ELEC3104: Digital Signal Processing

Chapter 5: Filters

Only LTI systems of the following form can be implemented in practice:

$$y[n] = \sum_{k=-A}^{B} a_k x[n-k] + \sum_{m=1}^{C} b_m y[n-m]$$

can not use the future value to determine the current value

Where A, B and C are finite positive integers. i.e., the system can be implemented with a finite number of additions and multiplications. LTI systems that can be realised in practice are commonly referred to as LTI filters.

Note: For causal filter, $A = 0$ and the output only depends on present and past values of the input and past values of the output.

5.1 Block Diagram Representation

LTI filters can be represented as block diagrams that make use of three basic elements.

5.1.1 Adder

A system that performs the addition of two signal sequences to form another sequence, which we denote as $y[n]$.

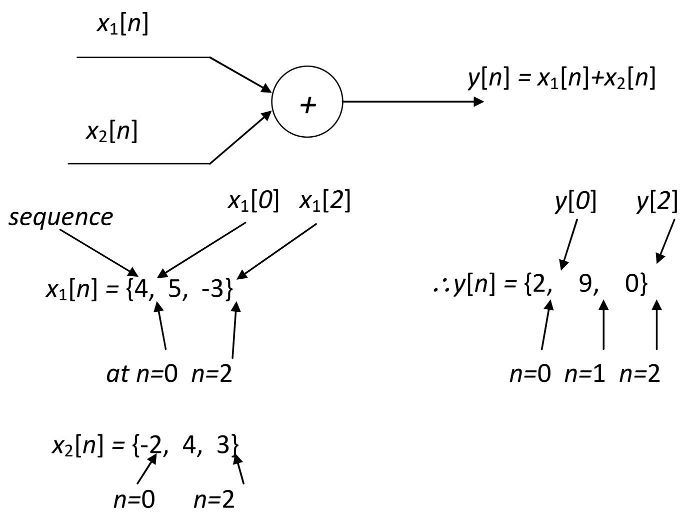


Figure 5.1: Block diagram representation of an adder, $x_1[n]$ and $x_2[n]$ denote discrete-time input signals and $y[n]$ denotes a discrete time output signal.

Note: It is not necessary to store either one of the sequences in order to perform the addition. In other words, the addition operation is **memoryless**.

5.1.2 Constant Multiplier

This operation simply represents applying a scale factor on the input *x[n]*. Note that this operation is also **memoryless**.

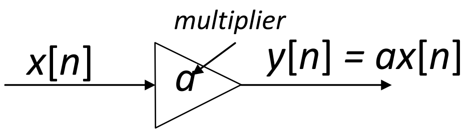
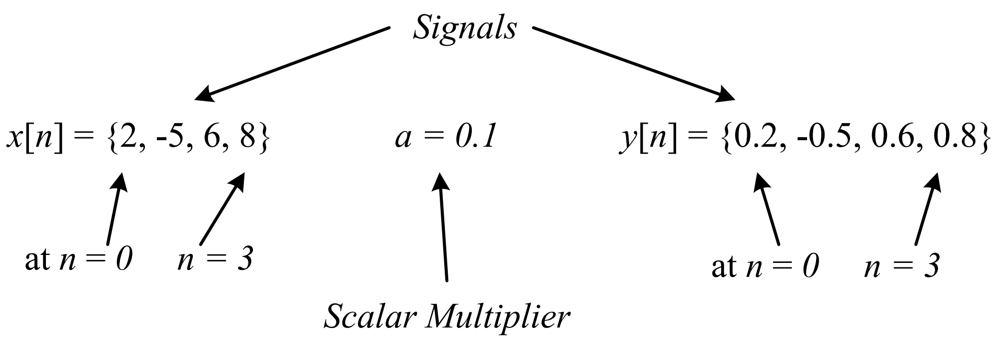


Figure 5.2: Block diagram representation of a multiplier. $x[n]$ and $y[n]$ denote discrete time input and output signals respectively. a denotes a scalar multiplier.

<u>Example 5.1</u>



5.1.3 Unit Delay Element

The unit delay is a special system that simply delays the signal passing through it by one sample.

If the input signal is $x[n]$, the output is $x[n-1]$. In fact, the sample $x[n-1]$ is stored in memory at time $n-1$ and it is recalled from memory at time n to form $y[n] = x[n-1]$.

Thus this basic building block requires memory. We use the symbol T or z^{-1} to denote the unit of delay.

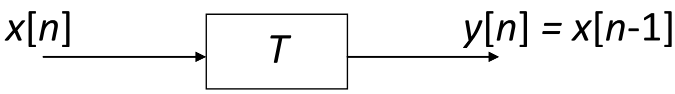
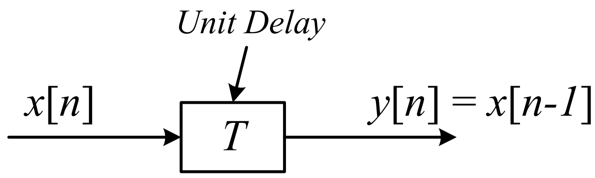


Figure 5.3: Block diagram representation of a unit delay. *T* denotes the sampling period.

Example 5.2



at $n = -1$ $\quad n = 0$ $\quad n = 1$

$$x[n] = \{0, 1, 0, 5, 7, -2, -1, 0\}$$

$$y[n] = \{0, 1, 0, 5, 7, -2, -1, 0\}$$

$y[0]$ $\quad y[1]$ $\quad y[2]$

$$\{y[n] = x[n-1];\ y[0] = x[0-1] = x[-1];\ y[1] = x[1-1] = x[0];\ \ldots\}$$

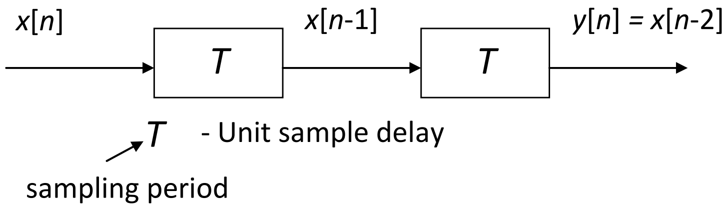
Note: The use of T, the sampling period to denote the unit delay is obvious since T is the interval between consecutive samples. The reason for using z^{-1} should be clear based on the z-transform of a delayed sequence. In general, z^{-1} is used much more often than T.

Note: Normally a combination of adders, multipliers and unit delays form a complex discrete-time system.

Note: A discrete-time system consisting of combinations of adders, multipliers and unit delays can always be described by a set of difference equations. The equations would be ordinary algebraic equations if no delays were present.

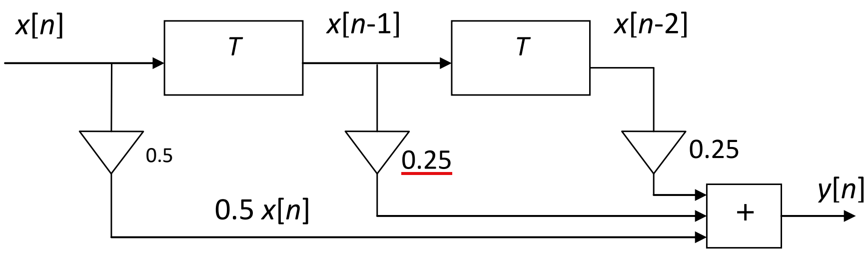
Example 5.3

$$y[n] = x[n-2]$$



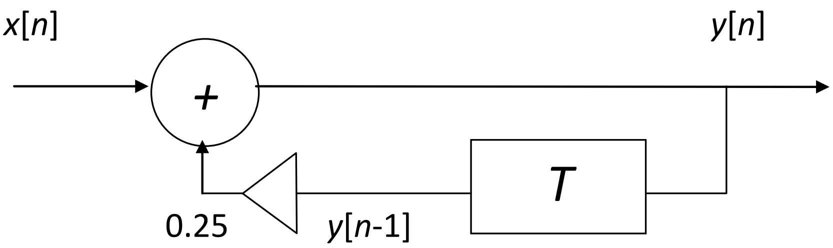
Example 5.4

$$y[n] = \frac{1}{2}x[n] + \frac{1}{4}x[n-1] + \frac{1}{4}x[n-2]$$



Example 5.5

$$y[n] = x[n] + 0.25y[n-1]$$



5.2 Non-Recursive Digital Filter

If $b_m = 0$, (for all m) then the calculation of $y[n]$ does not require the use of previously calculated samples of the output and filters of this type are referred to as non-recursive filters.

$$y[n] = \sum_{k=0}^{M} a_k x[n-k]$$
$$= a_0 x[n] + a_1 x[n-1] + \cdots + a_M x[n-M]$$

This is equivalent to the convolution sum

$$\boxed{y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]}$$

Where $h[k]$, the impulse response of the filter, is defined as

$$h[n] = \begin{cases} a_n, & 0 \leq n \leq M \\ 0, & otherwise \end{cases}$$

i.e., a non-recursive digital filter is a finite impulse response (FIR) filter.

Property:

An FIR filter is always stable. Finite Impulse Response

(a) Since $b_k = 0$, the impulse response has finite support, i.e., it is non-zero only at a finite number of samples. Hence the absolute sum of any impulse response is finite and consequently all FIR filters are stable.

Note: FIR (non-recursive) filters can be made with exactly linear phase characteristics. The ability to have an exactly linear phase response is one of the most important properties of a FIR-LTI system (filter).

Note: Only FIR filters can have linear phase characteristics.

5.3 Recursive Filters (IIR)

Every recursive digital filter must contain at least one closed loop. Each closed loop contains at least one delay element.

$$y[n] = \sum_{k=0}^{M} a_k x[n-k] - \sum_{k=1}^{L} b_k y[n-k]$$

For recursive filters $b_k \neq 0$, for some k.

Let $a_0 \neq 0$ and $a_k = 0, k \geq 1$ and also $b_1 \neq 0$ and $b_k = 0$, for all $k > 1$.

$$y[n] = a_0 x[n] - b_1 y[n-1]$$

$$H(z) = \frac{a_0}{1 + b_1 z^{-1}}$$

The corresponding impulse response is infinitely long.

Note: A recursive filter is an infinite impulse response (IIR) filter.

