ELEC3104: Digital Signal Processing

Chapter 1: Signals and Spectra

1.1 Introduction

The interpretation and modification of signals plays a key role in almost all systems and in most cases this takes the form of digital signal processing. i.e., the signals are represented and manipulated digitally as sequences of numbers. Think about what this means in the context of commonly used devices such as smartphones – everything from reducing background noise during phone calls, to playing music, to recording videos are all examples of signal processing systems and all of them are ultimate implemented in terms of additions, multiplications and comparisons of numbers! Understanding the fundamentals of how mathematical manipulation of sequences of numbers relate to understanding and modifying signals is the first step in designing these sorts of systems and introducing you to these concepts is the aim of this course.

The presentation of this material may seem mathematically involved – this is the nature of this subject – but the key aim of this material is to help you develop insights connecting the mathematics to signal properties. The concepts and ideas presented may initially seem daunting but over the course of the semester my aim is to show you that as you start to connect the dots, the concepts reinforce each other and paint a cohesive picture. The field of signal processing is vast and this course only provides an introduction to the field but if you find the topic interesting I strongly encourage you to pursue more advanced topics in other signal processing courses or on your own.

Lecture notes will be provided regularly over the course of the semester but the order and style of presentation of material will be somewhat different in the lectures compared to the notes. This is intentional as the succinct presentation style of notes is not well suited for lectures. More importantly, the lectures and the lecture notes are designed to complement each other (and not as replacements for each other), with the lectures intended for the presentation and discussion of new ideas and concepts, and the notes to support them with mathematical detail. My recommendation is that you also take your own notes in the lectures to remind yourself of the ideas and how they connect with each other but not write down every equation and derivation since most of these will be repeated in the notes. It is much more important and useful to develop an intuitive understanding of the concepts at the lectures.

Finally, the suggested textbooks and reference books cover all the topics included in this course and more but the style and order of presentation will differ from the lectures (and each other). Learning a topic from multiple perspectives helps solidify your understanding of it and I strongly encourage you to follow these books in addition to the lecture material.

1.2 Signals

The term 'signal' can have a number of interpretations depending on the context, but generally the term refers to a quantity that conveys some sort of information. For example, the time varying voltage fed to a loudspeaker is a signal that carries information about the audio to be played through the loudspeaker. Similarly, the two dimensional array of numbers representing a digital photograph is an example of a 2D signal and if these values vary with time, that could represent a video signal. It is mathematically convenient to represent signals as functions of one or more independent variables. Signals may be categorised as:

- Continuous signals
- Discrete signals

In the case of a continuous signal, $x(t)$, the independent variable t is continuous and thus $x(t)$ is defined for all t. Continuous signals are also known as analogue signals.

Note: t is a continuous independent variable ($t \in \mathbb{R}$) and may not necessarily indicate time.

On the other hand, discrete signals, $x[n]$ are defined only at discrete points and consequently the independent variable takes on only a discrete set of values. A discrete signal is thus a sequence of numbers.

Note: n is a discrete independent variable taking integer values ($n \in \mathbb{Z}$)

Note: Discrete signals are distinct from digital signals in that the digital signal is obtained by quantizing the values taken by the discrete signal in order to represent it with a fixed number of bits.

