Z-Transform & Filters

Consider two polynomials,

$$A(z) = a_0 + a_1 z + a_2 z^2$$
$$B(z) = b_0 + b_1 z + b_2 z^2$$

$$\text{Let } C(z) = A(z)B(z) = (a_0 + a_1 z + a_2 z^2)(b_0 + b_1 z + b_2 z^2)$$

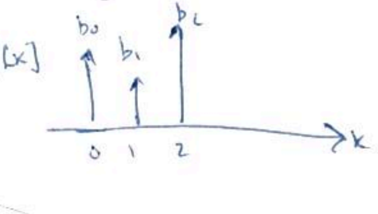
$$= a_0 b_0 + (a_1 b_0 + a_0 b_1) z + (a_2 b_0 + a_1 b_1 + a_0 b_2) z^2 + (a_2 b_1 + a_1 b_2) z^3 + a_2 b_2 z^4$$

polynomial multiplication = convolution

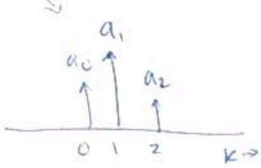
Now consider two discrete-time sequences (signals).

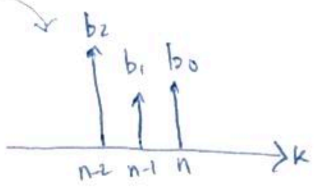
$$x[n] = \{.0, 0, \underset{\uparrow}{a_0}, a_1, a_2, 0, 0, 0..\} \text{ ie, } x[n] = \begin{cases} a_0, & n=0 \\ a_1, & n=1 \\ a_2, & n=2 \\ 0, & \text{otherwise} \end{cases}$$

$$y[n] = \{\ldots 0, 0, \underset{\uparrow}{b_0}, b_1, b_2, 0, 0 \ldots\} \quad y[n] = \begin{cases} b_0, & n=0 \\ b_1, & n=1 \\ b_2, & n=2 \\ 0, & \text{otherwise} \end{cases}$$



$$p[n] = x[n] * y[n] = \sum_k x[k]\, y[n-k]$$





for $n \le -1$, $x[k]$ & $y[n-k]$ ~~never~~ don't overlap $\therefore p[n] = 0,\ n \le -1$

for $n > 4$, $x[k]$ & $y[n-k]$ don't overlap $\therefore p[n] = 0,\ n > 4$

for $0 \le n \le 4$ ~~at n=0~~, $p[0] = a_0 b_0$

$p[1] = x[0]y[1] + x[1]y[0] = a_0 b_1 + a_1 b_0$

$p[2] = x[0]y[2] + x[1]y[1] + x[2]y[0] = a_0 b_2 + a_1 b_1 + a_2 b_0$

$p[3] = x[1]y[2] + x[2]y[1] = a_1 b_2 + a_2 b_1$

$p[4] = x[2]y[2] = a_2 b_2$.

Z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]\, \underbrace{z^{-n}}_{z \text{ is complex valued}}$$

Why z^{-1}? we'll see later.

$\downarrow$

z-transform of a sequence, $x[n]$ is a power series of $\overset{\uparrow}{z^{-1}}$

For finite sequence of length, $x[n]$, i.e., $x[n] = 0$ outside the range $0 \leq n \leq N$ the z-transform is a polynomial of degree N.

$$X(z) = x[0] + x[1]z^{-1} + x[2]z^{-2} + \dots + x[N]z^{-N}$$

Why are we interested in this?

