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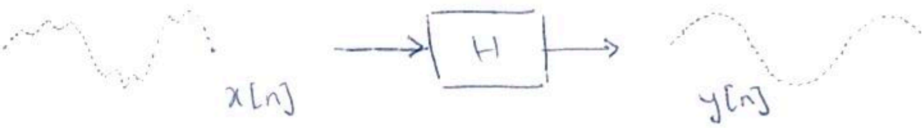
Discrete-Time Systems



input signal
(sequence of numbers)

output signal
(sequence of numbers)

$$y[n] = H\{x[n]\}$$

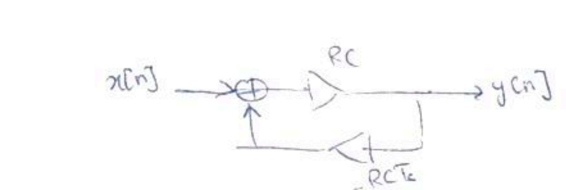
← Discrete-time system.

~~at any y n=k.~~

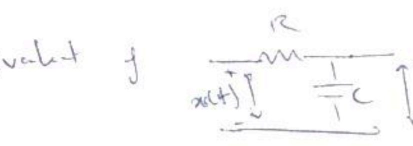
~~y[n] can depend only on~~

Must be implemented as some combination of multiplication & addition (only).

For e.g.,



is the DSP equivalent of



i.e., $y[n] = RC\,x[n] + e^{-RCT_s}\,y[n-1]$.

at any $n=k$

$y[k]$ depends on $x[k]$ & $y[k-1]$ → Dynamic system (system with memory).

On the other hand, $y[n] = a_1 x[n] + a_2 (x[n])^3$ → Static / Memoryless system.

→ Also, <u>Causal</u> → i.e., $y[k]$ only depends on $x[k]$ & past values.

$y[n] = a_1 x[n] + a_2 x[n-1]$ is also causal

$y[n] = a_1 x[n-1] + a_2 x[n+1]$ is non causal → common in

Linear system

If $y_1[n] = H\{x_1[n]\}$
& $y_2[n] = H\{x_2[n]\}$

H is linear iff $H\{a x_1[n] + b x_2[n]\} = a y_1[n] + b y_2[n]$

Time-Invariant / Shift-Invariant system

Let $x_k = x[n-k]$ denote $x[n]$ delayed by k samples.

If $y[n] = H\{x[n]\}$

H is Time-invariant iff $H\{x_k\} = y_k$
↳ $y[n-k]$.

ie, if input is delayed by k ~~n0~~ then output is also delayed by same amount 'k'.

Implication → System behaviour does not change with time.
~~↳ ie, does the same thing~~

$H\{x[n]\} = y[n] = \cos \omega_0 n \cdot x[n]$ → not time-invariant
(time variant)

since $\underbrace{H\{x[n-k]\}}_{H\{x_k\}} = \cos \omega_0 n \cdot x[n-k] \neq y[n-k]$
↳ $\cos \omega_0 (n-k) \cdot x[n-k]$.

We will predominantly focus on LTI system
↳ Linear & Time Invariant

- Ease of analysis → Lots of established maths theory.
- Covers a lot of what we might want to do!

Stability

$$x[n] \longrightarrow \boxed{H} \longrightarrow y[n] = H\{x[n]\}$$

Bounded if $\max_n |x[n]| < A$

↳ some finite value

If $y[n]$ is always bounded ~~if~~ given any $x[n]$ that is bounded then H is BIBO stable.

eg. $y[n] = n\,y[n-1] + x[n]$ → unstable

$y[n] = 0.5\,y[n-1] + x[n]$ → stable

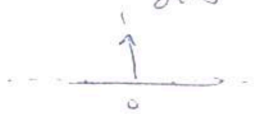
~~come back to~~ How to determine stability?

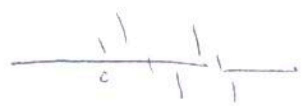
↳ Easy for LTI system

↳ come back after poles & zeros

LTI Systems

$$\underset{\text{impulse}}{\delta(n)} \longrightarrow \boxed{H} \longrightarrow h[n] \leftarrow \text{impulse response.}$$





$$h[n] = H\{\delta[n]\}$$

↑

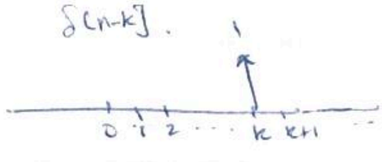
* This completely describes the LTI system → you can determine the output given any input, $x[n]$ if you know $h[n]$.

