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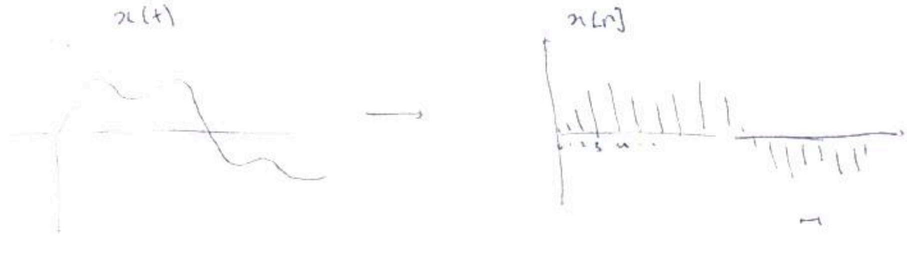
~~Nyquist Sa~~ Discrete Time Signals

~~Nyquist Sampling Theorem~~



$x(t)$ $\qquad$ $x[n]$ $\qquad$ Sample period - T_s

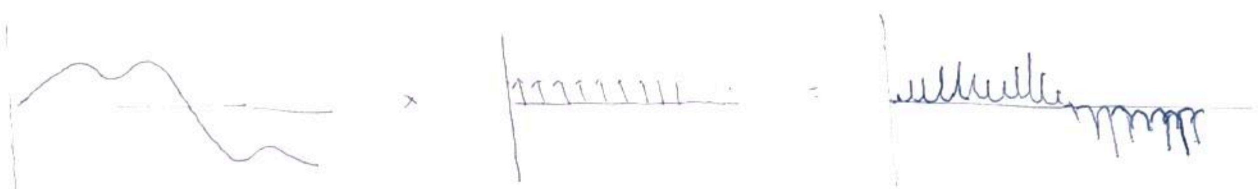
Equivalent if frequency components are preserved.

Sampling Frequency $F_s = 1/T_s$.

How do you determine the frequency components of a discrete signal?

↳ What is the discrete time equivalent of FT?

Consider



$$x(t) \times \psi(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) = x_d(t)$$

Dirac delta.

$$x[n] \stackrel{def}{=} x(nT)$$

$x[n] \neq x_d(t)$ ⟶ Fourier Transform can be applied.

Sequence of numbers (discrete signal)

Continuous signal (non-zero values only at nT)

$$\cancel{X_d(\omega) = \mathcal{F}\{x_d(t)\} = \mathcal{F}\{\sum x[n]\delta(t-nT)\}}$$

Notes about Dirac-delta:

1. $f(x)\,\delta(x-a) = f(a)\cdot\delta(x-a)$.
2. $\int_{-\infty}^{\infty} f(x)\,\delta(x-a) = f(a)$.
3. $f(x) * \delta(x-a) = f(x-a)$.
4. $\delta(ax) = \frac{1}{a}\,\delta(x)$

$$\cancel{X_d(\omega)}\quad x_d(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) = \sum_{n=-\infty}^{\infty} \underbrace{x(t)}_{\text{function.}}\,\delta(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} \underbrace{x[n]}_{\text{number.}}\,\delta(t-nT)$$

$$X_d(\omega) = \mathcal{F}\{x_d(t)\} = \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} x[n]\,\delta(t-nT)\right\}$$

$$= \sum_{n=-\infty}^{\infty} \underbrace{x[n]}_{\text{Scalar w.r.t. } \mathcal{F}}\,\mathcal{F}\{\delta(t-nT)\}$$

$$= \sum_{n=-\infty}^{\infty} x[n]\,e^{-j\omega nT}$$

We define this as Discrete Time Fourier Transform ($\theta = \omega T$)

i.e. $$\boxed{\hat{x}(\theta) = \sum_{n=-\infty}^{\infty} x[n]\,e^{-j\theta n}}$$

DTFT Analysis equation.

Also defines magnitude & phase spectra.

But, ~~φ~~

$$x_d(\omega) = \mathcal{F}\{x_d(t)\} = \mathcal{F}\{x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT)\} \rightarrow \text{also!}$$

$$= \frac{1}{2\pi}\mathcal{F}\{x(t)\} * \mathcal{F}\{\sum_{n=-\infty}^{\infty}\delta(t-nT)\}$$

$$\Rightarrow x_d(\omega) = \frac{1}{2\pi}X(\omega) * \mathcal{F}\{\psi(t)\}$$

← Leads to Relationship between FT & DTFT spectra.

$$\Psi(\omega) = \mathcal{F}\{\psi(t)\} = \mathcal{F}\{\sum_{n=-\infty}^{\infty}\delta(t-nT)\}$$

but $\psi(t) = \sum_{n=-\infty}^{\infty}\delta(t-nT)$

← Periodic & can be written as Fourier Series.