$$x[n] \rightarrow \underset{LTI}{\boxed{h[n]}} \rightarrow y[n].$$

$$y[n] = x[n] * h[n] = \sum_{k} x[k]\, h[n-k]$$

If $x[n]$ & $h[n]$ are both finite,

~~Now~~ $X(z)H(z) = \left(x[0] + x[1]z^{-1} + \dots x[N]z^{-N}\right)\left(h[0] + h[1]z^{-1} + \dots h[M]z^{-M}\right)$

$$= \sum_{k} \sum_{m} x[k]\, h[m]\, z^{-(k+m)}$$

$n=0$, $x[0]\left(h(0) + h(1)z^{-1} + h\right.$
$+ x[1]\left(h[0]z^{-1} + h[1]z^{-2} + h[2]z^{-3} +\right.$
$+ x[2]\left(h[0]z^{-2} + h[1]z^{-3} + \dots\right.$
$\vdots$

$$= \sum_{k} x[k] \sum_{n} h[n-k]\, z^{-n}$$

$$= \sum_{n} \underbrace{\sum_{k} x[k]\, h[n-k]}_{x[n]*h[n] = y[n]}\, z^{-n}$$

$$= \sum_{n} y[n] z^{-n} = Y(z) \qquad \text{i.e., } Y(z) = X(z)\overset{\text{Transfer function}}{\overbrace{H(z)}}$$

For infinite sequences,

~~$X(z)H(z) = x[0](h[0] + h[1]z^{-1} + h[2]z^{-2} \dots\dots$~~

$$X(z)H(z) = \begin{aligned} &+ \vdots \\ &+ x[-1](\dots + h[-1]z^2 + h[0]z + h[1] + h[2]z^{-1} + \dots) \\ &\ \ x[0](\dots + h[-1]z^{+1} + h(0) + h[1]z^{-1} + \dots) \\ &+ x[1](\dots + h[-2]z^1 + h[-1] + h[0]z^{-1} + h[1]z^{-2} + \dots) \\ &+ \vdots \end{aligned}$$

Manipulating polynomials (z-transform) is convenient.

For eg. $y[n] = x[n] + \alpha y[n-1]$ (LTI system)

What is the impulse response?

$$Y(z) = \sum_n y[n]z^{-n} = \sum (x[n] + \alpha y[n-1])z^{-n}$$

$$= \sum_n x[n]z^{-n} + \sum_n \alpha y[n-1]z^{-n}$$

$$= X(z) + \alpha\left(\sum_m y[m]z^{-m}\right)z^{-1} \quad m = n-1$$

$$= X(z) + \alpha z^{-1}Y(z)$$

$$\Rightarrow Y(z)(1-\alpha z^{-1}) = X(z)$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-\alpha z^{-1}} = 1 + (\alpha z^{-1}) + (\alpha z^{-1})^2 + (\alpha z^{-1})^3 + \dots$$

$$= 1 + \alpha z^{-1} + \alpha^2 z^{-2} + \alpha^3 z^{-3} + \dots$$

Comparing with

$$H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + \dots$$

$$h[n] = \begin{cases} \alpha^n, & n \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

$$\text{In general } \underset{\substack{\uparrow \\ \text{Impulse response}}}{h[n]} \xleftrightarrow{\text{z transform}} \underset{\substack{\uparrow \\ \text{Transfer function.}}}{H(z)}$$

Properties of Z-transform

i. Linear : if $x[n] \xrightarrow{Z} X(z)$

if $y[n] \xrightarrow{Z} Y(z)$.

$$p[n] = \alpha x[n] + \beta y[n] \xrightarrow{Z} \alpha X(z) + \beta Y(z).$$

$$\hookrightarrow P(z) = \sum_n p[n] z^{-n} = \sum_n \left(\alpha x[n] + \beta y[n]\right) z^{-n} = \alpha \sum_n x[n] z^{-n} + \beta \sum_n y$$

$$= \alpha X(z) + \beta Y(z)$$

Time shift :

