ELEC3104: Digital Signal Processing

Chapter 7: Filter Design

7.3 IIR Filter Design

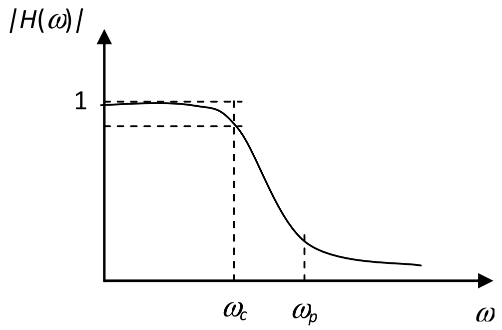
A commonly followed approach to IIR filter design is to first design an analogue filter that satisfies the requirements and then convert it into a digital filter.

7.3.1 Analogue Filter Design

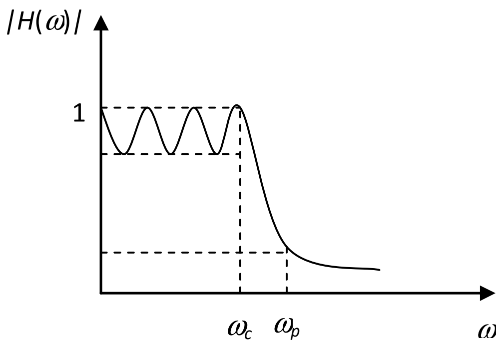
The technique used to design analogue filters is to specify a prototype low-pass filter function which is normalised to provide a Cut-off frequency (ω_c) at 1 *rad/sec* and then apply transformations to achieve the actual desired cut-off frequencies and filter type. Therefore, prime consideration will be given to low-pass filter design using the following design procedures:

- Butterworth Low-pass filter

- Maximally flat in passband.

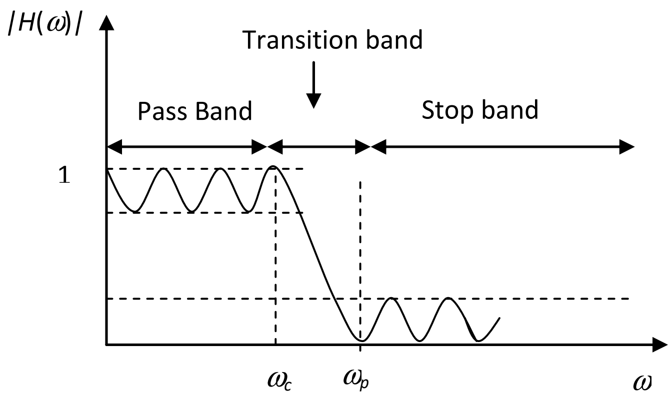


- Chebyshev I low-pass filter
    - Maximal ripple in passband.



- Chebyshev II low-pass filters
    - Maximal ripple in stopband

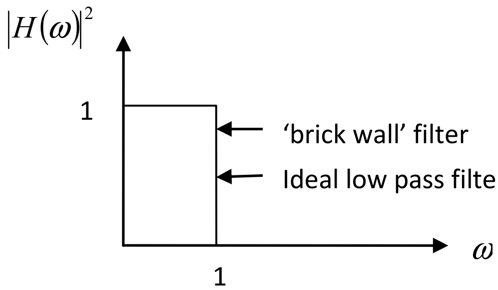
- Elliptic low pass filter
    - minimal transition width
    - ripple in both pass band and stopband



- Bessel filter

  - Maximally constant group delay.

The ideal, brick-wall, low pass filter prototype is one, which has a unit amplitude frequency response from dc to 1 *rad/sec* with response dropping to zero thereafter.



This may be defined in term of the following squared magnitude transfer function:

$$H(j\omega)\cdot H(-j\omega) = |H(j\omega)|^2 = \frac{1}{1+F(\omega^2)}$$

Where,

$$F(\omega^2) = \begin{cases} 0, & 0 < \omega < 1 \\ \infty, & \omega > 1 \end{cases}$$

Since $F(\omega^2)$ is a polynomial in ω^2, it will satisfy the ideal conditions only when the order of the polynomial, n, is infinite. However, it is not practically feasible to form a polynomial of this type (∞ order). Therefore approximations to $F(\omega^2)$ are made and these are the different approaches to analogue filter design (Butterworth, Chebyshev type I, etc.) mentioned earlier.

Butterworth Filters:

The Butterworth approximation to $F(\omega^2)$ is given by $F(\omega^2) = \omega^{2n}$. This yields,

$$|H(j\omega)|^2 = \frac{1}{1+\omega^{2n}}$$

$$|H(j\omega)| = \frac{1}{\sqrt{1+\omega^{2n}}}$$

where n is a positive integer

The magnitude response of a Butterworth filter of order n and cut-off frequency ω_c is represented in the form

$$|H(j\omega)|^2 = \frac{1}{1+\left(\frac{\omega}{\omega_c}\right)^{2n}}$$

The following properties are easily determined:

1. $|H(j\omega)|^2_{\omega=0} = 1$ for all n.
2. $|H(j\omega)|^2_{\omega=\omega_c} = 1/2$ for all n; this implies $|H(j\omega)| = 1/\sqrt{2} = 0.707$ and $20\log|H(j\omega)|_{\omega=\omega_c} = -3.0dB$
3. $|H(j\omega)|^2$ is a monotonically decreasing function ω (throughout pass band and stop band).
4. As n gets larger $|H(j\omega)|^2$ approaches an ideal low-pass frequency response.
5. $|H(j\omega)|^2$ is called maximally flat at the origin since all order derivatives exist with respect to ω and are zero at the origin.

Note:

$$|H(j\omega)| = \frac{1}{\left[1+\left(\frac{\omega}{\omega_c}\right)^{2n}\right]^{\frac{1}{2}}} = \left[1+\left(\frac{\omega}{\omega_c}\right)^{2n}\right]^{-\frac{1}{2}}$$

$$= 1 - \frac{1}{2}\left(\frac{\omega}{\omega_c}\right)^{2n} + \frac{3}{8}\left(\frac{\omega}{\omega_c}\right)^{4n} - \frac{5}{16}\left(\frac{\omega}{\omega_c}\right)^{6n} + \cdots$$

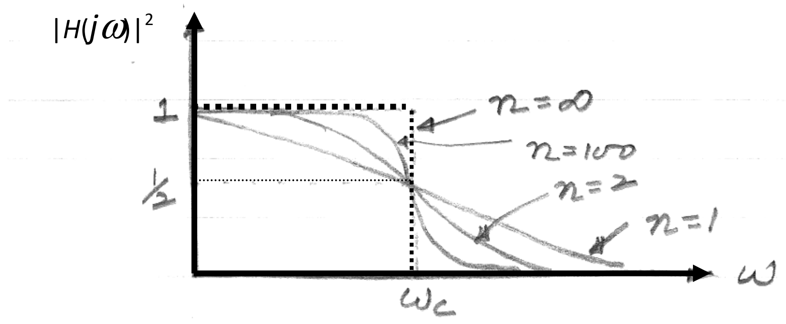


Figure 7.1: The magnitude response of a Butterworth filter

It is convenient in many cases to look at the magnitude response in decibels, i.e., plot $20\log|H(j\omega)|$ versus ω.

Figure 7.2 is a straight line approximation of the frequency response in decibels (dB) for the Butterworth filters.

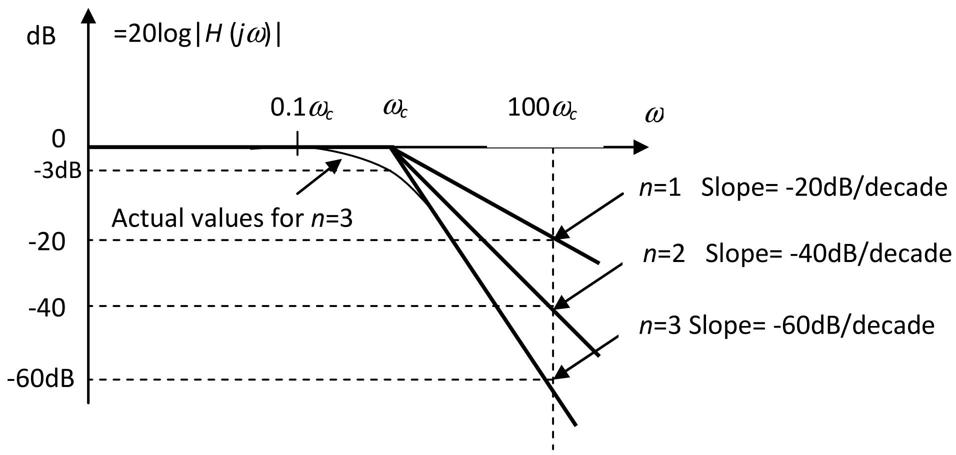


Figure 7.2: Filter gain plot for analogue Butterworth filters of various orders of n

Starting with the magnitude square frequency response we would like to find the system function $H(s)$ that gives the Butterworth magnitude squared response.

$$|H(j\omega)|^2 = \frac{1}{[1+\omega^{2n}]} \quad = 1/(1+(-(j^*\omega)^\wedge 2)^\wedge n)$$

$$= H(s) \cdot H(-s)$$

where, $s = j\omega$ ⟶ ω = s / j = -j s

H(s)*H(-s) = 1/(1+(-s^2)^n)

i.e.,

$$\boxed{H(s) \cdot H(-s) = \begin{cases} \dfrac{1}{1+s^{2n}}, & n - even \\ \dfrac{1}{1-s^{2n}}, & n - odd \end{cases}}$$

When n is even, the poles of this product (roots of the denominator) are given by

$$1 + s^{2n} = 0$$

$$s^{2n} = -1 = e^{\pm j(2k+1)\pi}$$

$$\boxed{\therefore s = e^{\pm j\frac{(2k+1)\pi}{2n}}}$$

where, $k = 0, 1, \ldots, n-1$, to give $2n$ unique roots (poles)

For instance, when $n = 2$, $k = 0, 1$ and

$$s = e^{\pm j\frac{\pi}{4}}, e^{\pm j\frac{3\pi}{4}}$$

When n is odd, the poles of the product $H(s) \cdot H(-s)$ are given by

$$1 - s^{2n} = 0$$

$$s^{2n} = 1 = e^{\pm j(2k)\pi}$$

$$\boxed{\therefore s = e^{\pm j\frac{2k\pi}{2n}} = e^{\pm j\frac{k\pi}{n}}}$$

where, $k = 1, 2, \dots, n-1$, to give $2n-2$ unique poles, also $k = 0\ and\ n$ to give $s = e^{j0}$ and $s = e^{j\pi}$ (i.e., $+1$ and -1) as poles. This gives a total of $2n$ poles.

Note: Both when n is even and odd, the poles of the product $H(s) \cdot H(-s)$ are arrayed on the unit circle separated by $\pi/2n$ radians. Moreover, for every pole s_k, its reflection about the origin $-s_k$ is also a pole.

The system $H(s) \cdot H(-s)$ is a combination (cascade) of two systems $H(s)$ and $H(-s)$, where the poles of one system is obtained by reflecting the pole of the other system about the origin. i.e., if s_1 is a pole of $H(s)$ then $-s_1$ is a pole of $H(-s)$.

Hence if $H(s)$ is chosen such that its poles are the poles of $H(s) \cdot H(-s)$ that are on the left half of the s-plane then the poles on the right half (which are reflections of the ones on the left half) must be the poles of $H(-s)$.

Since the poles of $H(s)$ are on the left half of the s-plane, it is stable.
the left half of the s-plane is the stable region, correspond to the inside of the unit circle when mapped into z-plane.

Note: Since the poles (s_k) of $H(-s)$ are on the right half of the s-plane, the poles of $H(p)$ where $p = -s$ are given by $p_k = -s_k$ and are present on the left half and hence this system is also stable. In fact this system is the same as $H(s)$.

