ELEC3104: Digital Signal Processing

Chapter 6: Discrete Fourier Transform

6.1 Discrete Spectrum

The sampling theorem states that any bandlimited continuous signal, $x(t)$, whose spectrum is zero outside the interval $[-B, B]$ (for a real signal, the spectrum is symmetric),i.e.,

$$X(\omega) = \mathcal{F}\{x(t)\}$$

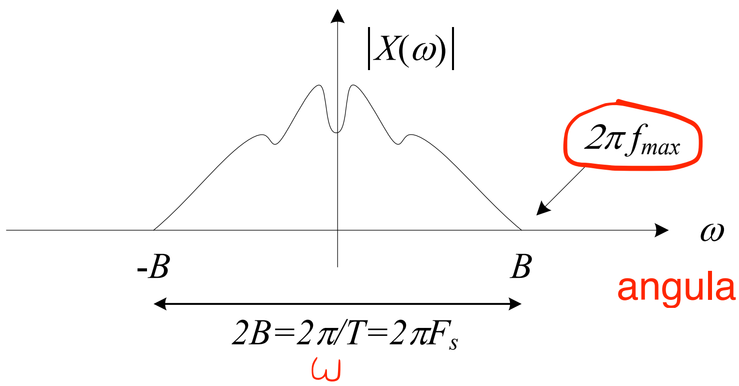
$$X(\omega) = 0, \qquad |\omega| > B$$

can be represented by a discrete signal, $x[n]$, obtained by taking samples of $x(t)$ spaced T apart, where $T = 2\pi/2B$.

$$x[n] \stackrel{\text{def}}{=} x(nT)$$

$$T = \frac{2\pi}{2B}$$

notice the 2π here



angular speed, not frequency

Figure 6.1: Spectrum of bandlimited signal

The spectral analysis of the sampled signal lead to the DTFT which was defined as

$$\hat{x}(\theta) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\theta n}$$

continuous discrete

If the signal $x(t)$ is also *timelimited*, i.e., $x(t) = 0,\ t < 0\ and\ t > L$. Then the number of samples of $x[n]$ is also limited to some value N.

total time

$$N = \frac{L}{\left(\frac{2\pi}{2B}\right)} = \frac{2BL}{2\pi}$$

data quantity

step size, equally spaced

Thus limits of the sum in the DTFT change to,

$$X_D(\theta) = \sum_{n=0}^{N-1} x[n]e^{-j\theta n}$$

Note: The points are indexed from 0 to $N-1$ rather than 1 to N.

Note: Mathematically a signal cannot be both timelimited and bandlimited, however the approximation is not an unreasonable one in practical situations. Moreover, the final definition of the DFT will not be dependent on these conditions and can be considered independent of the continuous time case.

The sampling theorem shows that a signal with finite support in the frequency domain (bandlimited) can be represented by discrete samples in the time domain. Due to the duality of the Fourier transform, it can be shown that the spectrum of a signal limited in the time domain can be represented by discrete samples of the spectrum.

If the signal $x(t)$ is timelimited to the interval $[0, L]$, it can be shown that the spacing between samples of the spectrum should be $2\pi/L$ (or less). This is left as an **exercise**.

spectral points

The discrete spectrum, $\hat{x}[k]$, is then defined as the samples of the continuous spectrum, $\hat{x}(\theta)$ at the points $\omega = k2\pi/L$ which corresponds to $\theta = \omega T = (k2\pi/L)(2\pi/2B) = k2\pi/N$. (since, $T = 2\pi/2B$ and $2BL/2\pi = N$) i.e.,

$$\boxed{\hat{x}[k] = \hat{x}\left(k\frac{2\pi}{L}\right) = \sum_{n=0}^{N-1} x[n]e^{j\frac{2\pi k}{N}n}}$$

This is defined as the **discrete Fourier transform (DFT)**.

The number of points required to represent the spectrum can be calculated as the length of the spectrum divided by the spacing between points (the same way it was done in the time domain earlier). This turns out to be,

$$\frac{2B}{\left(\frac{2\pi}{L}\right)} = \frac{2BL}{2\pi} = N$$

i.e., both the time domain representation, $x[n]$, and the frequency domain representation, $\hat{x}[k]$, require N samples. Thus the DFT can be viewed as an operation takes in a N-dimensional vector as the input and gives another N-dimensional vector as the output.

Note: The signal/vector $x[n]$ can be real of complex (in most practical applications the signal under consideration and consequently $x[n]$ will be real). The DFT coeffcients $\hat{x}[k]$ are generally complex. Thus, $DFT: \mathbb{C}^N \to \mathbb{C}^N$ in general but in most applications, $DFT: \mathbb{R}^N \to \mathbb{C}^N$.

Often the complex exponential is represented as

$$\omega_N = e^{j\frac{2\pi}{N}}$$

Note: Sometimes people use $\omega_N = e^{-j\frac{2\pi}{N}}$. (Not here though)

Note: ω_N is an N-th root of unity, meaning

$$\omega_N^N = (\omega_N)^N = e^{j2\pi N/N} = e^{j2\pi} = 1$$

There are N distinct, N-th roots of unity corresponding to the N powers of ω_N. i.e.,

$$\sqrt[N]{1} = \omega_N^0, \omega_N^1, \omega_N^2, \dots, \omega_N^{N-1}$$

The complex exponential vector is then written as,

$$\boldsymbol{\omega} = [1, \omega_N, \omega_N^2, \dots, \omega_N^{N-1}]$$

Consisting of the N distinct powers of ω_N. (The subscript N of ω_N will be dropped and only ω will be used from henceforth as long as there is no ambiguity.)

The notation $\boldsymbol{\omega}^k$ is used to represent the vector of the k-th powers of the elements of $\boldsymbol{\omega}$. i.e.,

$$\boldsymbol{\omega}^k = \left[1, \omega^k, \omega^{2k}, \dots, \omega^{(N-1)k}\right]$$

And hence,

$$\boldsymbol{\omega}^N = \left[1^N, e^{j\frac{2\pi N}{N}}, e^{j\frac{4\pi N}{N}}, \dots, e^{j\frac{2\pi(N-1)N}{N}}\right] = [1,1, \dots, 1]$$

Note: Powers of vectors are not defined mathematically and the above is a choice of notation and not an operation.