$$x[n] \xrightarrow{Z} X(z)$$

$$x[n-k] \xrightarrow{Z} \sum_n x[n-k] z^{-n} \qquad m = n-k.$$

$$= \sum_m x[m] z^{-m} z^{-k} = z^{-k} X(z).$$

Time reversal

$$x[n] \xrightarrow{Z} X(z)$$

$$x[-n] \xrightarrow{Z} \sum_n x[-n] z^{-n} = \sum_m x[m] (z^{-1})^{-m}$$

$n = -m$

$$= X(z^{-1}) = X(1/z).$$

Multiplication by power series

~~$a^n x[n]$~~ $x[n] \xrightarrow{Z} X(z)$

$$a^n x[n] \xrightarrow{Z} \sum_n a^n x[n] z^{-n} = \sum_n x[n] (a^{-1} z)^{-n}$$

$$= X(a^{-1} z) = X(z/a)$$

⑤

Comparing Z-transform & DTFT → what is z?

$$X(z) = \sum_n x[n]\, \underset{=}{z^{-n}}$$

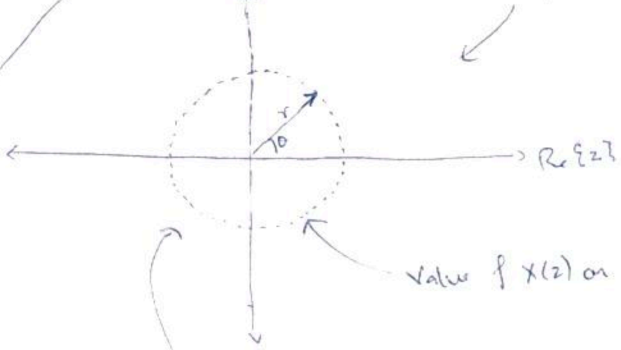
→ z is complex, can be written as $\underline{re^{j\theta}}$ → polar form.

$$= \sum_n \underbrace{\underbrace{x[n]\,\underset{=}{r^{-n}}}_{x_r[n]}\, e^{-j\theta n}}_{\text{DTFT of } x_r[n]}$$

→ real

$$= \hat{x}_r(\theta)$$

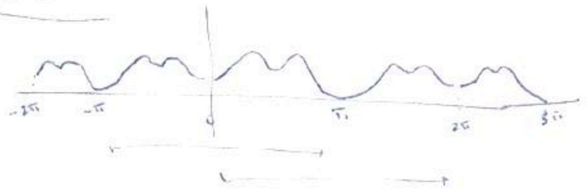
X(z) is defined on this plane



Value of X(z) on circle of radius 'r' is $\hat{x}_r(\theta)$.

— defined for $0 \le \theta \le$ or $-\pi \le \theta \le \pi$

wrapped around this circle



Periodic with period 2π as we know.

May not be defined for all values of 'r'.

Sufficient condition: $x_r[n]$ is absolutely summable.

ie $\sum_n |x_r[n]| < \infty$.

$\Rightarrow \sum_n |r^{-n} x[n]| < \infty$

→ If x[n] is finite valued ~~&~~ of finite duration, this is true for all ~~finite value~~ 'r'. ie, X(z) is defined for all z

→ If x[n] is infinite long but defined for n≥0. ie x[n] is $\underline{\text{a causal sequence}}$ → Note.
then this is true for all r > r₀ for some value of r

Region of convergence (ROC) of $X(z)$

↳ The region on the z-plane where $X(z)$ is defined given a sequence $x[n]$.

One sided sequences.
↓
also true if shifted by finite amount $x[n-k]$.

If $x[n]$ takes non-zero values for $n \geq 0$ ROC typically takes the form $r > r_0$

If $x[n]$ takes non-zero values for $n \leq 0$ ROC typically takes $r < r_0$.

Note on causal sequences

$$x[n] \rightarrow \underset{\text{LTI system}}{\boxed{h[n]}} \rightarrow y[n].$$

$$y[n] = x[n] * h[n] = \sum_k x[k]\,h[n-k] = \sum_k h[k]\,x[n-k]$$

why? $\hat{y}(\theta) = \hat{x}(\theta)\hat{h}(\theta) = \hat{h}(\theta)\hat{x}(\theta)$

Both identical.