A plot of the poles $(2n = 6)$ of the Butterworth function for $n = 3$ (odd) is shown in Figure 7.3.

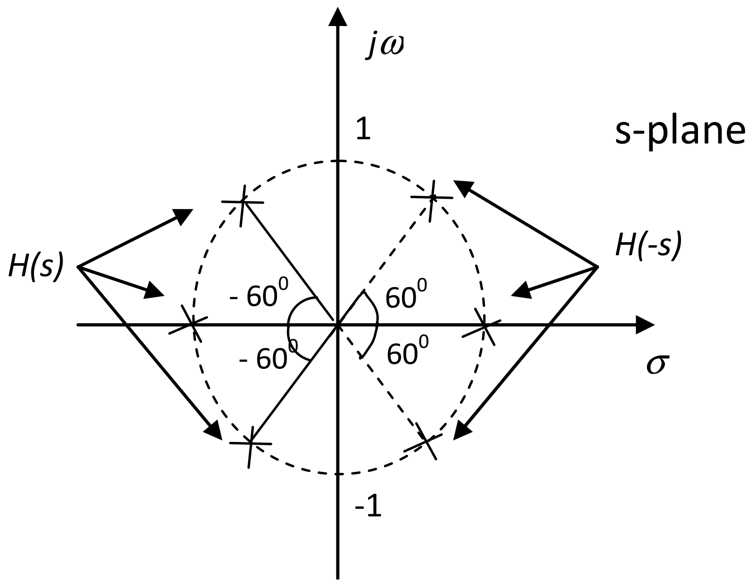


Figure 7.3: The s-plane of a Butterworth filter with n = 3.

A plot of the poles of the Butterworth function for *n = 4* (even) is shown in *Figure 7.6.*

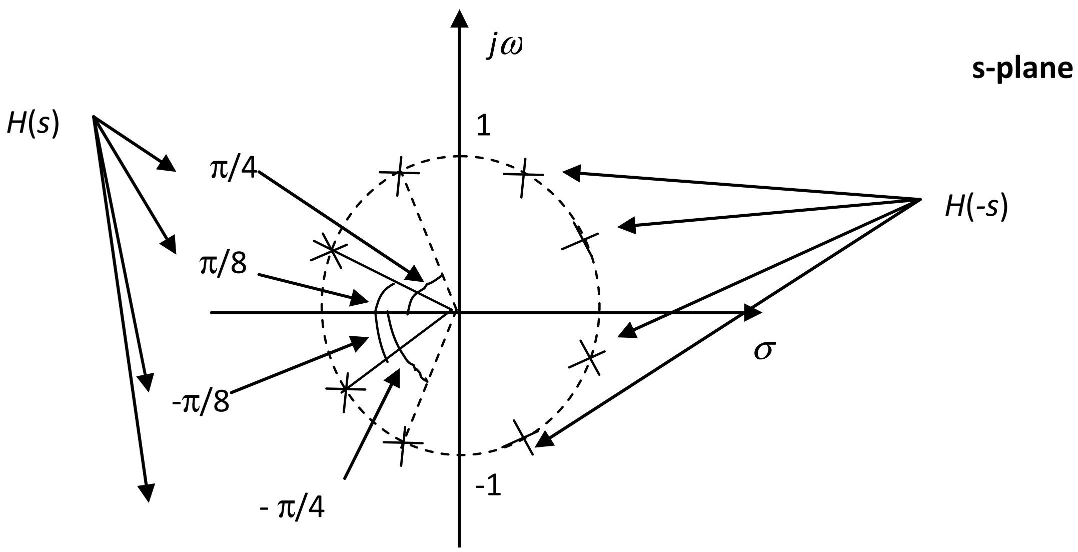
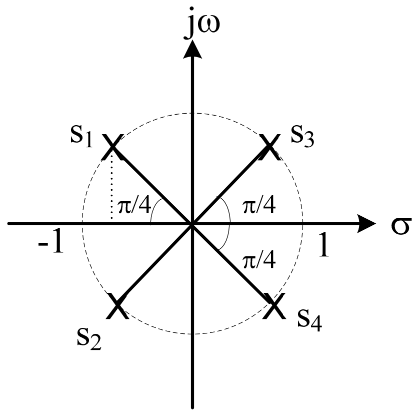


Figure 7.4: The s-plane of a Butterworth filter with n = 4.

Example 7.1

Find the transfer function $H(s)$ for the normalised (i.e., $\omega_c = 1\ rad/sec$) Butterworth filter of order 2.

Since $n = 2$, the poles of $H(s) \cdot H(-s)$ are



Therefore,

$$H(s) = \frac{1}{(s - s_1)(s - s_2)}$$

$$= \frac{1}{\left[s - \left(\frac{-1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)\right]\left[s - \left(\frac{-1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)\right]}$$

$$= \frac{1}{s^2 + \sqrt{2}s + 1}$$

Example 7.2

Determine the transfer function of a Butterworth filter of the low-pass type with order, $n = 3$. Assume that the 3dB cut-off frequency $\omega_c = 1\ rad/sec$.

For $n = 3$, the $2n = 6$ poles of $H(s)H(-s)$ are located on a circle of unit radius with angular spacing $\pi/3$. Allocating the left-half plane poles to $H(s)$, we obtain

$$s_1 = -1, \qquad s_2 = \frac{-1}{\sqrt{2}} + j\frac{\sqrt{3}}{2}, \qquad s_3 = \frac{-1}{\sqrt{2}} - j\frac{\sqrt{3}}{2}$$

The transfer function of a Butterworth filter of order 3 is therefore,

$$H(s) = \frac{1}{(s+1)\left(s+\frac{1}{2}-j\frac{\sqrt{3}}{2}\right)\left(s+\frac{1}{2}+j\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{1}{(s+1)(s^2+s+1)}$$

$$= \frac{1}{s^3+2s^2+2s+1}$$

Chebyshev Filters

There are two types Chebyshev filters, one containing a ripple in the pass-band (*type 1*) and the other containing a ripple in the stop-band (*type 2*).

A type 1 – Low-pass normalised (unit bandwidth) Chebyshev filter with a ripple in the pass-band is characterised by the following magnitude squared frequency response:

$$|H(j\omega)|^2 = \frac{1}{1+\varepsilon^2[T_n(\omega)]^2}$$

where, $T_n(\omega)$ is the n^{th} order Chebyshev polynomial.

The Chebyshev polynomials can be generated and thus defined from the following recursive formula:

$$T_n(\omega) = 2\omega T_{n-1}(\omega) - T_{n-2}(\omega), \qquad n \geq 2$$

with, $T_0(\omega) = 1$ and $T_1(\omega) = \omega$

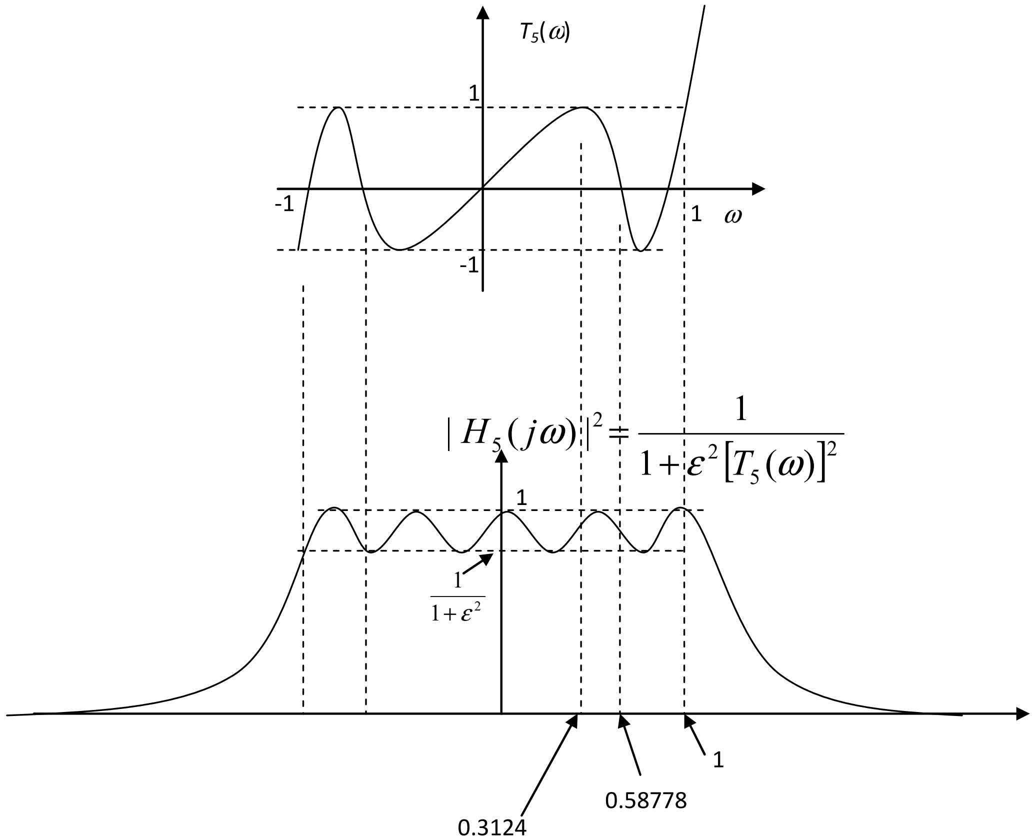
Exercise:

Based on the above recursive expression, $T_0(\omega)$ and $T_1(\omega)$, generate the Chebyshev polynomials of order 2 to 6.

Note: The Chebyshev polynomials can also be defined using cosines as

$$T_n(\omega) = \begin{cases} \cos(n\cos^{-1}\omega), & for\ |\omega| \le 1 \\ \cosh(n\cosh^{-1}\omega), & for\ |\omega| > 1 \end{cases}$$

For $n = 5$



It is noticed that for $n = 5$, the Chebyshev polynomial oscillates between +1 and -1 while outside the interval it grows towards $+\infty$.

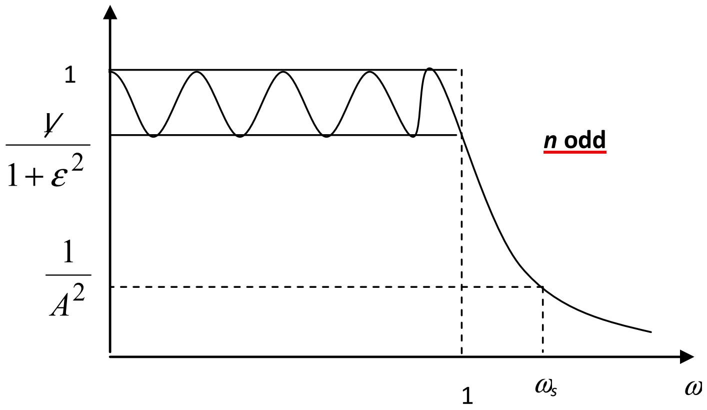
Hence, when $n = 5$, the magnitude squared response of the Chebyshev filter given by

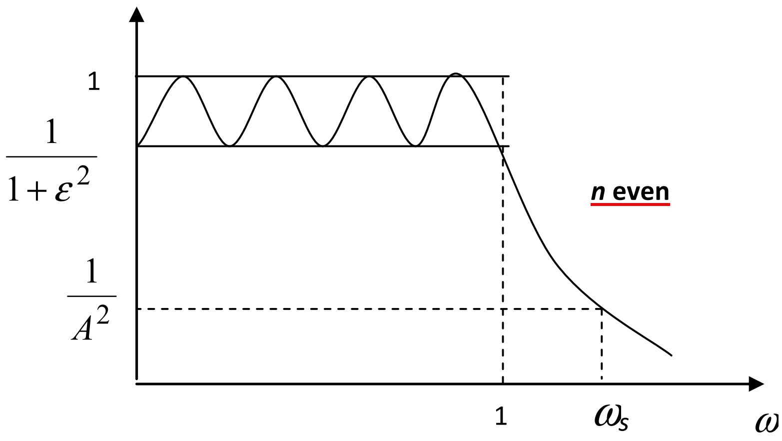
$$|H(j\omega)|^2 = \frac{1}{1 + \varepsilon^2[T_5(\omega)]^2}$$

oscillates between 1 and $\frac{1}{1+\varepsilon^2}$ when $-1 \le \omega \le 1$ (as $T_5(\omega)$ oscillates between -1 and 1). Outside this interval, $T_5(\omega) \to \infty$ and hence $|H(j\omega)|^2 \to 0$. This oscillation of the Chebyshev polynomial also causes the equal magnitude ripple in the passband of $|H(j\omega)|^2$.

