ELEC3104: Digital Signal Processing

Chapter 2: Discrete Time Signals

2.1 Discrete Signals

Digital Signal Processing (DSP) is a rapidly developing technology for scientists and engineers. In the 1990s the digital signal processing revolution started, both in terms of the consumer boom in digital audio, digital telecommunications and the wide used of technology in industry.

Due to the availability of low cost digital signal processors, manufacturers are producing plug-in DSP boards for PCs, together with high-level tools to control these boards. There are many areas where DSP technology is now being used and the current proliferation of such technology will open up further applications.

In the medical field, DSP systems are widely utilized for recording data analysis and the interpretation of ECG signals.

Audiologists and speech therapists are exposed to DSP systems for both testing a person's level of hearing and subsequently DSP hearing aid filtering.

The professional music industry uses spectrum analysers, digital filtering, sampling conversion filters, etc. and is one of the biggest users and exploiters of DSP technology.

In summary, DSP is applied in the area of control and power systems, biomedical engineering, instrumentation (test and measurement), automotive engineering, telecommunications, mobile communication, speech analysis and synthesis, audio and video processing, seismic, radar and sonar processing and neural computing.

There are many advantages to using DSP techniques for variety of applications, these include:

- high reliability and reproducibility
- flexibility and programmability
- the absence of component drift problem
- compressed storage facility

DSP hardware allows for programmable operations. Through software, one can easily modify the signal processing functions to be performed by the hardware. For all these reasons, there has been vast growth in DSP theory & applications over the past decade.

2.1.1 Analogue to Digital Conversion

Before any DSP algorithm can be performed, the signal must be in a digital form. The A/D conversion process involves the following steps:

- The signal (Band-limited) is first sampled, converting the analogue signal into a discrete-time signal
- The amplitude of each sample is quantised into one of 2^B levels (where B is the number of bits used to represent a sample in the A/D converter)
- The discrete amplitude levels are represented or encoded into distinct binary words each of length B bits.

A practical representation of the A/D conversion process is shown below:

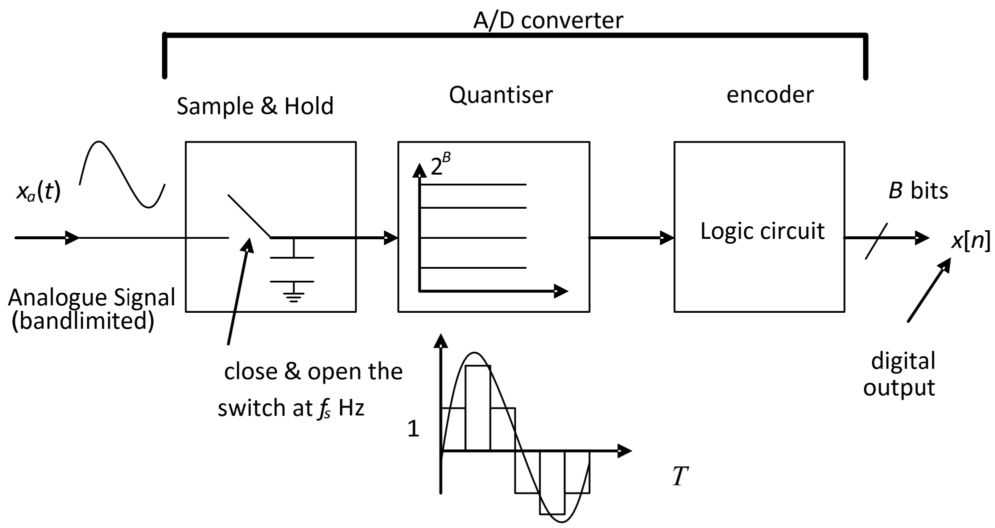


Figure 2.1: Analogue to Digital conversion process

Sample and hold (S/H) takes a snapshot of the analogue signal every T sec and then holds that value constant for T secs until the next snapshot is obtained.

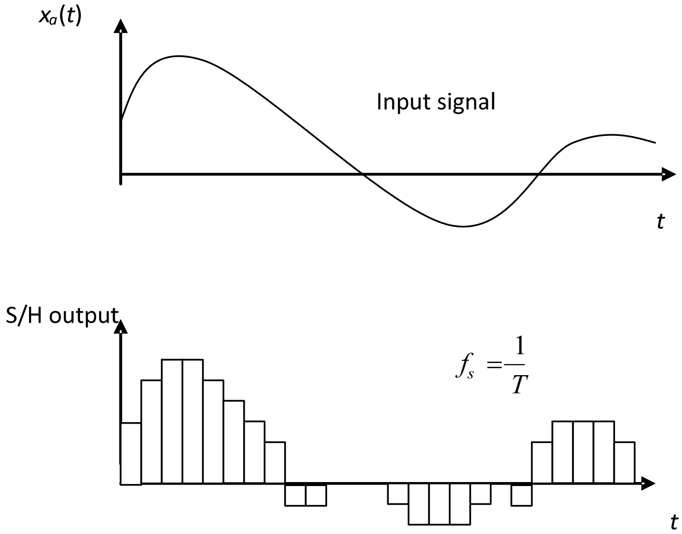


Figure 2.2: An example of "sample and hold" process to convert analogue signals into digital signals

