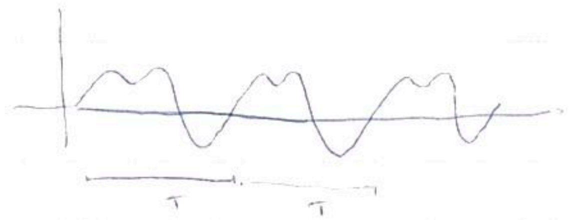
Fourier Series (continuous Time).

Periodic Signals/functions

$$f(t) = f(t+T) \quad \forall t \qquad \text{for some value of } T.$$

Smallest such T is the fundamental Period

eg.



Fourier Series

If a periodic function $f(t)$ with period (fundamental) T_0 satisfies the "Dirichlet" conditions, i.e.,

1. $f(t)$ is single valued everywhere.
2. Absolutely Integrable over one period:
$$\int_a^{a+T_0} |f(t)|\, dt < \infty.$$
3. Has a finite number of extrema within each period,
4. Has at most a finite number of finite discontinuities within each period.

then $f(t)$ can be written as a Fourier Series, i.e., a sum of sinusoids:

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \phi_n)$$ ← NB. the most useful form.

where $\omega_0 = \frac{2\pi}{T}$

Note on Sinusoids

$$A\cos(\omega t + \phi) = A\cos\omega t\cos\phi - A\sin\omega t\sin\phi$$

Generic sinusoid

Amplitude – A

Frequency – ω (or $f = \omega/(2\pi)$)

Phase shift – ϕ

$$= a\cos\omega t + b\sin\omega t$$

where $a = A\cos\phi$

$b = -A\sin\phi$.

Also, $a\cos(\omega t + \phi_1) + b\sin(\omega t + \phi_2)$

$$= a\cos\phi_1\cos\omega t - a\sin\phi_1\sin\omega t + b\sin\phi_2\cos\omega t + b\cos\phi_2\sin\omega t$$

$$= \underbrace{(a\cos\phi_1 + b\sin\phi_2)}_{\hat{a}}\cos\omega t + \underbrace{(b\cos\phi_2 - a\sin\phi_1)}_{\hat{b}}\sin\omega t$$

All equivalent representation of any sinusoid

Finally, $a\cos\omega t + b\sin\omega t = a\left(\frac{e^{j\omega t} + e^{-j\omega t}}{2}\right) + b\left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right)$

$$= \underbrace{\left(\frac{a - jb}{2}\right)}_{c_+} e^{j\omega t} + \underbrace{\left(\frac{a + jb}{2}\right)}_{c_+^*} e^{-j\omega t}$$

← complex conjugate of c

$$= c_+ e^{j\omega t} + c_+^* e^{-j\omega t}$$

Note: z^* & $\bar{z}$ both represent complex conjugate of z

Representation of Fourier Series

$$f(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 t + \Phi_n)$$

fundamental period
$T_0 = \frac{2\pi}{\omega_0}$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\omega_0 t + b_n \sin n\omega_0 t\right)$$

$$= C_0 + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t} + C_{-n} e^{-jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$ ⟵ Most useful form of Fourier Series.

Does not have to be continuous or differentiable (just satisfy Dirichlet conditions)

Each term of summation is ~~both~~ continuous & differentiable.

They reconcile because the sum is infinite.

Infinite sums → Cannot change order without justification
↳ Convergence issues may arise.

∴ If $x(t)$ is not ~~continuous~~ differentiable then ~~sum~~ Fourier series **must** have infinite terms.

Fourier Series - synthesis & Analysis

$$f(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

↳ Synthesis Equation. → Tells you how to "synthesise" $f(t)$ as a sum of its complex sinusoid components.

How to determine the components? i.e., how to determine C_n?
↳ Fourier series coefficients

We make use of the following property of complex exponentials / sinusoids,

$$\frac{1}{T_0}\int_a^{a+T_0} e^{jn\omega_0 t} e^{-jk\omega_0 t}\,dt = \frac{1}{T_0}\int_a^{a+T_0} e^{j(n-k)\omega_0 t}\,dt$$

$$= \frac{1}{T_0\, j(n-k)\frac{2\pi}{T_0}}\left[e^{j(n-k)\frac{2\pi}{T_0}(a+T_0)} - e^{j(n-k)\frac{2\pi}{T_0}a}\right]$$

$$= \frac{e^{j(n-k)\frac{2\pi a}{T_0}}}{j(n-k)2\pi}\left[e^{j(n-k)2\pi} - 1\right]$$