Note:

- The magnitude squared frequency response $|H(j\omega)|^2$ oscillates between 1 and $\frac{1}{1+\varepsilon^2}$ within the pass-band, the so called equiripple, and has a value of $\frac{1}{1+\varepsilon^2}$ at $\omega_c = 1$, the cut-off frequency.





- $|H(j\omega)|^2$ is monotonic outside the pass-band , including both transition band and the stop-band.

$$|H(j\omega)|^2 = \frac{1}{1+\varepsilon^2[T_n(\omega)]^2}$$

as ω increases, $\varepsilon^2[T_n(\omega)]^2 \gg 1$. Therefore,

$$|H(j\omega)|^2 = \frac{1}{\varepsilon^2[T_n(\omega)]^2}$$

$$|H(j\omega)| = \frac{1}{\varepsilon T_n(\omega)}$$

$$\begin{aligned} 20\log_{10}|H(j\omega)| &= 20\log_{10}\left(\frac{1}{\varepsilon T_n(\omega)}\right) \\ &= -20\log_{10}\big(\varepsilon T_n(\omega)\big) \\ &= -[20\log_{10}\varepsilon + 20\log_{10}T_n(\omega)] \end{aligned}$$

For large values of ω, the Chebyshev polynomial can be approximated as $T_n(\omega) \approx 2^{n-1}\omega^n$. Therefore

$$\begin{aligned} -20\log_{10}|H(j\omega)| &= 20\log_{10}\varepsilon + 20\log_{10}2^{n-1} + 20\log_{10}\omega^n \\ &= 20\log_{10}\varepsilon + 20(n-1)\log_{10}2 + 20n\log_{10}\omega \end{aligned}$$

Therefore the attenuation is

$$-20\log_{10}|H(j\omega)| \approx 20\log_{10}\varepsilon + 6(n-1) + 20n\log_{10}\omega$$

Clearly the Chebyshev approximation depends on the values of ε and n.

The maximum permissible ripple fixes the value of ε, and once this value of ε has been determined the value of the attenuation in the stop-band fixes the value of the filter complexity n.

Example 7.3

A Chebyshev low-pass characteristic is required to have a maximum pass-band ripple of 1.2dB and an attenuation of atleast 25dB at $\omega = 2.5$. Determine the values of ε and n.

$$|H(j\omega)|^2 = \frac{1}{1+\varepsilon^2[T_n(\omega)]^2}$$

At $\omega = 1$, the ripple is $1.2dB$ and $T_n(\omega) = 1$

$$-1.2dB = 10\log_{10}\left(\frac{1}{1+\varepsilon^2}\right)$$

Therefore,

$$\frac{1}{1+\varepsilon^2} = 10^{-\frac{1.2}{10}}$$

$$\Rightarrow 1+\varepsilon^2 = 10^{\frac{1.2}{10}}$$

$$\Rightarrow \varepsilon^2 = 10^{\frac{1.2}{10}} - 1$$

$$\Rightarrow \varepsilon = 0.5641$$

At $\omega = 2.5$, namely in the stop-band, attenuation is -25dB

$$25 = 20\log_{10}\varepsilon + 6(n-1) + 20n\log_{10}\omega$$

$$25 = 20\log_{10}0.5641 + (n-1)6 + 20 * n\log_{10}2.5$$

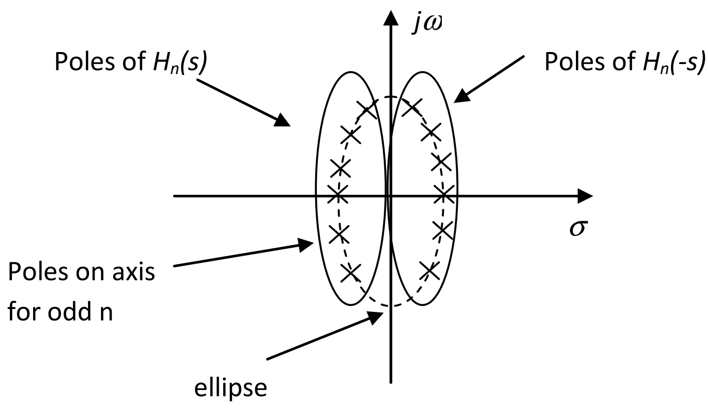
This gives, $n = 2.577$ and we choose $n = 3$.

Once ε and n have been obtained, we continue to determine $H(s)$.

$$|H(j\omega)|^2 = \frac{1}{1+\varepsilon^2 T_n^2(\omega)}$$

It can be shown that $H(s)$ can be obtained by finding the roots of $1+\varepsilon^2 T_n^2\left(\frac{s}{j}\right) = 0$.

It can further be shown that the poles of $\frac{1}{1+\varepsilon^2 T_n^2\left(\frac{s}{j}\right)}$ fall on an ellipse as shown below.



Note: It can be shown that a comparison of the normalised Chebshev pole locations with the normalised Butterworth pole locations reveals that the imaginary parts are identical and the real part of the Butterworth pole times a factor “$\tanh(A_k)$” is equal to the real part of the Chebyshev pole. Where,

$$A_k = \frac{1}{n}\sinh^{-1}\left(\frac{1}{\varepsilon}\right)$$

Hence knowing the normalized Chebyshev poles can be derived from the normalized Butterworth poles. The denormalised Chebyshev poles are obtained by multiplying the normalised Chebshev poles by a denormalising–factor equal to $\cosh(A_k)$.

Example 7.4

Determine $H(s)$ for the previous example.

$$A_k = \frac{1}{3}\sinh^{-1}\left(\frac{1}{0.5641}\right) = 0.4457$$

$$\tanh A_k = 0.4184$$

$$\cosh A_k = 1.1009$$

for $n = 3$, the Butterworth poles are (selecting in the left-hand of the s-plane)

$$p_1 = \cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3} = -0.5 + j0.866$$

$$p_2 = \cos\pi + j\sin\pi = -1 + j0$$

$$p_3 = \cos\frac{4\pi}{3} + j\sin\frac{4\pi}{3} = -0.5 - j0.866$$

Multiplying the real parts of p_1,p_2 and p_3 by $\tanh(A_k)$ we obtain the normalised Chebyshev poles.

$$p_1' = -0.2092 + j0.866$$

$$p_2' = -0.4184 + j0$$

$$p_3' = -0.2092 - j0.866$$

Multiplying p_1', p_2' and p_3' by $\cosh(A_k)$ we obtain the de-normalised Chebyshev poles

$$p_1'' = -0.2303 + j0.9534$$

$$p_2'' = -0.4606 + j0$$

$$p_3'' = -0.2303 - j0.9534$$

Therefore the de-normalised Chebyshev transfer function is

$$H(s) = \frac{k}{(s + 0.4606)(s + 0.2303 - j0.9534)(s + 0.2303 + j0.9534)}$$

Also,

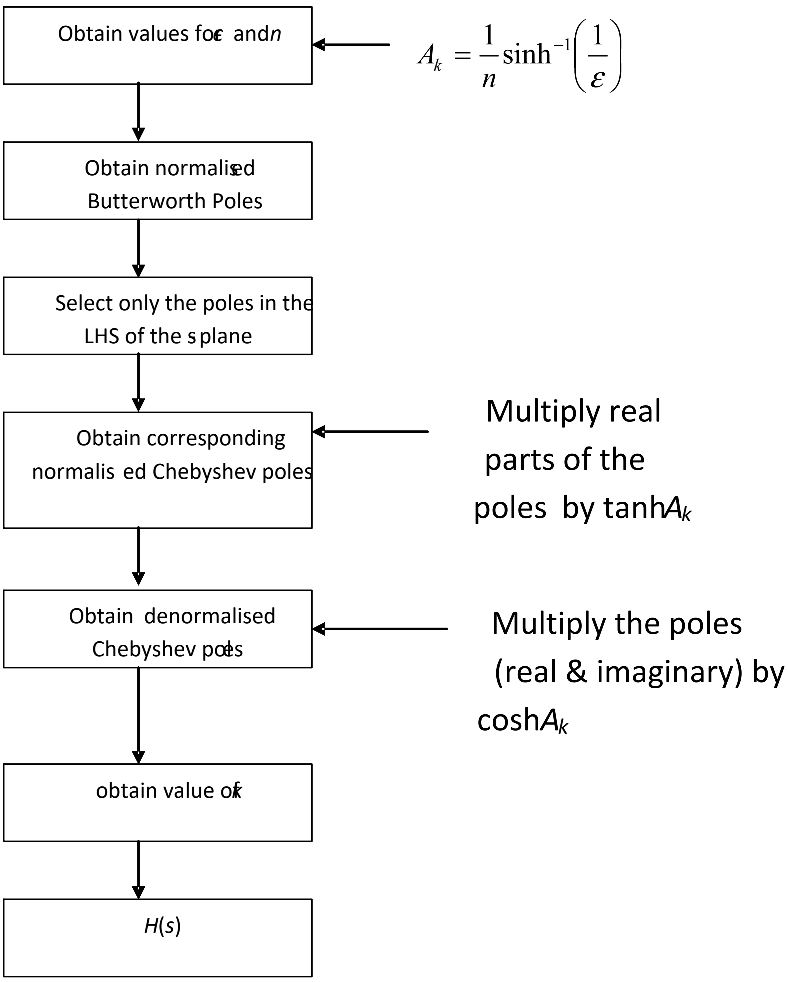
$$H(s)|_{s=j\omega=0} = 1$$

Hence,

$$\frac{k}{(0.4606)(0.2303 - j0.9534)(0.2303 + j0.9534)} = 1$$

Therefore, $k = 0.4431$ (DC gain)

Recommended design process for deriving $H(s)$ for a Chebyshev low-pass filter



Analogue to Analogue Filter Transformations

To transform analogue low-pass filter $H(s)$ with unity cut-off frequency to a **low-pass filter** $H'(s)$ with cut-off frequency ω_c, we substitute

$$\boxed{s \to \frac{s}{\omega_c}}$$

Example 7.5

First order Butterworth prototype filter is given by

$$H(s) = \frac{1}{s+1} \leftarrow normalised$$

To transform to a new cut-off frequency, $\omega_c = 5$, we replace s with s/ω_c .

$$H'(s) = \frac{1}{\frac{s}{\omega_c} + 1} = \frac{\omega_c}{s + \omega_c} = \frac{5}{s+5}$$

To transform analogue low-pass filter $H(s)$ with unity cut-off frequency to a **high-pass filter** $H'(s)$ with cut-off frequency ω_c, we substitute

$$\boxed{s \to \frac{\omega_c}{s}}$$

Example 7.6

First order Butterworth prototype filter is given by

$$H(s) = \frac{1}{s+1} \leftarrow normalised$$

To transform to a new cut-off frequency, $\omega_c = 2$, we replace s with ω_c/s .

$$H'(s) = \frac{1}{\frac{\omega_c}{s} + 1} = \frac{s}{s+2}$$

Note: High-pass filter contains zeros as well as poles.