Example 2.1

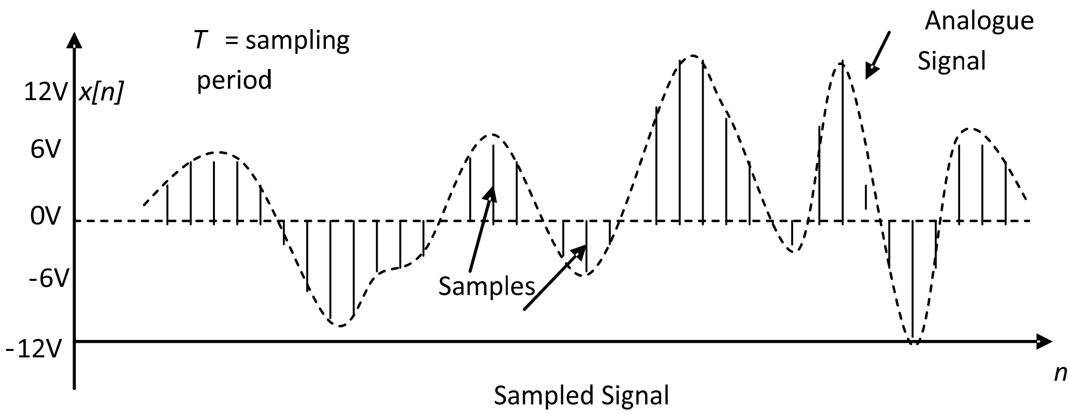


Figure 2.3: An example of sampling analogue signals to dicrete signals. The sampling period is *T*.

Example 2.2

A 4-bit (B = 4) A/D converter (bipolar).

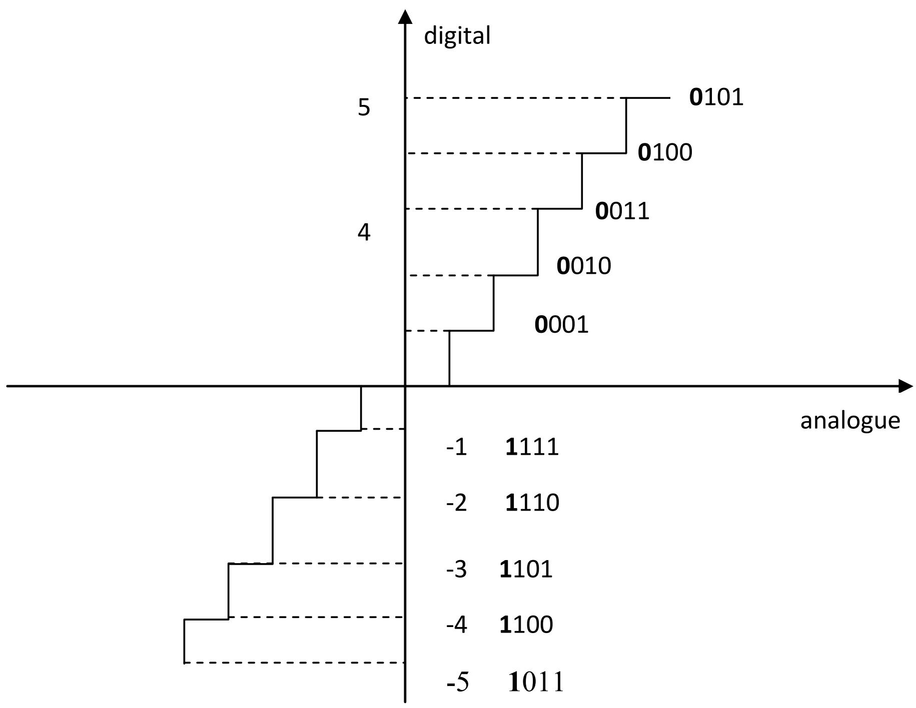


Figure 2.4: Input-output characteristic of 4-bit quantiser (linear) - (two's complement notation)

2.1.2 Sampling Theorem

The process of converting an analogue signal, $x(t)$, into a discrete signal, $x[n]$, involves taking the values (samples) of $x(t)$ at discrete points $t = nT$, spaced T apart.

$$x[n] \stackrel{\text{def}}{=} x(nT)$$

i.e., samples of $x(t)$ at $1/T$ points in each unit interval (1 sec if the independent variable t denotes time) are taken. In other words, the sampling rate, F_s, is $1/T\ Hz$.

$$F_s = \frac{1}{T}\ Hz$$

The sampling theorem gives condition (imposed on F_s) that allows for perfect recovery of $x(t)$ from $x[n]$.

