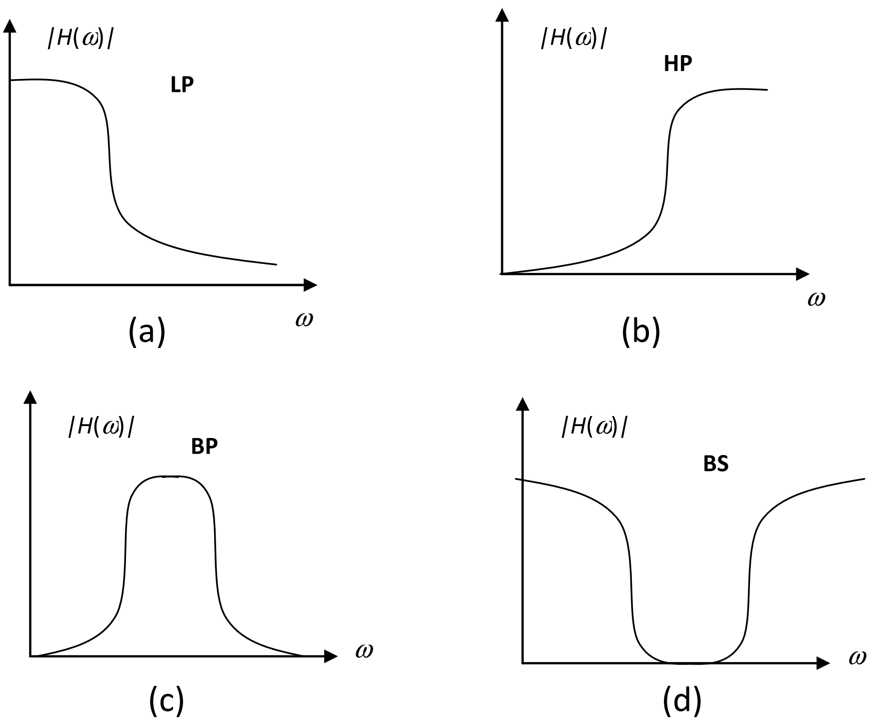
ELEC3104: Digital Signal Processing

Chapter 7: Filter Design

The commonly required types of frequency filter fall into four main groups:

- Low-Pass Filter (LP)
- High-Pass Filter (HP)
- Band-Pass Filter (BP)
- Band-Stop Filter (BS)

The frequency responses of these filters are shown in Figure 7.1 (a to d). Also shown are the frequency responses for the **ideal** LP, HP, BP and BS filters.



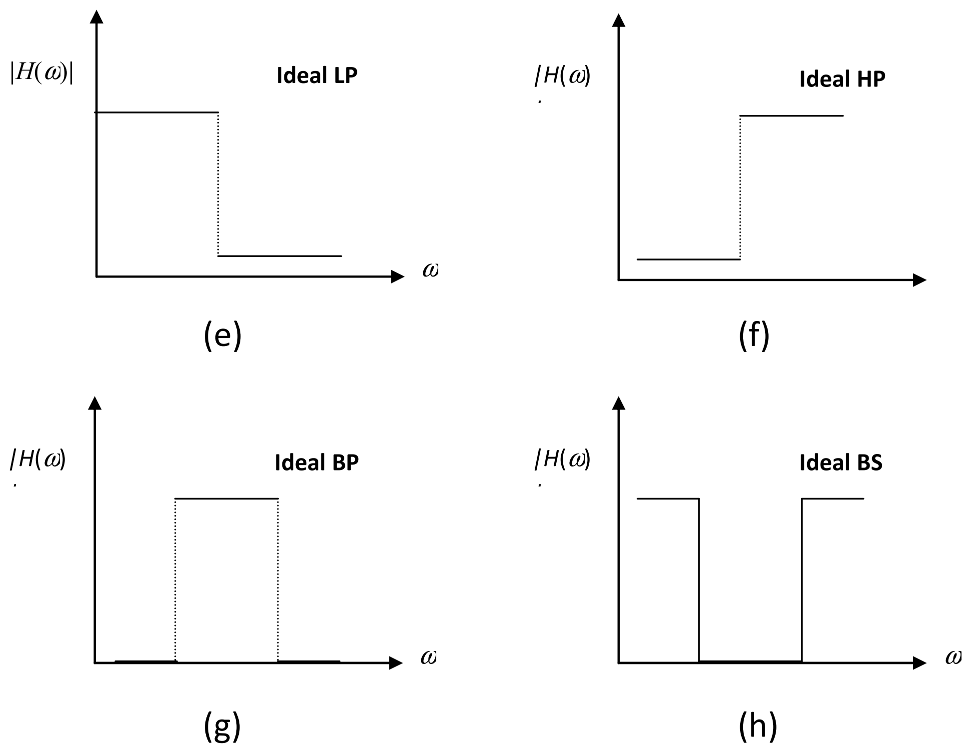
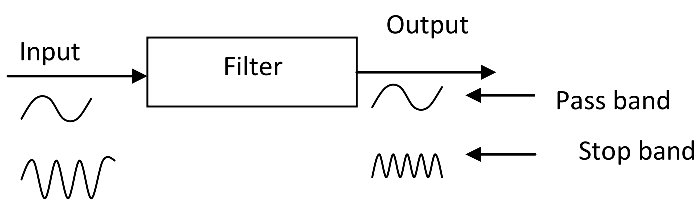


Figure 7.1: Basic types of frequency Responses. (a)-(d) show examples of practical frequency response of lowpass, highpass, bandpass and bandstop filters, respectively. (e)-(h) show the ideal frequency responses of the above filters.

The filter design procedure normally begins with a specification of the frequency response for the filter describing how the filter reacts to sinusoid inputs. If an input sinusoid is not attenuated or attenuated less than a specific tolerance as it goes through the system, it is said to be in a **pass band** of the filter.



If it is attenuated more than a specified value it is said to be stopped and within the **stop band** of the filter.

Input sinusoids with neither a little nor a large amount of attenuation are said to be in the **transition band**. A typical frequency response is shown in Figure 7.2, showing the pass band, transition band and stop band.