~~when n-k~~

$$= 0 \quad \text{when } n-k \neq 0$$

When $n-k=0$,

$$\frac{1}{T_0}\int_a^{a+T_0} e^{jn\omega_0 t} e^{-jk\omega_0 t}\,dt = \frac{1}{T_0}\int_a^{a+T_0} dt$$

$$= 1.$$

ie,
$$\frac{1}{T_0}\int_a^{a+T_0} e^{jn\omega_0 t} e^{-jk\omega_0 t} = \begin{cases} 0, & n \neq k \\ 1, & n = k. \end{cases}$$

$$\frac{1}{T_0}\int_a^{a+T_0} f(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T_0}\int_a^{a+T_0} e^{-jk\omega_0 t} \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}\, dt.$$

$$= \sum_{n=-\infty}^{\infty} C_n \frac{1}{T_0}\int_a^{a+T_0} e^{-jk\omega_0 t}\, e^{jn\omega_0 t}\, dt.$$

$$= C_k.$$

ie, $$C_n = \frac{1}{T_0}\int_a^{a+T_0} f(t)\, e^{-jn\omega_0 t}\, dt$$

Analysis equation.

↓

Analyse $f(t)$ to determine its components.

<u>Fourier Series Coefficient</u>

As long as $f(t)$ is real valued..

$$C_n = C_{-n}^{*} \quad \text{(complex conjugate pair)}$$

Remembering, $A\cos(\omega t + \phi) = a\cos\omega t + b\sin\omega t$

where, $a = A\cos\phi$ & $b = -A\sin\phi$.

or $\phi = \tan^{-1}(-b/a)$ ←

Also, $a\cos\omega t + b\sin\omega t = C_+ e^{j\omega t} + C_- e^{-j\omega t}$

where $C_+ = \frac{a-jb}{2}$, $C_- = \frac{a+jb}{2}$

$$\angle C_+ = \tan^{-1}(-b/a) = \phi \;\leftarrow$$

Finally, $|C_+|^2 = |C_-|^2 = \dfrac{a^2+b^2}{4}$

$$= \frac{A^2}{4}(\cos^2\phi + \sin^2\phi)$$

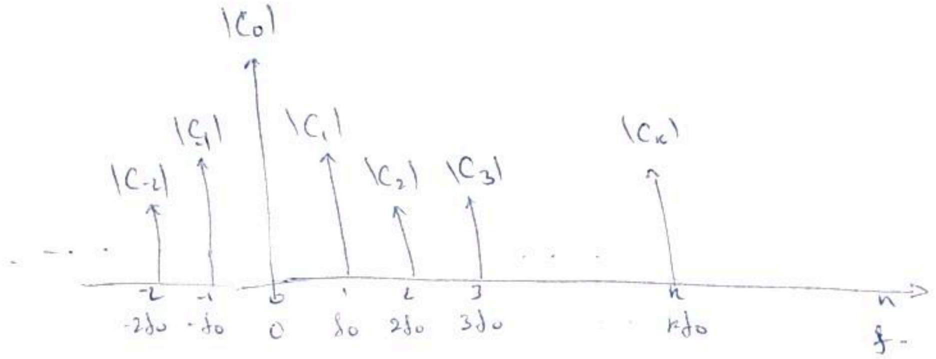
$$= A^2/4. \leftarrow$$ Amplitude square of sinusoid (scaled by 1/4).

or $|C_+| = |C_-| = \dfrac{|A|}{2}$.

$\leftarrow$ Sinusoid is split equally into two complex exponentials

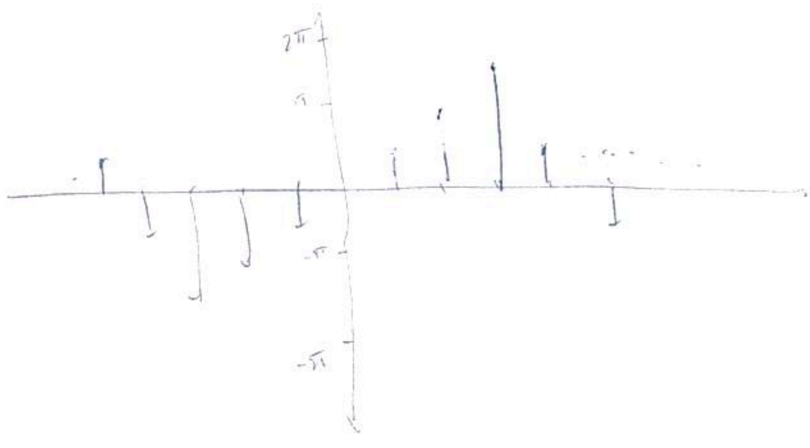
Magnitude Spectrum

$|C_n|$ vs n. or $|C_n|$ vs frequency ($n\omega_0$). (nf_0)



Phase Spectrum

$\angle C_n$ vs n or $\angle C_n$ vs frequency (nf_0 or $n\omega_0$)



Fourier Transform (continuous Time).