Note: The following properties of the Dirac delta, $\delta(t)$, are used to derive this theorem. Given a function, $f(x)$:

1. $$f(x) \cdot \delta(x-a) = f(a) \cdot \delta(x-a)$$
2. $$\int_{-\infty}^{\infty} f(x)\delta(x-a)dx = f(a)$$
3. $$f(x) * \delta(x-a) = f(x-a)$$
4. $$\delta(ax) = \frac{1}{a}\delta(x)$$

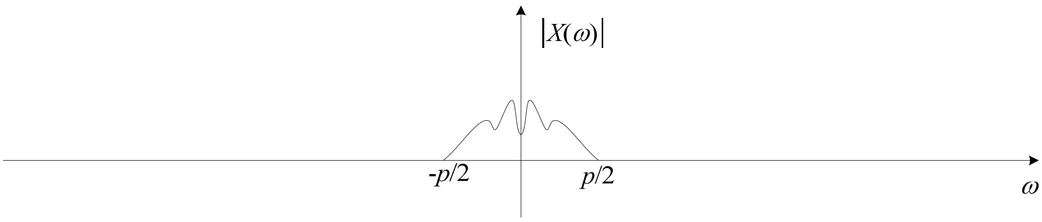
Consider a signal, $x(t)$, that is *bandlimited*. i.e., for some positive value p,

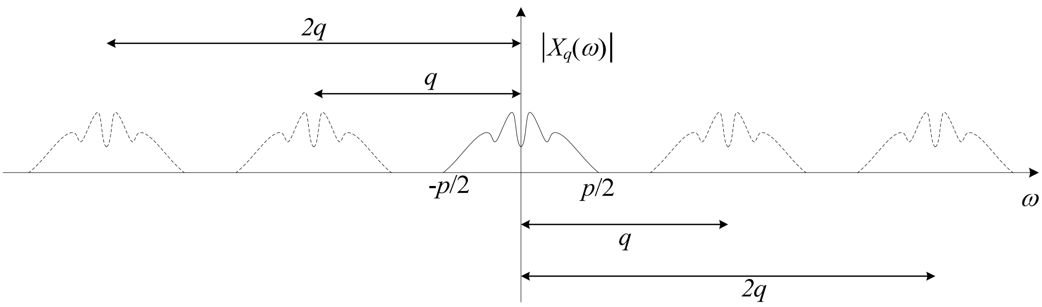
$$\mathcal{F}\{x(t)\} = X(\omega) = 0, \qquad |\omega| > \frac{p}{2}$$

and p is the lowest such value.

In other words, the spectrum $X(\omega)$ is uniformly zeros outside of the interval $[-p/2, p/2]$ and can be periodised with a period q as follows:

$$X_q(\omega) = \sum_{k=-\infty}^{\infty} X(\omega - kq)$$

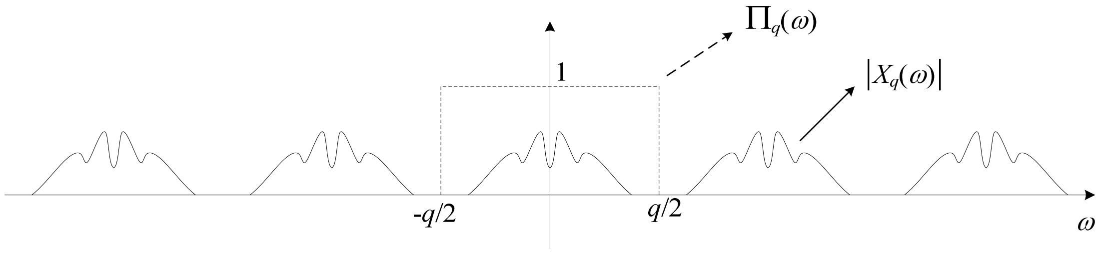




Note: If $x(t)$ is real valued, $X(\omega) = \overline{X(-\omega)}$ and hence $|X(\omega)| = |X(-\omega)|$. i.e., $|X(\omega)|$ is symmetrical about the vertical axis, $\omega = 0$.