The components of the complex exponential vector are then

$$\boldsymbol{\omega}^k[m] = \omega^{km} = e^{j\frac{2\pi}{N}km}$$

Note: The bold face $\boldsymbol{\omega}$ is used to represent the vector and the ω is used to represent the scalar complex exponent $e^{j\frac{2\pi}{N}}$.

Thus the DFT in vector form is,

$$\hat{x}[k] = \sum_{n=0}^{N-1} x[n]\boldsymbol{\omega}^{-n}[k]$$

$$\hat{\boldsymbol{x}} = \sum_{n=0}^{N-1} x[n]\boldsymbol{\omega}^{-n}$$

Where, $\hat{\boldsymbol{x}} = \left[\hat{x}[0], \hat{x}[1], \dots, \hat{x}[N-1]\right]$

The **DFT transforms one vector into another vector and is a *linear* transformation**. i.e., it follows the principle of superposition. Using $\underline{\mathcal{F}}$ to represent the DFT,

$$\underline{\mathcal{F}}\{ax[n] + by[n]\} = a\underline{\mathcal{F}}\{x[n]\} + b\underline{\mathcal{F}}\{y[n]\}$$

Exercise 6.1

Show that the DFT is a linear transform.

As a linear transformation from $\mathbb{C}^N$ to $\mathbb{C}^N$, the DFT, $\boldsymbol{X} = \underline{\mathcal{F}}\{\boldsymbol{x}\}$ can be represented by the matrix equation

$$\underbrace{\begin{bmatrix} \hat{x}[0] \\ \hat{x}[1] \\ \hat{x}[2] \\ \vdots \\ \hat{x}[N-1] \end{bmatrix}}_{\hat{\boldsymbol{x}}} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1\cdot 1} & \omega^{-1\cdot 2} & \cdots & \omega^{-1\cdot(N-1)} \\ 1 & \omega^{-2\cdot 1} & \omega^{-2\cdot 2} & \cdots & \omega^{-2\cdot(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega^{-(N-1)\cdot 1} & \omega^{-(N-1)\cdot 2} & \cdots & \omega^{-(N-1)^2} \end{bmatrix}}_{\underline{\mathcal{F}}} \underbrace{\begin{bmatrix} x[0] \\ x[1] \\ x[2] \\ \vdots \\ x[N-1] \end{bmatrix}}_{\boldsymbol{x}}$$

6.1.1 Orthogonality

Consider the vectors $\boldsymbol{\omega}^k$ and $\boldsymbol{\omega}^l$ where k and l are integers. The inner product (dot product) of these vectors is

$$\begin{aligned} \boldsymbol{\omega}^k \cdot \boldsymbol{\omega}^l &= \sum_{n=0}^{N-1} \boldsymbol{\omega}^k[n]\, \overline{\boldsymbol{\omega}^l[n]} \\ &= \sum_{n=0}^{N-1} \omega^{kn} \omega^{-ln} \\ &= \sum_{n=0}^{N-1} \omega^{(k-l)n} \end{aligned}$$

which is a geometric series and can be written as

$$\begin{aligned} \boldsymbol{\omega}^k \cdot \boldsymbol{\omega}^l &= \frac{1-\left(\omega^{(k-l)}\right)^N}{1-\omega^{(k-l)}} \\ &= 0, \qquad k \neq l \end{aligned}$$

(ω^N)^(k-l)=1^(k-l)=1

When $k = l$ the summation and hence the inner product reduces to

$$\boldsymbol{\omega}^k \cdot \boldsymbol{\omega}^l = \sum_{n=0}^{N-1} \omega^0 = N$$

Hence,

$$\boldsymbol{\omega}^k \cdot \boldsymbol{\omega}^l = \begin{cases} 0, & k \neq l \\ N, & k = l \end{cases}$$

i.e., $\boldsymbol{\omega}^k$ and $\boldsymbol{\omega}^l$ are orthogonal.

Now consider the inner product (dot product) between the DFT coefficients vector, $\hat{\boldsymbol{x}}$, and the complex exponential vector $\boldsymbol{\omega}^{-n}$:

$$\begin{aligned}\hat{\boldsymbol{x}} \cdot \boldsymbol{\omega}^{-n} &= \sum_{k=0}^{N-1} \hat{x}[k]\overline{\boldsymbol{\omega}^{-n}[k]} \\ &= \sum_{k=0}^{N-1} \hat{x}[k]e^{j\frac{2\pi}{N}kn}\end{aligned}$$

But,

$$\begin{aligned}\sum_{k=0}^{N-1} \hat{x}[k]e^{j\frac{2\pi}{N}kn} &= \sum_{k=0}^{N-1}\left(\sum_{m=0}^{N-1} x[m]e^{-j\frac{2\pi}{N}km}\right)e^{j\frac{2\pi}{N}kn} \\ &= \sum_{k=0}^{N-1}\sum_{m=0}^{N-1} x[m]e^{j\frac{2\pi}{N}k(m-n)} \\ &= \sum_{m=0}^{N-1} x[m]\left(\sum_{k=0}^{N-1} \omega^{k(m-n)}\right) \\ &= \sum_{m=0}^{N-1} x[m](\boldsymbol{\omega}^{m} \cdot \boldsymbol{\omega}^{n}) \\ &= Nx[n]\end{aligned}$$

(Based on orthogonality)

Thus,

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} \hat{x}[k]e^{j\frac{2\pi n}{N}k}$$

which is the inverse DFT. In vector notation, this is written as

$$\boldsymbol{x} = \frac{1}{N}\sum_{k=0}^{N-1} \hat{x}[k]\boldsymbol{\omega}^{k}$$

The DFT and its inverse are thus

$$\hat{x}[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi k}{N}n}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} \hat{x}[k]e^{j\frac{2\pi n}{N}k}$$

6.1.2 Periodicity and Negative Frequencies

Consider the definition of the DFT

$$\hat{x}[k] = \sum_{n=0}^{N-1} x[n]\omega^{-kn}$$

It can be seen that

$$\omega^{-n(k+N)} = \omega^{-nN}\omega^{-kn} = \omega^{-kn}$$

Suggesting periodicity in the DFT coefficients

$$\hat{x}[k + pN] = \hat{x}[k], \qquad p \in \mathbb{Z}$$

If the DFT is viewed as the DTFT spectrum sampled at discrete points, this periodicity can be viewed as a consequence of the periodicity of the DTFT spectrum (which if you recall was periodic with a period of 2π). In fact, the substitution $\theta = k2\pi/N$ made to initially obtain the expression for DFT indicates that the interval of 2π is divided into N points.