5.4 System Structures - Direct Form I and II

Consider a system given by the difference equation

$$y[n] = a_0 x[n] + a_1 x[n-1] - b_1 y[n-1]$$

This can be written as a set of two equations, separating the input and output signals as

$$v[n] = a_0 x[n] + a_1 x[n-1] - System\ 1$$

$$y[n] = v[n] - b_1 y[n-1] \quad - System\ 2$$

The simplest implementation of this is as follows

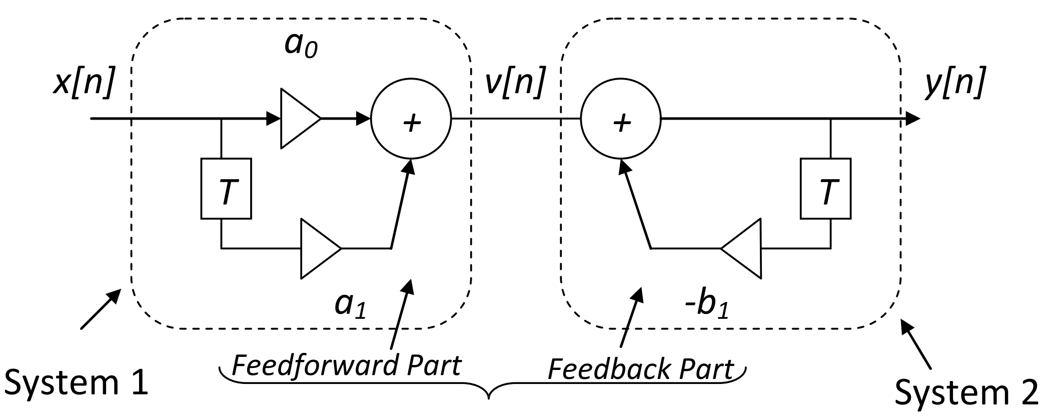


Figure 5.4: Direct Form I structure

Note: This is the direct form-I structure where the feedforward system involving only the inputs and the feedback system involving only delayed outputs are separated and implemented with the feedback system following the feedforward system.

Consider that without changing the input-output relationship, we can reverse the ordering of the two *linear* systems in the cascade representation. Prove this for a cascade of LTI systems as an **exercise**.

(Hint: Make use of the associative property of convolution).

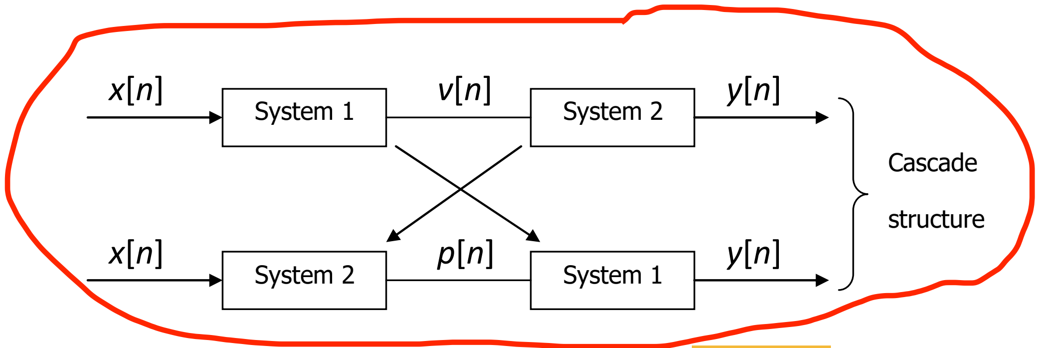


Figure 5.5: Two systems forming a cascade structure can be interchanged without affecting the final output signal.

Applying this to the above system gives the direct form-II structure.

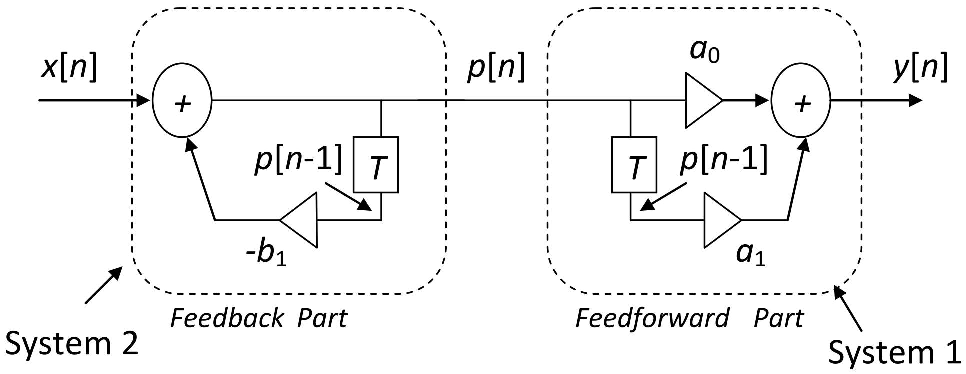


Figure 5.6: Direct Form-II structure

There is no need for two delay operations; they can be combined into a single delay as shown in Figure 5.7. Since delay operations are implemented with memory in a computer, the implementation of Figure 5.7 would require less memory compared to the implementation of Figure 5.6.

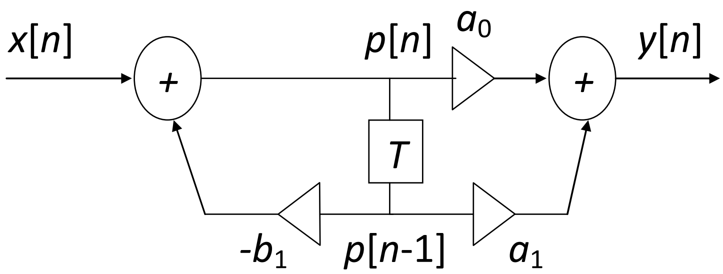


Figure 5.7: Canonical Form of Direct Form – II structure.

Both block diagrams Figure 5.4 and Figure 5.6/Figure 5.7 represent the same difference equation. Consider the Direct Form-II implementation in Figure 5.6 (or equivalently Figure 5.7)

$$p[n] = x[n] - b_1 p[n-1]$$

$$y[n] = a_0 p[n] + a_1 p[n-1]$$

Substituting $n \to n-1$ in the above expression,

$$y[n-1] = a_0 p[n-1] + a_1 p[n-2]$$

Multiplying by b_1 on both sides,

$$b_1 y[n-1] = a_0 b_1 p[n-1] + a_1 b_1 p[n-2]$$

Hence the sum $y[n] + b_1 y[n-1]$ becomes

$$\begin{aligned} y[n] + b_1 y[n-1] &= \underbrace{a_0 p[n] + a_1 p[n-1]}_{y[n]} + \underbrace{a_0 b_1 p[n-1] + a_1 b_1 p[n-2]}_{b_1 y[n-1]} \\ &= a_0 \underbrace{(p[n] + b_1 p[n-1])}_{x[n]} + a_1 \underbrace{(p[n-1] + b_1 p[n-2])}_{x[n-1]} \end{aligned}$$

Hence,

$$y[n] + b_1 y[n-1] = a_0 x[n] + a_1 x[n-1]$$

i.e.,

$$y[n] = a_0 x[n] + a_1 x[n-1] - b_1 y[n-1]$$

Which is the original difference equation (also representing Direct form-I structure in Figure 5.4).

5.5 Pole-Zero Description

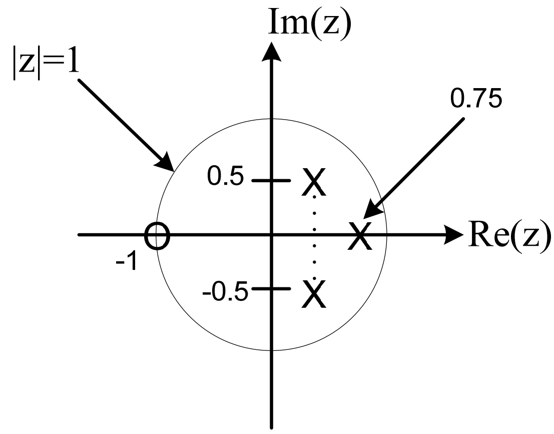
The **zeros** of a z-transform $H(z)$ are the values of z for which $H(z) = 0$. The **poles** of a z-transform are the values of z for which $H(z) = \infty$. If $H(z)$ is a rational function , then

$$H(z) = \frac{Y(z)}{X(z)} = \frac{a_0 + a_1 z^{-1} + \cdots + a_M z^{-M}}{1 + b_1 z^{-1} + \cdots + b_L z^{-L}}$$

$$= a_0 \frac{(z - z_1)(z - z_2) \ldots (z - z_M)}{(z - p_1)(z - p_2) \ldots (z - p_L)}$$

The complex quantities (or may be real) $z_1, z_2, z_3, \ldots$ are called zeros of $H(z)$ and the complex quantities (or may be real) $p_1, p_2, p_3, \ldots$ are called the poles of $H(z)$. We thus see that $H(z)$ is completely determined, except for the constant a_0, by the values of poles and zeros.

The information contained in the z-transform can be conveniently displayed as a pole-zero diagram (see figure below).



In the diagram, 'X' marks the position of a pole and 'O' denotes the position of a zero.

The poles are located at $z = 0.5 \pm 0.5j$ and $z = 0.75$, a single zero is at $z = -1$.

An important feature of the pole-zero diagram is the unit circle $|z| = 1$. The pole-zero diagram provides an insight into the properties of a given discrete-time system.

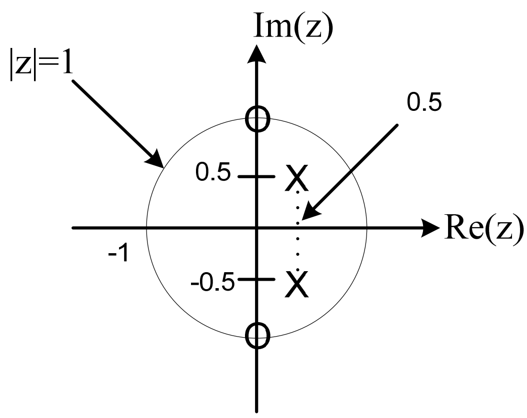
From the locations of the poles d zeros we can infer the frequency response of the system as well as its degree of stability.

For a stable system, all the poles must lie inside the unit circle. Zeros may lie inside, on, or outside the unit circle.

Note: It was earlier noted (when comparing the Laplace transform with the z-transform) that the left half of the s-plane is mapped onto the inside of the unit circle. For analogue systems to be stable, all the poles must lie on the left half of the s-plane. The digital equivalent is that all poles must be inside the unit circle.