If $q > p$, the repeated images do not overlap and the original spectrum, $X(\omega)$ can be recovered from the periodised version, $X_q(\omega)$ by multiplying with the rectangular function, $\Pi_q(\omega)$.

$$\Pi_q(\omega) = \begin{cases} 1, & |\omega| \le \dfrac{q}{2} \\ 0, & |\omega| > \dfrac{q}{2} \end{cases}$$



i.e.,

$$X(\omega) = \Pi_q(\omega) \cdot X_q(\omega), \qquad q > p$$

Now, $X_q(\omega)$ can be written as

$$\begin{aligned} X_q(\omega) &= \sum_{k=-\infty}^{\infty} X(\omega - kq) \\ &= \sum_{k=-\infty}^{\infty} X(\omega) * \delta(\omega - kq) \\ &= X(\omega) * \left(\sum_{k=-\infty}^{\infty} \delta(\omega - kq) \right) \end{aligned}$$

The term in the bracket occurs frequently and is commonly referred to as the dirac comb or the shah function. Denoting it by

$$\rho_q(\omega) = \sum_{k=-\infty}^{\infty} \delta(\omega - kq)$$

We obtain

$$\boxed{X(\omega) = \Pi_q(\omega) \cdot \big(X(\omega) * \rho_q(\omega)\big)}$$

f1(t)*f2(t)=F^-1*(F1(w).F2(w))

Taking the inverse Fourier transform

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\}$$
$$= \mathcal{F}^{-1}\left\{\Pi_q(\omega)\cdot\left(X(\omega) * \rho_q(\omega)\right)\right\}$$
$$= \mathcal{F}^{-1}\{\Pi_q(\omega)\} * \mathcal{F}^{-1}\{X(\omega) * \rho_q(\omega)\}$$
$$= \mathcal{F}^{-1}\{\Pi_q(\omega)\} * 2\pi\left(\mathcal{F}^{-1}\{X(\omega)\}\cdot\mathcal{F}^{-1}\{\rho_q(\omega)\}\right)$$

The last two steps are due to the convolution property of Fourier transforms.

Now,

$$\mathcal{F}^{-1}\{X(\omega)\} = x(t)$$
$$\mathcal{F}^{-1}\{\Pi_q(\omega)\} = \frac{q}{2\pi} sinc\left(\frac{qt}{2}\right)$$

(The second relationship can be obtained using the integral formula of the inverse Fourier transform and is left as an **exercise.**)

In order to evaluate $\mathcal{F}^{-1}\{\rho_q(\omega)\}$, it must first be noted $\rho_q(\omega)$ is a periodic function (with period q) and can be written as a Fourier series. i.e.,

$$\rho_q(\omega) = \sum_{k=-\infty}^{\infty} c_k e^{jk\left(\frac{2\pi}{q}\right)\omega}$$

Note: In this expansion, ω is the independent variable and the fundamental frequency is $2\pi/q$.

Now,

$$c_k = \frac{1}{q}\int_{-\frac{q}{2}}^{\frac{q}{2}} \rho_q(\omega)e^{-jk\left(\frac{2\pi}{q}\right)\omega}\,d\omega$$

In the interval, $\left[-\frac{q}{2},\frac{q}{2}\right]$, the dirac comb is only a single delta function. i.e.,

$$\rho_q(\omega) = \delta(\omega), \qquad \omega \in \left[-\frac{q}{2},\frac{q}{2}\right]$$

This reduces the above integral to,

$$c_k = \frac{1}{q} \int_{-\frac{q}{2}}^{\frac{q}{2}} \delta(\omega) e^{-jk\left(\frac{2\pi}{q}\right)\omega} d\omega$$

$$= \frac{1}{q}$$

due to the second property of the delta function.

Thus,

$$\rho_q(\omega) = \frac{1}{q} \sum_{k=-\infty}^{\infty} e^{jk\left(\frac{2\pi}{q}\right)\omega}$$

Hence,

$$\mathcal{F}^{-1}\{\rho_q(\omega)\} = \mathcal{F}^{-1}\left\{\frac{1}{q} \sum_{k=-\infty}^{\infty} e^{jk\left(\frac{2\pi}{q}\right)\omega}\right\}$$

$$= \frac{1}{q} \sum_{k=-\infty}^{\infty} \mathcal{F}^{-1}\left\{e^{jk\left(\frac{2\pi}{q}\right)\omega}\right\}$$

$$= \frac{1}{q} \sum_{k=-\infty}^{\infty} \delta\left(t + k\left(\frac{2\pi}{q}\right)\right)$$

Note: The inverse Fourier transform of a Dirac comb is another Dirac comb. Similarly, the Fourier transform of a Dirac comb is another Dirac comb. Also, note that the fundamental periods of the Dirac combs are not the same.