Causal system → output depends only on present & past values of input.

ie, $y[n]$ depends on $x[n], x[n-1] \ldots$ only

⇒ ie, in $y[n] = \sum_k h[k]\,x[n-k]$

↳ $h[k] = 0$, $k < 0$ for causal system.

↓

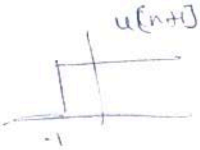
Causal sequence ⇒ $h[n]$ is zero for $n <$

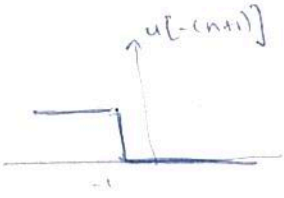
Examples of Z-transform

① $x[n] = u[n] - u[n-10]$.

$$X(z) = \sum_{n=0}^{9} (1)\, z^{n} = \frac{1-z^{-10}}{1-z^{-1}}$$

② $y[n] = -\alpha^{n} u[-n-1]$





$$Y(z) = ? = \sum_{n} -\alpha^{n} u[-n-1]\, z^{-n}$$

$$= \sum_{n=-\infty}^{-1} (\alpha/z)^{n}$$

$$= \sum_{k=1}^{\infty} (z/\alpha)^{k} + 1 - 1$$

$$= \sum_{k=0}^{\infty} (z/\alpha)^{k} - 1$$

$$= \underbrace{\frac{1}{1-(z/\alpha)}}_{} - 1 = \frac{\alpha}{\alpha - z} - 1 = \frac{z}{\alpha - z}$$

↳ as long as $|z/\alpha| < 1$ i.e. $|z| < |\alpha|$ ← region of convergence (ROC)

Note on unit step signal/function

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

↑ used to denote one-sided sequences generally

or sequences of finite duration

↳ eg. $u[n] - u[n-N] = \begin{cases} 1, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise.} \end{cases}$

$x[n]u[n]$ is typically used to denote that $x[n]$ is a causal sequence (one sided)

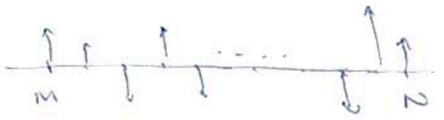
↓ equivalent to saying $x[n] = 0,\ n < 0$.

Inverse Z-transform

$$x[n] \xrightarrow{\text{Z-transform}} X(z) \quad + \text{ROC}$$

If $x[n]$ is a finite sequence, then inversion is trivial. (comparison)

ie, $x[n] = 0, \quad n < M \;\&\; n > N.$



$$\xrightarrow{\text{Z-T}} X(z) = \underline{\underline{x[M]}}\, z^{-M} + x[M+1]\, z^{-(M+1)} + \dots x[N]$$

↓ Coefficients are $x[n]$.

What if $x[n]$ is not a finite sequence?

$$X(z) + \text{ROC} \longrightarrow x[n]\,?$$

~~If you choose an 'r'~~ such

If you choose a circular contour around the origin that lies in the ROC. ~~[illegible]~~

then IDTFT around that circle gives $x_r[n]$ from which $x[n]$ can be determined.

$$\text{ie, } \quad x[n] = r^n x_r[n] = r^n \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{X(re^{j\theta})}_{X(z)} e^{j\theta}\, d\theta$$

↳ Not very useful, if $r = 1$ is in ROC then just use IDTFT if ROC is $r > 1$ then r^n can grow very rapidly.

More generally,

$$x[n] = \frac{1}{2\pi j} \oint_C X(z)\, z^{n-1}\, dz$$

↳ Not very easy to compute.

Z-transform very useful for analysis.
Generally inversion is not ~~[illegible]~~ required.