Any input, $x[n]$ can be written as

$$x[n] = \dots + x_{-2}\delta[n+2] + x_{-1}\delta[n+1] + x_0\delta[n] + x_1\delta[n-1] + x_2\delta[n-2] + \dots$$

signal / function $\delta[n-k] \longrightarrow$ impulse delayed by k samples

$\delta[n-k]$.



Value $x_k \rightarrow$ value of $x[n]$ at $n=k$, i.e., $x[k]$.

$$\therefore\ y[n] = H\{x[n]\} = H\{\dots + x_{-2}\delta[n+2] + x_{-1}\delta[n+1] + x_0\delta[n] + x_1\delta[n-1] +$$

Since $H\{\}$ is linear, $H\{x_1\delta[n-1] + x_2\delta[n-2]\} = x_1 H\{\delta[n-1]\} + x_2 H\{\delta[$

Since $H\{\}$ is Time invariant, $H\{\delta[n-k]\} = h[n-k]$, if $H\{\delta[n]\} = h[n$

$$\therefore\ y[n] = \dots + x_{-2}h[n+2] + x_{-1}h[n+1] + x_0 h[n] + x_1 h[n-1] + x_2 h[n-2] + \dots$$

i.e., $$y[n] = \sum_{k=-\infty}^{\infty} x_k h[n-k]$$

since x_k is the value of $x[n]$ at $n=k$, i.e. $x_k = x[k]$.

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]\,h[n-k]$$

This is convolution.
denoted as $x[n] * h[n]$

i.e. $y[n] = x[n] * h[n]$

~~Stability of~~ LTI systems in the freq. domain - What do they do?

$$x[n] \longrightarrow \underset{h[n]}{\boxed{H}} \longrightarrow y[n]$$

$$y[n] = x[n] * h[n]$$

$$\hat{y}(\theta) = \sum_{n=-\infty}^{\infty} y[n] e^{j\theta n} = \sum_{n=-\infty}^{\infty} \left(x[n] * h[n]\right) e^{j\theta n}$$

$$= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k] h[n-k] e^{j\theta n}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \underbrace{\sum_{n=-\infty}^{\infty} h[n-k] e^{j\theta n}}_{e^{j\theta k} \hat{h}(\theta)}$$

$$= \sum_{k=-\infty}^{\infty} x[k] e^{j\theta k} \hat{h}(\theta)$$

$$\Rightarrow \hat{y}(\theta) = \hat{x}(\theta) \hat{h}(\theta)$$

$\hat{h}(\theta)$ is referred to as the frequency response of the system.

$$\underset{\text{impulse response}}{h[n]} \overset{\text{DTFT}}{\longleftrightarrow} \underset{\text{freq. response}}{\hat{h}(\theta)}$$

~~Spectrum of output is~~ ~~Spectrum~~ Energy at each frequency (θ) of output is energy at θ of input times gain of θ of the system ($\hat{h}(\theta)$).

Every LTI system is a filter with a fixed gain at each frequency.

$$\Rightarrow |\hat{y}(\theta)| e^{j\angle\hat{y}(\theta)} = |\hat{x}(\theta)| |\hat{h}(\theta)| e^{j(\angle\hat{x}(\theta) + \angle\hat{h}(\theta))}$$

i.e. $|\hat{y}(\theta)| = |\hat{x}(\theta)| |\hat{h}(\theta)|$

$$\angle\hat{y}(\theta) = \angle\hat{x}(\theta) + \angle\hat{h}(\theta)$$

Magnitude Response

$$x[n] \rightarrow \boxed{H} \rightarrow y[n] \qquad \hat{y}(\theta) = \hat{x}(\theta)\,\hat{h}(\theta)$$

$\hat{x}(\theta)$ $h[n]$, $\hat{h}(\theta)$ $\hat{y}(\theta)$

$$|\hat{y}(\theta)| = |\hat{x}(\theta)|\,|\hat{h}(\theta)|$$

Energy in each freq in output

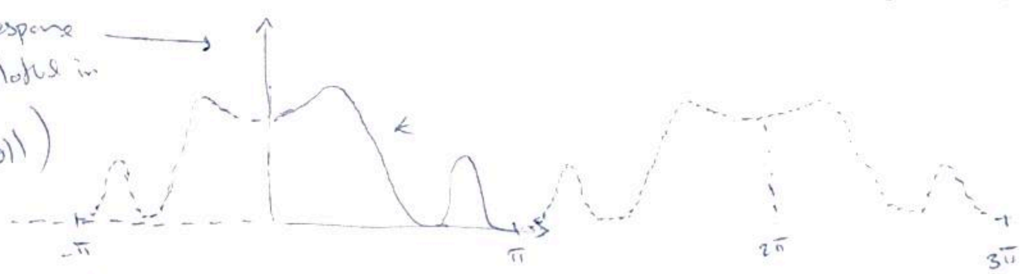
Energy in each freq in input

gain at each frequency

Mathematically identical only interpretation differs.