Let, $T = 2\pi/q$, then

$$\mathcal{F}^{-1}\{\rho_q(\omega)\} = \frac{T}{2\pi} \sum_{k=-\infty}^{\infty} \delta(t + kT)$$

By substituting $k = -n$

$$\mathcal{F}^{-1}\{\rho_q(\omega)\} = \frac{T}{2\pi} \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Previously it was shown that,

$$x(t) = \mathcal{F}^{-1}\{\Pi_q(\omega)\} * 2\pi\left(\mathcal{F}^{-1}\{X(\omega)\} \cdot \mathcal{F}^{-1}\{\rho_q(\omega)\}\right)$$

Substitution the corresponding inverse Fourier transforms and using $q = 2\pi/T$,

$$x(t) = \frac{1}{T} sinc\left(\frac{\pi t}{T}\right) * 2\pi\left(x(t) \cdot \frac{T}{2\pi} \sum_{n=-\infty}^{\infty} \delta(t - nT)\right)$$

$$= \frac{1}{T} sinc\left(\frac{\pi t}{T}\right) * 2\pi\left(\frac{T}{2\pi} \sum_{n=-\infty}^{\infty} x(t)\delta(t - nT)\right)$$

$$= sinc\left(\frac{\pi t}{T}\right) * \left(\sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)\right)$$

Since, $x(t)\delta(t - nT) = x(nT)\delta(t - nT)$ by the first property of the delta function.

But, $x(nT)$ are the values of the discrete signal, $x[n]$, obtained by taking evenly spaced samples (spaced T apart) from $x(t)$. Thus,

$$x(t) = sinc\left(\frac{\pi t}{T}\right) * \left(\sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)\right)$$

$$= \sum_{n=-\infty}^{\infty} x[n]\left(sinc\left(\frac{\pi t}{T}\right) * \delta(t - nT)\right)$$

$$= \sum_{n=-\infty}^{\infty} x[n] \cdot sinc\left(\frac{\pi(t - nT)}{T}\right)$$

x(t) <==>x[n]

The $sinc$ function is completely independent of $x(t)$. Therefore the above equation indicates that the original signal, $x(t)$, can be recovered from the discrete signal $x[n]$ as long as the following expression (repeated from earlier) holds.

$$X(\omega) = \Pi_q(\omega) \cdot \left(X(\omega) * \rho_q(\omega)\right)$$

This expression in turn is true as long as $\boldsymbol{q > p}$.

no overlaps

Now, $X(\omega) = 0$ for all $|\omega| > p/2$. Therefore $p/2$ corresponds to the maximum frequency, f_{max} , present in the signal. Since, $\omega = 2\pi f$,

$$\frac{p}{2} = 2\pi f_{max}$$

$$p = 4\pi f_{max}$$

Also, $q = 2\pi/T = 2\pi F_s$, where $F_s = 1/T$ is the sampling rate. Substituting for q and p in $q > p$ results in the commonly used expression for the *Nyquist Sampling Theorem.*

$$\boxed{F_s > 2f_{max}}$$

-> q > p
-> 2*pi*Fs > 4*pi*fmax
-> Fs > 2*fmax

2.2 Spectrum of a Discrete Signal

As previously mentioned, a discrete signal is obtained by taking regular samples of a continuous signal.

$$x[n] \stackrel{\text{def}}{=} x(nT), \qquad n \in \mathbb{Z}$$

While this discrete signal is defined only at a discrete set of points (i.e., when n is an integer), it can be represented "continuously" with the aid of an impulse train (Dirac comb) at the sample points. i.e.,

$$\begin{aligned} x_d(t) &= x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT) \\ &= \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT) \\ &= \sum_{n=-\infty}^{\infty} x[n]\delta(t - nT) \end{aligned}$$

Taking the Fourier transform

$$X_d(\omega) = \mathcal{F}\{x_d(t)\} = \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} x[n]\delta(t - nT)\right\}$$

Interchanging the summation and the Fourier transform

$$\begin{aligned} X_d(\omega) &= \sum_{n=-\infty}^{\infty} x[n] \cdot \mathcal{F}\{\delta(t - nT)\} \\ &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega nT} \end{aligned}$$

Letting $\theta = \omega T$, gives the discrete time Fourier transform (DTFT).

very important

$$\hat{x}(\theta) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\theta n}$$

Note: $X_d(\omega)$ is the continuous Fourier transform of the signal $x_d(t)$ which, while representing a discrete signal, is defined at all values of t. $\hat{x}(\theta)$ is the discrete time Fourier transform (DTFT) which is evaluated on the actual discrete signal $x[n]$, which is defined only at discrete integer points, $n \in \mathbb{Z}$.

While $x[n]$ can be either real or complex, in most cases it is real. $\hat{x}(\theta)$ on the other hand is complex valued and similar to the continuous Fourier transform can be represented in terms of magnitude and phase as follows.

$$\hat{x}(\theta) = |\hat{x}(\theta)| e^{j\angle\hat{x}(\theta)}$$

This gives rise to the magnitude spectrum, $|\hat{x}(\theta)|$, and the phase spectrum, $\angle\hat{x}(\theta)$, just as in the case of the continuous Fourier transform.