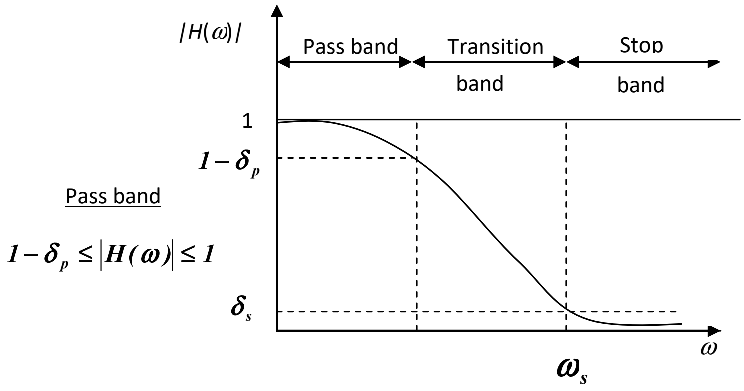
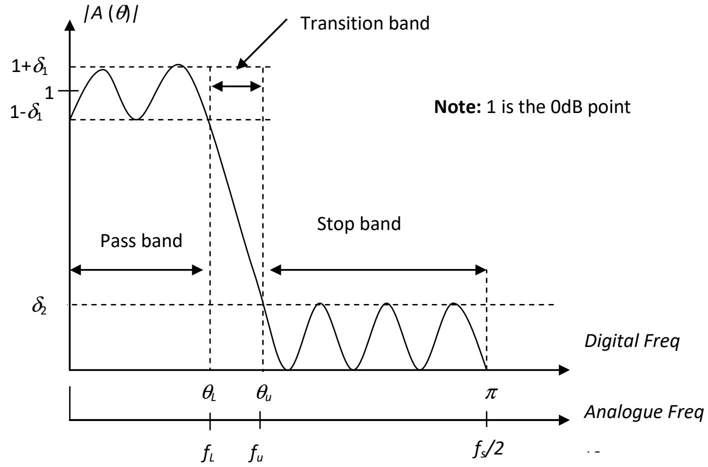


Figure 7.2: Low-pass filter frequency response

The filter with this type of frequency response is called a low-pass filter as it passes all frequencies less than a certain value ω_p (or ω_c), called the cut-off frequency and attenuates or stops all frequencies past ω_s, the stop band frequency.

The design of a digital filter is the task of determining a transfer function which is a rational function of z^{-1} (e.g., $\frac{a_0+a_1z^{-1}+a_2z^{-2}}{1+b_1z^{-1}+b_2z^{-2}}$) in the case of a recursive filter (IIR) or a polynomial in z^{-1} (e.g., $a_0 + a_1z^{-1} + a_2z^{-2}$) in the case of a non-recursive filter. A typical magnitude characteristic of a low-pass filter is shown below.



The goal of the design is to determine a transfer function $H(z)$ such that its magnitude characteristics, $|H(\theta)|$, satisfies the conditions:

$$1 - \delta_1 \leq |H(\theta)| \leq 1 + \delta_1, \qquad 0 \leq \theta \leq \theta_L$$
$$|H(\theta)| \leq \delta_2, \qquad \theta_U \leq \theta \leq \pi$$

7.1.1 Choosing between FIR and IIR filters

FIR Filters	IIR Filters
System function contain only zeros	Contain poles and zeros (normally)
Non-recursive or recursive structures are both possible; the best known is the non-recursive (transversal) structure.	Only recursive structure is possible; the most widely used form is the cascade connection of first-order and second order sections.
FIR Filters can have an exactly linear phase response. The implication of this is that no phase distortion is introduced into the signal by the filter.	The phase responses of IIR filters are nonlinear, especially at the band edges.
The effects of using a limited number of bits to implement filters such as round off noise and quantization errors are much less severe in FIR than in IIR.	Because of quantization of the filter coefficients, a pole can in principle move from a position inside the unit circle to a position outside the unit circle and hence cause instability.
FIR requires more coefficients for sharp cut-off filters than IIR. Thus for a given amplitude response specification, more processing time and storage will be required for FIR implementation.	IIR requires fewer coefficients for sharp cut off filters than FIR.
Complexity is proportional to the length of the impulse response.	No direct relation between the complexity and the length of the impulse response (which is infinite by definition) Filters with high selectivity can be realized with relatively low complexity.
FIR filters have no analogue counterpart. FIR design procedures are normally iterative procedures. Design equations do not exist.	Analogue filters can be readily transformed into equivalent IIR digital filters meeting similar specifications. IIR filters can be designed using design formulae.

Example 7.1

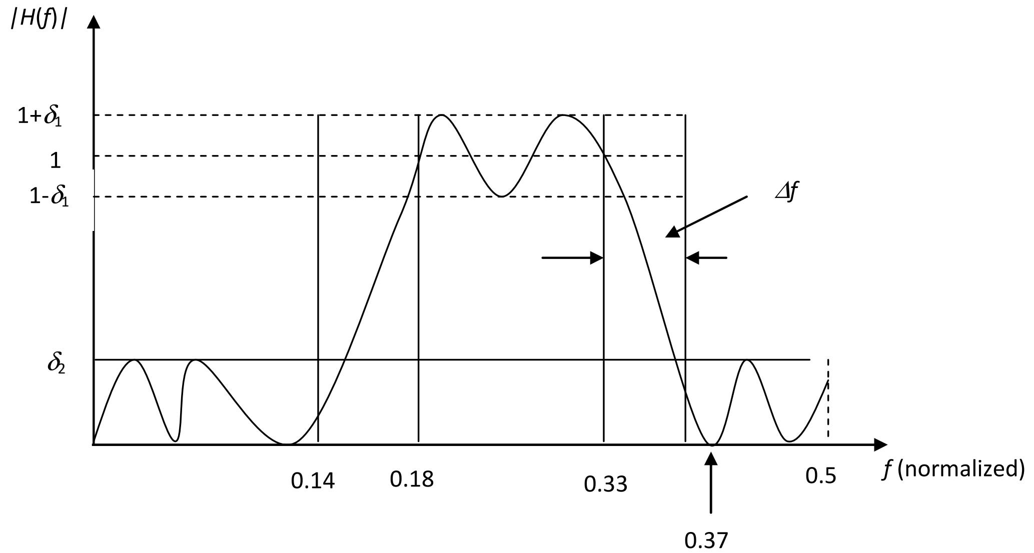
An FIR filter is to be designed to meet the following frequency response specifications.