$|\hat{h}(\theta)|$

Magnitude response typically plotted in dB scale. $\left(20\log_{10}|\hat{h}(\theta)|\right)$



← symmetric, defn in $(-\pi,\pi]$ as long as $h[n]$ is real.

Phase Response & Group Delay

$$\angle\hat{y}(\theta) = \angle\hat{x}(\theta) + \angle\hat{h}(\theta) \rightarrow \text{what does this mean?}$$

Consider a single freq $-\theta_1$

if $x[n] = A_1\cos(\theta_1 n + \phi_1)$

$$\hat{x}(\theta) = \underbrace{\frac{A_1}{2}e^{j\phi_1}\delta(\theta-\theta_1)}_{\theta_1 \text{ freq component}} + \underbrace{\frac{A_1}{2}e^{-j\phi_1}\delta(\theta+\theta_1)}_{-\theta_1 \text{ freq component}}$$

at θ_1, $\hat{y}(\theta_1) = \hat{x}(\theta_1)\hat{h}(\theta_1) = \frac{\overbrace{A_1|\hat{h}(\theta_1)|}^{B_1}}{2}e^{j\phi_1}e^{j\angle\hat{h}(\theta)}\delta(\theta-\theta_1)$

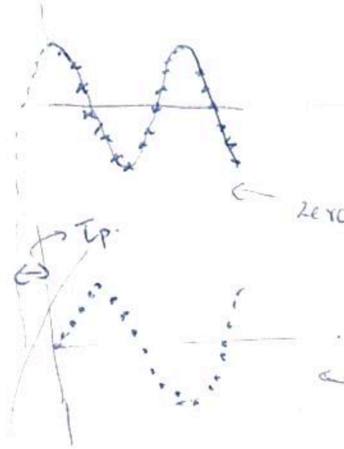
$$= \frac{B_1}{2}e^{j(\phi_1+\angle\hat{h}(\theta_1))}\delta(\theta-\theta_1)$$

similarly at $-\theta_1$ $y(-\theta_1) = \frac{B_1}{2}e^{-j(\phi_1+\angle\hat{h}(\theta_1))}\delta(\theta+\theta_1)$ since $|\hat{h}_1(\theta)| = |\hat{h}(-\theta)|$

$\hat{h}(\theta)$ $\hat{h}(-\theta)$

$$\therefore \hat{y}(\theta) = \frac{B_1}{2} e^{j(\phi_1 + \angle\hat{h}(\theta_1))}\,\delta(\theta-\theta_1) + \frac{B_1}{2} e^{-j(\phi_1 + \angle\hat{h}(\theta_1))}\,\delta(\theta+\theta_1)$$

$$\Rightarrow y[n] = B_1 \cos\Big(\theta_1 n + \phi_1 + \underbrace{\angle\hat{h}(\theta_1)}_{\text{due to phase response of } \hat{H}_2}\Big) = B_1 \cos\left(\left(n + \frac{\angle\hat{h}(\theta_1)}{\theta_1}\right)\theta_1 + \phi_1\right)$$



← zero phase $\cos(\theta_1 n)$

← $\cos(\theta_1 n + \pi/2)$. Delayed by $\pi/2$.

→ Phase delay $\tau_p = -\dfrac{\angle\hat{h}(\theta)}{\theta}$

Group delay measures the time delay experienced by a group of sinusoidal components (close to each other, as in AM)

$$\text{Group delay, } \tau_g = -\frac{d}{d\theta}\angle\hat{h}(\theta)$$

- Constant group delay indicates all components experience the same time delay.

  → shape of the envelope remains the same.

→ Group delay is the delay experienced by a modulated pulse.

Stability of LTI system - Based on impulse response

$$y[n] = x[n] * h[n]$$

$$x[n] \rightarrow \boxed{h[n]} \rightarrow y[n]$$

Bounded signal

Let $\max_n |x[n]| = A$

~~System is stable~~ If $|y[n]| = \left| \sum_{k=-\infty}^{\infty} x[k]h[n-k] \right|$

$$\leq \sum_{k=-\infty}^{\infty} |x[n]||h[n-k]|$$

$$\leq A \underbrace{\sum_{k=-\infty}^{\infty} |h[n-k]|}$$

As long as this term is finite, $y[n]$ is bounded

$\therefore$ System is BIBO stable iff

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty.$$

Impulse response of stable system must decay to zero otherwise this will grow to be infinite

should decay fast enough

otherwise sum will still not be bounded.