ie, $\psi(t) = \sum_{m=-\infty}^{\infty} C_m e^{jm\omega_0 t}$ where $\omega_0 = 2\pi/T$

$$C_m = \frac{1}{T}\int_{-T/2}^{T/2}\psi(t)\,e^{-jn\omega_0 t}\,dt = \frac{1}{T}\int_{-T/2}^{T/2}\delta(t)\,e^{-jn\omega_0 t}\,dt.$$

$$= 1/T$$

$$\Rightarrow \psi(t) = \frac{1}{T}\sum_{m=-\infty}^{\infty} e^{jm\frac{2\pi}{T}t}$$

$$\Rightarrow \Psi(\omega) = \mathcal{F}\{\psi(t)\} = \mathcal{F}\{\frac{1}{T}\sum_{m=-\infty}^{\infty} e^{jm\frac{2\pi}{T}t}\}$$

$$= \frac{1}{T}\sum_{m=-\infty}^{\infty}\mathcal{F}\{e^{jm\frac{2\pi}{T}t}\} = \frac{2\pi}{T}\sum_{m=-\infty}^{\infty}\delta(\omega - m\frac{2\pi}{T}$$

$$\therefore \quad x_d(\omega) = \frac{1}{2\pi} X(\omega) * \mathcal{F}\{\psi(t)\}$$

$$= \cancel{\frac{1}{2\pi}} X(\omega) * \frac{\cancel{2\pi}}{T} \sum_{m=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi m}{T}\right)$$

$$= \frac{2\pi}{T} \sum_{m=-\infty}^{\infty} X(\omega) * \delta\left(\omega - \frac{2\pi m}{T}\right)$$

$$= \frac{2\pi}{T} \sum_{m=-\infty}^{\infty} X(\omega) * \delta\left(\frac{1}{T}(\omega T - 2\pi m)\right)$$

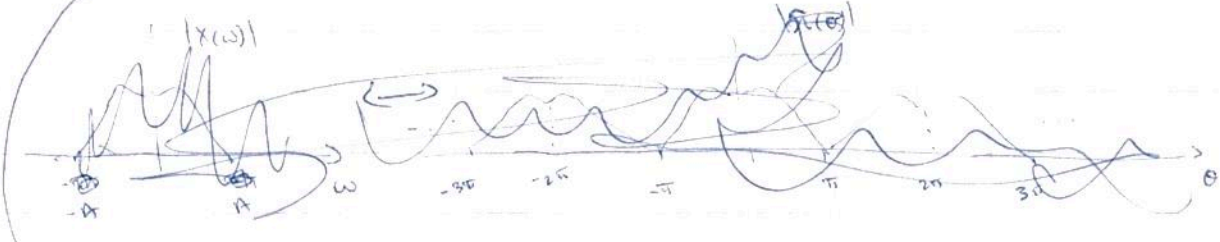
$$= \frac{\cancel{2\pi}}{\cancel{T}} \sum_{m=-\infty}^{\infty} X(\omega) * \cancel{T}\delta(\omega T - 2\pi m)$$

$$\int_{-\infty}^{\infty} x(t) e^{j\theta t} \, dt = \int_{-\infty}^{\infty} x(t) e^{j\frac{\theta}{T}t} \, dt = X(\theta).$$

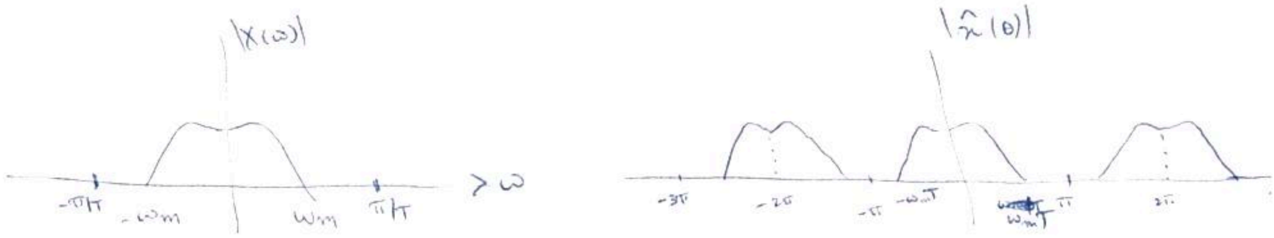
$$\Rightarrow \quad \hat{x}(\theta) = \cancel{2\pi} \sum_{m=-\infty}^{\infty} X(\theta) * \delta(\theta - 2\pi m)$$

$$= \cancel{2\pi} \sum_{m=-\infty}^{\infty} X(\theta - 2\pi m)$$

DTFT spectrum is set of repetition of the continuous time Fourier transform spectrum



Repeats with period 2π, $\therefore$ any interval of length 2π is adequate to describe the entire signal. Typically this interval is $[-\pi, \pi]$



This $x[n]$ has all the informat in $x(t)$ as long as T is small enough such that $\omega_m T < \pi$

ie $\frac{2\pi f_m}{f_s} < \pi$

$$\Rightarrow \boxed{f_s > 2f_m}$$

↳ Nyquist Theorem!