- Pass band 0.18-0.33 (normalized)
- Transition band 0.04 (normalized)
- Stop-band deviation 0.001
- passband deviation 0.05

(i) Sketch the tolerance scheme for the filter.

(ii) Express the filter band edge frequencies in the standard unit of kHz, assuming a sampling frequency of 10 kHz and the stop band and pass band deviation in dBs.

The tolerance diagram for the filter is shown below:



Since the sampling rate, $F_S = 10kHz$

- Pass band: 1.8 – 3.3 kHz
- Stop band: 0-1.4kHz and 3.7 – 5 kHz
- Stop band attenuation: $-20\log_{10}(0.001) = -60dB$
- Pass band ripple: $20\log_{10}(1 + 0.05) = 0.42dB$

Example 7.2

The following transfer functions represent two different filters that meet the same magnitude response specifications.

$$H_1(z) = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{1 + b_1 z^{-1} + b_2 z^{-2}}$$

where, $a_0 = 0.4981819$; $a_1 = 0.9274777$; $a_2 = 0.4981819$; $b_1 = -0.6744878$ and $b_2 = -0.3633482$.

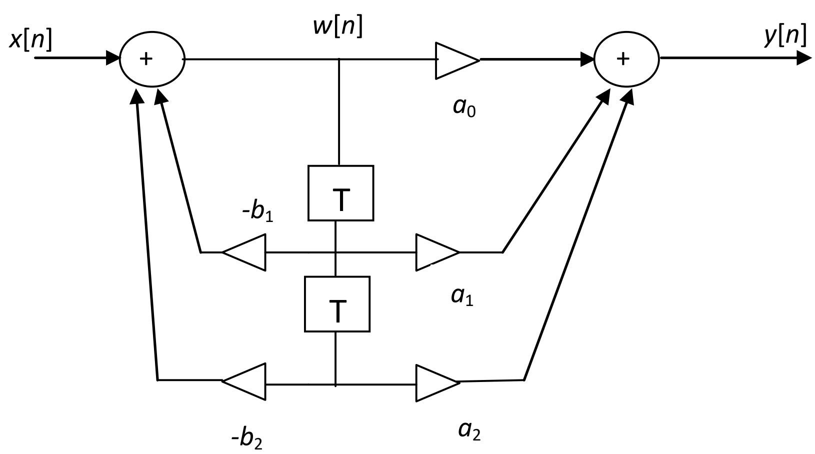
$$H_2(z) = \sum_{k=0}^{11} h[k] z^{-k}$$

where,

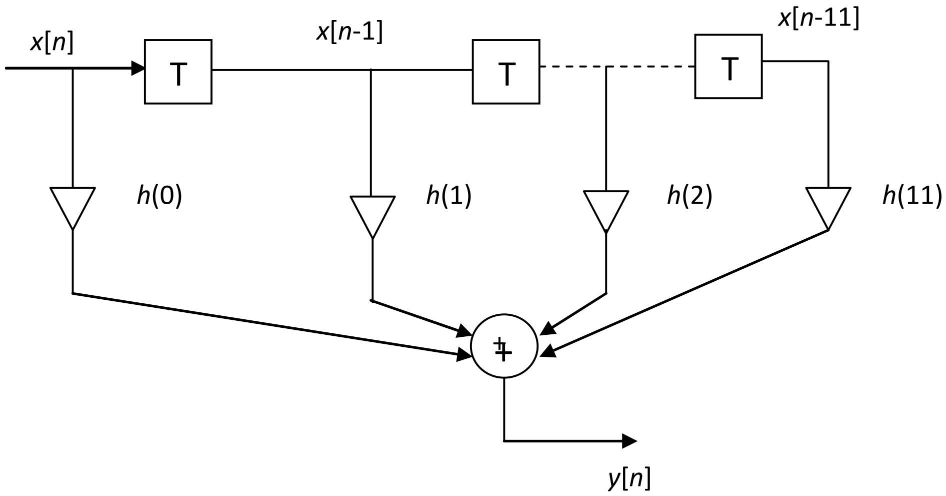
$$h[0] = 0.54603280 \times 10^2 = h[11]$$
$$h[1] = -0.45068750 \times 10^{-1} = h[10]$$
$$h[2] = 0.69169420 \times 10^{-1} = h[9]$$
$$h[3] = -0.55384370 \times 10^{-1} = h[8]$$
$$h[4] = -0.63428410 \times 10^{-1} = h[7]$$
$$h[5] = 0.57892400 \times 10^{0} = h[6]$$

Determine and comment on the computations and storage requirements.

Filter 1:



Filter 2:



	FIR (Filter 2)	IIR (Filter 1)
Number of Multiplications	12	5
Number of additions	11	4
Storage locations (coefficients and data)	24	8

It is evident that the IIR filter is more economical in both computations and storage requirements than the FIR filter. In this chapter we will look at

- FIR filter design directly from the specification in the frequency domain
- IIR filter design by first designing a suitable analogue filter and transforming it to a digital filter.

7.1.2 Design Techniques

The method used to calculate the filter coefficients depends on whether the filter is an IIR or a FIR type filter. There are several methods of calculating filter coefficients of which the following are the most widely used.

FIR digital filters	IIR digital filters
Window	Impulse invariant transformation
Frequency sampling	Bilinear transformation
Optimisation method (e.g Remez Algorithm)	Pole-zero placement method

We chose the method that best suits out particular application.

In most cases, if the FIR properties are vital then a good candidate is the optimization method, where as, if IIR properties are desirable, then the bilinear method will in most cases suffice.

7.2 FIR Filter Design

The most essential feature of FIR filters is, by definition, the finite length of the impulse response. Another important point is that it can be seem directly from the impulse response of an FIR filter whether we have a linear phase characteristic or not.