Example 5.6

Determine the transfer function *H(z)* of a discrete-time system with the pole-zero diagram shown below:



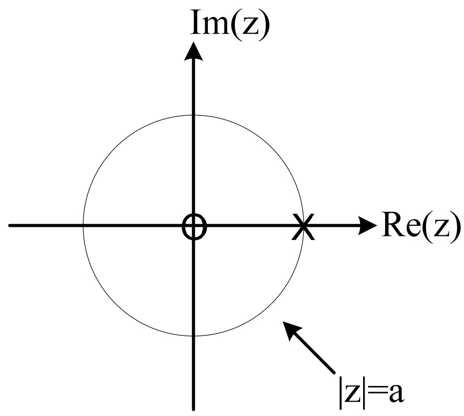
$$H(z) = \frac{K(z-1j)(z+1j)}{(z-0.5-0.5j)(z-0.5+0.5j)}$$

$$= \frac{K(1+z^{-2})}{1-z^{-1}-0.5z^{-2}}$$

Example 5.7

Determine the pole-zero plot of $H(z) = \frac{z}{z-a}$

zero: 0
pole: a

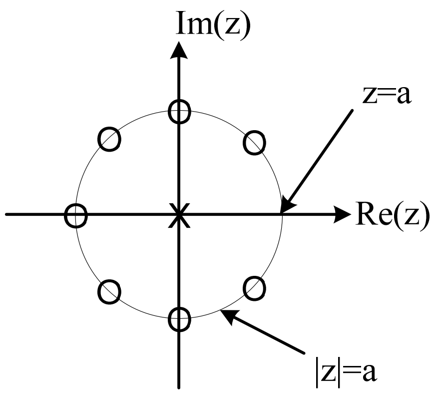


Example 5.8

Determine the pole-zero plot:

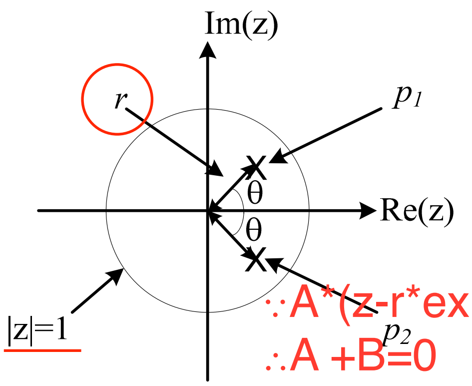
$$H(z) = \frac{z^M - a^M}{z^{M-1}(z-a)} = \frac{(z-z_1)(z-z_2)\ldots(z-z_M)}{z^{M-1}}$$

The zero $z = a$ cancels the pole at $z = a$. Thus, $H(z)$ has $M - 1$ zeros and $M - 1$ poles as shown in the diagram below for $M = 8$.



5.6 Second Order Systems

Consider a system, $H(z)$, with two complex conjugate poles in the z-plane:



Poles: $p_1 = re^{j\theta}$
$p_2 = re^{-j\theta}$

Zero: $z_1 = 0$

∵A*(z-r*exp(-j*theta))+B*(z-r*exp(j*theta))=1

∴A +B=0

B*r*(cos()-j*sin())+B*r*(cos()+j*sin())

A typical transfer function might be:

$$H(z) = \frac{z}{(z - re^{j\theta})(z - re^{-j\theta})} = z\left[\frac{A}{z - re^{j\theta}} + \frac{B}{z - re^{-j\theta}}\right]$$

$$= z\left[\frac{\frac{1}{2jr\sin\theta}}{z - re^{j\theta}} - \frac{\frac{1}{2jr\sin\theta}}{z - re^{-j\theta}}\right]$$

$$H(z) = \frac{1}{j(2r\sin\theta)}\left[\frac{1}{1 - re^{j\theta}z^{-1}} - \frac{1}{1 - re^{-j\theta}z^{-1}}\right]$$

Of the form z/(z-a),which is a
a geometric series in time domain

Taking the inverse z-transform,

h[n] = inverse z of H(z) ?

$$h[n] = \frac{1}{j2r\sin\theta}\left[\left(re^{j\theta}\right)^n - \left(re^{-j\theta}\right)^n\right]$$

$$= \frac{r^n}{j2r\sin\theta}\left[e^{j\theta n} - e^{-j\theta n}\right]$$

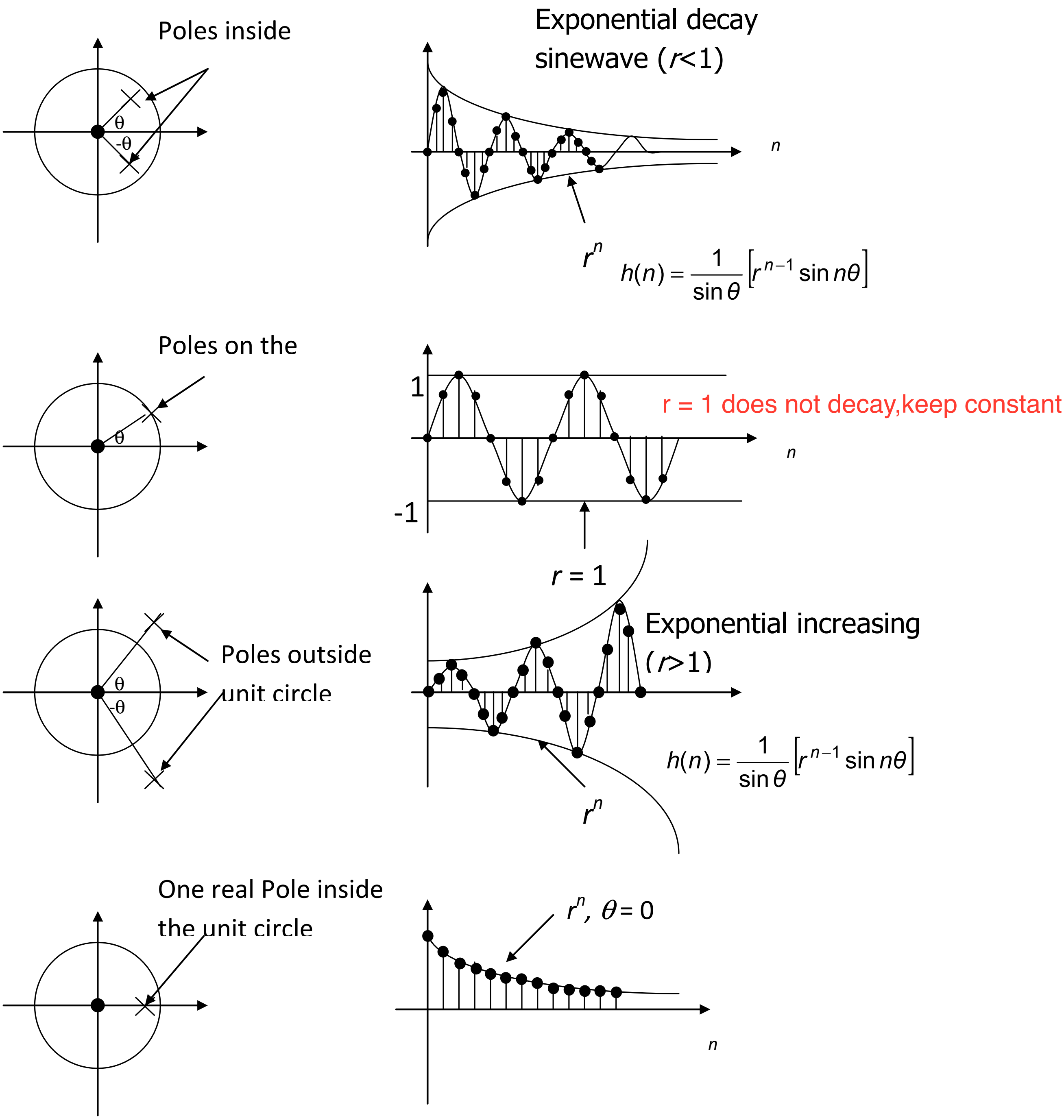
how come h[n] contains theta?

$$h[n] = \frac{1}{\sin\theta}\left[r^{n-1}\sin n\theta\right]$$

where, $h[n]$ is the impulse response of the 2nd order system with complex poles and θ is the frequency of oscillation of the impulse response.

We note that the impulse response will decay away to zero provided r is less than one. $(r < 1)$

Recall that r is also the distance from the origin in the z-plane to the poles p_1 or p_2, so that system will be stable if the poles in the z-plane lie inside the unit circle.

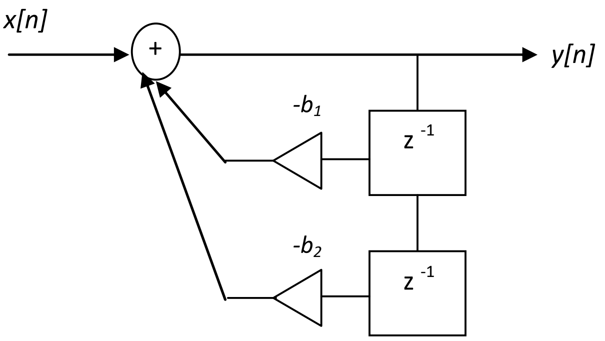


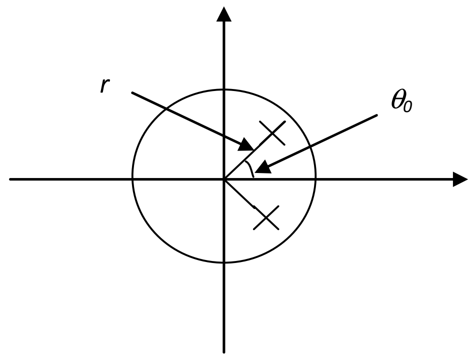
Poles inside
θ
-θ
Exponential decay
sinewave (r<1)
n
r^n
$h(n) = \frac{1}{\sin\theta}\left[r^{n-1}\sin n\theta\right]$
Poles on the
θ
1
-1
r = 1 does not decay,keep constant
n
$r = 1$
Poles outside
unit circle
θ
-θ
Exponential increasing
(r>1)
$h(n) = \frac{1}{\sin\theta}\left[r^{n-1}\sin n\theta\right]$
r^n
One real Pole inside
the unit circle
r^n, $\theta = 0$
n

Note: A system that is both stable and causal must have all its poles inside the unit circle on the z-plane.

Note: A pole outside the unit circle will contribute either a right sided increasing exponential term, which is not stable, or a left sided decaying exponential term that is not causal.