To determine the relationship between the spectrum of a discrete signal, $x[n]$, and the spectrum of the corresponding continuous time signal, $x(t)$, consider:

$$X_d(\omega) = \mathcal{F}\{x_d(t)\} = \mathcal{F}\left\{x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)\right\}$$

$$= \mathcal{F}\{x(t)\} * \mathcal{F}\left\{\sum_{n=-\infty}^{\infty} \delta(t - nT)\right\}$$

Exercise 2.1

Show that

$$\mathcal{F}\left\{\sum_{n=-\infty}^{\infty} \delta(t - nT)\right\} = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\omega - k\frac{2\pi}{T}\right)$$

(Hint: A similar result for the inverse Fourier transform is shown in the sampling theorem section)

Now,

$$X_d(\omega) = X(\omega) * \left(\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta\left(\omega - k\frac{2\pi}{T}\right)\right)$$

$$= \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} X(\omega) * \delta\left(\omega - k\frac{2\pi}{T}\right)$$

$$= \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} X(\omega) * \delta\left(\frac{1}{T}(\omega T - k2\pi)\right)$$

$$= \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} X(\omega) * T\delta(\omega T - k2\pi)$$

Since $\delta(px) = \frac{1}{p}\delta(x)$.

$$X_d(\omega) = 2\pi \sum_{k=-\infty}^{\infty} X(\omega) * \delta(\omega T - k2\pi)$$

But we earlier defined the DTFT as:

$$\hat{x}(\theta) = X_d(\omega)|_{\omega=\frac{\theta}{T}}$$

And

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{-j\frac{\theta}{T}t}dt = X(\theta)$$

Thus,

$$\hat{x}(\theta) = \sum_{k=-\infty}^{\infty} X(\theta) * \delta(\theta - k2\pi)$$

$$\underline{\hat{x}(\theta) = \sum_{k=-\infty}^{\infty} X(\theta - k2\pi)}$$

This indicates that $\hat{x}(\theta)$ is periodic with a period of 2π and is obtained as the sum of the analogue spectrum, $X(\theta)|_{\theta=\omega T}$ repeated at every 2π interval.

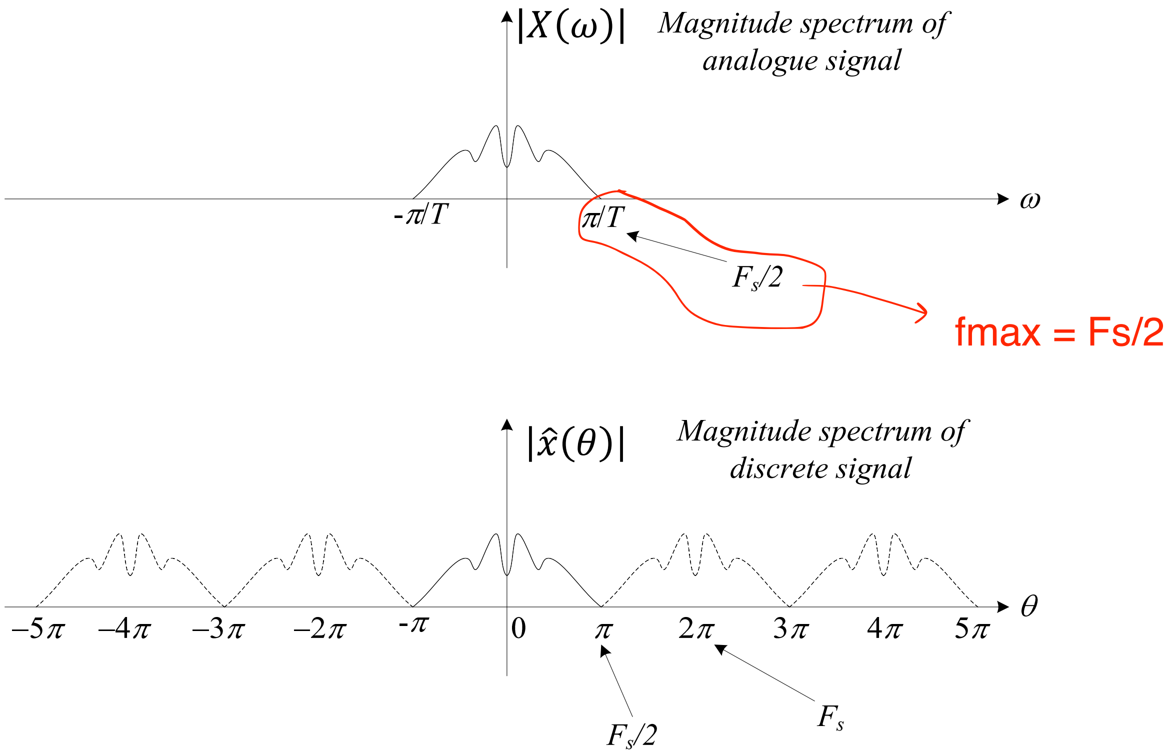


Figure 2.5: Magnitude spectra of continuous and discrete signals ($\omega = 2\pi f$ and $\theta = 2\pi f / F_s$). Here the lowest possible F_s is chosen

Note: This is not surprising since we observed that periodisation of the spectrum resulted in sampling by the Dirac comb when looking at the sampling theorem. This is just the reverse showing that sampling be a Dirac comb results in periodisation of the spectrum.