A filter is said to have a linear phase response of its response satisfies one of the following relationships.

$$\phi(\theta) = -a\theta$$
$$\phi(\theta) = b - a\theta$$

where a and b are constants.

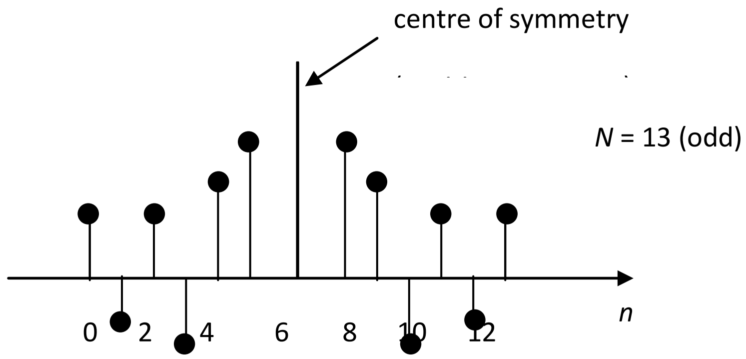
It can be shown that the first condition is satisfied when the impulse response of the filter has positive symmetry. i.e.,

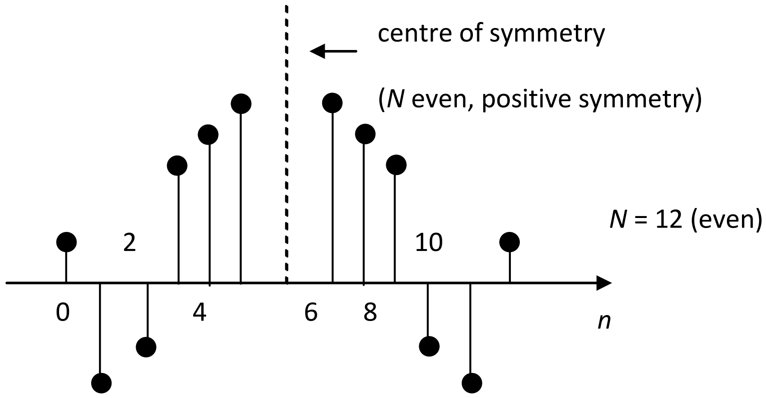
$$h[n] = h[N - n - 1], \qquad a = \frac{N-1}{2}$$

where N denotes the filter length

The second condition is satisfied when the impulse response has negative symmetry. i.e.,

$$h[n] = -h[N-n-1], \qquad a = \frac{N-1}{2} \; and \; b = \frac{\pi}{2}$$





7.2.1 Design of FIR filters using windows

The easiest way to obtain an FIR filter is to simply truncate the impulse response of an IIR filter. If $h_d[n]$ represents the impulse response of the desired IIR filter then a FIR filter, with impulse response $h[n]$, can be obtained as follows:

$$h[n] = \begin{cases} h_d[n], & N_1 \le n \le N_2 \\ 0, & otherwise \end{cases}$$

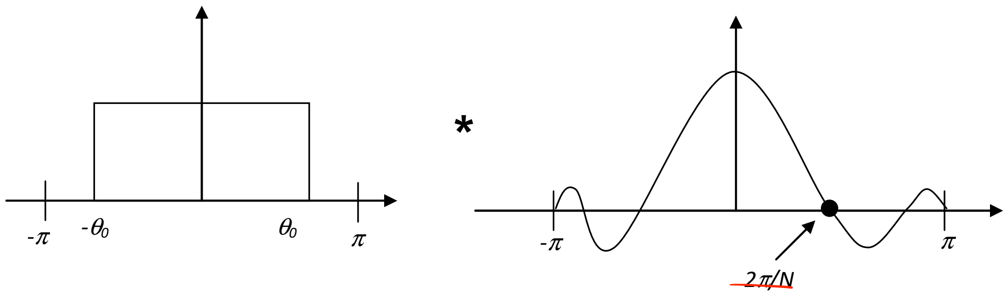
In general, $h[n]$ can be though of as being formed by the product of $h_d[n]$ and a window function $w[n]$.

$$h[n] = h_d[n] \cdot w[n]$$

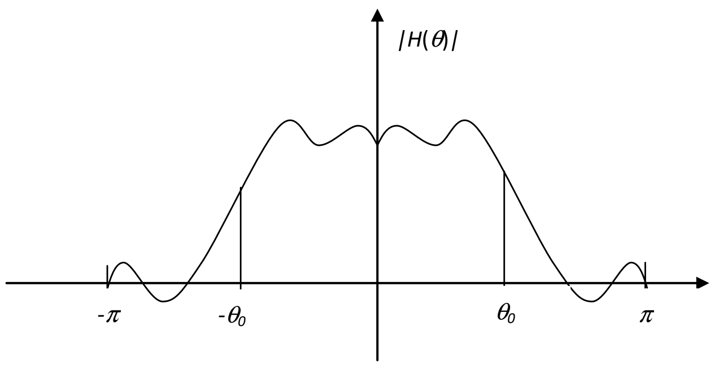
Equivalently,

$$\hat{h}(\theta) = \hat{h}_d(\theta) * \hat{w}(\theta)$$

Now, if $h_d[n]$ was an ideal low-pass filter with cut-off frequency θ_0 and $w[n]$ was a rectangular window,



an extreme case is that the main lobe is reduced to 0 so the sinc is now more like a dirac delta.which will give us an ideal FIR filter



Therefore it is seen that the convolution produces a smeared version of the ideal low pass frequency response $\hat{h}_d(\theta)$.

In general, the wider the main lobe of $\hat{w}(\theta)$, the more spreading, where as the narrower the main lobe (larger N) the closer $|\hat{h}(\theta)|$ comes to $|\hat{h}_d(\theta)|$.

We are left with a trade off of making N large enough so that smearing is minimized, yet small enough to allow reasonable implementation.