Recall from the analysis of the periodicity of the DTFT spectrum, $\hat{x}(\theta)$, that the spectrum from in the interval $[\pi, 2\pi]$ is identical to that in the interval $[-\pi, 0]$. In terms of the DFT index the interval $\theta \in [0,2\pi]$ corresponds to $k \in [0, N-1]$, (shifted by 2π) thus the indices from $N/2$ (or $(N+1)/2$ if N is odd) to N represent the negative frequencies.

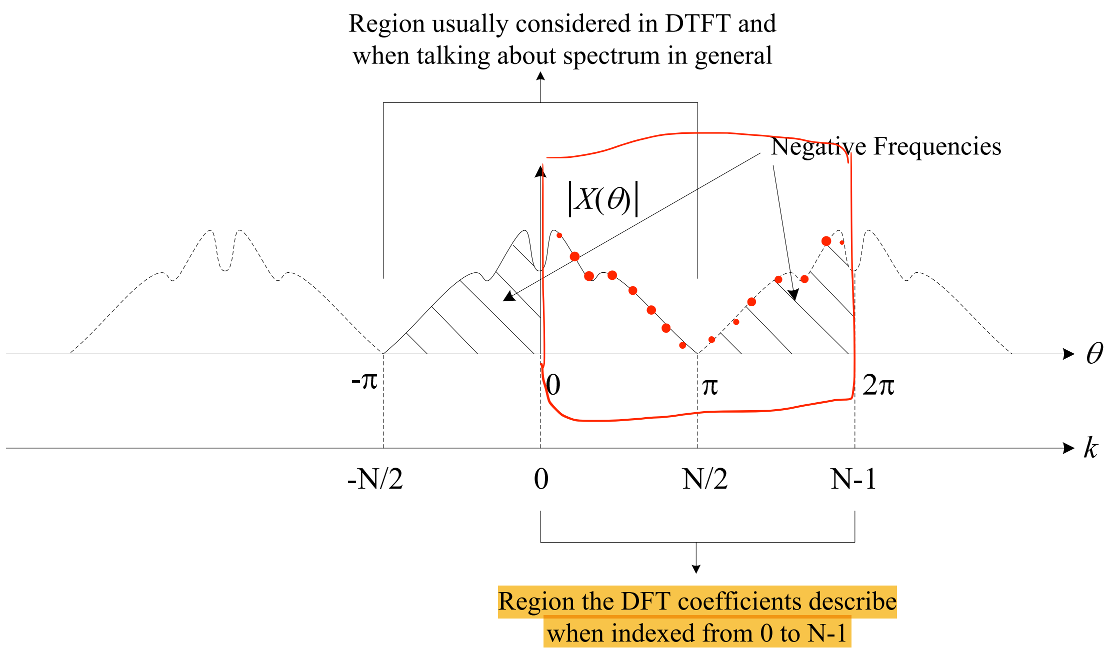


Figure 6.2: Region of the spectrum addressed by DFT and periodicity

Now consider the inverse DFT,

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} \hat{x}[k]\omega^{kn}$$

Once again, since $\omega^{k(n+N)} = \omega^{kn}$,

$$x[n + pN] = x[n]$$

Thus there is an implicit assumption of periodicity in the input signal $x[n]$ as well when using the DFT which is not the case for the DFT.

6.1.3 The FFT Algorithm

- The Fast Fourier Transform (FFT) is simply a mathematical technique to accelerate the calculation of the DFT. It was developed by Cooley and Tukey (1965) requires N to be a power of 2.
- Typically, if the DFT is calculated for a block of $2n$ samples e.g. 512 or 1024 samples (N) it would make the calculation of the DFT quite demanding.
- The FFT simply uses repetition and redundancy in the calculation to speed it up.
- The FFT is simply a TECHNIQUE to calculate the DFT, NOT a different transform.

A comparison of the number of complex multiplications required for direct evaluation of the DFT and the number needed for the Cooley-Tukey FFT is given below

No. of Points	Complex Multiplications in DFT (N^2)	FFT algorithm (N/2)log2(N)	Times faster than direct evaluation
16	256	32	8
256	65,536	1024	64
1024	1,048,576	5120	205

6.1.4 Zero Padding

As previously mentioned, the FFT algorithm for computing the DFT (and the IDFT) operates on signals (vectors) of length that is a power of 2. While this is not necessarily true in all implementations (for instance the FFT implementation in MATLAB does NOT require the length of the input to be a power of 2), it may be the case in many of them. In such cases, it is common to add zeros at the end of the signal to make it up to the required length. This is called *zero padding*. (Many programs will do this automatically if needed).

In order to see how zero padding affect the DFT spectrum, let $\boldsymbol{x} = \left[x[0], x[1], \ldots, x[N-1]\right]$ be the original input. For an integer, $M > N$, define

$$y[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ 0, & N \le n \le M-1 \end{cases}$$

Then

$$\hat{y}[k] = \underline{\mathcal{F}}_M\{y[n]\} = \sum_{n=0}^{M-1} y[n]\omega_M^{-kn} = \sum_{n=0}^{N-1} x[n]\omega_M^{-kn}$$

Note: In this case the complex exponentials are used with their subscripts and be aware that $\underline{\omega_N \neq \omega_M}$. The last summation is NOT the N-point DFT of $x[n]$ for precisely this reason.