5.6.1 Second Order Resonant System (All-pole)





$$p_1 = re^{j\theta_0} = r\cos\theta_0 + jr\sin\theta_0$$
$$p_2 = re^{-j\theta_0} = r\cos\theta_0 - jr\sin\theta_0$$

if you only have two poles then they must be complex conjugate to each other

All pole systems have poles only (without counting the zeros at the origin). Any second order all pole system can be written in the form:

$$H(z) = \frac{1}{1 + b_1 z^{-1} + b_2 z^{-2}} = \frac{z^2}{z^2 + b_1 z + b_2}$$

If the two poles are given by p_1 and p_2, the system can be written as

$$H(z) = \frac{z^2}{(z - p_1)(z - p_2)} = \frac{z^2}{(z - re^{j\theta_0})(z - re^{-j\theta_0})}$$
$$H(z) = \frac{z^2}{z^2 - r(e^{j\theta_0} + e^{-j\theta_0})z + r^2} = \frac{z^2}{z^2 - (2r\cos\theta_0)z + r^2}$$

Comparing this to the general form of a second order all pole system given earlier, we obtain

$$b_1 = -2r\cos\theta_0\,, \qquad b_2 = r^2$$

$$\therefore \cos\theta_0 = -\frac{b_1}{2\sqrt{b_2}}$$

$$\theta_0 = \frac{2\pi f_0}{f_s}$$

Where, θ_0 is the resonant frequency. We can derive $H(\theta)$ and its magnitude from

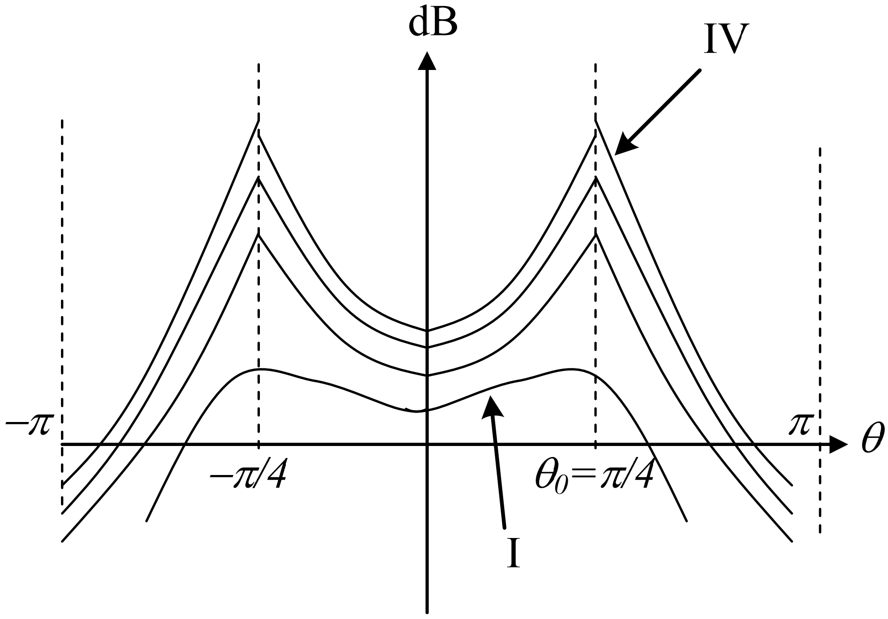
$$H(z) = \frac{1}{1 + b_1 z^{-1} + b_2 z^{-2}}$$

	b_1	$b_2 = r^2$
I	-0.94	0,5
II	-1.16	0.7
III	-1.34	0.9
IV	-1.41	0.99

$$b_2 = r^2$$

$$b_1 = -2r\cos(\theta_0)$$

$$\theta_0 = \cos^{-1}\left[\frac{-b_1}{2\sqrt{b_2}}\right]$$



?

Example 5.9

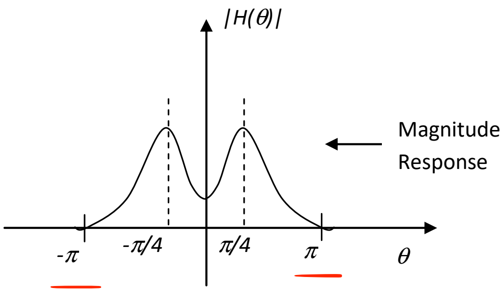
Sketch the magnitude response for the system having the transfer function

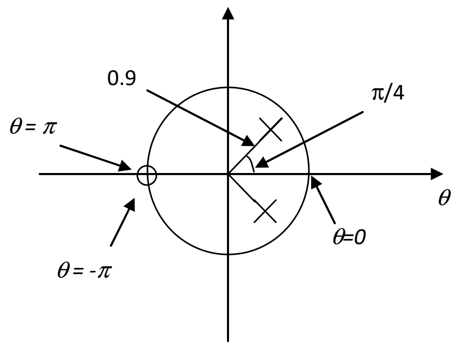
$$H(z) = \frac{1+z^{-1}}{\left(1-0.9e^{j\frac{\pi}{4}}z^{-1}\right)\left(1-0.9e^{-j\frac{\pi}{4}}z^{-1}\right)}$$

The system has a zero at $z = -1$ and poles at $z = 0.9e^{\pm j\frac{\pi}{4}}$.

z = 1*exp(j*pi)

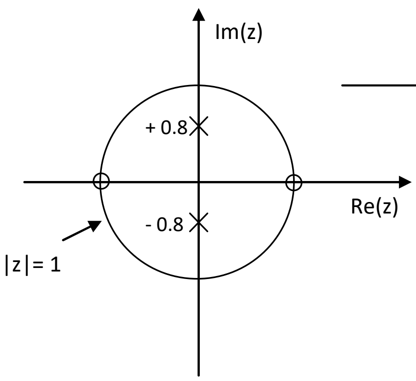
$\therefore$ Magnitude response will be zero at $\theta = \pi$ and large at $\theta_0 = \pm\frac{\pi}{4}$ because the poles are close to the unit circle.

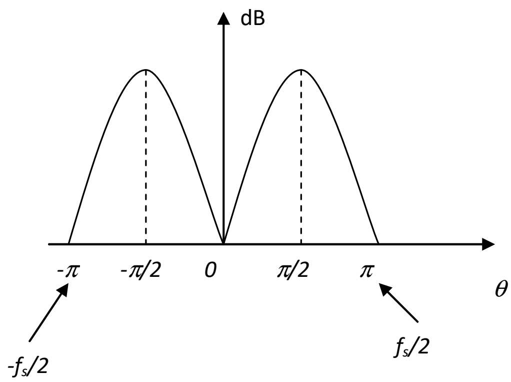




Example 5.10

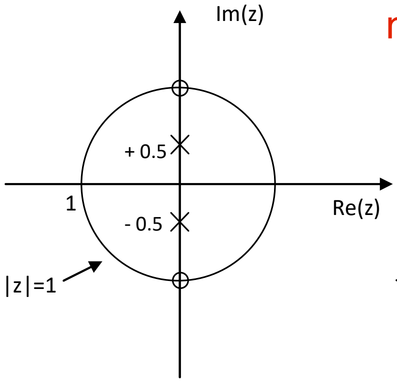
Sketch the approximate magnitude response from the pole-zero map given below:



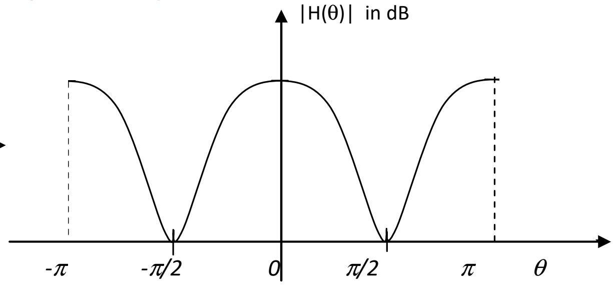


Example 5.11

Sketch the approximate magnitude response from the pole-zero map given below:



notch filter



5.6.2 Stability of a second-order system

Consider a two-pole resonant system given by

$$H(z) = \frac{z^2}{z^2 + b_1 z + b_2} = \frac{1}{1 + b_1 z^{-1} + b_2 z^{-2}}$$

This system has two zeros at the origin and poles at

$$p_1, p_2 = -\frac{b_1}{2} \pm \frac{\sqrt{b_1 - 4b_2}}{2}$$

For the system to be stable the poles must lie inside the unit circle. i.e., $|p_1| < 1$ and $|p_2| < 1$.

$$|p_1, p_2| < 1$$

$$\Rightarrow \left| -\frac{b_1}{2} \pm \frac{\sqrt{b_1 - 4b_2}}{2} \right| < 1$$

If the poles, p_1 and p_2 are complex, then $b_1 < 4b_2$. This gives,

$$\left| \frac{-b_1 \pm j\sqrt{4b_2 - b_1}}{2} \right| < 1$$

magnitude

$$\Rightarrow \sqrt{\left(-\frac{b_1}{2}\right)^2 + \left(\frac{\sqrt{4b_2 - b_1}}{2}\right)^2} < 1$$

$$\Rightarrow \frac{b_1^2 + (4b_2 - b_1)}{4} < 1$$

$$\boxed{\Rightarrow b_2 < 1}$$

If the poles, p_1 and p_2 are real, then $b_1 > 4b_2$. This gives

$$-1 < \frac{-b_1 \pm \sqrt{b_1 - 4b_2}}{2} < 1$$

$$\Rightarrow -2 + b_1 < \pm\sqrt{b_1 - 4b_2} < 2 + b_1$$

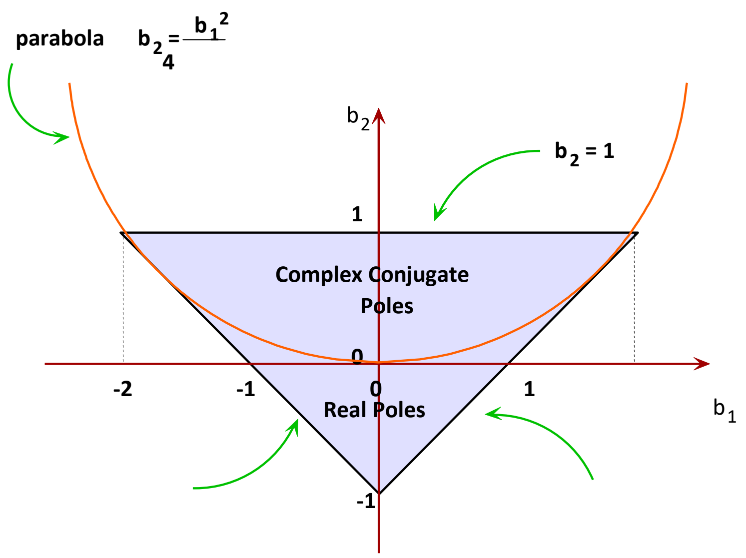
Therefore

$$-2 + b_1 < -\sqrt{b_1 - 4b_2} \text{ and } \sqrt{b_1 - 4b_2} < 2 + b_1$$

$$\text{i.e., } (-2 + b_1)^2 > b_1^2 - 4b_2 \text{ and } b_1^2 - 4b_2 < (2 + b_1)^2$$

$$\text{i.e., } \boxed{b_1 - b_2 - 1 < 0} \text{ and } \boxed{b_1 + b_2 + 1 > 0}$$

These three conditions define a triangle and $b_1^2 = 4b_2$ defines a parabola as shown below.



