\omega_0}\rho_0^n \sin[(n+1)\omega_0]$$

When $\rho_0 = 1$, the output, $y[n]$ to an impulse input $x[n] = \delta[n]$ is

$$y[n] = \frac{1}{\sin\omega_0} \sin[(n+1)\omega_0]\, u[n]$$

↑ Constant scaling factor (A).

$$= \frac{u[n]}{A} \sin(n\omega_0 + \omega_0)$$

frequency

$$= \frac{u[n]}{A} \cos\left[n\omega_0 + \omega_0 - \frac{\pi}{2}\right]$$

Amplitude → $1/A$ phase

or when $x[n] = A\delta[n] = \sin\omega_0\, \delta[n]$

$$y[n] = u[n]\cos\left[n\omega_0 + (\omega_0 - \pi/2)\right]$$

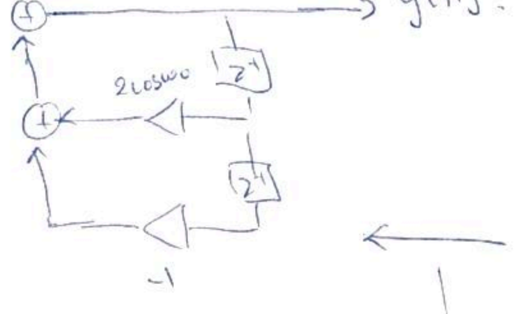
$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - 2z\cos\omega_0 + z^{-2}}$$

$$\Rightarrow y[n] = x[n] + 2\cos\omega_0\, y[n-1] - y[n-2]$$

at $n=0$, $y[n-1]=0$, $y[n-2]=0$ $x[n] = A = \sin\omega_0$.

$\therefore$ Oscillator is

$$y[n] = \underbrace{(2\cos\omega_0)}_{\text{scalar}}\, y[n-1] - y[n-2] \quad \text{with } y[0] = \underbrace{\sin\omega_0}_{\text{constant}}.$$

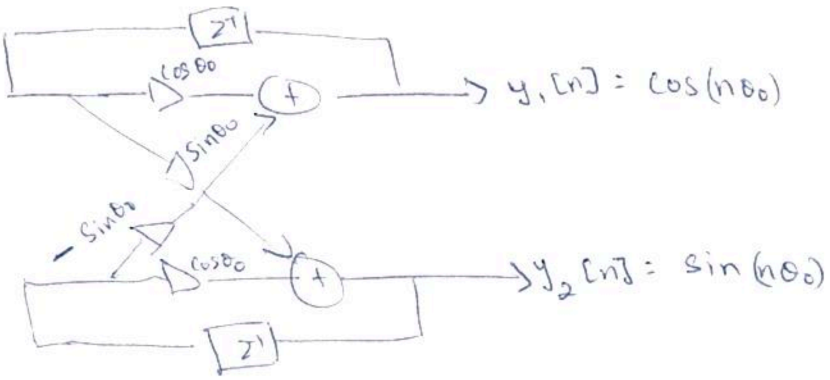


Cheaper than direct computation of $\cos(\overbrace{\omega_0 n + \phi_0}^{\text{wrap or}})$

trig, multiply

Cheaper than look up table.

Sine & Cosine Oscillator



How?

Consider,

$$\underbrace{\cos[(n+1)\theta_0]}_{y_1[n+1]} = \underbrace{\cos n\theta_0}_{y_1[n]} \cos\theta_0 - \underbrace{\sin n\theta_0}_{y_2(n)} \sin\theta_0 .$$

$$\therefore\ y_1[n] = y_1[n-1]\cos\theta_0 - \sin\theta_0\, y_2[n]$$

Similarly,

$$\underbrace{\sin\left[(n+1)\theta_0\right]}_{y_2[n+1]} = \sin\theta_0 \underbrace{\cos(n\theta_0)}_{y_1[n]} + \cos\theta_0 \underbrace{\sin(n\theta_0)}_{y_2[n]}$$

$$\therefore y_2[n] = \sin\theta_0\, y_1[n-1] + \cos\theta_0\, y_2[n-1].$$

putting the two difference equation together we get the structure.