Partial Fraction

Sometimes you can split $X(z)$ into a sum of expression all of which are previously known ~~be~~ z-transforms of sequences & the inversion of $X(z)$ is a sum of these sequences.

ie if $X(z) = a_1 X_1(z) + a_2 X_2(z) + \dots$

& if
$$X_1(z) \xleftarrow{Z-T} x[n]$$
$$X_2(z) \xleftarrow{Z-T} x_2[n]$$
$$\vdots$$

Then $x[n] = a_1 x_1[n] + a_2 x_2[n] + \dots$

Example.

$$X(z) = \frac{z^{-1}}{1-0.25z^{-1}-0.375z^{-2}}, \text{ROC } |z|>0.5 \quad \to \text{what is } x[n]?$$

$$X(z) = \frac{z}{z^2-0.25z-0.375} = \frac{z}{(z-0.75)(z+0.5)}$$

$$= z\left[\frac{A}{z-0.75} + \frac{B}{z+0.5}\right]$$

$$= \frac{(4/5)z}{z-0.75} - \frac{(4/5)z}{z+0.5}$$

$$= \left(\frac{4}{5}\right)\left(\frac{1}{1-0.75z^{-1}}\right) - \left(\frac{4}{5}\right)\left(\frac{1}{1+0.5z^{-1}}\right)$$

but we know $\alpha^n u[n] \xrightarrow{ZT} \dfrac{1}{1-\alpha z^{-1}}$, ROC: $|z|>|\alpha|$

$$\therefore \frac{1}{1-0.75z^{-1}} \xrightarrow{IZT} 0.75^n u[n] \qquad \frac{1}{1+0.5z^{-1}}, |z|>0.5 \xrightarrow{IZT} (-0.5)^n u($$

ROC $|z|>0.75$

$$\therefore x[n] = \frac{4}{5}\left[(0.75)^n - (-0.5)^n\right], \ n \geq 0.$$

LTI systems + Filters & Z-transform

Not all LTI systems can be implemented in practice.

Only those of the form.

$$y[n] = \sum_{k=-A}^{B} a_k x[n-k] + \sum_{m=-C}^{D} b_m y[n-m]$$

Finite sums.

→ Can be implemented with finite number of ops & finite memory.

referred to as digital filters.

For causal digital filter $A=0$ & $C=0$.

If $b_m = 0 \;\forall m$ then $y[n]$ only depends on ~~x[n]~~ finite number of values of $x[n]$, i.e., $x[n]$, $-A \leq n \leq B$.

→ Non-recursive filter.

If $b_m \neq 0$ for some m then $y[n]$ depends on past (or future) values of y (i.e., $y[n-k]$)

→ Recursive filter

Z-transform.

$$Y(z) = \sum_{k=-A}^{B} a_k z^{-k} X(z) + \sum_{m=-C}^{D} b_m z^{-m} Y(z).$$

$$\Rightarrow Y(z)\left[1 - \sum_{m-C}^{D} b_m z^{-M}\right] = X(z) \sum_{k=-A}^{B} a_k z^{-k}.$$

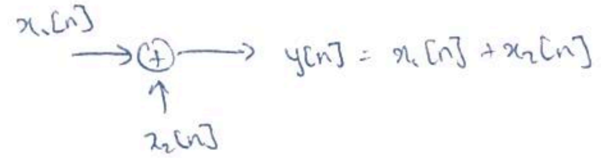
$$\Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=-A}^{B} a_k z^{-k}}{}$$

← Polynomial

Transfer fn of any LTI filter is a → Rational function

(causal)
All LTI filters ~~req~~ requir only 3 operatn.

Addition :



Multiplication (with scalar)

delay.