7.2.2 Design Procedure

An ideal low pass filter with linear phase of slope -β and cut-off ω_c can be characterized in the frequency domain by

unit step function with phase shift

$$\hat{h}_d(\theta) = \begin{cases} e^{-j\theta\beta}, & |\theta| < \theta_c \\ 0, & \theta_c < |\theta| \leq \pi \end{cases}$$

The corresponding impulse response $h_d[n]$ can be obtained by taking the inverse Fourier transform of $H_d(\theta)$ and easily shown to be

$$h_d[n] = \frac{\sin[\theta_c(n-\beta)]}{\pi(n-\beta)}$$

A causal FIR filter with impulse response $h[n]$ can be obtained by multiplying $h_d[n]$ by a window beginning at the origin and ending at $N-1$.

$$h[n] = \frac{\sin[\theta_c(n-\beta)]}{\pi(n-\beta)} w[n]$$

For $h[n]$ to be a linear phase filter, β must be selected so that the resulting $h[n]$ is symmetric. As $\frac{\sin[\theta_c(n-\beta)]}{\pi(n-\beta)}$ is symmetrical about $n = \beta$ and the window symmetric about $n = \frac{N-1}{2}$, $h[n]$ can be made symmetric by setting

$$\beta = \frac{N-1}{2}$$

Example 7.3

(a) Determine the impulse response $h_d[n]$ of the lowpass filter whose frequency response is given by

$$H(\theta) = \begin{cases} 1, & 0 \leq |\theta| \leq \frac{\pi}{3} \\ 0, & \frac{\pi}{3} < |\theta| \leq \pi \end{cases}$$

(b) To obtain a finite impulse response from $h_d[n]$ a rectangular window of length $N = 9$ is used. Compute the coefficients of the FIR filter with a linear phase characteristic ands with this finite impulse response.

(a)

$$h_d[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\theta)e^{j\theta n}d\theta$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\theta n}}{jn}\right]_{-\frac{\pi}{3}}^{\frac{\pi}{3}} = \frac{1}{\pi}\left(\frac{\sin n\frac{\pi}{3}}{n}\right), \qquad n \neq 0$$

$$= \begin{cases} \frac{1}{3}, & n = 0 \\ \frac{\sin\left(n\frac{\pi}{3}\right)}{n\pi}, & n \neq 0 \end{cases}$$

Hopitail's law

(b) $h_d[n]$ is symmetric about $n = 0$ and the window is symmetric about $\frac{N-1}{2} = 4$. Hence a linear phase filter is obtained by shifting $h_d[n]$ by 4 samples prior to multiplying with the window function. (**Note:** This changes the phase characteristics of $h_d[n]$ but not the magnitude characteristics).

$$h[n] = h_d[n-4] \cdot w[n]$$

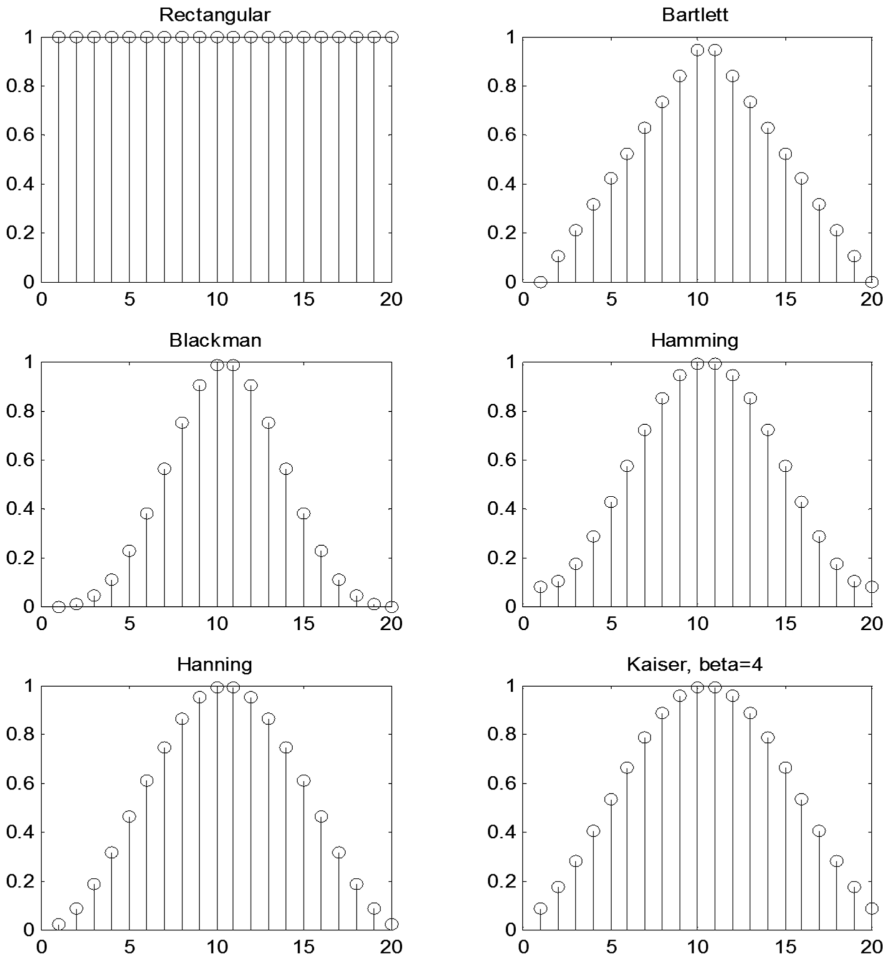
The coefficients are

	$-\frac{\sqrt{3}}{8\pi}$	0	$\frac{\sqrt{3}}{4\pi}$	$\frac{\sqrt{3}}{2\pi}$	$\frac{1}{3}$	$\frac{\sqrt{3}}{2\pi}$	$\frac{\sqrt{3}}{4\pi}$	0	$-\frac{\sqrt{3}}{8\pi}$
n	0	1	2	3	4	5	6	7	8

This is symmetric about $n = 4$.