The system is only stable if and only if the point (b_1, b_2) lies inside the **stability triangle**.

5.6.3 Minimum, Maximum and Mixed Phase systems

Consider two systems:

$$H_1(z) = 1 + \frac{1}{2} z^{-1}$$ zero inside the unit circle

$$H_2(z) = \frac{1}{2} + z^{-1}$$ zero outside the unit circle

$H_2(z)$ is the reverse of the system $H_1(z)$. This is due to the reciprocal relationship between the zeros of $H_1(z)$ and $H_2(z)$.

$$H_1(\theta) = 1 + \frac{1}{2} e^{-j\theta} \ \& \ H_2(\theta) = \frac{1}{2} + e^{-j\theta}$$ $\rightarrow$ $|z| = 1$

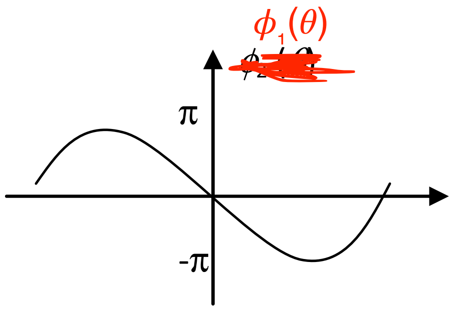
$$|H_1(\theta)| = |H_2(\theta)| = \sqrt{\frac{5}{4} + \cos\theta}$$

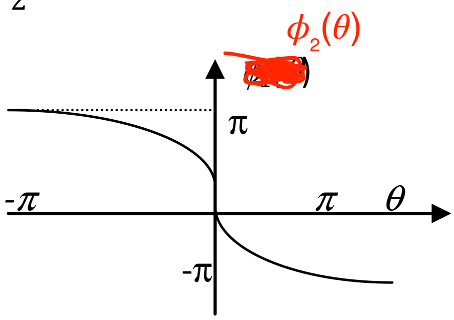
The magnitude characteristics of the two filters are identical because the roots of $H_1(z)$ and $H_2(z)$ are reciprocal.

The phase responses of the two systems are given by

$$\phi_1(\theta) = \tan^{-1} \frac{\sin\theta \times 0.5 \rightarrow y}{2 + \cos\theta \times 0.5 \rightarrow x}$$

$$\phi_2(\theta) = \tan^{-1} \frac{\sin\theta}{\frac{1}{2} + \cos\theta}$$





Note: Reflecting a zero from $z = \rho$ to $z = \frac{1}{\rho}$ (from inside the unit circle to outside or vice versa) does not change the magnitude response of the system but changes the phase response.

We observe that the phase response of the system with its zero inside the unit circle, $\phi_1(\theta)$, begins at zero phase at frequency $\theta = 0$ and terminates at zero phase at frequency $\theta = \pi$. Hence the net phase change is:

$$\phi_1(\pi) - \phi_1(0) = 0$$ bounded

On the other hand the net phase change for the system with the zero outside the unit circle is

$$\phi_2(\pi) - \phi_2(0) = -\pi$$ explode

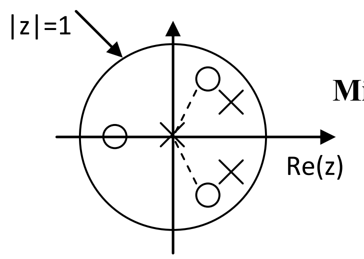
The first system, $H_1(z)$, is an example of a **minimum-phase** system and the second system, $H_2(z)$ is an example of a **maximum-phase** system.

If a filter with M zeros has some of its zeros inside the unit circle and the remaining outside the unit circle, it is called a **mixed-phase** system.

A minimum-phase property of FIR filter carries over to IIR filter. A system given by

$$H(z) = \frac{B(z)}{A(z)}$$

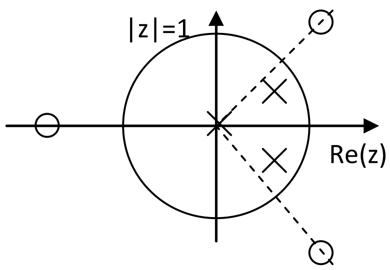
is called **minimum-phase** if all its poles and zeros are inside the unit circle.



Minimum phase

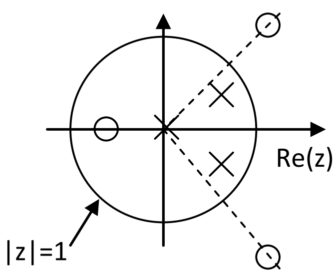
Magnitudes are the same

If all the zeros lie outside the unit circle, the system is called **maximum-phase**.



Maximum phase

If zeros lie both inside and outside the unit circle, the system is called **mixed-phase**.

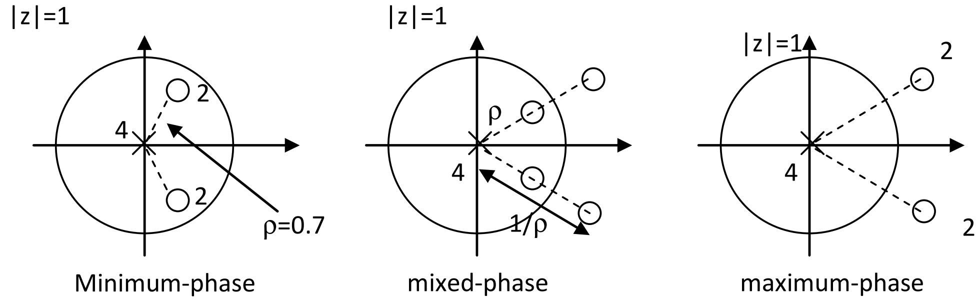


Mixed phase

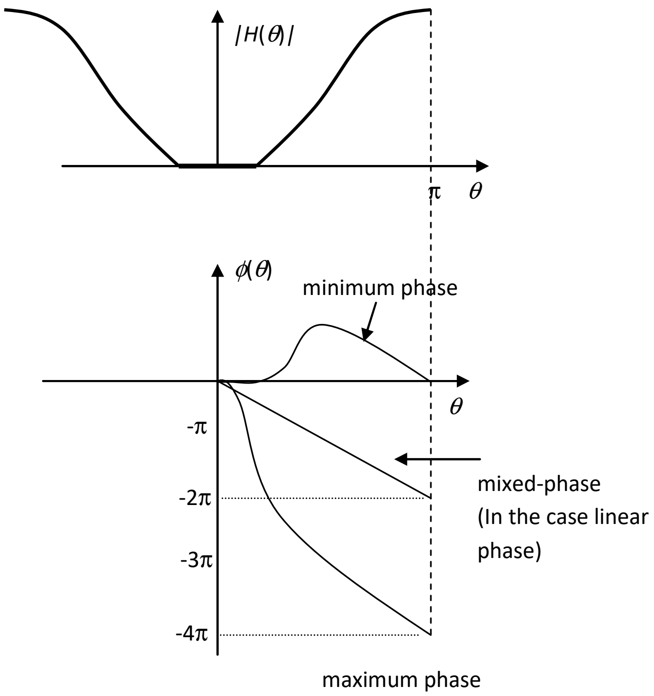
Note: For a given magnitude response, the minimum-phase system is the causal system that has the smallest absolute value of phase at every frequency (θ). That is, in the set of causal and stable filters having the same magnitude response, the minimum-phase response exhibits the smallest deviation from zero phase.

Example 5.12

Consider a fourth-order all-zero filter containing a double complex conjugate set of zeros located at $z = 0.7e^{\pm j\frac{\pi}{4}}$. The minimum-phase, mixed phase and maximum phase system pole-zero patterns having identical magnitude response are shown below.



The magnitude response and the phase response of the three systems are shown below: The minimum-phase system seems to have the phase with the smallest deviation from zero at each frequency.



the numerator is the reverse of the denominator
so they have equal value in magnitude, so H =1

5.6.4 All-Pass Filters

An all-pass filter is one whose magnitude response is constant for all frequencies, but whose phase response is not identically zero.

[The simplest example of an all-pass filter is a pure delay system with system function $H(z) = z^{-k}$]

A more interesting all-pass filter is one that is described by

$$H(z) = \frac{a_L + a_{L-1}z^{-1} + \cdots + a_1 z^{-L+1} + a_0 z^{-L}}{1 + a_1 z^{-1} + \cdots + a_L z^{-L}}$$ coefficients are swaped

where $a_0 = 1$ and all coefficients are real. If we define the polynomial $A(z)$ as

$$A(z) = \sum_{k=0}^{L} a_k z^{-k}, \qquad a_0 = 1$$

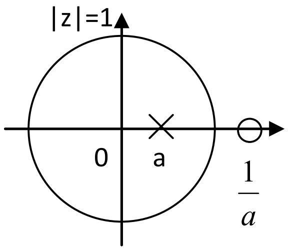
$$\therefore H(z) = z^{-L}\frac{A(z^{-1})}{A(z)}$$ this is what you want

$$\Rightarrow |H(\theta)|^2 = H(z) \cdot H(z^{-1})|_{z=e^{j\theta}} = 1$$

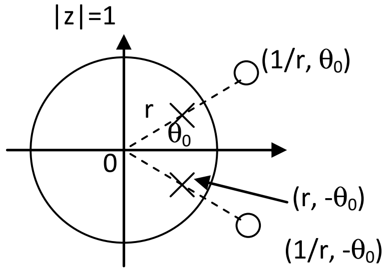
i.e., all-pass filter

IH(z)I^2 = H(z)*H(z*)=H(z)*H(z^-1)

Furthermore, if z_0 is a pole of $H(z)$, then $1/z_0$ is a zero of $H(z^{-1})$ (i.e. the poles and zeros are reciprocals of one another). The figure shown below illustrates typical pole-zero patterns for a single-pole, single-zero filter and a two-pole, two-zero filter.