$x[n] \rightarrow \boxed{z^{-1}} \rightarrow y[n] = x[n-1]$

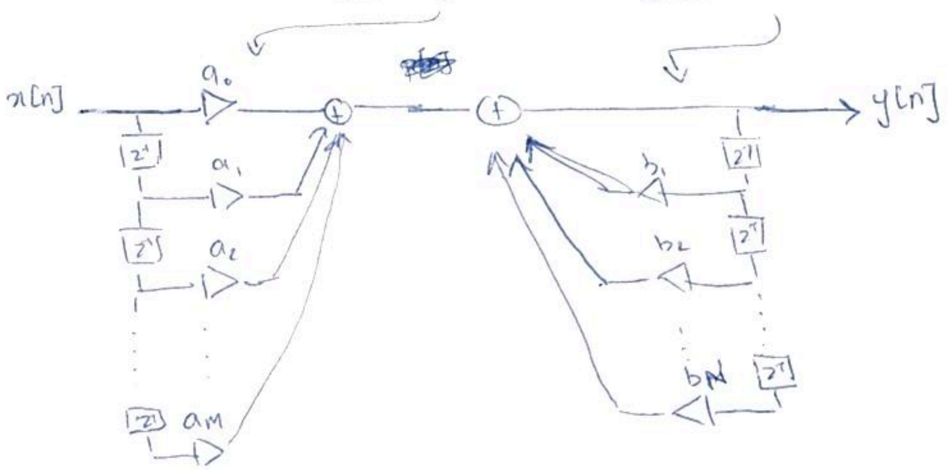
or

$x[n] \rightarrow \boxed{z^{-k}} \rightarrow y[n] = x[n-k]$

Since $x[n-k] \xrightarrow{ZT} z^{-k} X(z)$.

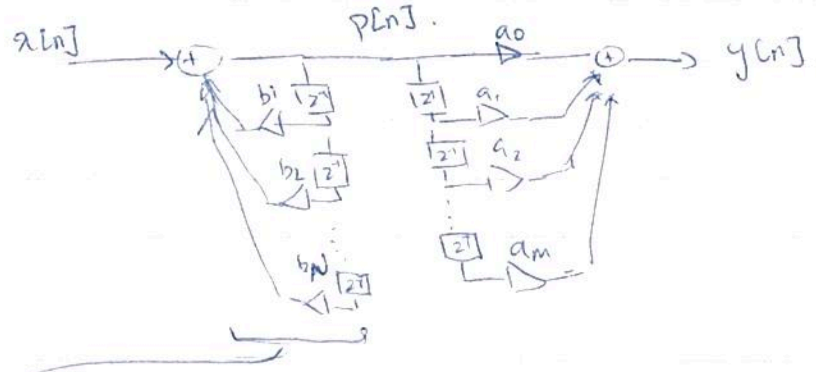
General causal LTI system

$$y[n] = \underbrace{\sum_{k=0}^{M} a_k x[n-k]}_{} + \underbrace{\sum_{m=1}^{N} b_m y[n-m]}_{}.$$

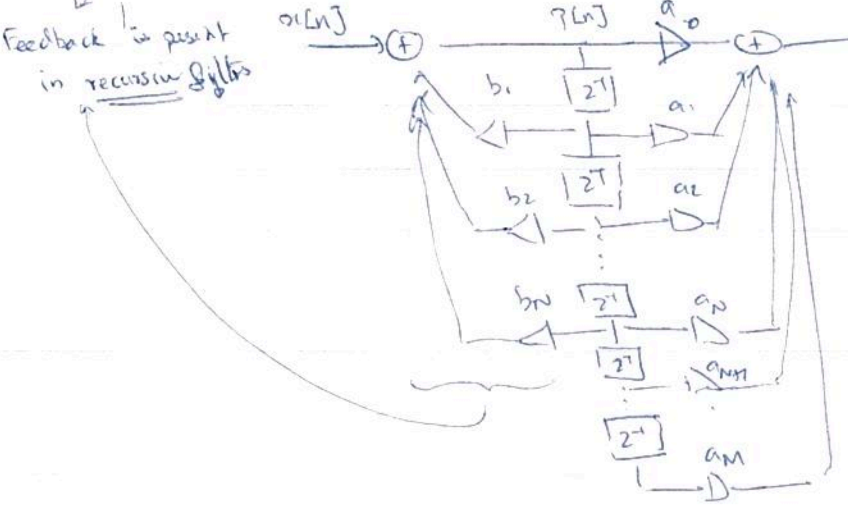


Direct Form-I structure.