Example 7.7

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

Determine the transfer function of the high-pass filter with cut-off frequency $\omega_c = 1$ corresponding to the above 3rd order Butterworth filter.

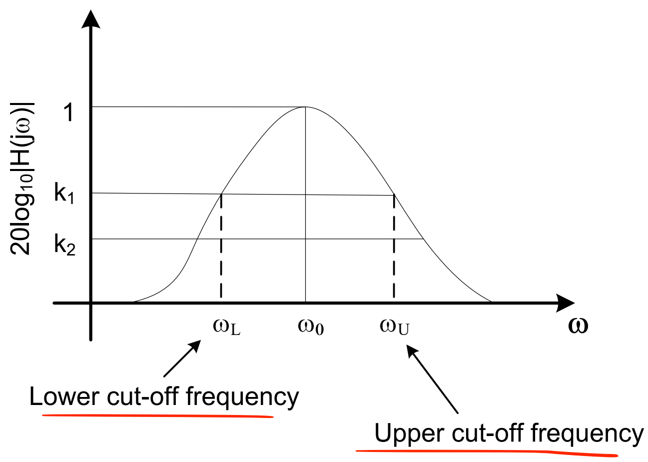
$$s \rightarrow \frac{\omega_c}{s}, \qquad i.e., \qquad s \rightarrow \frac{1}{s}$$

Making this substitution gives

$$H'(s) = \frac{1}{\left(\frac{1}{s}+1\right)\left(\left(\frac{1}{s}\right)^2 + \frac{1}{s} + 1\right)} = \frac{s^3}{(s+1)(s^2+s+1)}$$

By definition, a band pass filter rejects both low and high frequency components and passes a certain band of frequencies some where between them. Thus the frequency response $H(j\omega)$, of a band-pass filter has the following properties.

1. $|H(j\omega)| = 0$ at both $\omega = 0$ & $\omega = \infty$.
2. $|H(j\omega)| = 1$ for a frequency band centered on ω_0, where ω_0 is the mid frequency of the filter.



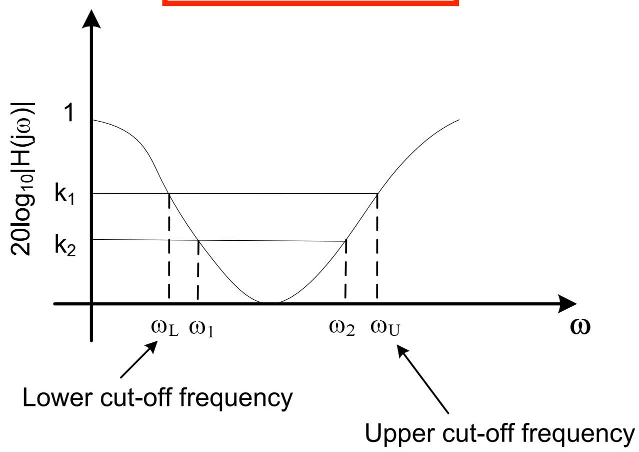
The bandwidth of the band-pass filter is given by, $B = \omega_U - \omega_L$

To convert unity cut off low-pass filter $H(s)$ into a **band-pass filter,** $H'(s)$ with lower cut-off frequency ω_L and the upper cut-off frequency ω_U, we make the substitution

$$\boxed{s \to \frac{s^2 + \omega_L \omega_U}{s(\omega_U - \omega_L)}}$$

Similarly to convert unity cut-off lowpass filter $H(s)$ into a bandstop filter $H'(s)$ with lower cutoff frequency ω_L and upper cut-off frequency ω_U, we make the substitution:

$$\boxed{s \to \frac{s(\omega_U - \omega_L)}{s^2 + \omega_L \omega_U}}$$



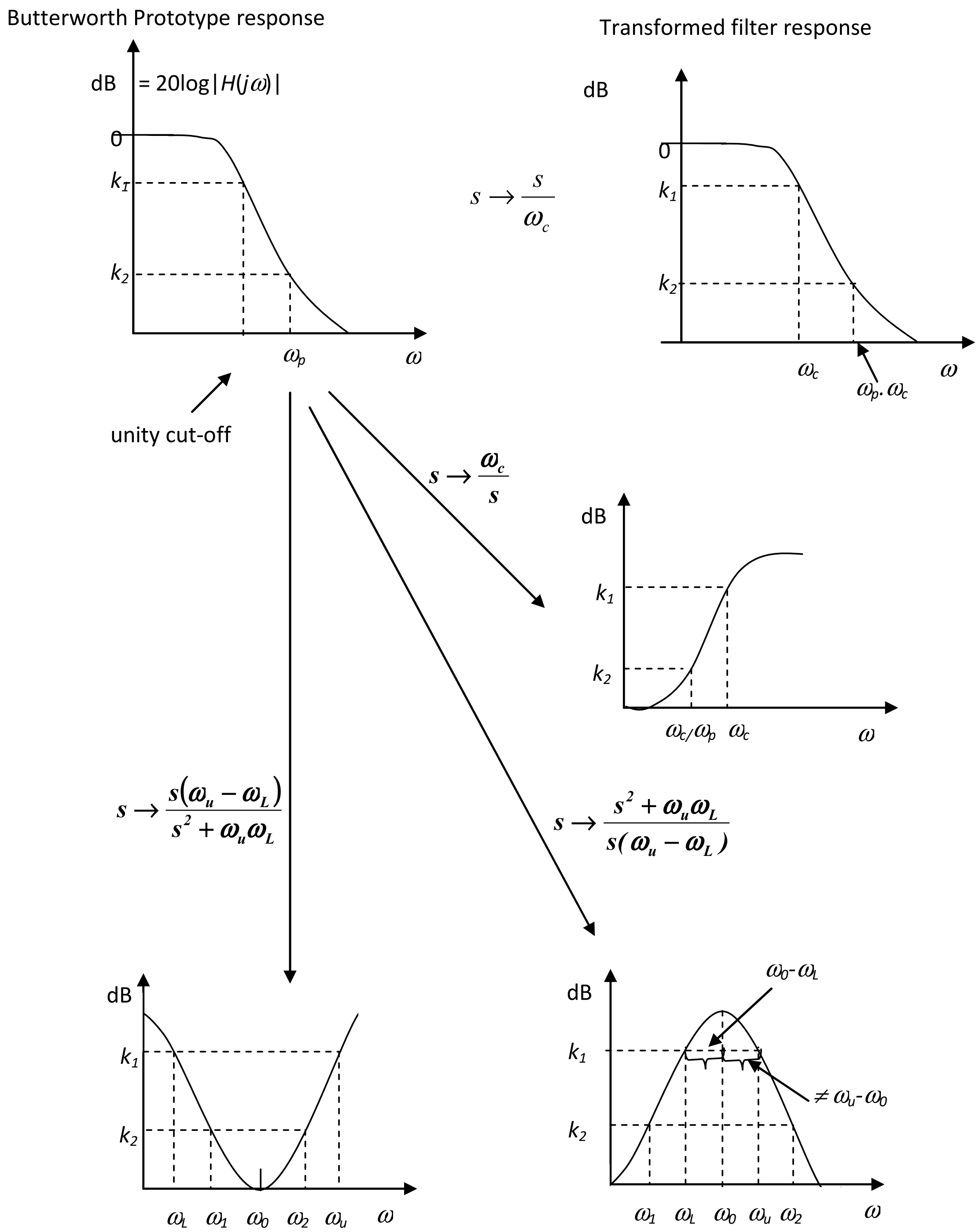
Note: (Second Method): A low-pass to band-pass transformation can be performed by

$$s \to \frac{s^2 + \omega_0^2}{Bs}$$

where, $B = \omega_U - \omega_L$ is the bandwidth of the band-pass filter, and ω_0 is the centre frequency.

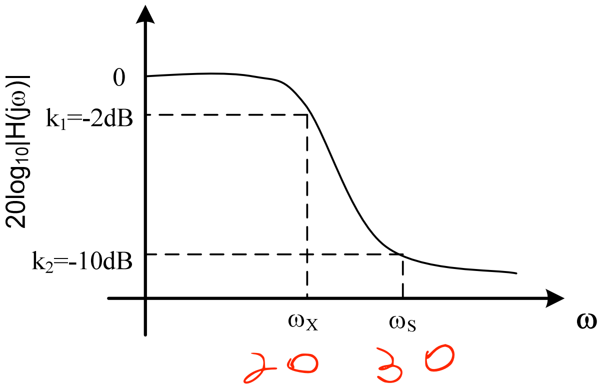
ω0^ 2 = ωu * ωl

Summary: Analogue to Analogue Transformation



Example 7.8

Design an analogue Butterworth filter that has -2dB or better cut-off frequency of 20 rad/sec and at least 10dB of attenuation at 30 rad/sec.



$$|H(j\omega)|^2 = \frac{1}{1+\left(\frac{\omega}{\omega_c}\right)^{2n}}$$

$$10\log_{10}|H(j\omega)|^2 = 10\log_{10}\left(\frac{1}{1+\left(\frac{\omega}{\omega_c}\right)^{2n}}\right)$$

From given data,

$$10\log_{10}|H(j\omega)|^2 = \begin{cases} k_1, & \omega = \omega_x \\ k_2, & \omega = \omega_s \end{cases}$$

Considering $\omega = \omega_x$,

$$k_1 = 10\log_{10}\left(\frac{1}{1+\left(\frac{\omega_x}{\omega_c}\right)^{2n}}\right)$$

$$\frac{k_1}{10} = -\log_{10}\left[1+\left(\frac{\omega_x}{\omega_c}\right)^{2n}\right]$$

$$1+\left(\frac{\omega_x}{\omega_c}\right)^{2n} = 10^{-\frac{k_1}{10}}$$

$$\therefore \left(\frac{\omega_x}{\omega_c}\right)^{2n} = 10^{-\frac{k_1}{10}} - 1$$

Similarly,

$$\left(\frac{\omega_s}{\omega_c}\right)^{2n} = 10^{-\frac{k_2}{10}} - 1$$

Taking the ration gives,

$$\frac{\left(\frac{\omega_x}{\omega_c}\right)^{2n}}{\left(\frac{\omega_s}{\omega_c}\right)^{2n}} = \frac{10^{-\frac{k_1}{10}} - 1}{10^{-\frac{k_2}{10}} - 1}$$

$$\left(\frac{\omega_x}{\omega_s}\right)^{2n} = \frac{10^{-\frac{k_1}{10}} - 1}{10^{-\frac{k_2}{10}} - 1}$$

$$\log_{10}\left(\frac{\omega_x}{\omega_s}\right)^{2n} = \log_{10}\left(\frac{10^{-\frac{k_1}{10}} - 1}{10^{-\frac{k_2}{10}} - 1}\right)$$

$$2n\log_{10}\left(\frac{\omega_x}{\omega_c}\right) = \log_{10}\left(\frac{10^{-\frac{k_1}{10}} - 1}{10^{-\frac{k_2}{10}} - 1}\right)$$

Hence,

$$\boxed{\boxed{n = \frac{\log_{10}\left(\frac{10^{-\frac{k_1}{10}} - 1}{10^{-\frac{k_2}{10}} - 1}\right)}{2\log_{10}\left(\frac{\omega_x}{\omega_s}\right)}}}$$

If n is an integer we use that value, otherwise we use the next largest integer.

In this case,

$$n = \frac{\log_{10}\left(\frac{10^{-\frac{-2}{10}} - 1}{10^{-\frac{-10}{10}} - 1}\right)}{2\log_{10}\left(\frac{20}{30}\right)}$$

$$= 3.3709$$

Hence the order, n is chosen as 4.