All-pass



All pass filter

$$H(z) = \frac{1 - \frac{1}{a}z^{-1}}{1 - az^{-1}}, \qquad |a| < 1 \; for\ stability$$ pole has to lie inside unit circle

zero: z = 1/a
pole: z= a

We can easily show that the magnitude response is constant. 错误多的想报警。。。

$$|H(\theta)|^2 = H(\theta)\cdot H^*(\theta) = H(z)\cdot H(z^{-1})|_{z=e^{j\theta}}$$

$$= \frac{1-\frac{1}{a}e^{-j\theta}}{1-ae^{-j\theta}}\cdot\frac{1-\frac{1}{a}e^{j\theta}}{1-ae^{j\theta}}$$

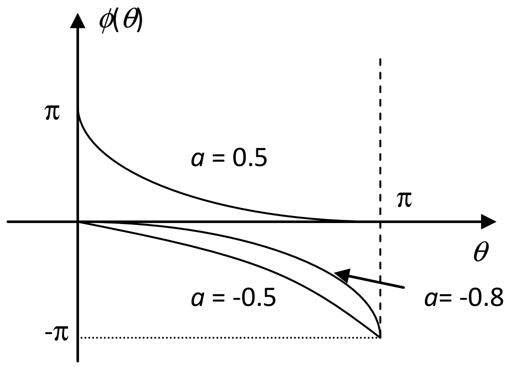
$$= \frac{1-\frac{2}{a}\cos\theta+\frac{1}{a^2}}{1-2a\cos\theta+a^2} = a^2$$

Phase response:

$$H(\theta) = \frac{1-\frac{1}{a}e^{-j\theta}}{1-ae^{-j\theta}}\cdot\frac{1-ae^{j\theta}}{1-ae^{j\theta}}$$

$$= \frac{2-(a+a^{-1})\cos\theta - j(a-a^{-1})\sin\theta}{1-2a\cos\theta+a^2}$$

$$\therefore\ \phi(\theta) = \tan^{-1}\left[\frac{-(a-a^{-1})\sin\theta}{2-(a+a^{-1})\cos\theta}\right]$$



When $0 < a < 1$, the zero lies on the positive real axis. The phase over $0 \le \theta \le \pi$ is positive, at $\theta = 0$ it is equal to π and decreases until $\theta = \pi$, where it is zero.

When $-1 < a < 0$, the zero lies on the negative real axis. The phase over $0 \le \theta \le \pi$ is negative, starting at 0 for $\theta = 0$ and decreases to $-\pi$ at $\theta = \pi$.

Example 5.13

Determine the DC gain of the filter described by

$$H(z) = \frac{1}{1 - az^{-1}}, \qquad 0 < a < 1$$

The DC gain is obtained by substituting $\theta = 0$ in the frequency response (which is obtained by making the substitution $z = e^{j\theta}$).

$$H(\theta)|_{\theta=0} = \frac{1}{1 - ae^{-j(0)}} = \frac{1}{1 - a}$$

If this DC gain is undesirable, a constant gain factor, $1 - a$ can be introduced so that $H(z)$ becomes:

$$H(z) = \frac{1 - a}{1 - az^{-1}}, \qquad DC\ gain = 1$$

Note: The original difference equation is $y[n] = x[n] + ay[n-1]$ and the DC gain normalised difference equation is $y[n] = (1 - a)x[n] + ay[n-1]$.

Example 5.14

Consider a low-pass filter.

$$y[n] = ay[n-1] + bx[n], \qquad 0 < a < 1$$

Note: A low pass filter is a system with high gain at all frequencies from zero up to a cut-off frequency and very low gain above the cut-off frequency. Hence it passes frequencies lower than the cut-off frequency.

i. Determine b such that $|H(0)| = 1$
ii. Determine the 3dB bandwidth (here) for the normalised filter obtained in (i).

$$Y(z) = aY(z)z^{-1} + bX(z)$$

z domain???

$$H(z) = \frac{b}{1 - az^{-1}}$$

$$H(\theta) = \frac{b}{1 - ae^{-j\theta}}$$

$$H(0) = \frac{b}{1 - a}$$

For $|H(0)| = 1$ we get $b = 1 - a$.

Thus the normalised filter is given by

$$H(\theta) = \frac{1-a}{1-ae^{-j\theta}} = \frac{1-a}{(1-a\cos\theta) + \ ja\sin\theta}$$

$$|H(\theta)| = \frac{1-a}{\sqrt{(1-a\cos\theta)^2 + (a\sin\theta)^2}}$$

$$|H(\theta)| = \frac{1-a}{\sqrt{1+a^2-2a\cos\theta}}$$

Note: The above relationship can also be obtained from $|H(\theta)|^2 = H(\theta)\cdot H^*(\theta)$. This is left as an **exercise**.

The 3dB cut-off frequency is the frequency at which power (square of the magnitude response) is half the maximum power. For a low-pass filter, the maximum power is at $\theta = 0$. Thus,

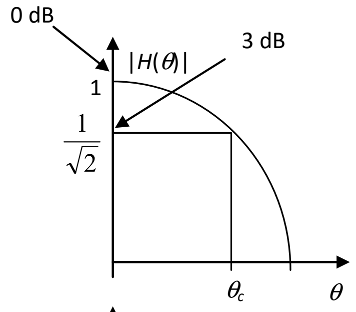
$$|H(\theta)|^2_{\theta=\theta_c} = \frac{1}{2}|H(0)|^2$$

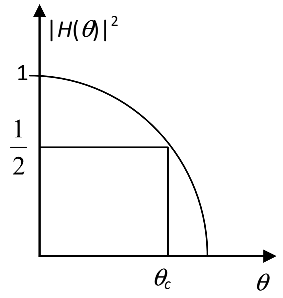
Hence, for the normalised filter

$$|H(\theta)|^2_{\theta=\theta_c} = \frac{1}{2}$$

$$\frac{(1-a)^2}{1+a^2-2a\cos\theta_c} = \frac{1}{2}$$

$$\theta_c = \cos^{-1}\left[\frac{-a^2+4a-1}{2a}\right]$$





Example 5.15

numerator is the reverse of the denominator so they have the same magnitude

Consider a filter described by

$$H(z) = \frac{a + cz^2}{c + az^2}$$

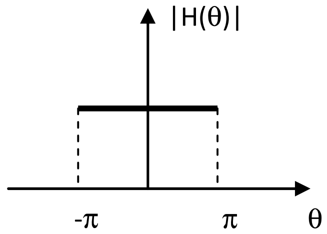
where, a and c are constants.

Show that the magnitude response $|H(\theta)|$ is unity for all θ.

$$|H(\theta)|^2 = H(\theta) \cdot H^*(\theta) = \frac{c + ae^{-j2\theta}}{a + ce^{-j\theta}} \cdot \frac{c + ae^{j2\theta}}{a + ce^{j2\theta}}$$

$$= \frac{c^2 + a^2 + ac\left[e^{j2\theta} + e^{-j2\theta}\right]}{a^2 + c^2 + ac\left[e^{j2\theta} + e^{-j2\theta}\right]} = 1$$

This is an all-pass filter



5.6.5 Notch Filters

When a zero is placed at a given point on the z-plane, the frequency response will be zero at the corresponding point. A pole on the other hand produces a peak at the corresponding frequency point.

Poles that are close to the unit circle give rise large peaks, where as zeros close to or on the unit circle produces troughs or minima. Thus, by strategically placing poles and zeros on the z-plane, we can obtain sample low pass or other frequency selective filters (**notch filters**).

Example 5.16

Obtain, by the pole-zero placement method, the transfer function of a sample digital notch filter (see figure below) that meets the following specifications:

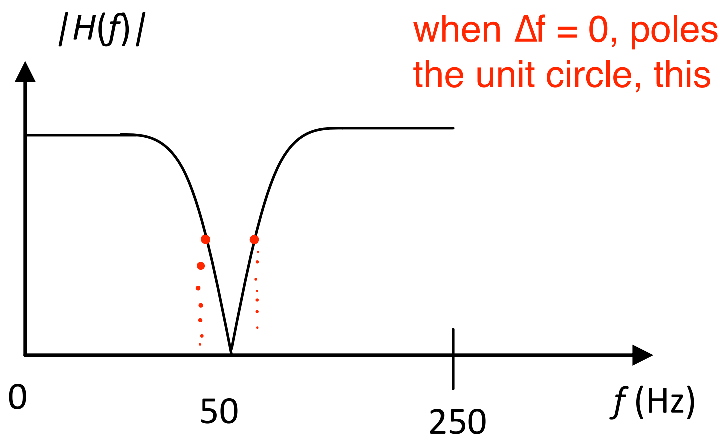
Notch Frequency: 50Hz

3db width of the Notch: ±5Hz

Sampling frequency: 500 Hz

The radius, r of the poles is determined by: $r = 1 - \left(\frac{\Delta f}{f_s}\right)\pi$

when Δf = 0, poles are put at the edge of the unit circle, this is a bit dangerous.



To reject the component at 50Hz , place a pair of complex zeros at points on the unit circle corresponds to 50Hz.

notch frequency

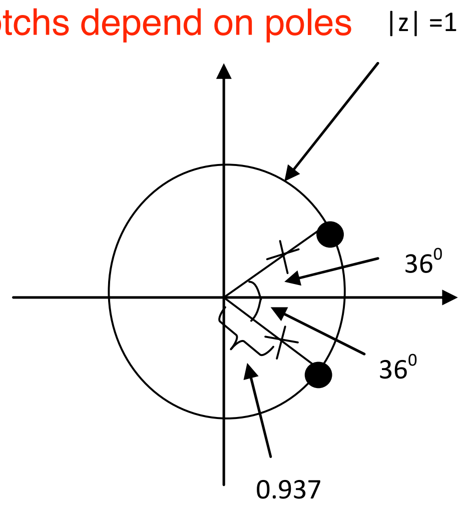
i.e. at angles of $360^0 \times \frac{50}{500} = \pm 36^0 = \pm 0.2\pi\, rad$

sampling frequency

To achieve a sharp notch filter and improved amplitude response on either side of the notch frequency, a pair of complex conjugate zeros are placed at a radius $r < 1$.

$$r = 1 - \left(\frac{\Delta f}{f_s}\right)\pi = 1 - \left(\frac{10}{500}\right)\pi = 0.937$$

bandwidth of Notchs depend on poles



$$H(z) = \frac{\left(z - e^{-j0.2\pi}\right)\left(z - e^{j0.2\pi}\right)}{(z - 0.937e^{-j0.2\pi})(z - 0.937e^{j0.2\pi})}$$

$$= \frac{z^2 + 1 - \left(e^{j0.2\pi} + e^{-j0.2\pi}\right)z}{z^2 + 0.878 - 0.937(e^{j0.2\pi} + e^{-j0.2\pi})z}$$

$$= \frac{z^2 + 1 - 2z\cos(0.2\pi)}{z^2 + 0.878 - 2z \times 0.937\cos(0.2\pi)}$$

$$= \frac{1 - 1.6180z^{-1} + z^{-2}}{1 - 1.5161z^{-1} + 0.878z^{-2}}$$

5.7 System Structures – Parallel and Cascade

5.7.1 Parallel Structure

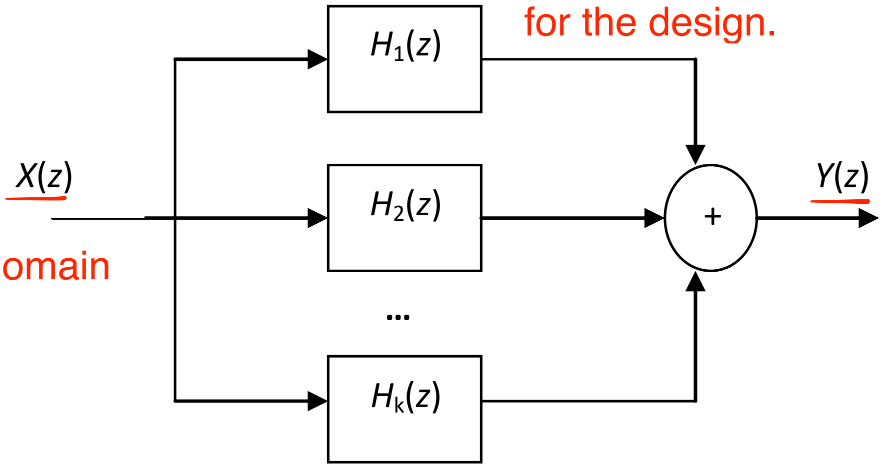
Consider,

$$H(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2} \ldots + a_M z^{-M}}{1 + b_1 z^{-1} + b_2 z^{-2} + \cdots b_L z^{-L}}$$

$$= \sum_{i=1}^{k} H_i(z) = H_1(z) + H_2(z) + \cdots + H_k(z)$$

we are using sum rather than multiplication (convolution case),as do it this way is easier for the design.