Consider the DTFT,

$$\hat{x}(\theta) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\theta n}$$

↳ This is a Fourier Series with θ as the independent variable (instead of 't')

↳ $c_n \Leftrightarrow x[n]$

$\underset{\text{fundamental period}}{\underline{\omega_0} t} \Leftrightarrow \theta$ → what's the fundamental period? ↓ we know its 2π

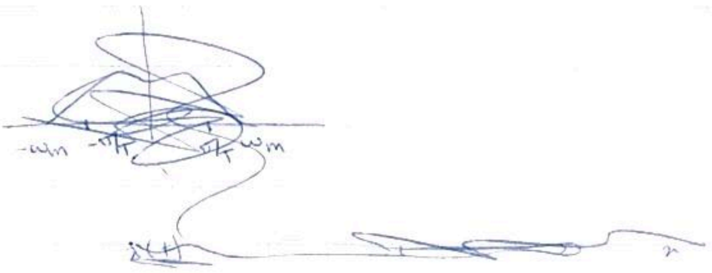
$$\therefore\ x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{x}(\theta) e^{j\theta n} d\theta$$

→ Inverse DTFT

(AKA synthesis equation).

→ Discuss properties of DTFT

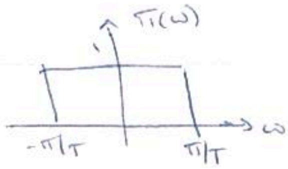
~~What happens when $f_s < 2f_m$?~~



$$
\begin{tikzcd}
x(t) \arrow[r, leftrightarrow, "FT"] \arrow[d, "\text{Sampling}"'] & X(\omega) \arrow[d, leftrightarrow, "\hat{x}(\theta) = \sum_{k=-\infty}^{\infty} X(\theta + 2\pi k)"] \\
x[n] \arrow[r, leftrightarrow, "DTFT"] \arrow[u, bend left=60, "\text{Interpolation}"] & \hat{x}(\theta) \arrow[u, bend right=60, "\text{Multiply by rectangle}"']
\end{tikzcd}
$$

$\to \hat{x}(\theta)$

$$X(\omega) = X_d(\omega)\, \Pi(\omega)$$



$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} X_d(\omega)\, e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\pi/T}^{\pi/T} \left(\sum_{n=-\infty}^{\infty} x[n]\, e^{-j\omega n T}\right) e^{j\omega t}\, d\omega$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} x[n] \int_{-\pi/T}^{\pi/T} e^{j\omega(t-nT)}\, d\omega$$

$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} x[n]\left(\frac{e^{j\frac{\pi}{T}(t-nT)} - e^{-j\frac{\pi}{T}(t-nT)}}{j(t-nT)}\right)$$

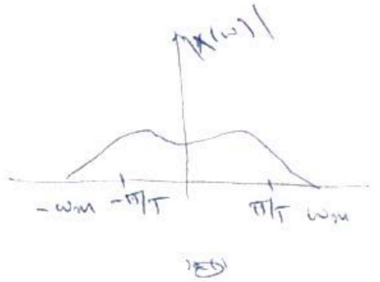
$$= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} x[n]\ \frac{2\pi}{T}\ \mathrm{sinc}\left(\frac{\pi}{T}(t-nT)\right)$$

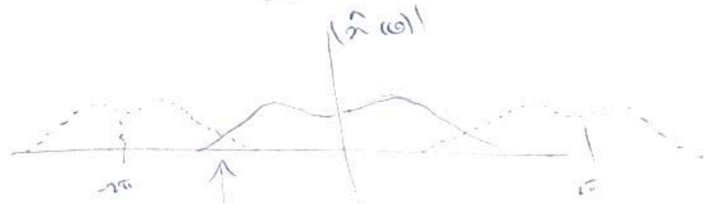
$$x(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} x[n]\ \mathrm{sinc}\left(\frac{\pi}{T}(t-nT)\right)$$

Sinc Interpolation.

What happens if $f_s < 2f_{max}$?