We have,

$$k_1 = 10\log_{10}\left(\frac{1}{1+\left(\frac{\omega_x}{\omega_c}\right)^{2n}}\right)$$

Substituting the values for k_1, ω_x and n we obtain

$$\omega_c = \frac{\omega_x}{\left(10^{-\frac{k_1}{10}}-1\right)^{\frac{1}{2n}}} = \frac{20}{\left(10^{-\frac{-2}{10}}-1\right)^{\frac{1}{2\times 4}}}$$

$$= 21.3868\, rad/sec$$

The transfer function of the normalised low-pass Butterworth filter ($\omega_c = 1$) for $n =$ 4, can be found from its poles as shown previously.

$$H(s) = \frac{1}{(s^2+0.76536s+1)(s^2+1.84776s+1)}$$ see page 5

The De-normalised Butterworth filter is obtained by making the substitution $s \to s/\omega_c$.

$$H'(s) = \frac{0.209210\times 10^6}{(s^2+16.3886s+457.394)(s^2+39.5176s+457.394)}$$

Example 7.9

Design an analogue Butterworth filter (the filter is monotonic in the pass and stop bands (i.e. no ripples)) which meets the following specifications:

- Pass Band: 0 to 10 kHz
- Transition Band: 10 to 20 kHz
- Stop Band attenuation: -10 dB (starts at 20 kHz)

$$|H(j\omega)|^2 = \frac{1}{1+\left(\frac{\omega}{\omega_c}\right)^{2n}}$$

At $\omega = \omega_p$,

$$10\log_{10}|H(j\omega)|^2 = -10$$

i.e.,

$$-10 = 10\log_{10}1 - 10\log_{10}\left(1+\left(\frac{\omega_p}{\omega_c}\right)^{2n}\right)$$

$$-10 = -10\log_{10}\left(1+\left(\frac{\omega_p}{\omega_c}\right)^{2n}\right)$$

$$1 = \log_{10}\left(1+\left(\frac{\omega_p}{\omega_c}\right)^{2n}\right)$$

$$1+\left(\frac{\omega_p}{\omega_c}\right)^{2n} = 10$$

assume pass band edge to be cut-off frequency

$$\left(\frac{\omega_p}{\omega_c}\right)^{2n} = 9$$

Substituting ω_p and ω_c,

$$\left(\frac{2\pi\times 20}{2\pi\times 10}\right)^{2n} = 9$$

$$2^{2n} = 9$$

This gives,

$$n = \frac{\log_{10}9}{2\log_{10}2} = 1.5849$$

Thus we choose $n = 2$ (even).

The normalised 2nd order Butterworth filter is

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

Substituting $s \to s/\omega_c$ gives the denormalised transfer function as

$$H'(s) = \frac{1}{\left(\frac{s}{\omega_c}\right)^2 + \sqrt{2}\left(\frac{s}{\omega_c}\right) + 1}$$

$$= \frac{(\omega_c)^2}{s^2 + \sqrt{2}\omega_c s + (\omega_c)^2}$$

7.3.2 Analogue to Digital Transformation

The ***impulse invariant transformation*** and the ***bilinear transformation*** are two ways of converting an analogue system into a digital system. In transforming an analogue filter to digital filter, we must obtain either $H(z)$ or $h[n]$ from the analogue filter design. In such transformations, we generally require that the essential properties of the analogue frequency response be preserved in the frequency response of the resulting digital filter. This implies that we want the imaginary axis of the s-plane to map into the unit circle of z-plane.

A second condition is that a stable analogue filter should be transformed to a stable digital filter. That is if the analogue system has two poles only in the left half s-plane, then the digital filter must have poles inside the unit circle.

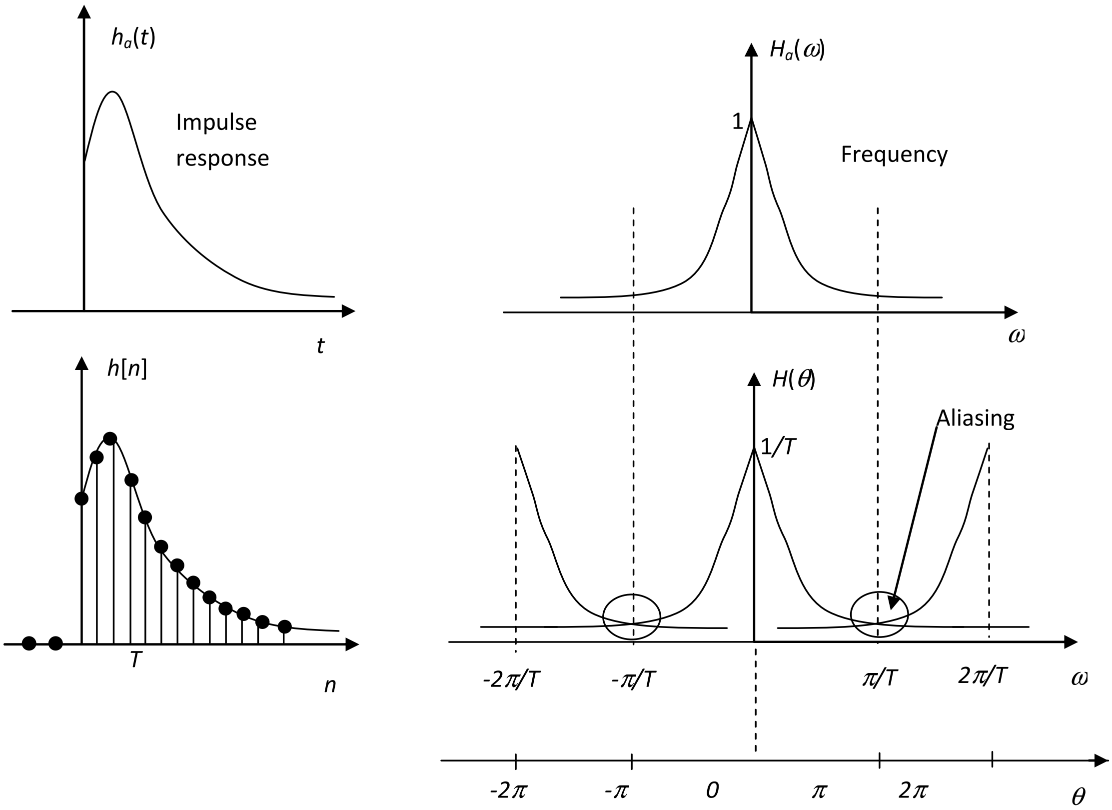
Impulse invariant transformation

In this method we start from an analogue filter of impulse response $h_a(t)$ and the system function $H_a(s)$. The objective of the transformation is to realise an IIR filter with an impulse response $h[n]$ which satisfies:

$$\boxed{h[n] = h_a(nT)}$$

where, T is the sampling period.

The characteristic property preserved by this transformation is that the impulse response of the resulting digital filter is a sampled version of the impulse response of the analogue filter.



aliasing is unavoidable here

We see that with this method there are problems to a greater or lesser extent depending on the choice of T. The sampling frequency affects the frequency response of the impulse invariant discrete filter. A sufficient high sampling frequency is necessary for the frequency response to be close to that of the equivalent analogue filter. Thus due to aliasing, the frequency response of the digital filter will not be identical to that of the analogue filter.

So how do we find the filter coefficients of the IIR filter in this design method?

To obtain the mapping let,

$$H_a(s) = \frac{1}{s+b}, \qquad b > 0$$

Taking the inverse Laplace transform

$$h_a(t) = e^{-bt}$$

If the analogue impulse response is sampled at period T,

$$h[n] \stackrel{\text{def}}{=} h_a(nT) = e^{-bnT}$$

Considering only causal systems,

$$h[n] = \begin{cases} e^{-bnT}, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

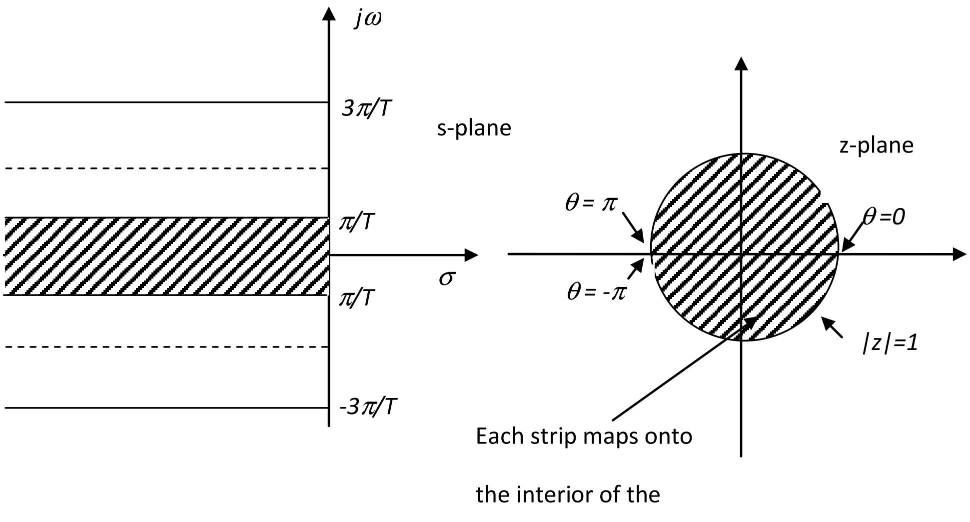
Taking the z-transform

$$H(z) = \frac{1}{1 - e^{-bT} z^{-1}}$$

It can be seen that $H(z)$ is obtained from $H_a(s)$ by using the mapping relationship

$$\boxed{\frac{1}{s+b} \rightarrow \frac{1}{1 - e^{-bT} z^{-1}}, \qquad b > 0}$$

where, T is the sampling period.



In this kind of mapping, the perimeter is the imaginary axis.

Note: The mapping is for poles only and not for zeros.

Example 7.10

Find $H(z)$ using the impulse invariant transform from an analogue system given by

$$H(s) = \frac{2}{(s+1)(s+3)}$$

Using partial fractions this can be written as

$$H(s) = \frac{1}{s+1} - \frac{1}{s+3}$$

Using the mapping, $\frac{1}{s+b} \to \frac{1}{1-e^{-bT}z^{-1}}$,

$$H(z) = \frac{1}{1-e^{-T}z^{-1}} - \frac{1}{1-e^{-3T}z^{-1}}$$

$$= \frac{(e^{-T}-e^{-3T})z^{-1}}{1-(e^{-T}-e^{-3T})z^{-1}+e^{-4T}z^{-2}}$$

Example 7.11

Use the impulse invariant method to design digital filter from an analogue prototype that has a system function.

$$H(s) = \frac{s+a}{(s+a)^2+b^2}$$

To design a filter using the impulse invariant method, expand $H(s)$ using partial fractions.

$$H(s) = \frac{s+a}{[s+(a+jb)][s+(a-jb)]}$$

$$= \frac{1/2}{s+(a+jb)} + \frac{1/2}{s+(a-jb)}$$

Substituting $\frac{1}{s+c} \to \frac{1}{1-e^{-cT}z^{-1}}$,

$$H(z) = \frac{1}{2}\left[\frac{1}{1 - e^{-(a+jb)T}z^{-1}}\right] + \frac{1}{2}\left[\frac{1}{1 - e^{-(a-jb)T}z^{-1}}\right]$$

$$= \frac{\frac{1}{2}\left[1 - e^{-(a-jb)T}z^{-1}\right] + \frac{1}{2}\left[1 - e^{-(a+jb)T}z^{-1}\right]}{(1 - e^{-(a+jb)T}z^{-1})(1 - e^{-(a-jb)T}z^{-1})}$$

$$= \frac{1 - e^{-aT}\cos(bT)\, z^{-1}}{1 - 2e^{-aT}\cos(bT)\, z^{-1} + e^{-2aT}z^{-2}}$$

Hence,

$$H(z) = \frac{1 + a_1 z^{-1}}{1 + b_1 z^{-1} + b_2 z^{-2}}$$

where,

$a_1 = e^{-aT}\cos(bT)$, $b_1 = -2e^{-aT}\cos(bT)$ and $b_2 = e^{-2aT}$.