2.2.1 Aliasing

From the above discussion it can be seen that if the sampling rate, F_s, is not high enough neighbouring repetitions overlap (and add) and distort the signal. i.e., if $q < p$ in the discussion of the sampling theorem (section 2.1.2), periodisation causes overlap between the copies of the spectra and consequently multiplying, $X_q(\omega)$, with the rectangular function. $\Pi_q(\omega)$ does not return the spectrum of the original signal.

$$\Pi_q(\omega) \cdot X_q(\omega) \neq X(\omega), \qquad q < p$$

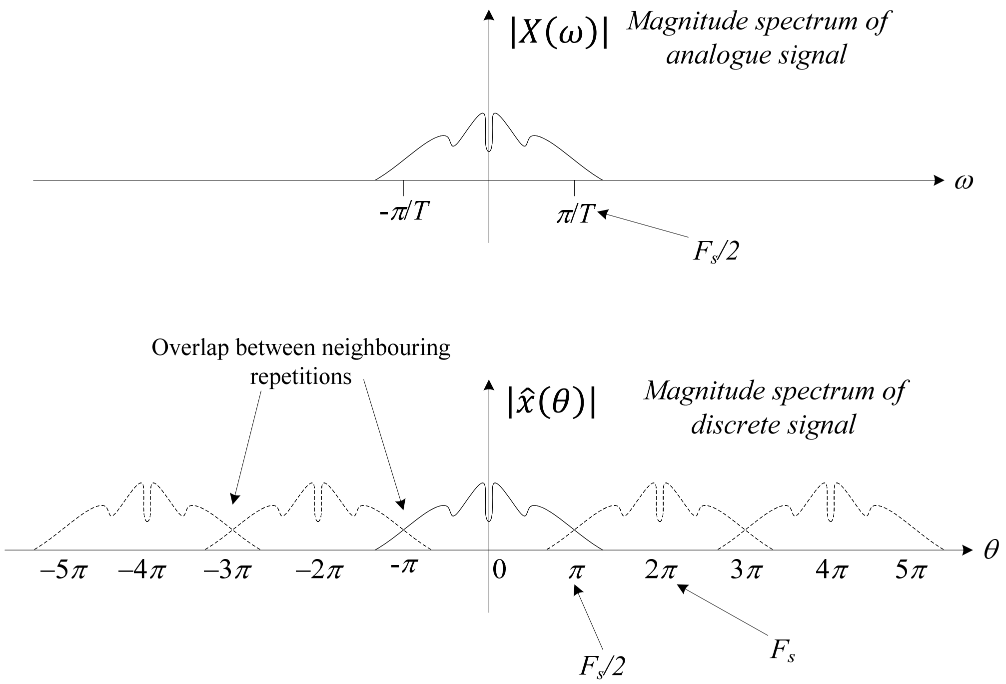


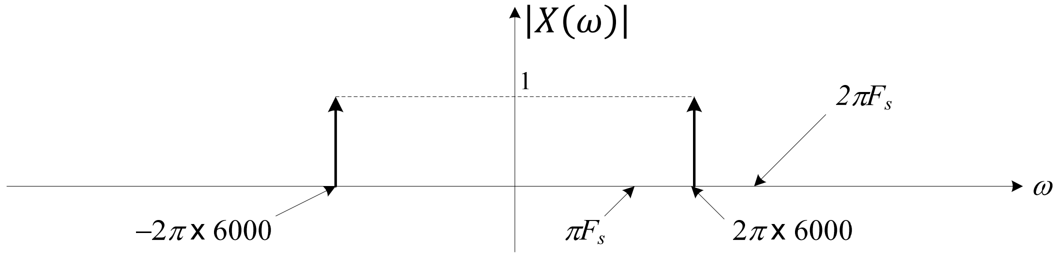
Figure 2.6: An example of Aliasing

In this case, the overlap in the spectrum causes part of it to fold over and alias back into itself. This is elaborated further after the discussion on the spectrum of discrete signals. This is known as aliasing.