Now,

$$\omega_M^{-kn} = e^{-j\frac{2\pi kn}{M}} = e^{-j\frac{2\pi knN}{MN}} = \underbrace{e^{-j\frac{2\pi n\left(\frac{kN}{M}\right)}{N}}}_{\omega_N} = \omega_N^{-n\left(\frac{kN}{M}\right)}$$

Thus whenever kN/M is an integer,

$$\hat{y}[k] = \sum_{n=0}^{N-1} x[n]\omega_N^{-n\left(\frac{kN}{M}\right)} = \hat{x}[kN/M]$$

with zero padding

without zero padding

Alternatively,

$$\hat{x}[k] = \hat{y}[kM/N]$$

Since we choose M, let $\underline{M = pN}$, giving

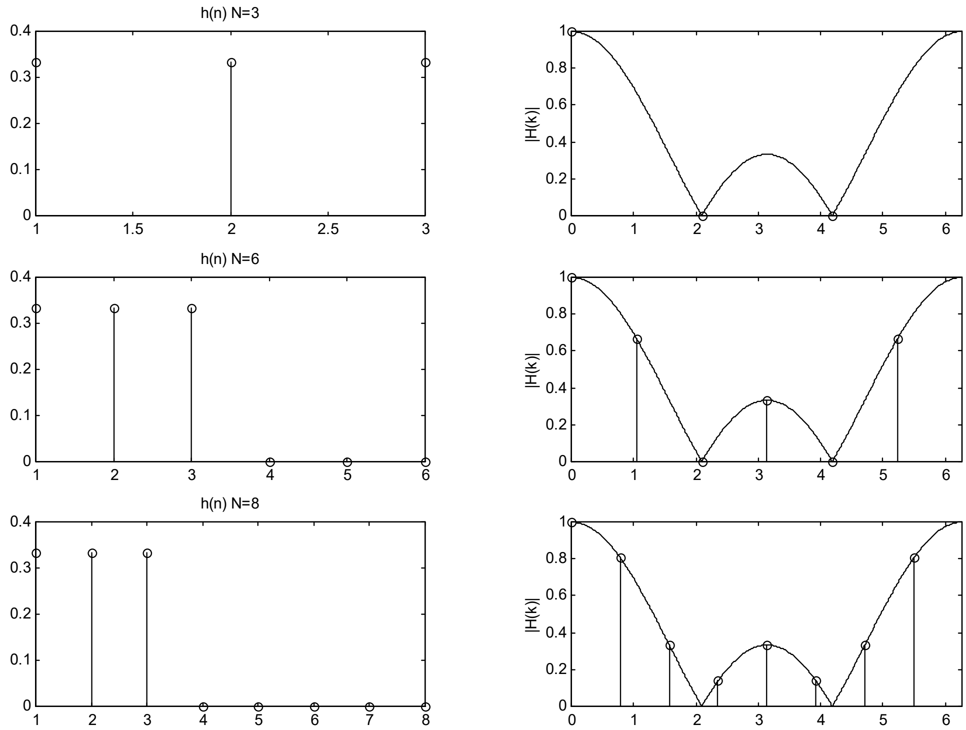
$$\hat{x}[k] = \hat{y}[pk]$$

i.e., If $x[n]$, a signal of length N is zero padded to make another signal $y[n]$ of length M, the k-th component of $\hat{x}[k] = \underline{\mathcal{F}}\{x[n]\}$ is the pk-th component of $\hat{y}[k] = \underline{\mathcal{F}}\{y[n]\}$.

Note: The zero-valued elements contribute nothing to the sum in the above equation, but act to decrease the frequency spacing (from $2\pi/N$ to $2\pi/M$).

Note: The zero padding gives us a high-density spectrum and provided a better displayed version for plotting. But it does not give us a high resolution spectrum because no new information is added to the signal, only additional zeros.

Example 6.1



6.2 Spectral Analysis

The different tools developed so far, namely the Fourier series, the Fourier transform, the discrete time Fourier transform (DTFT) and the discrete Fourier transform allow for the spectral analysis of signals (among other things). The different tools are used under different conditions:

- Continuous periodic signals – Fourier Series
- Continuous signals – Fourier transform
- Continuous spectral analysis of discrete signals – DTFT
- Discrete spectral analysis of discrete signals - DFT

Important ideas about Fourier series were:

- The Fourier series coefficients were complex valued
- They represented how much of each complex exponential (or equivalently each sinusoid) was present in the actual signal and also contained information about their onset.
- In particular, the magnitude of the Fourier series coefficients represented how much of the corresponding sinusoid was present (recall that n-th coefficient is associated with the n-th harmonic component)
- The phase of the coefficients determine the relative phase among the different components.

Note: Each period of a sinusoid corresponds to a total phase change of 2π and any phase can be viewed as a shift in the sinusoid (when compared to a phase of zero). The following plot a Fourier sum of three sinusoids of frequencies 5, 10 and 15 Hz with the Fourier coefficients having a magnitude of 1 in all cases and a phase of zero for all three components in the first case, and phases of $2\pi/3$, π and $-3\pi/4$ in the second case. Note how the resultant sums have different shapes due to this.

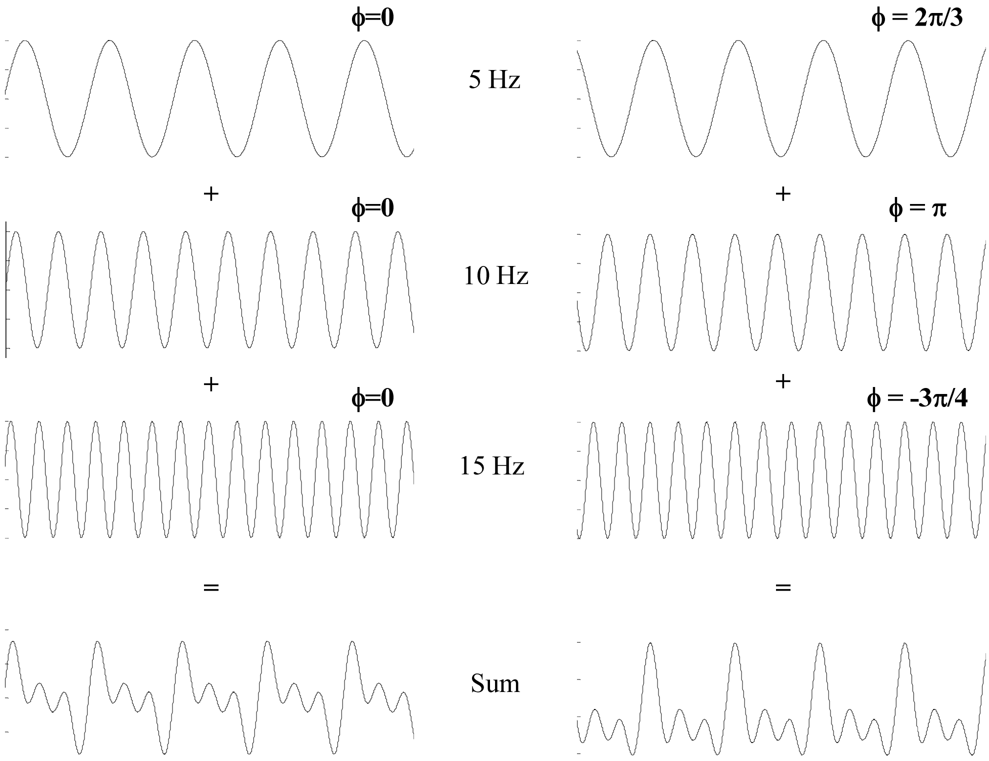


Figure 6.3: Fourier sum of three components highlighting the impact of phase.