Note: The zero at $s = -a$ is mapped to a zero at $z = e^{-aT}\cos(bT)$. Thus, the location of the zero in the discrete time filter obtained using the impulse invariant transform depends on the position of the poles as well as the zero in the analogue filter.

Example 7.12

Using impulse invariant method design a digital filter to approximate the following normalised analogue transfer function:

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

Assume that the 3dB cut-off frequency of the digital filter is required to be 150Hz and the sampling frequency is 1.28kHz.

Before applying the impulse invariant transform, we need to de-normalise the transfer function.

$$H'(s) = H(s)|_{s \to \frac{s}{\omega_c}}$$

$$= \frac{1}{\left(\frac{s}{\omega_c}\right)^2 + \sqrt{2}\left(\frac{s}{\omega_c}\right) + 1}$$

$$= \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2}$$

Where, $\omega_c = 2\pi \times 150 = 942.4778$

Using partial fractions $H'(s)$ can be written as

$$H'(s) = \frac{\omega_c^2}{s^2 + \sqrt{2}\omega_c s + \omega_c^2}$$

$$= \frac{A}{s - p_1} + \frac{B}{s - p_2}$$

We get,

$$p_1, p_2 = \frac{-\sqrt{2}\omega_c \pm \sqrt{2\omega_c^2 + 4\omega_c}}{2}$$

$$= \frac{-\sqrt{2}\omega_c \pm j\sqrt{2}\omega_c}{2}$$

$$= \frac{-\sqrt{2}\omega_c(1 \pm j)}{2}$$

Therefore,

$p_1 = -666.4324(1 + j)$ and $p_2 = -666.4324(1 - j)$

Also,

$A = -\frac{\omega_c}{\sqrt{2}} j$ and $B = \frac{\omega_c}{\sqrt{2}} j$

Using the mapping $\frac{1}{s+a} \to \frac{1}{1-e^{-aT}z^{-1}}$ we get

$$H'(z) = \frac{A}{1 - e^{p_1T}z^{-1}} + \frac{B}{1 - e^{p_2T}z^{-1}}$$

$$= \frac{A(1 - e^{p_2T}z^{-1}) + B(1 - e^{p_1T}z^{-1})}{(1 - e^{p_1T}z^{-1})(1 - e^{p_2T}z^{-1})}$$

$$= \frac{(A+B) - (Ae^{p_2T} + Be^{p_1T})z^{-1}}{1 - (e^{p_1T} + e^{p_2T})z^{-1} + e^{(p_1+p_2)T}z^{-2}}$$

Substituting for A, B, p_1 and p_2 we get

$$H(z) = \frac{393.9264z^{-1}}{1 - 1.0308z^{-1} + 0.3530z^{-2}}$$

Therefore,

$$H(\theta) = \frac{393.9264e^{-j\theta}}{1 - 1.0308e^{-j\theta} + 0.3530e^{-j2\theta}}$$

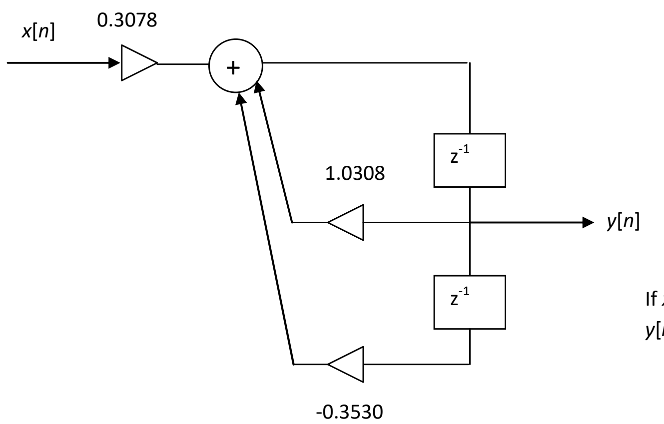
This gives,

$$H(\theta)|_{\theta=0} = \frac{393.9264}{1 - 1.0308 + 0.3530} \approx 1223$$

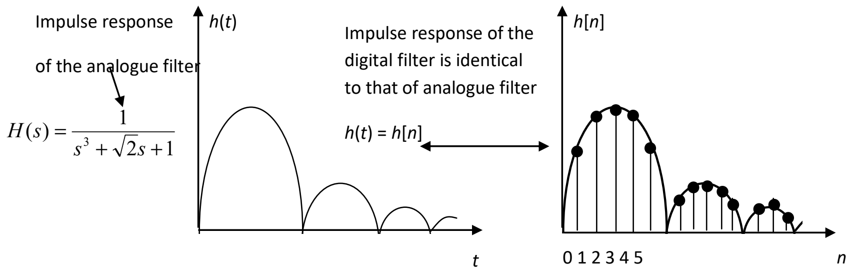
This is approximately equal to the sampling frequency.

Such a large gain is characteristic of impulse invariant filters.To keep the gain down (and to avoid overflows when the filter is implemented). It is common practice to divide the gain by f_S. Thus the new transfer function becomes

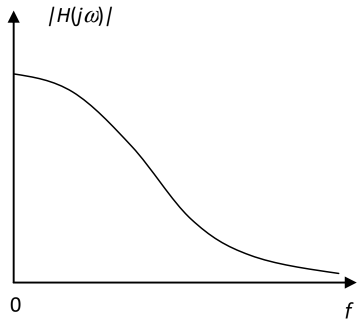
$$H(z) = \frac{0.3078z^{-1}}{1 - 1.0308z^{-1} + 0.3530z^{-2}}$$

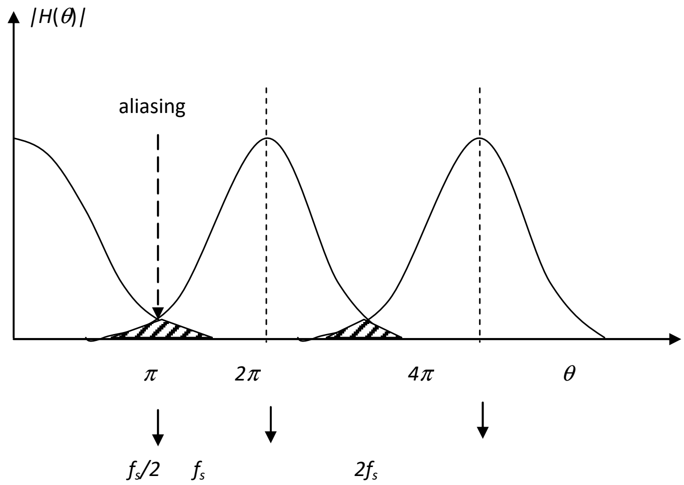


If $x[n]=\delta[n]$ then $y[n]=h[n]$.



Note: The sampling frequency affects the frequency response of the digital filter obtained using impulse invariant transformation. A sufficiently high sampling frequency is necessary for the frequency response to be closer to that of the equivalent analogue filter (see below)

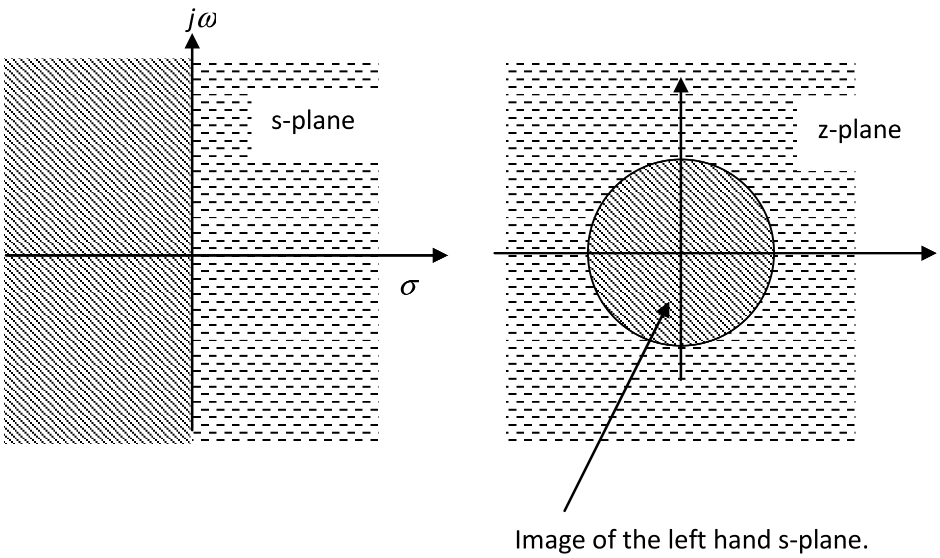




Low degree of aliasing can be achieved by making the sampling frequency high.

Bilinear Transformation

The bilinear transformation yields stable digital filters from stable analogue filters (the impulse invariant technique may not). Also the bilinear transformation avoids the problem of aliasing encountered with the use of the impulse invariant transformation, because it maps the entire imaginary axis in the s-plane on to the unit circle in the z-plane.



$$z = e^{sT} = \frac{e^{\frac{sT}{2}}}{e^{-\frac{sT}{2}}} = \frac{1 + \frac{sT}{2} + \left(\frac{sT}{2}\right)^2 \cdot \frac{1}{2} + \cdots}{1 - \frac{sT}{2} + \left(\frac{sT}{2}\right)^2 \cdot \frac{1}{2} + \cdots}$$

taylor serise

Dropping the higher order terms,

$$z \approx \frac{1 + \frac{sT}{2}}{1 - \frac{sT}{2}}$$

i.e.,

$$\boxed{s = \frac{2}{T} \frac{1 - z^{-1}}{1 + z^{-1}}}$$

The price paid for the avoidance of aliasing is an introduction of distortion in the frequency axis. Consequently, the design of digital filters using the bilinear transformation is only useful when the distortion can be compensated.

Example 7.13

We have a system function $H_a(s)$ such that

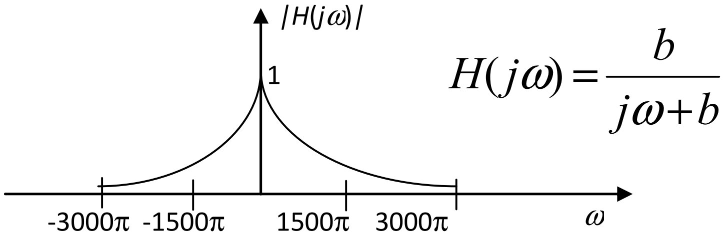
$$H_a(s) = \frac{b}{s+b}$$

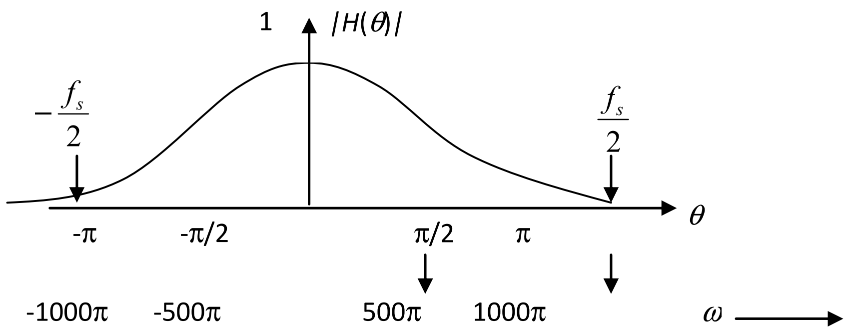
Applying bilinear transformation,

$$H(z) = \frac{b}{\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{bT(1+z^{-1})}{bT+2+(bT-2)z^{-1}}$$

Let $b = 1000$ and $T = \frac{1}{1000}$

The magnitude responses, $|H(\omega)|$ and $|H(\theta)|$ are shown below:





There is a very important property of the bilinear transformation that can be seen in the above example. The entire frequency range $(-\infty \leq \omega_a \leq \infty)$ of the continuous system maps into the fundamental interval $(-\pi \leq \theta \leq \pi)$ of the discrete system, where $\omega = 0$ corresponds to $\theta = 0$, $\omega = \infty$ to $\theta = \pi$ and $\omega = -\infty$ to $\theta = -\pi$.