7.2.3 Window Functions

Window functions are used to truncate a signal or impulse response to produce a signal or impulse response of finite duration. Some of the most commonly used window functions are:



7.2.4 Frequency-Sampling Filter

Although it is implied that all FIR filters are non-recursive, this is not the case. To illustrate the approach, let us consider the following FIR filter having a casual finite-duration unit-sample response containing *N* elements of constant value.

$$h[n] = \begin{cases} \dfrac{g_0}{N}, & 0 \le n \le N-1 \\ 0, & otherwise \end{cases}$$

The corresponding filter transfer function is

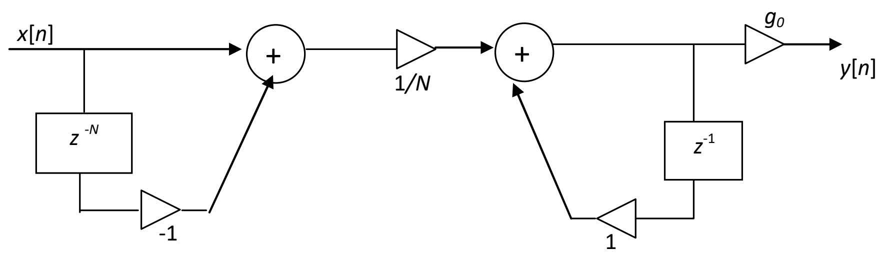
$$\begin{aligned} H(z) &= \frac{g_0}{N}\left[1 + z^{-1} + z^{-2} + \cdots + z^{-(N-1)}\right] \\ &= \frac{g_0}{N}\sum_{p=0}^{N-1} z^{-p} \\ &= \frac{g_0}{N}\frac{1 - z^{-N}}{1 - z^{-1}} \\ &= H_c(z) \cdot H_{R0}(z) \end{aligned}$$

where,

$$H_c(z) = \left(\frac{1 - z^{-N}}{N}\right)$$

$$H_{R0}(z) = \left(\frac{g_0}{1 - z^{-1}}\right)$$

This analytic form of the system function suggests a novel way to implement the above filter, as the two-stage cascade structure shown below.

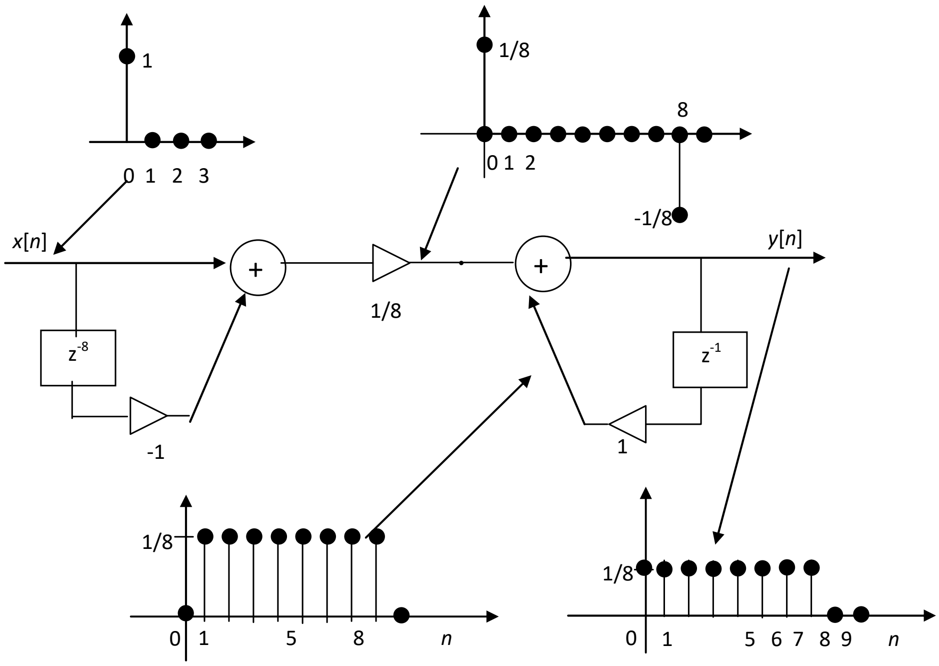


Comb filter Resonator

Example 7.4

Let $N = 8$ and $g_0 = 1$

$$H(z) = \frac{1 - z^{-8}}{8} \cdot \frac{1}{1 - z^{-1}}$$



Filter implementation using the comb filter resonator structure.

The comb filter, $H_c(z)$ is given by

$$H_c(z) = \frac{1 - z^{-N}}{N}$$

Therefore,

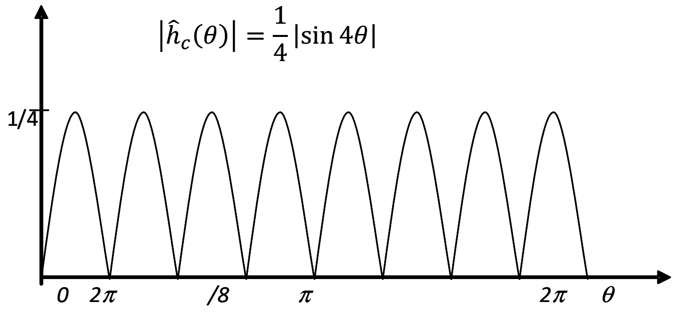
$$\hat{h}_c(\theta) = \frac{1 - e^{-j\theta N}}{N}$$

$$= \frac{2j}{N} e^{-j\frac{N\theta}{2}} \sin \frac{N\theta}{2}$$

$\hat{h}_c(\theta)$ has N zeros equally spaced over the frequency range $0 \le \theta \le 2\pi$. Because the magnitude response resembles a comb, this filter has become known as a **comb filter**.