Both the Fourier transform and the DTFT both produce continuous spectra (which can be separated as continue magnitude and phase spectra) which can be interpreted in the same manner as the Fourier series coefficients. The difference being that instead of discrete spectral components whose frequencies are multiples of a fundamental frequency, all possible frequencies (a continuum) is present.

The DFT transforms a discrete signal onto a discrete spectrum and can be implemented on digital processors. The DFT coefficients are complex and are typically represented as magnitude and phase spectra. It should be noted that while the DFT spectrum is discrete, it is different from the Fourier series coefficients in that the related sinusoids are not harmonics. i.e., their frequencies are not integer multiples of a fundamental frequency.

Example 6.2

Let $x(t)$ be a cosine wave of frequency 1000 Hz, if a discrete signal, $x[n]$, is obtained by sampling 1sec of this signal it at a rate of 4000 Hz, plot the magnitude spectrum of $x[n]$.

$$x(t) = \cos(2\pi 1000t), \qquad 0 \le t \le 1, t \in \mathbb{R}$$

$$x[n] \stackrel{\text{def}}{=} x(nT) = \cos(2\pi 1000nT), \qquad n \in \mathbb{Z}$$

$$T = \frac{1}{F_s} = \frac{1}{4000}$$

$$0 \le n \le N-1$$

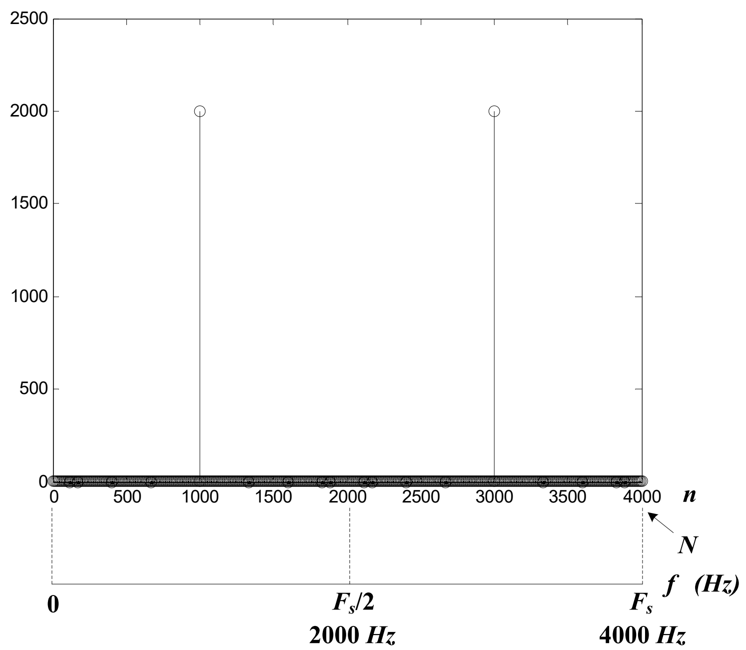
$$N = \frac{t_{max}}{T} = \frac{1sec}{\frac{1}{4000}} = 4000$$

$$\therefore x[n] = \cos\left(\frac{2\pi 1000n}{4000}\right), \qquad 0 \le n \le 3999$$

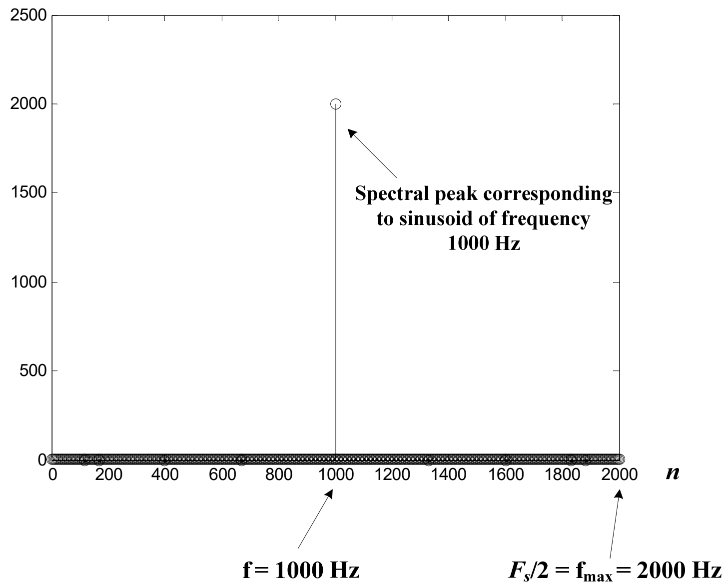
$$\hat{x}[k] = \underline{\mathcal{F}}\{x[n]\} = \sum_{n=0}^{N-1} x[n]\omega^{-kn}, \qquad 0 \le k \le N-1$$

$$\hat{x}[k] = \sum_{n=0}^{3999} \cos\left(\frac{2\pi 1000n}{4000}\right) e^{-j\frac{2\pi k}{4000}n}, \qquad 0 \le k \le 3999$$