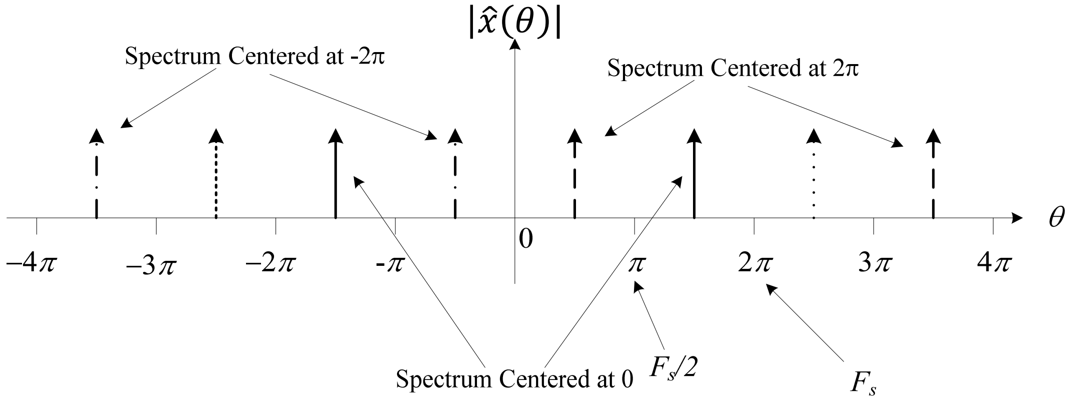
Example 2.3

If a cosine wave of frequency, $f = 6000\ Hz$ is sampled at a rate of $F_s = 8000\ Hz$. Sketch the spectrum of the analogue and discrete signals.

Magnitude spectrum of analogue signal



Magnitude spectrum of discrete signal



2.2.2 Inverse DTFT

In order to find the inverse DTFT, consider the integral,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{x}(\theta)e^{jn\theta}\,d\theta$$

and substitute the DTFT for $\hat{x}(\theta)$

$$\hat{x}(\theta) = \sum_{m=-\infty}^{\infty} x[m]e^{-jm\theta}$$

This gives,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{x}(\theta)e^{jn\theta}d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\sum_{m=-\infty}^{\infty}x[m]e^{-jm\theta}\right)e^{jn\theta}d\theta$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}x[m]\left(\int_{-\pi}^{\pi}e^{j\theta(m-n)}d\theta\right)$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}x[m]\left(\left[\frac{e^{j\theta(m-n)}}{j(m-n)}\right]_{-\pi}^{\pi}\right)$$

$$= \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}x[m]\left(\frac{e^{j\pi(m-n)}-e^{-j\pi(m-n)}}{j(m-n)}\right)$$

Since m and n are integers, $m-n$ is an integer and hence,

$$\frac{e^{j\pi(m-n)}-e^{-j\pi(m-n)}}{j(m-n)} = 0, \qquad m \neq n$$

when $m = n$, the L'hospitals rule is applied and

$$\frac{e^{j\pi(m-n)}-e^{-j\pi(m-n)}}{j(m-n)} = 2\pi, \qquad m = n$$

Thus,

$$\frac{e^{j\pi(m-n)}-e^{-j\pi(m-n)}}{j(m-n)} = 2\pi \cdot \delta_{m,n}$$

Where,

$$\delta_{m,n} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$$

and is referred to as the Kronecker delta.

This gives the inverse DTFT

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{x}(\theta)e^{jn\theta}d\theta = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty}x[m](2\pi\delta_{m,n})$$

$$= x[n]$$

Thus, the DTFT and its inverse are

$$\hat{x}(\theta) = \sum_{n=-\infty}^{\infty} x[n]e^{-jn\theta}$$

$$x[n] = \frac{1}{2\pi}\int_{\pi}^{\pi} \hat{x}(\theta)e^{jn\theta}d\theta$$

2.2.3 Properties of the DTFT

Linearity	$ax[n] + by[n] \overset{DTFT}{\longleftrightarrow} a\hat{x}(\theta) + b\hat{x}(\theta)$
Frequency - shift	$e^{jkn}x[n] \overset{DTFT}{\longleftrightarrow} \hat{x}(\theta - k)$
Time – shift	$x[n - n_0] \overset{DTFT}{\longleftrightarrow} e^{-j\omega n_0}\hat{x}(\theta)$

2.2.4 Anti-Aliasing Filter

In practice aliasing is avoided by choosing a suitable high sampling rate based on your knowledge of the signals involved. For example, the highest audible frequency is around 20kHz (for children and generally this reduces with age), consequently in most audio signal processing system frequencies above this need not be preserved, which in turn suggests a suitable choice of sampling rate would be a value greater than 40kHz. However, increasing the sampling rate also increases computational costs and a trade-off needs to be made. Finally, in most systems an analogue low-pass filter is employed to limit the maximum frequency of the input signal to the Nyquist rate (half the sampling rate). This filter is referred to as an anti-aliasing filter.