$x(t) \xrightarrow{\text{Sample}} x[n]$





Overlap causes distortion

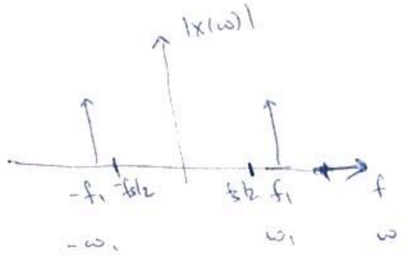
Cannot recover original signal

Aliasing!

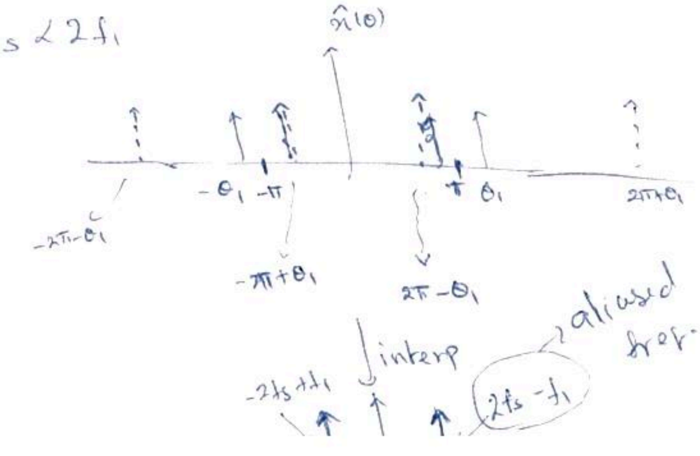
Why is it called Aliasing?

Consider $x(t) = \cos(2\pi f_1 t)$.

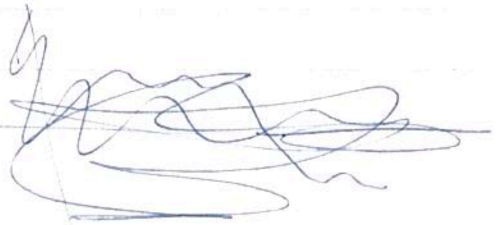
if $f_s < 2f_1$

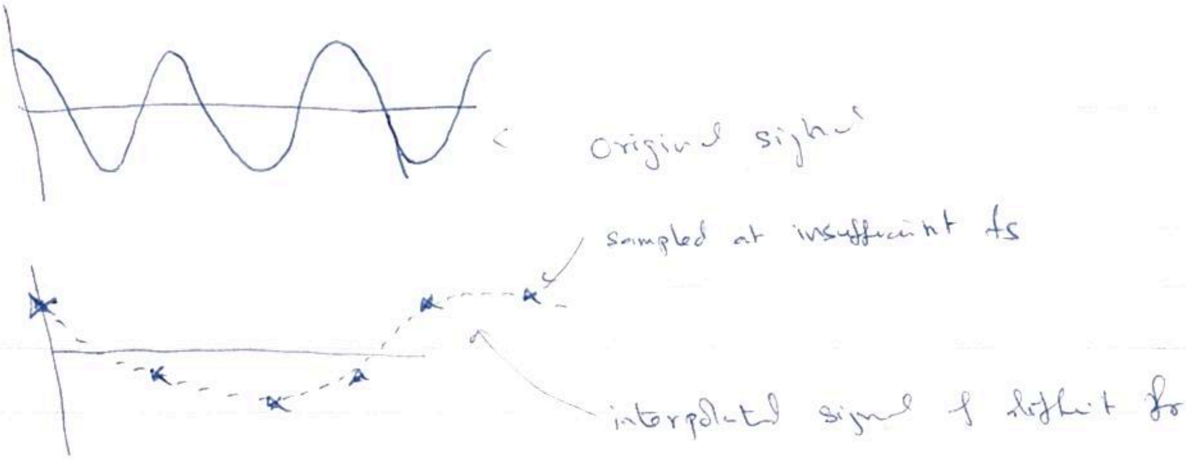


Sample



Time domain story of Aliasing



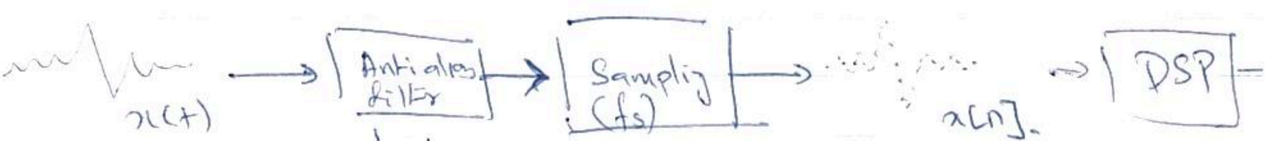


Dealing with Aliasing in DSP systems

↳ converting analogue to digital

→ Choose fs high enough.

→ Use anti-aliasing filter



Analogue filter
Low Pass filter
cut-off freq < fs/2