Plotting the Magnitude spectrum in MATLAB,

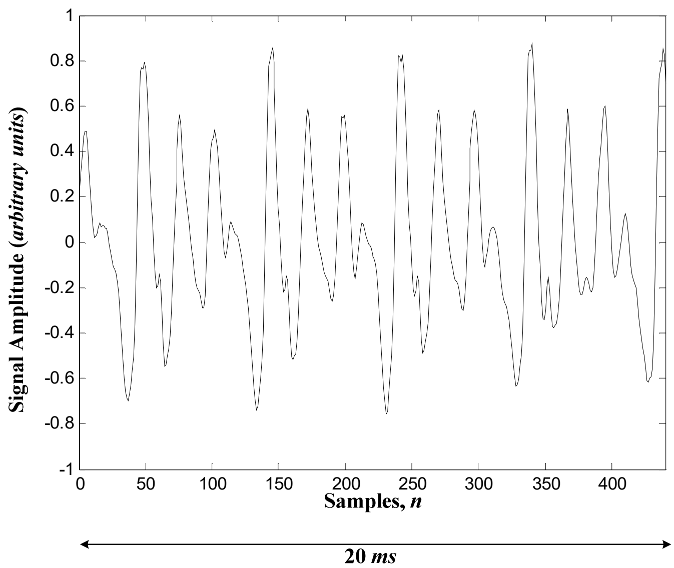


Note: The DFT spectrum goes from 0 to $N-1$, which corresponds to $\theta = 0\,rad$ to $\theta = 2\pi\,rad$. Recall that the region $[\pi, 2\pi]$ is the same as the region $[-\pi, 0]$ due to the periodicity of the spectrum and this region in turn corresponds to the negative frequencies of the Fourier transform. Thus it is sufficient to plot the DFT spectrum in the interval $[0, N/2]$.



Example 6.3

Consider a more complex signal such as the one shown below, which is 20ms of voiced speech samples at 22.05 kHz.

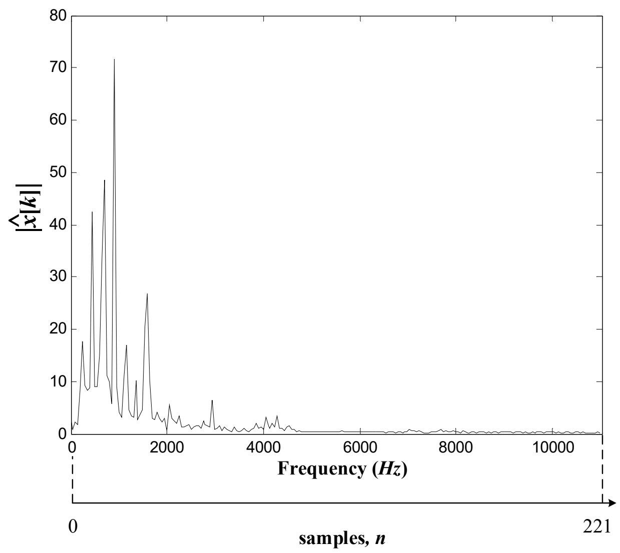


The spectral content of such a signal is much richer than that of simple sinusoids, but the DFT can still be made use of to ascertain its spectrum.

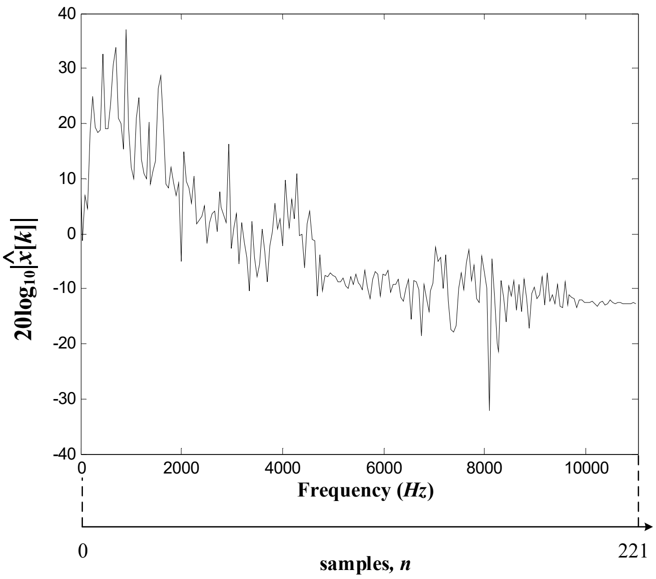
The number of samples in the signal, N is

$$N = \frac{duration}{\frac{1}{F_s}} = \frac{20ms}{\frac{1}{22050}} = 441$$

Thus a 441-point DFT was performed and the magnitude of the first 221 ($\approx N/2$) coefficients is plotted as the magnitude spectrum.



Often the dynamic range of $|X[k]|$ is large, making it hard to obtain useful information from its plot. Hence $20\log|X[k]|$, which is the power spectrum in dB is plotted instead.



6.2.1 Time varying spectral content and spectrograms

The DFT does not require the spectral content of the signal to be unchanging with time. However, it should be noted that when viewing the spectrum as the contribution of sinusoids of different frequencies to the signal, the sinusoids themselves exist throughout the region of analysis. To illustrate this, consider two signals, $x[n]$ and $y[n]$, defined as

$$x[n] = \sin\left(\frac{2\pi(300)}{F_s}n\right) + \sin\left(\frac{2\pi(700)}{F_s}n\right), 0 \leq n \leq 7999$$

$$y[n] = \begin{cases} \sin\left(\frac{2\pi(300)}{F_s}n\right), & 0 \leq n \leq 3999 \\ \sin\left(\frac{2\pi(700)}{F_s}n\right), & 4000 \leq n \leq 7999 \end{cases}$$

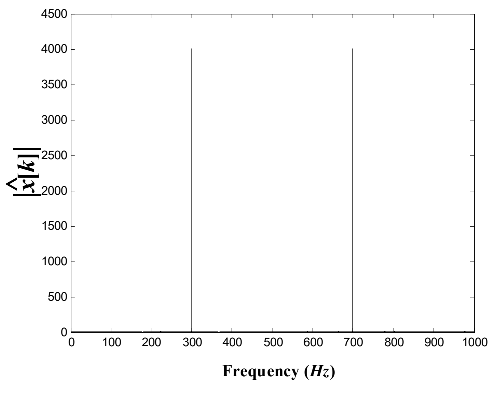
$$F_s = 2000\, Hz$$

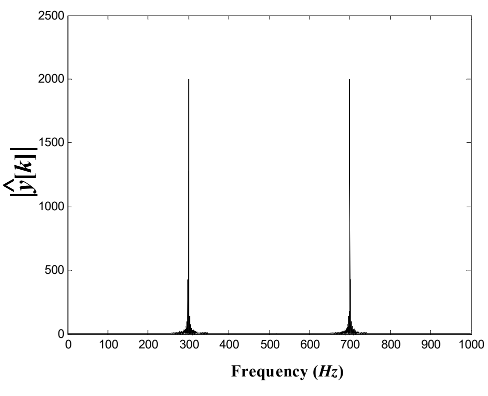
Observe that both signals have the same number of samples, $N = 8000$ and hence the same duration, t_{dur}

$$t_{dur} = \frac{N}{F_s} = \frac{8000}{2000} = 4\, secs$$

The first signal, $x[n]$, is the sum of two sinusoidal components of frequencies 300 Hz and 700 Hz, both existing at all times in the 4 sec duration. The second signal, $y[n]$, also consists of the two sinusoidal components of the same frequencies, however, the 300Hz component exists only in the first 2 secs and the 700 Hz component only in the last 2 secs.