To demonstrate this mapping, consider $s = \sigma + j\omega$ results in:

$$z = e^{(\sigma+j\omega)T} = e^{\sigma T} e^{j\omega T}$$

On the imaginary axis of the s-plane, $\sigma = 0$. i.e.,

$$z = e^{j\omega T} = e^{j\theta}$$

where, $\theta = \omega T$

Since the bilinear mapping is given by $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$,

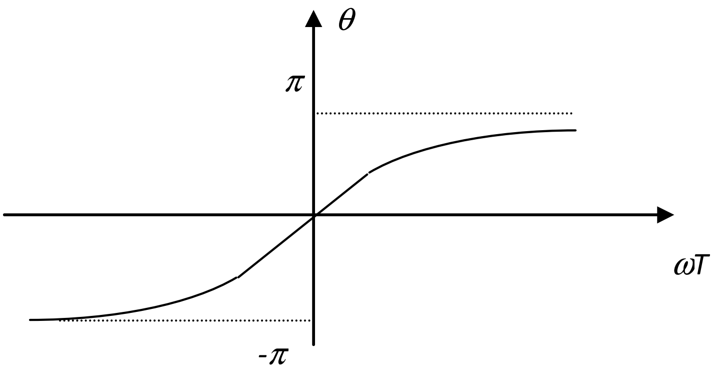
$$j\omega = \frac{2}{T}\frac{1-e^{-j\theta}}{1+e^{-j\theta}}$$

$$= \frac{2}{T} j \tan\left(\frac{\theta}{2}\right)$$

Hence,

$$\boxed{\begin{aligned} \omega &= \frac{2}{T}\tan\left(\frac{\theta}{2}\right) \\ \theta &= 2\tan^{-1}\left(\frac{\omega T}{2}\right) \end{aligned}}$$

We see that a nonlinear relation exists between ω and θ. This effect is called 'Warping' and is shown below.



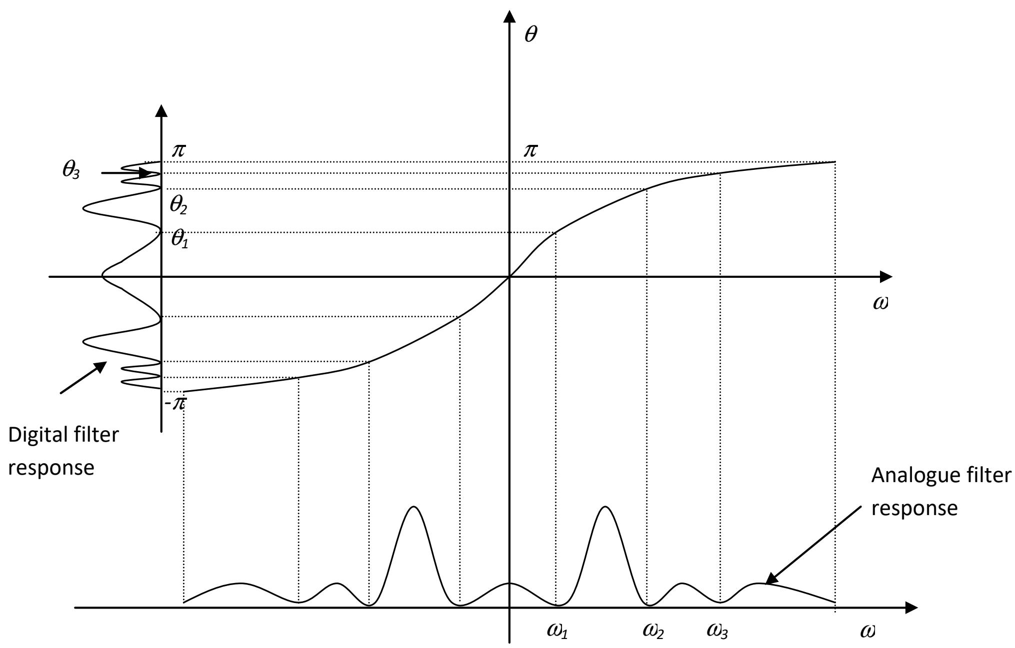
At low frequencies, $\tan^{-1}\left(\frac{\omega T}{2}\right) \approx \frac{\omega T}{2}$. Therefore,

$$\theta \approx 2 \times \frac{\omega T}{2} = \omega T$$

The great advantage of warping is that no aliasing of the frequency characteristic can occur in the transformation of an analogue filter to a discrete filter, which we encountered in the impulse-invariant method.

We must however check carefully just how the various characteristic frequencies of the continuous characteristic frequencies of the discrete filter.

We can illustrate this with the aid of a diagram (below) for a band-pass filter.



The effect of "warping" in the conversion of $|H(\omega)| \rightarrow |H(\theta)|$ is seen from the above diagram.

In designing a digital filter by this method we must first **pre-wrap** the given filter specifications to find the continuous filter to which we are going to apply the bilinear transformation.

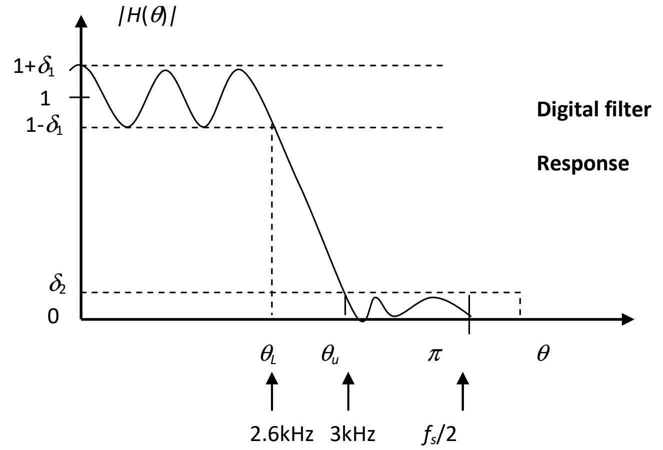
Example 7.14

The specification of a desired digital low pass filter is shown below. Find the specifications for the corresponding analogue low pass filter that will lead to this digital filter after bilinear transformation.

Sampling frequency, $f_s = 8\ kHz$, $(T = 1/f_s = 125\mu s)$

A pass band upto $f_L = 2.6\ kHz$, $(\theta_L = \omega_L T = 2\pi \frac{f_L}{f_s} = 0.65\pi)$

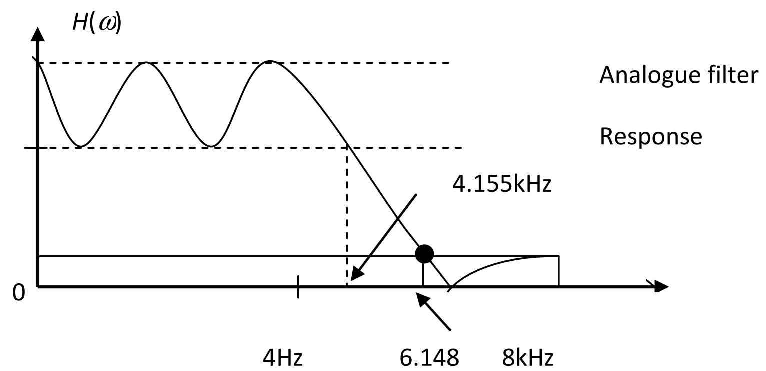
and a stop band above $f_U = 3\ kHz$, $(\theta_U = 2\pi \frac{f_U}{f_s} = 0.75\pi)$



To obtain this filter using bilinear transformation, we must start from an analogue filter with

$$\omega_L = 2\pi f_l = \frac{2}{T}\tan\frac{\theta_L}{2} = 2\pi(4155) \Rightarrow f_l = 4.155\ kHz$$

$$\omega_u = 2\pi f_h = \frac{2}{T}\tan\frac{\theta_U}{2} = 2\pi(6148) \Rightarrow f_h = 61.48\ kHz$$



Example 7.15

Determine, using bilinear transformation method, the transfer function and difference equation for the digital equivalent of the RC filter. The normalized transfer function for the RC filter is

$$H(s) = \frac{1}{s+1}$$

Assume a sampling frequency of 150Hz and a cut-off frequency of 30Hz

Desired cut-off frequency is 30Hz. Therefore,

$$\theta_c = \omega_c T = 2\pi f_c \cdot \frac{1}{f_s}$$

$$= 2\pi(30) \cdot \frac{1}{150} = 0.4\pi$$

The analogue frequency is determined after pre-warping,

$$\omega_c' = \frac{2}{T}\tan\left(\frac{0.4\pi}{2}\right) = 217.95\, rad/sec$$

The frequency in Hz is $f_c' = \frac{\omega_c'}{2\pi} = 34.68\, Hz.$

Note: The pre-warped frequency, f', is always greater than the one obtained without pre-warping.

The de-normalised analogue filter transfer function is obtained from $H(s)$ by making the substitution, $s \to s/\omega_c'$.

$$H'(s) = H(s)|_{s\to\frac{s}{\omega_c'}} = \frac{\omega_c'}{s+\omega_c'} = \frac{\left(\frac{2}{T}\right)0.7265}{s+\left(\frac{2}{T}\right)0.7265}$$

$$H(z) = H'(s)|_{s=\frac{2z-1}{Tz+1}} = \frac{0.4208(1+z^{-1})}{1-0.1584z^{-1}}$$

Example 7.16

It is required to design a digital filter to approximate the analogue transfer function

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1}$$

Using the bilinear transformation method obtain the transfer function, $H(s)$ of the digital filter assuming a 3dB cut-off frequency of 150 Hz and a sampling frequency of 1.28 kHz.

$$\theta_c = 2\pi\left(\frac{f_s}{f_c}\right) = 2\pi\left(\frac{150}{1280}\right) = \frac{15}{64}\pi$$