When $N = 8$,

$$\hat{h}_c(\theta) = \frac{2j}{8} e^{-j\frac{8\theta}{2}} \sin \frac{8\theta}{2} = \frac{1}{4} j e^{-j4\theta} \sin 4\theta$$

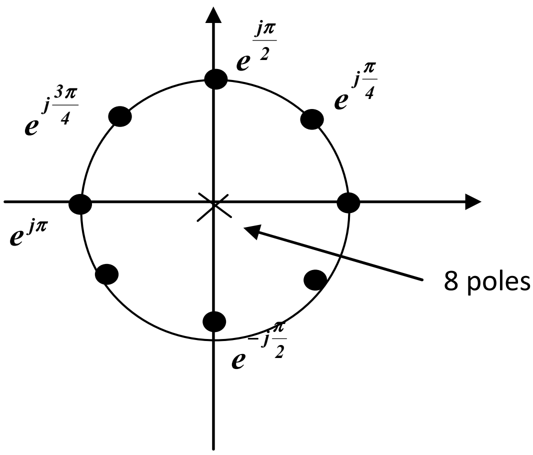


$|\hat{h}_c(\theta)|$ vs θ for a Comb filter

The pole/zero pattern for the comb filter can be obtained using the transfer function

$$H_c(z) = \frac{1 - z^{-N}}{N} = \frac{1}{N}\frac{z^N - 1}{z^N} = \prod_{k=0}^{N-1}\left(1 - e^{-j\frac{2\pi k}{N}}z^{-1}\right)$$

The above equation explicitly indicates that there are N equally spaced zeros over the unit circle with the first zero at $z = 1$.



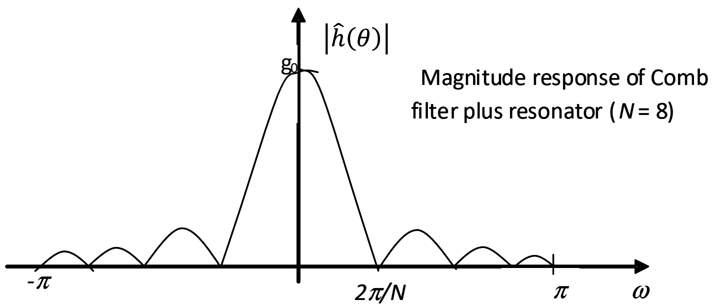
Pole/zero pattern for comb filter N = 8

$H_{R0}(z)$ is the system function of a resonator, or a filter that has poles on the unit circle at frequency $\omega = 0$ i.e. at $z = 1$. Since this pole is not strictly within the unit circle, the filter is not stable. When this resonator is cascaded with the comb filter, however, the zero of the comb filter located at $z = 1$ cancels the resonator pole, making the combination stable.

The frequency response of this two-stage filter is

$$\hat{h}(\theta) = \frac{g_0}{N}\frac{1 - e^{-j\theta N}}{1 - e^{-j\theta}} = \frac{g_0}{N}e^{-j\frac{(N-1)\theta}{2}}\frac{\sin\left(\frac{N\theta}{2}\right)}{\sin\frac{\theta}{2}}$$

For $N = 8$, $\left|\hat{h}(\theta)\right|$ has a maximum at $\theta = 0$, equal to g_0 and is equal to zero at $\theta_k = \frac{2\pi k}{N}$.



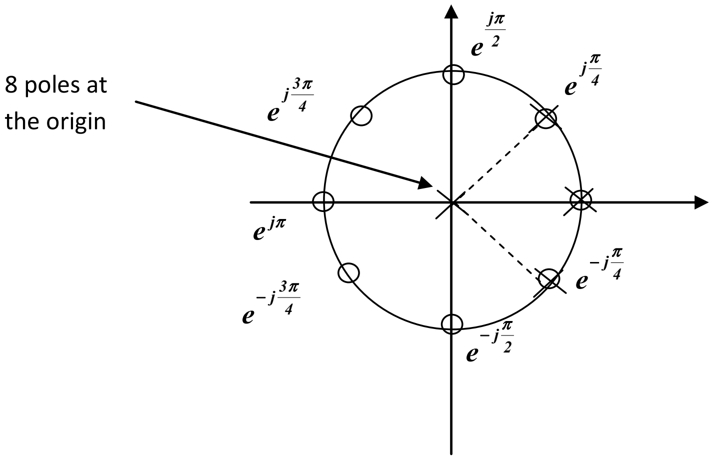
Let us consider a second-order resonator, whose coefficients are real-valued, and the poles are situated at the following zero locations of the comb filter.

$$\theta_1 = \frac{2\pi}{N}, \qquad \theta_2 = -\frac{2\pi}{N} = -\theta_1$$

The system function of the second-order resonator can be written in parallel form as

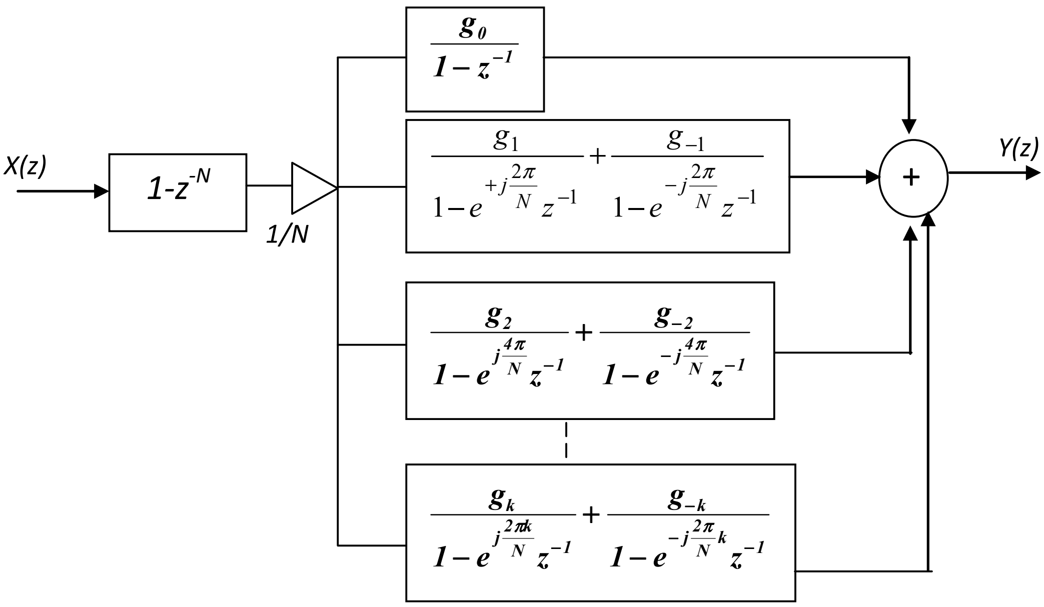
$$H_{R1}(z) = \frac{g_1}{1 - e^{\left(-j\frac{2\pi}{N}\right)}z^{-1}} + \frac{g_{-1}}{1 - e^{\left(j\frac{2\pi}{N}\right)}z^{-1}}$$

For $N = 8$, the pole zero plot of the system comprising of $H_{R0}(z)$ and $H_{R1}(z)$ in parallel cascaded with the comb filter, $H_c(z)$ is shown below.