Note: Use partial fractions to get $H_i(z)$



z domain

5.7.2 Cascade Realisation

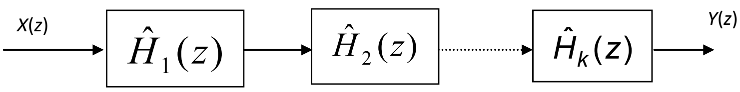
The system,

$$H(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2} \ldots + a_M z^{-M}}{1 + b_1 z^{-1} + b_2 z^{-2} + \cdots b_L z^{-L}}$$

can also be written in the form

$$H(z) = \prod_{i=1}^{k} H_i(z) = H_1(z) \cdot H_2(z) \cdot \ldots \cdot H_k(z)$$

i.e., a product of lower order (1st or 2nd) transfer functions. Each of which is a separate section. The cascade structure is a popular form



Example 5.17

Determine the parallel realisation of the system given by

$$H(z) = \frac{23 + 40z^{-1} + 36z^{-2} + 19z^{-3}}{(2 + z^{-1})(5 + 2z^{-1} + 3z^{-2})}$$

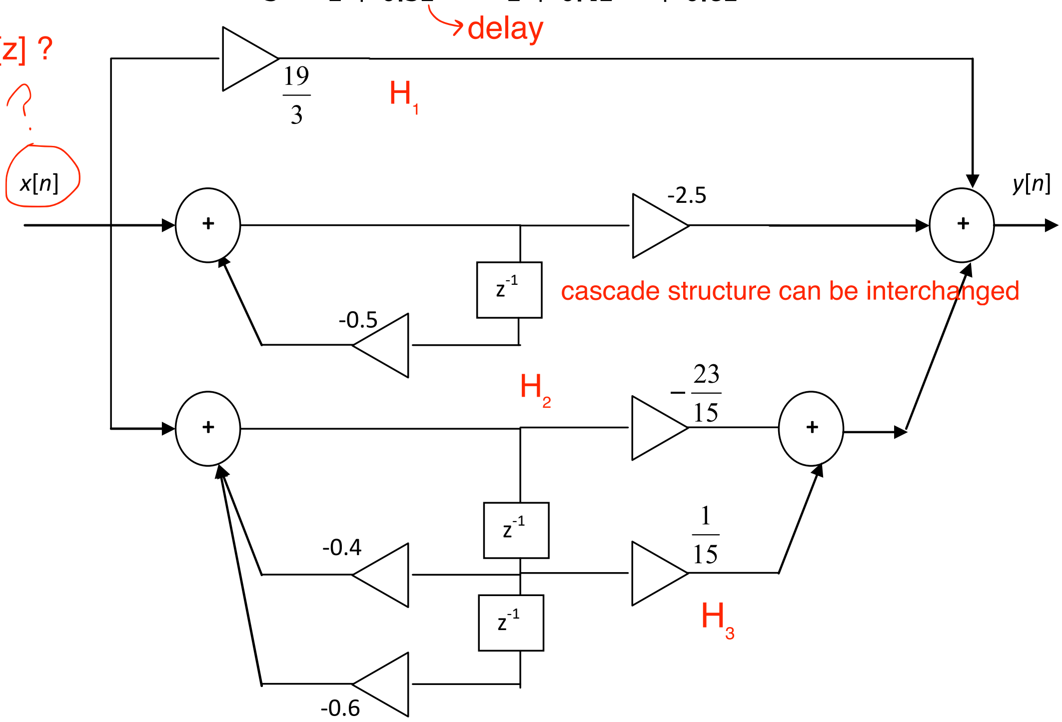
$$= A + \frac{B}{2 + z^{-1}} + \frac{C - Dz^{-1}}{5 + 2z^{-1} + 3z^{-2}}$$

$$H(z) = \frac{19}{3} - \frac{5}{2 + z^{-1}} - \frac{1}{3}\frac{23 - z^{-1}}{5 + 2z^{-1} + 3z^{-2}}$$

$$H(z) = \frac{19}{3} - \frac{5}{2}\frac{1}{1 + 0.5z^{-1}} - \frac{1}{3}\frac{23 - z^{-1}}{5(1 + 0.4z^{-1} + 0.6z^{-2})}$$

$$= \frac{19}{3} + \frac{-2.5}{1 + 0.5\underbrace{z^{-1}}_{\text{delay}}} + \frac{-\frac{23}{15} + \frac{1}{15}z^{-1}}{1 + 0.4z^{-1} + 0.6z^{-2}}$$

should be x[z] ?



Example 5.18

Determine the cascade realisation of the system given by

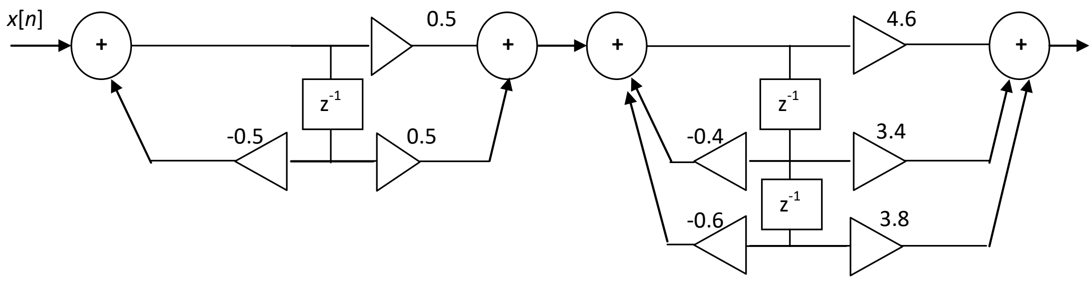
$$H(z) = \frac{23 + 40z^{-1} + 36z^{-2} + 19z^{-3}}{10 + 9z^{-1} + 8z^{-2} + 3z^{-3}}$$

$$= \frac{(1 + z^{-1})}{(2 + z^{-1})}\frac{23 + 17z^{-1} + 19z^{-2}}{5 + 2z^{-1} + 3z^{-2}}$$

$$= \frac{(1 + z^{-1})}{2(1 + 0.5z^{-1})}\frac{23 + 17z^{-1} + 19z^{-2}}{5(1 + 0.4z^{-1} + 0.6z^{-2})}$$

$$= \frac{(0.5 + 0.5z^{-1})}{1 + 0.5z^{-1}}\frac{4.6 + 3.4z^{-1} + 3.8z^{-2}}{1 + 0.4z^{-1} + 0.6z^{-2}}$$

$$= \underbrace{\left(\frac{(0.5 + 0.5z^{-1})}{1 + 0.5z^{-1}}\right)}_{H1}\underbrace{\left(\frac{4.6 + 3.4z^{-1} + 3.8z^{-2}}{1 + 0.4z^{-1} + 0.6z^{-2}}\right)}_{H2}$$



5.8 Digital Oscillators

A digital oscillator can be made using a second order discrete-time system, by using appropriate coefficients. A difference equation for an oscillating system is given by

$$p[n] = A\cos(n\theta)$$

We know that the z-transform of $p[n]$ is

$$P(z) = \frac{1 - \cos\theta\, z^{-1}}{1 - 2\cos\theta\, z^{-1} + z^{-2}}$$

Let $P(z) = Y(z)/X(z)$. This gives

$$Y(z)(1 - 2\cos\theta\, z^{-1} + z^{-2}) = X(z)(1 - \cos\theta\, z^{-1})$$

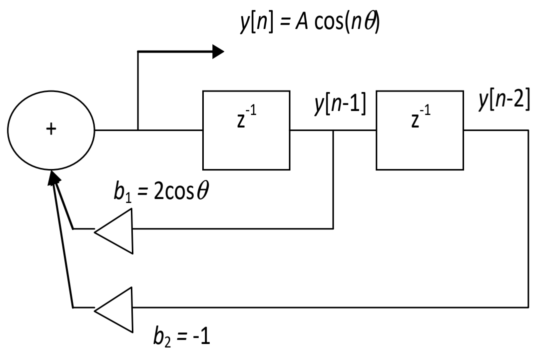
Taking the inverse z-transform,

$$y[n] - 2\cos\theta\, y[n-1] + y[n-2] = x[n] - \cos\theta\, x[n-1]$$

An oscillator has no input. i.e., $x[n] = 0$ and $x[n-1] = 0$. So the equation of a digital oscillator becomes

$$y[n] = 2\cos\theta\, y[n-1] - y[n-2]$$

and its structure is shown below.