Interchanging the two parts of the expression.



No need to double up delays.



Feedback is present in recursive filters

Direct Form - II Struct

To see why it's equivalent;

$$p[n] = x[n] + \sum_{m=1}^{N} b_m p[n-m]$$

$$y[n] = \sum_{k=0}^{M} a_k p[n-k].$$

$$= \sum_{k=0}^{M} a_k \left(x[n-k] + \sum_{m=1}^{N} b_m p[n-k-m] \right)$$

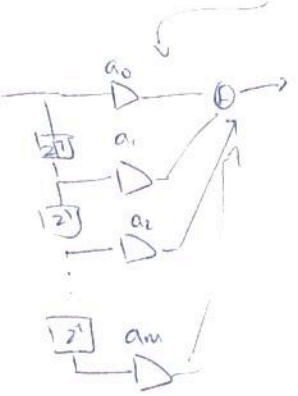
$$= \sum_{k=0}^{M} a_k x[n-k] + \sum_{m=1}^{N} b_m \underbrace{\sum_{k=0}^{M} a_k p[n-k-m]}_{y[n-m]}$$

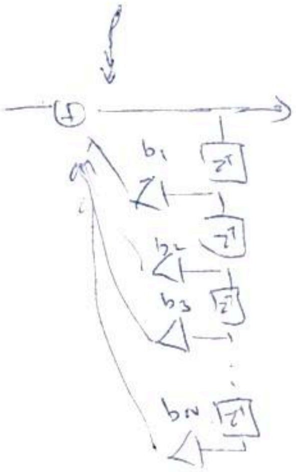
$$= \sum_{k=0}^{M} a_k x[n-k] + \sum_{m=1}^{N} b_m y[n-m].$$

Alternatively,

we know

$$H(z) = \frac{\sum_{k=0}^{M} a_k z^{-k}}{1 - \sum_{m=1}^{N} b_m z^{-m}} = \underbrace{\left(\sum_{k=0}^{M} a_k z^{-k}\right)}_{H_1(z)} \underbrace{\left(\frac{1}{1 - \sum_{m=1}^{N} b_m z^{-M}}\right)}_{H_2(z)}$$

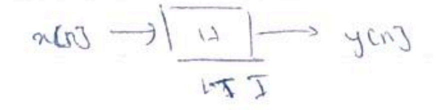




$$H(z) = H_1(z)\, H_2(z)$$

$$= H_2(z)\, H_1(z) \longrightarrow$$ order of LTI system can be interchanged.

LTI Syst respn



~~For a stable LTI~~

Impulse respn

$$\begin{tikzcd}
 & h[n] \arrow[dl, leftrightarrow, "ZT"'] \arrow[dr, leftrightarrow, "DTFT"] & \\
H(z) \arrow[rr, leftrightarrow, "z=e^{j\theta}"'] & & \hat{h}(\theta)
\end{tikzcd}$$

Transfer funct

Frequency respnse

For any stable LTI system.

$$\sum_n |h[n]| < \infty$$

$\therefore$ ROC includes $r=1$

hence $z=e^{j\theta}$ leads to DTFT.

Impulse respn, h[n] ~~can be~~

Infinite length sequnc
IIR filter

Must be recursive since all non-recursive filters are FIR.

Can be stable or unstable.

Finite length sequnc
FIR filter

Must be stable.

Let length of sequnc be N+1 → $0 \le n \le N$

~~h[n]~~

the $H(z) = h[0] + h[1]z^{-1} + h[2]z^{-2} + \ldots h[N]z^{-N}$

Non-recursive.

polynomial of order N

N^{th} order FIR filter.