This procedure can be generalized to include each second-order resonator cancelling a pair of zeros of the comb filter. These resonators are all connected in parallel and this parallel combination is then connected in cascade with the comb filter to produce the total filter $H_T(z)$ given by

$$H_T(z) = \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{g_k}{1 - e^{j\frac{2\pi k}{N}} z^{-1}}$$



$$\hat{h}\left(\frac{2\pi k}{N}\right) = g_k$$

$$\hat{h}\left(-\frac{2\pi k}{N}\right) = g_{-k}$$

Example 7.5

Lets us consider implementing the linear interpolator with the comb and resonator structure. The impulse response of the linear interpolator is given by

$$h[0] = \frac{1}{2}, \qquad h[1] = 1, \qquad h[2] = \frac{1}{2}$$
$$h[n] = 0, \qquad otherwise$$

The transfer function is

$$H(z) = \frac{1}{2} + z^{-1} + \frac{1}{2}z^{-2}$$

Therefore,

$$\hat{h}(\theta) = e^{j\theta}(1 + \cos\theta)$$

Since the number of elements in the unit sample response is equal to 3, we choose $N = 3$. The comb filter system function is then

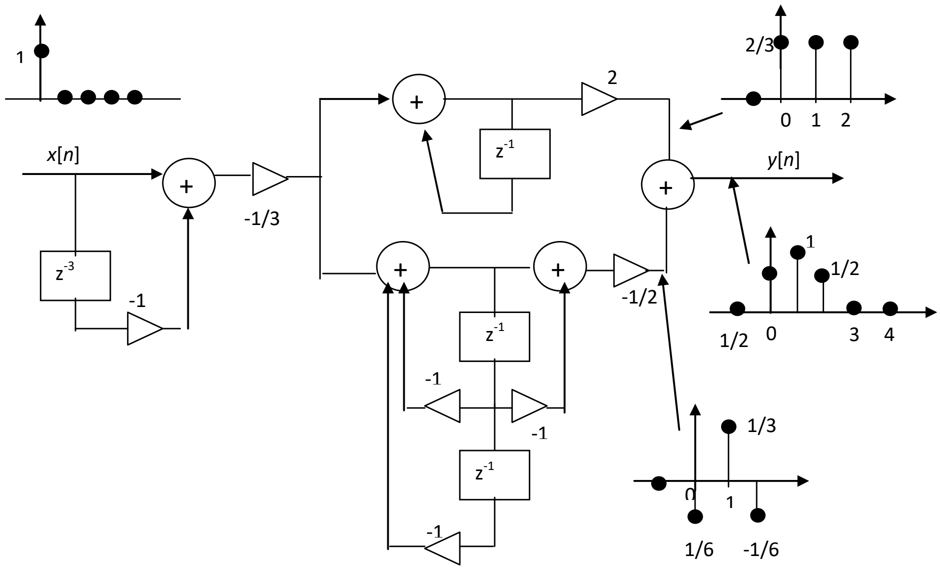
$$H_c(z) = \frac{1 - z^{-3}}{3}$$

$H_c(z)$ has three zeros located $z = e^{j\frac{2\pi k}{3}}$ for $k = 0, 1$ & 2. The zero at $z = 1$ will be cancelled by a real resonator and the zeros at $z = e^{\pm j\frac{2\pi}{3}}$ will be cancelled by a pair of complex resonators. The gains of the resonators are

$$g_0 = \hat{h}(\theta)|_{\theta=0} = 2$$
$$g_1 = \hat{h}(\theta)|_{\theta=\frac{2\pi}{3}} = \frac{1}{2}e^{-j\frac{2\pi}{3}}$$
$$g_2 = \hat{h}(\theta)|_{\left(\theta=\frac{4\pi}{3}\right)} = \frac{1}{2}e^{-j\frac{4\pi}{3}} = \frac{1}{2}e^{j\frac{2\pi}{3}} = g_1^*$$

Therefore,

$$H_{R0}(z) = \frac{2}{1 - z^{-1}}$$
$$H_{R1}(z) = \frac{\frac{1}{2}e^{-j\frac{2\pi}{3}}}{1 - e^{j\frac{2\pi}{3}}z^{-1}} + \frac{\frac{1}{2}e^{j\frac{2\pi}{3}}}{1 - e^{-j\frac{2\pi}{3}}z^{-1}} = \frac{-\frac{1}{2}(1 - z^{-1})}{1 + z^{-1} + z^{-2}}$$



Note: If the comb filter is followed by a recursive network that has a number of poles that coincide exactly with the same number of zeros of the comb filter, we obtain a **frequency-sampling** filter. The frequency-sampling filter can be particularly attractive solution if we want to make a narrow-band filter; the *N* is large, while the number of recursive sections can be very small. This means that the filter contain only a few multipliers and adders.

For practical applications we have to keep an eye on a number of things. In theory we start by assuming that a number of poles and zeros coincide exactly on the unit circle in the z-plane. This requires that we should be able to realize the filter coefficients to an accuracy of 100%. However, apart from a few obvious exceptions such as -1,0 and +1 this is never the case.

This means that while we can locate the zeros of the comb-filter sections in exactly the right position, we cannot do so far the corresponding poles. The best we can do is to get them in the vicinity. This can lead to a very erratic local variation of the frequency response, or even an unstable system if one of the poles lies just outside the unit circle. In practice both zeros and poles are therefore deliberately located just inside the unit circle. For comb filter the system function is then chosen as

$$H_c(z) = 1 - (az^{-1})^N$$

where a is made slightly less than 1