To obtain $y[n] = A\cos(n\theta)$, use the following initial conditions:

$$y[0] = A\cos(0\cdot\theta) = A$$

$$y[-1] = A\cos(-1\cdot\theta) = A\cos\theta$$

The frequency can be tuned by changing the coefficient b_1 (b_2 is a constant). The resonant frequency θ of the oscillator is given by,

$$\cos\theta = -\frac{b_1}{2\sqrt{b_2}} = -\frac{b_1}{2}$$

$b_2 = r^2 = 1$

Z transform of p[n]

$$P(z) = \sum_{n=-\infty}^{\infty} p[n]\, z^{-n} \quad , \quad (z = r e^{j\theta})$$

$$= \sum_{n=-\infty}^{\infty} A \cos(n\theta)\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} A\left(\frac{e^{jn\theta} + e^{-jn\theta}}{2}\right) z^{-n}$$

$$= \frac{A}{2} \sum_{n=-\infty}^{\infty} (e^{j\theta} z^{-1})^{n} + \frac{A}{2} \sum_{n=-\infty}^{\infty} (e^{-j\theta} z^{-1})^{n}$$

$$= \frac{A}{2} \cdot \frac{1}{1 - e^{j\theta} z^{-1}} + \frac{A}{2} \cdot \frac{1}{1 - e^{-j\theta} z^{-1}}$$

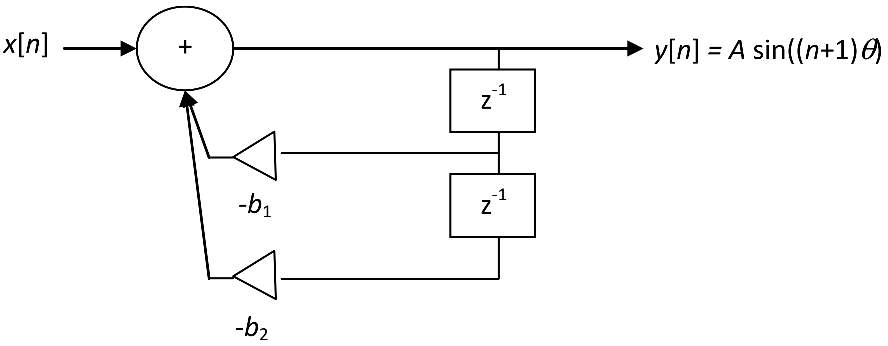
$$= \frac{A}{2} \cdot \frac{(1 - e^{-j\theta} z^{-1}) + (1 - e^{j\theta} z^{-1})}{(1 - e^{j\theta} z^{-1})(1 - e^{-j\theta} z^{-1})}$$

$$= \frac{A}{2} \cdot \frac{2 - 2\cos\theta \cdot z^{-1}}{1 - e^{j\theta} z^{-1} - e^{-j\theta} z^{-1} + z^{-2}}$$

$$= A \cdot \frac{1 - \cos\theta\, z^{-1}}{1 - 2\cos\theta\, z^{-1} + z^{-2}}$$

Example 5.19

A digital sinusoidal oscillator is shown below.



(a) Assuming θ_0 is the resonant frequency of the digital oscillator, find the values of b_1 and b_2 for sustaining the oscillation.

$$
\begin{aligned}
H(z) &= \frac{Kz^2}{z^2 + b_1 z + b_2} \\
&= \frac{K}{(z - re^{j\theta})(z - re^{-j\theta})} \\
&= \frac{K}{z^2 - r(e^{j\theta} + e^{-j\theta})z + r^2} \\
&= \frac{K}{z^2 - 2r\cos\theta_0\, z + r^2}
\end{aligned}
$$

Therefore, $b_1 = -2r\cos\theta_0$ and $b_2 = r^2$ where $r = 1$ for an oscillator

(b) Write the difference equation for the above figure. Assuming $x[n] = (A\sin\theta_0)\delta[n]$ and $y[-1] = y[-2] = 0$. Show, by analysing the difference equation, that the application of an impulse at $n = 0$ serves the purpose of beginning the sinusoidal oscillation, and prove that the oscillation is self-sustaining thereafter.

$$
\begin{aligned}
&y[n] = -b_1 y[n-1] - b_2 y[n-2] + x[n] \\
&\Rightarrow y[n] = 2\cos\theta_0\, y[n-1] - y[n-1] + A\sin\theta_0\, \delta[n]
\end{aligned}
$$

Assume, $y[n] = A\sin(n+1)\theta_0$ and $y[n-1] = A\sin n\theta_0$. Then for $n > 0$.

$$\begin{aligned} y[n+1] &= 2\cos\theta_0 \, A\sin(n+1)\theta_0 - A\sin n\theta_0 \\ &= A\big[2\cos\theta_0 \sin(n+1)\theta_0 - \sin\big((n+1)\theta_0 - \theta_0\big)\big] \\ &= A[2\sin(n+1)\theta_0 \cos\theta_0 - \sin(n+1)\theta_0 \cos\theta_0 \\ &\qquad + \cos(n+1)\theta_0 \sin\theta_0] \\ &= A[\sin(n+1)\theta_0 \cos\theta_0 + \cos(n+1)\theta_0 \sin\theta_0] \\ &= A\sin(n+2)\theta_0 \end{aligned}$$

Also, at $n = 0$,

$$y[0] = A\sin\theta_0 \, \delta[0] = A\sin\theta_0$$

And at $n = 1$,

$$y[1] = 2\cos\theta_0 \, A\sin\theta_0 = A\sin 2\theta_0$$

Thus the assumptions made above are satisfied for $n + 1 = 2$ giving, $y[2] = A\sin 3\theta_0$ and by mathematical induction $y[n] = A\sin(n+1)\theta_0$ for all $n > 0$.

(c) By setting the input to zero and under certain initial conditions, sinusoidal oscillation can be obtained using the structure shown above. Find these initial conditions.

$$y[n] = 2\cos\theta_0 \, y[n-1] - y[n-2] + x[n]$$

When, $x[n] = 0$, for oscillations it is necessary that $y[0] = A\sin\theta_0$. i.e.,

$$y[0] = 2\cos\theta_0 \, y[-1] - y[-2] = A\sin\theta_0$$

This can be satisfied by,

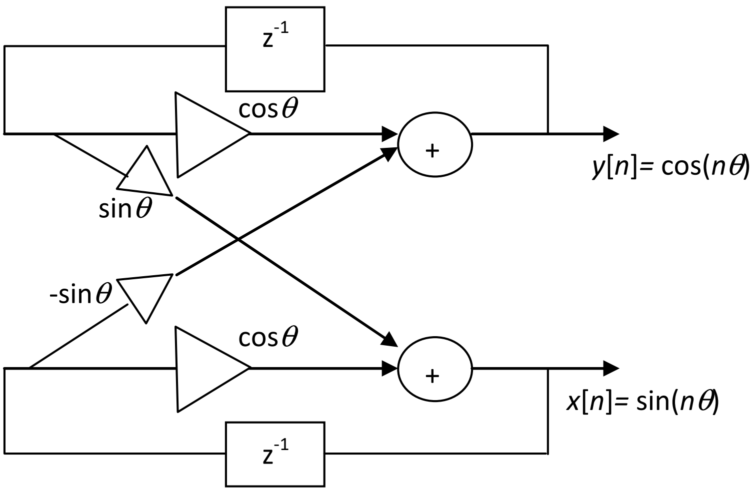
$$y[-1] = 0$$

$$y[-2] = A\sin\theta_0$$

These are the initial conditions.

5.8.1 Sine and Cosine oscillators

Sinusoidal oscillators can be used to deliver the carrier in modulators. In modulation schemes, both sines and cosines oscillators are needed. A structure that delivers sines and cosines simultaneously is shown below:



Trigonometric equation for $\cos(n+1)\theta$ is:

$$\cos(n+1)\theta = \cos n\theta \cos\theta - \sin n\theta \sin\theta$$

Let $y[n] = \cos(n\theta)$ and $x[n] = \sin(n\theta)$

$$\therefore y[n+1] = \cos\theta\, y[n] - \sin\theta\, x[n]$$

Replacing n by $n-1$

$$\boxed{y[n] = \cos\theta\, y[n-1] - \sin\theta\, x[n-1]}$$

Similarly,

$$\sin(n+1)\theta = \sin\theta \cos n\theta + \cos\theta \sin n\theta$$

$$\therefore x[n+1] = \sin\theta\, y[n] + x[n]\cos\theta$$

Replacing n by $n-1$

$$\boxed{x[n] = \sin\theta\, y[n-1] + \cos\theta\, x[n-1]}$$

Using these equations, the structure above can be obtained.

Example 5.20

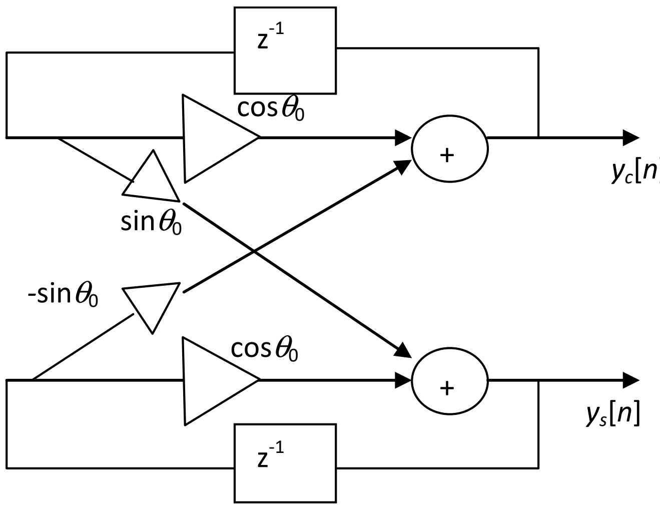
An oscillator is given by the following coupled difference equations expressed in matrix form.

$$\begin{bmatrix} y_c[n] \\ y_s[n] \end{bmatrix} = \begin{bmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{bmatrix} \begin{bmatrix} y_c[n-1] \\ y_s[n-1] \end{bmatrix}$$

Draw the structure for the realisation of this oscillator, where θ_0 is the oscillation frequency. If the initial conditions $y_c[-1] = A\cos\theta_0$ and $y_s[-1] = -A\sin\theta_0$, obtain the outputs $y_c[n]$ and $y_s[n]$ using the above difference equations.

$$y_c[n] = \cos\theta_0\, y_c[n-1] - \sin\theta_0\, y_s[n-1]$$

$$y_s[n] = \sin\theta_0\, y_c[n-1] + \cos\theta_0\, y_s[n-1]$$



$$y_s[0] = \sin\theta_0\,(A\cos\theta_0) + \cos\theta_0\,(-A\sin\theta_0) = 0$$

$$y_c[0] = \cos\theta_0\,(A\cos\theta_0) - \sin\theta_0\,(-A\sin\theta_0) = A$$

$$y_c[1] = \cos\theta_0 \cdot A - \sin\theta_0 \cdot 0 = A\cos\theta_0$$

$$y_s[1] = A\sin\theta_0 + 0 = A\sin\theta_0$$

$$y_c[2] = \cos\theta_0\, y_c[1] - \sin\theta_0\, y_s[1]$$

$$= \cos\theta_0\, A\cos\theta_0 - \sin\theta_0\, A\sin\theta_0 = A\cos(2\theta_0)$$

$$y_c[n] = A\cos(n\theta_0)$$

Similarly

$$y_s[n] = A\sin(n\theta_0)$$

Exercise

For the structure shown below, write down the appropriate difference equations and hence state the function of this structure.