Comparing the magnitude spectra of the two signals,



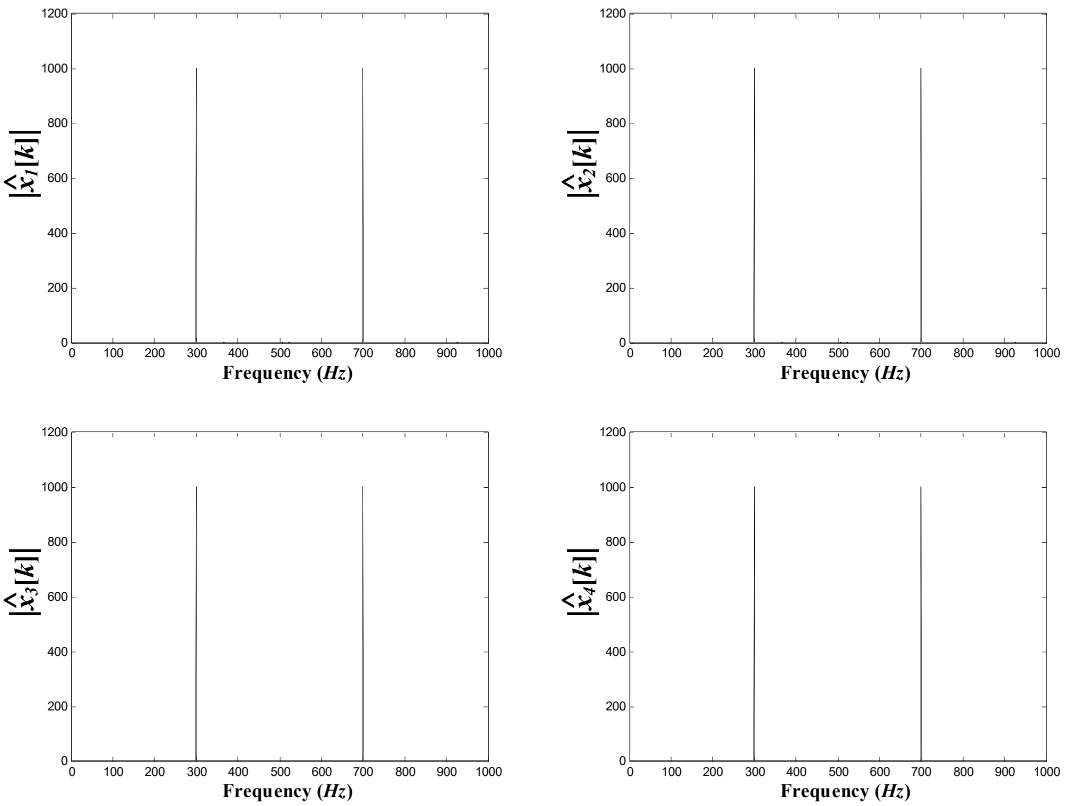


It can be seen that both magnitude spectra exhibit spectral peaks at 300 Hz and 700 Hz corresponding to the components present in them and the two signals cannot be distinguished based on the magnitude spectra even though they are very different signals.

This is because the sinusoidal components as given by the DFT exist at all points in the region of analysis even if the spectral composition of the original signal changes within this period. In such cases other components will also have non-zero amplitude and appropriate phase in order to cancel each other out at the periods where they do not exist.

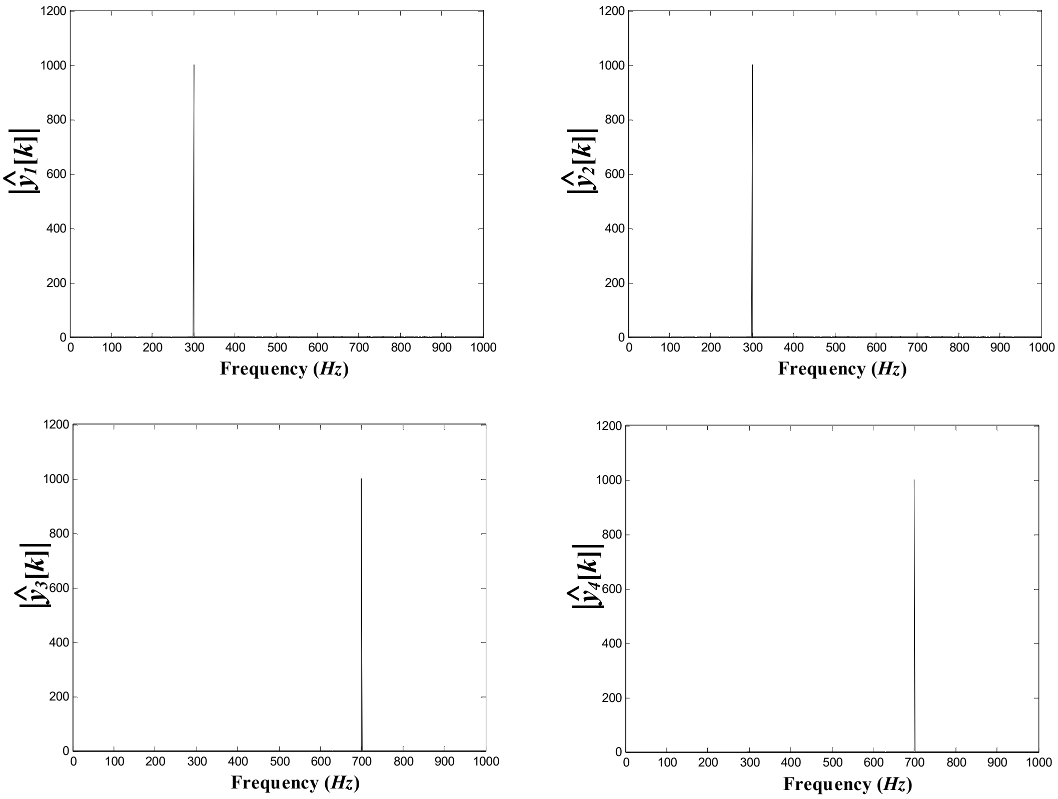
This information is present in the magnitude and phase spectra; however, it cannot be ascertained by visual observation thus reducing the usefulness of the DFT for spectral analysis when the spectral content of the signal changes with time.