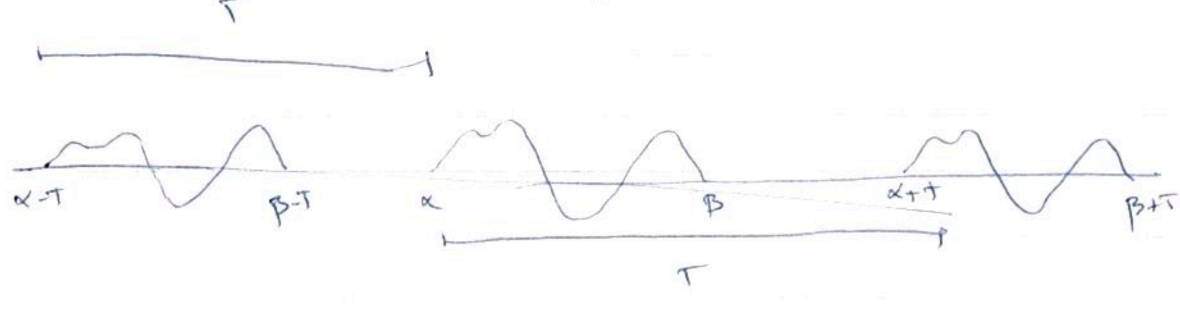
Consider any aperiodic signal/function with finite support

↳ function is zero outside some finite interval $[\alpha, \beta]$

$x(t)$.

'Periodise' by repeating at regular intervals T

$x_T(t)$.



$$x_T(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

Periodic with fundamental period T

$$\omega_0 = \frac{2\pi}{T}.$$

Consider $X(\omega_n) = T C_n = \int_a^{a+T} x_T(t)\, e^{-jn\omega_0 t}\, dt$.

where $\omega_n = n\omega_0$ represents the frequency.

$$\Rightarrow X(\omega_n) = T C_n = \int_{\left(\frac{\alpha+\beta}{2}\right) - \frac{T}{2}}^{\left(\frac{\alpha+\beta}{2}\right) + T/2} x_T(t)\, e^{-jn\omega_0 t}\, dt.$$

in $\left[\left(\frac{\alpha+\beta}{2}\right) - \frac{T}{2}, \left(\frac{\alpha+\beta}{2}\right) + \frac{T}{2}\right]$, $x_T(t) = x(t)$.

$$\therefore X(\omega_n) = \int_{\left(\frac{\alpha+\beta}{2}\right) - \frac{T}{2}}^{\left(\frac{\alpha+\beta}{2}\right) + \frac{T}{2}} x(t)\, e^{-jn\omega_0 t}\, dt$$

Let $T \to \infty \Rightarrow \omega_0 \to 0$.

$\Rightarrow$ spacing between consecutive $\omega_n \to$

ie $\omega_n - \omega_{n-1} \to 0$.

$\Rightarrow X(\omega_n)$ becomes continuous.

ie, as $T \to \infty$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$\hookrightarrow$ Fourier transform Analysis equation

Since

$$x_T(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}$$

$$= \sum_{n=-\infty}^{\infty} \frac{1}{T} X(\omega_n) e^{jn\omega_0 t} \qquad T = \frac{2\pi}{\omega_0}$$

$$= \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} X(\omega_n) e^{jn\omega_0 t} \omega_0 .$$

as $T \to \infty$, $\omega_0 \to 0$, $x_T(t) \to x(t)$ & $\omega_n \to \omega$
(as before ω becomes continuous)

$$\Rightarrow x(t) = \frac{1}{2\pi} \lim_{\omega_0 \to 0} \underbrace{\sum_{n=-\infty}^{\infty} f(\omega_n) \omega_0}_{\text{Reimann Integral.}}$$

$\hookrightarrow f(\omega_n) = X(\omega_n) e^{j\omega_n t}$ $(\omega_n = n\omega$

$$\Rightarrow x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

$\longrightarrow$ Fourier Transform Synthesis Equation

<u>Properties of Fourier Transform</u>

- Linearity $\mathcal{F}\{ax + by\} = a F_x + b F_y$.
- Freq Shifting $e^{j\omega_0 t} x(t) \longleftrightarrow X(\omega - \omega_0)$.
- Time Shift $x(t - t_0) \longleftrightarrow e^{-j\omega t_0} X(\omega)$
- Scaling $x(at) \longleftrightarrow \frac{1}{|a|} X(\omega/a)$
- Diff in time $\frac{d}{dt} x(t) \longleftrightarrow j\omega X(\omega)$.
- Diff in freq $-jt\, x(t) \longleftrightarrow \frac{d}{d\omega} X(\omega)$
- Convolution in Time $x * y \longleftrightarrow X(\omega) Y(\omega)$