The analogue frequency after pre-warping is

$$\omega_c' = \frac{2}{T}\tan\left(\frac{\theta_c}{2}\right) = \frac{2}{T} \times 1280\tan\left(\frac{\frac{15}{64}\pi}{2}\right)$$

$$= 987.5009 = \frac{2}{T} \times 0.3859\, rad/sec$$

$$f_c' = \frac{987.5009}{2\pi} = 157.1656\, Hz$$

Pre-warped analogue filter is given by,

$$H'(s) = \frac{\left(\frac{2}{T}\right)^2 (0.3857)^2}{s^2 + \frac{2}{T}(0.3857)\sqrt{2}s + \left(\frac{2}{T}\right)^2 (0.3857)^2}$$

$$H(z) = H'(s)\big|_{s=\frac{2}{T}\frac{z-1}{z+1}}$$

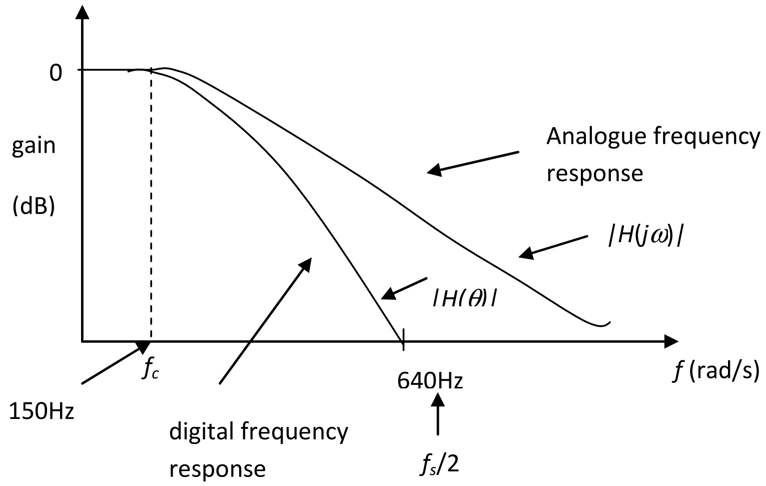
$$= \frac{\left(\frac{2}{T}\right)^2 (0.3957)^2}{\left(\frac{2}{T}\frac{z-1}{z+1}\right)^2 + \frac{2}{T}(0.3857)\sqrt{2}\left(\frac{2}{T}\frac{z-1}{z+1}\right) + \left(\frac{2}{T}\right)^2 (0.3957)^2}$$

$$= \frac{0.0878(1 + 2z^{-1} + z^{-2})}{1 - 1.0048z^{-1} + 0.3561z^{-2}}$$

$$\frac{(\omega_c')^2}{s^2 + \sqrt{2}\omega_c' s + (\omega_c')^2} \xrightarrow{Bilinear\ trans.} \frac{0.0878(1 + 2z^{-1} + z^{-2})}{1 - 1.0048z^{-1} + 0.3561z^{-2}}$$

analogue digital

The bilinear transformation maps an all-pole system to one that has both poles and zeros.



all sections in analogue are push into interval [0 pi] when converting into digital

Note: The cut-off frequency remains the same while roll off and attenuation in the stop band is increased.

Example 7.17

(a) An analogue transfer function can be converted to a digital transformation using the bilinear transformation. Derive this transform relationship using the following equation.

$$y[n] - y[n-1] = \frac{T}{2}\left[x[n] + x[n+1]\right] \leftarrow Digital\ integrator$$

$$H(s) = \frac{1}{s} \leftarrow Analogue\ integrator$$

where, T is the sampling period, $x[n]$ the input and $y[n]$ the output.

Taking the z-transform,

$$Y(z) - z^{-1}Y(z) = \frac{T}{2}[X(z) - z^{-1}X(z)]$$

$$H(z) = \frac{T}{2}\frac{1 + z^{-1}}{1 - z^{-1}}$$

Given that $H(s) = 1/2$ and comparing since they are equivalent gives,

$$\frac{1}{s} = \frac{T}{2}\frac{1+z^{-1}}{1-z^{-1}} \Rightarrow s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$$

(b) Convert the analogue filter $H(s)$

$$H(s) = \frac{s+0.1}{(s+0.1)^2 + 16}$$

into a digital IIR filter by means of the bilinear transformation.

The digital filter is to have a resonant frequency, $\theta_0 = \frac{\pi}{2}$.

$$H(s) = \frac{s+0.1}{s^2 + 0.2s + 16.01}$$

where, $\omega_0^2 = 16.01$

The analogue filter has a resonant frequency, $\omega_0 \approx 4\, rad/sec$.

The frequency is to be mapped into $\theta_0 = \frac{\pi}{2}$ by selecting the value of the parameter T.

$$\omega_0 = \frac{2}{T}\tan\frac{\theta_0}{2}$$

$$\Rightarrow 4 = \frac{2}{T}\tan\frac{\frac{\pi}{2}}{2} = \frac{2}{T}\tan\frac{\pi}{4}$$

$$\Rightarrow T = \frac{1}{2}$$

Thus the desired mapping is

$$s = 4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} = \frac{4\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.1}{4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)^2 + 0.2 \times 4\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 16.01}$$

$$= \frac{0.128 + 0.006z^{-1} - 0.122z^{-2}}{1 + 0.0006z^{-1} + 0.975z^{-2}}$$
$$\approx \frac{0.128 + 0.006z^{-1} - 0.122z^{-2}}{1 + 0.975z^{-2}}$$

The filter has poles at

$$p_1, p_2 = 0.987e^{\pm j\frac{\pi}{2}}$$

And zeros at $z_1 = -1$ and $z_2 = 0.95$.

Example 7.18

Convert the simple low pass filter $H(s) = \frac{1}{s+1}$ into an equivalent high pass discrete filter.

Assume $f_s = 150\ Hz$ and analogue cut-off, $f_c = 30\ Hz$.

$$\theta_c = 2\pi\left(\frac{30}{150}\right) = 0.4\pi$$
$$\omega_c' = \frac{2}{T}\tan\left(\frac{0.4\pi}{2}\right) = \frac{2}{T} \times 0.7265$$

always remember pre-warpping!!!

LPF to HPF transformation

$$H'(s) = H(s)|_{s \to \frac{\omega_c'}{s}} = \frac{1}{\frac{\omega_c'}{s} + 1} = \frac{s}{s + \omega_c'}$$

$$H(z) = H'(s)|_{z = \frac{2}{T}\frac{z-1}{z+1}} = \frac{\frac{2}{T}\left(\frac{z-1}{z+1}\right)}{\frac{2}{T}\left(\frac{z-1}{z+1}\right) + \frac{2}{T}0.7265}$$
$$= \frac{z - 1}{z - 1 + 0.7265(z + 1)}$$

$$H(z) = 0.5792\frac{1 - z^{-1}}{1 + 0.1584z^{-1}}$$

Example 7.19

Let us now apply this approach to the design of a digital low-pass filter. The magnitude response specification is

Sampling frequency	$f_s = 100\ kHz$
Passband:	$0\ to\ 10\ kHz$
Transition band:	$10kHz\ to\ 20kHz$
Stop band attenduation:	$-10\ dB$

The filter must be monotonic in the pass and stop bands (i.e., no ripple).

The monotonic requirement indicates a Butterworth filter.

Determine relevant digital frequencies,

$$\theta_c = 2\pi\frac{f_c}{f_s} = 2\pi\frac{10000}{100000} = 0.2\pi$$

$$\theta_h = 2\pi\frac{f_h}{f_s} = 0.4\pi$$

Determine relevant analogue frequencies (using pre-warping),

$$\omega_c = \frac{2}{T}\tan\frac{0.2\pi}{2} = 2f_s\tan 0.1\pi = 0.6498 \times 10^5$$

$$\omega_h = 2f_s\tan 0.2\pi = 1.4351 \times 10\text{^}5$$

The required Butterworth filter can be determined by using the following equations. Ensuring at least 10 dB attenuation at $\omega = \omega_h = 1.4351 \times 10^5$

assume pass band edge to be cut off frequency

$$|H_a(\omega)|^2 = \frac{1}{1 + \left(\frac{\omega_h}{\omega_c}\right)^{2N}}$$

$$\Rightarrow 10\log_{10}|H(\omega)|^2 = -10\log_{10}\left[1 + \left(\frac{1.4531}{0.6498}\right)^{2N}\right]$$

$$\Rightarrow 1 + \left(\frac{1.4531}{0.6498}\right)^{2N} = 10$$

$$\Rightarrow N = 1.367$$

Thus we choose $N = 2$. The poles occur in complex conjugate pairs and lie on the left-hand side of the s-plane.

$$p_1 = \omega_c e^{j\frac{3\pi}{4}} = 0.6498 \times 10^5(-0.7071 + j0.7071)$$

$$p_2 = \omega_c e^{j\frac{5\pi}{4}} = 0.6498 \times 10^5(-0.7071 - j0.7071)$$

$$H(s) = \frac{\boxed{p_1 p_2}}{(s - p_1)(s - p_2)} = \frac{4.223 \times 10^9}{s^2 + 0.919 \times 10^5 s + 4.223 \times 10^9}$$

ensure gain is 1 at DC

Applying the bilinear transform,

$$s = 2 \times 10^5 \frac{1 - z^{-1}}{1 + z^{-1}}$$

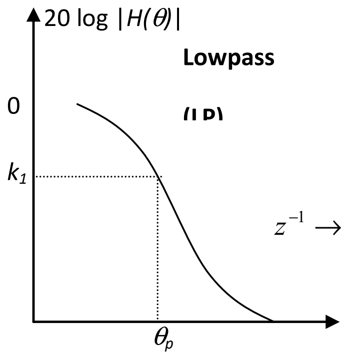
$$H(z) = \frac{0.0675 + 0.1349z^{-1} + 0.0675z^{-2}}{1 - 1.143z^{-1} + 0.4123z^{-2}}$$

Digital to Digital Filter Transformations

We have seen in the lecture notes that one method for the design of analogue filters relied on applying a transformation to an analogue low-pass filter with a unit bandwidth. It was shown that we could obtain low-pass, high-pass, band-pass and band-stop filters by selecting the appropriate transformation.

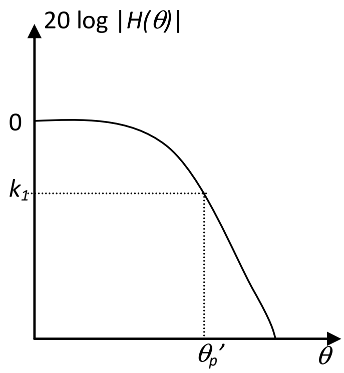
Similarly, a set of transformations can be formed that take a low-pass digital filter and turn it into high-pass, band-pass, and band-stop or another low-pass digital filter.

The transformations are given below:



$$z^{-1} \rightarrow \frac{z^{-1}-\alpha}{1-\alpha\, z^{-1}}$$

Lowpass (LP)

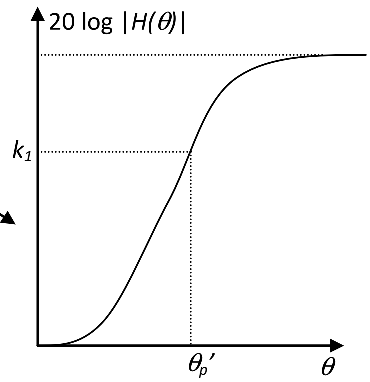


$$\alpha = \frac{\sin[(\theta_p - \theta_p')/2]}{\sin[(\theta_p + \theta_p')/2]}$$

Lowpass to Highpass

$$z^{-1} \rightarrow \frac{z^{-1}+\alpha}{1+\alpha\, z^{-1}}$$

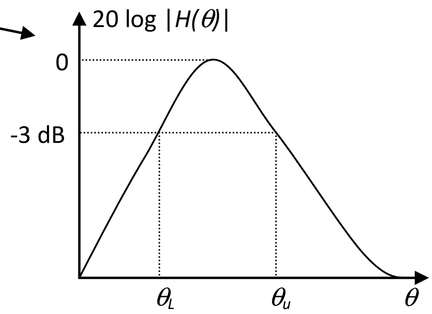
Simplest transformation is to change the signs of poles and zeros of the LP $\rightarrow$ LP transformation



$$\alpha = \frac{\cos[(\theta_p + \theta_p')/2]}{\cos[(\theta_p - \theta_p')/2]}$$

Lowpass to Bandpass

$$z^{-1} \rightarrow -\frac{z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha k}{k+1} z^{-1} + 1}$$



$$\alpha = \frac{\cos[(\theta_u + \theta_L)/2]}{\cos[(\theta_u - \theta_L)/2]}$$

$$k = \cot((\theta_u - \theta_L)/2) \cdot \tan\left(\frac{\theta_p}{2}\right)$$

Lowpass to Bandstop

$$z^{-1} = \frac{z^{-2} - \frac{2\alpha}{k+1} z^{-1} + \frac{k-1}{k+1}}{\frac{k-1}{k+1} z^{-2} - \frac{2\alpha}{k+1} z^{-1} + 1}$$