Instead of analysing the entire signal, consider dividing the 4 sec signals into 4 sections (frames) of 1 sec each and then plotting the magnitude spectra of these frames. In the case of $x[n]$, these are



All four are similar to each other and the magnitude spectrum of the entire signal. This is to be expected since the spectral content of $x[n]$ do not change with time and both components exist in all four frames.

Similarly dividing $y[n]$ into four 1 sec frames and plotting their magnitude spectra results in



Here the magnitude spectra of the first 2 frames indicate the presence of a 300 Hz component while the spectra of the next 2 indicate the presence of a 700 Hz component. These are exactly what one would expect based on the definition of $y[n]$.

Thus while using the DFT to analyse the spectral content of the entire signal did not result in useful results, dividing the signal into short frames, prior to DFT based spectral analysis of each frame gave more useful results when the spectral content of the signal was changing with time.

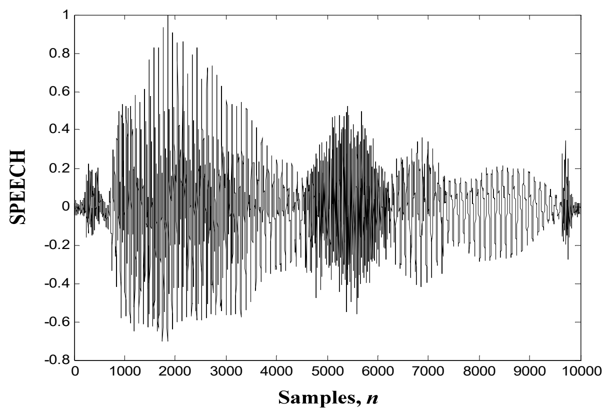
In general, the size of the frame would depend on the rate at which the spectral content was changing while being aware that a smaller size results in a smaller number of samples and hence a smaller number of points in the spectrum (and therefore a coarser frequency resolution).

- Pick a size that is small enough such that the spectral content does not vary much within the frames
- Pick a size that is large enough to provide the necessary frequency resolution

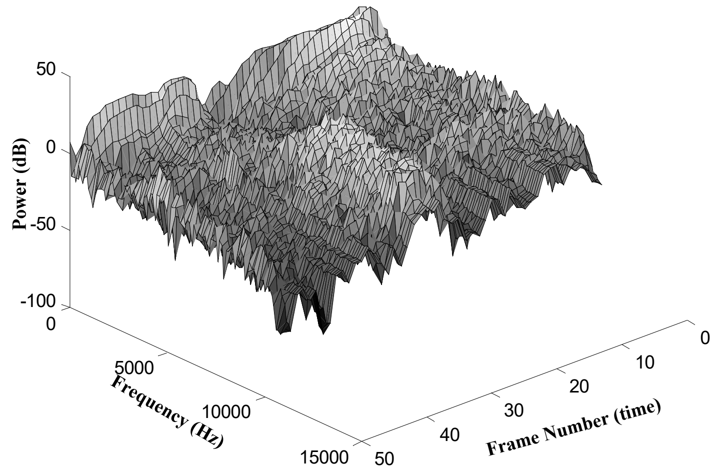
Often the magnitude spectra of consecutive frames are plotted together in a 3-D plot, with frame number (Time) on one axis, the DFT sample points (Frequency) on another axis and the magnitude of the DFT coefficients (Power) on the third axis. This is referred to as a spectrogram.

Example 6.4

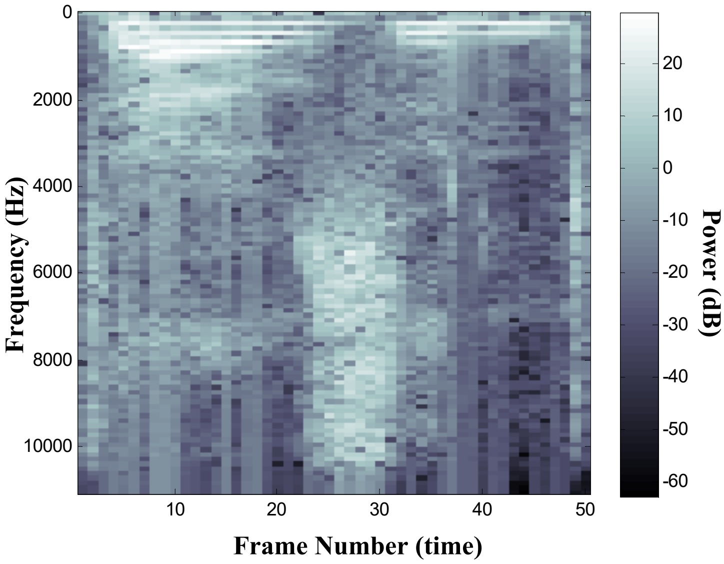
A speech signal consisting of 10000 samples (sampled at a rate of 22.05 kHz) is analysed by dividing it into frames of length 200 samples each and then creating a spectrogram. This is shown below



The spectrogram is



A 3-D plot like the one shown above is however hard to read (especially in black and white) and it is much more common (and readable) to plot the spectrogram as an image, with different shades (or colours) indicating different power levels as shown below.



Note: Each slice of the spectrogram across all frequencies corresponding to each frame number is the magnitude spectrum of that frame.

Note: The discussion on time varying spectral content was made in the context of DFT but it holds for the DTFT and the continuous Fourier transform as well.