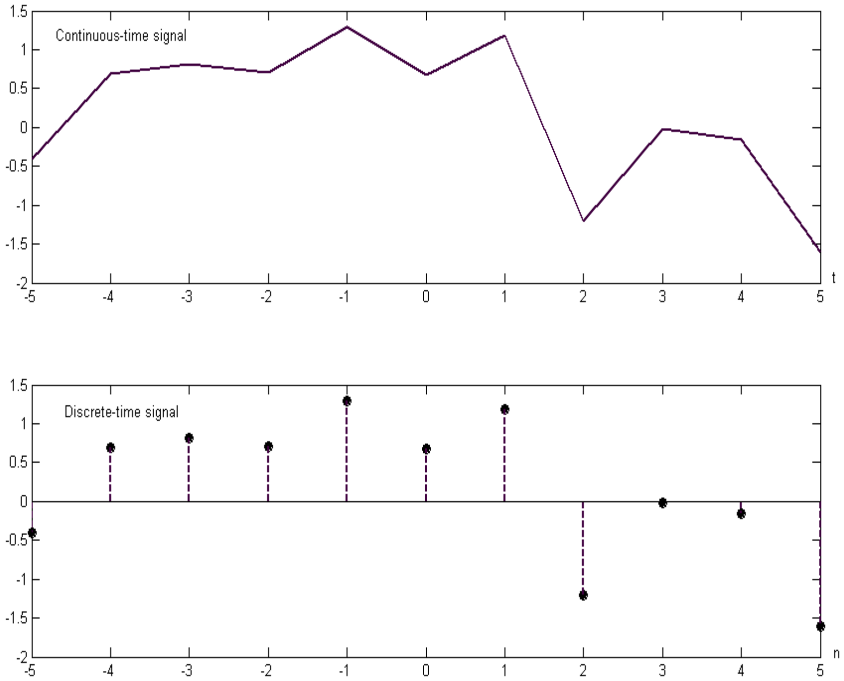


Figure 1.1: (a) An example of a continuous signal; (b) discrete signal

Note: If the independent variable denotes time, the terms continuous-time and discrete-time maybe used in place of continuous and discrete.

Note: In this course, unless otherwise mentioned, the independent variable, t, is taken to denote time with no loss of generality. The ideas hold even if the independent variable is not time.

Note: Mathematically signals are functions and may involve one or more independent variables. The terms signals and functions may be used interchangeably.

Exercise 1.1

Are the following ***continuous-time*** or ***discrete-time*** signals?

1. A person's body temperature
2. An hourly recording of a person's body temperature by a nurse in a hospital
3. The prices of stocks printed in the daily newspapers
4. Voltages and currents in a circuit
5. Voltages and currents specified only at certain values of t

Answers: 1 and 4 are continuous-time, 2, 3, and 5 are discrete-time

1.3 Periodicity and Fourier series

1.3.1 Periodic Signals

An important class of signals is periodic signals. A periodic continuous signal $x(t)$ has the property that there is a positive value of $\boldsymbol{P} \in \mathbb{R}$ for which

$$x(t) = x(t + \boldsymbol{P})$$

for all values of t. In other words, a periodic signal has the property that it is unchanged by a time shift of $\boldsymbol{P}$. In this case we say $x(t)$ is periodic with period $\boldsymbol{P}$.

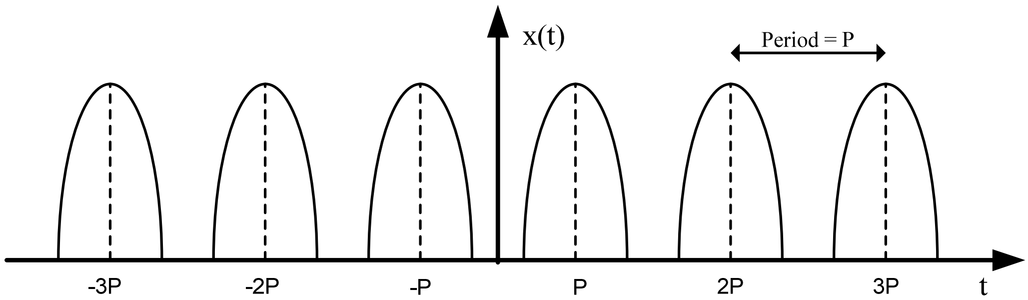


Figure 1.2: An example of a continuous periodic signal

1.3.2 Fourier series

Any periodic signal $x(t)$, of finite power, with a period of T seconds can be represented as a summation of sine waves and cosine weaves if it satisfies the "Dirichlet conditions", i.e.,

1. $x(t)$ is single valued everywhere
2. $x(t)$ is absolutely integrable over one period, i.e.,

$$\int_{a}^{a+T} |x(t)| dt < \infty$$

3. $x(t)$ has a finite number of extrema within each period
4. $x(t)$ has at most a finite number of finite discontinuities within each period.

This representation is known as the trigonometric Fourier series:

$$x(t) = \frac{A_0}{2} + \sum_{n=1}^{\infty} (A_n \cos(n\omega_0 t) + B_n \sin(n\omega_0 t))$$

The periodic signal, $x(t)$, is composed of a number of sinusoids (sines and cosines) of frequencies which are multiples of the fundamental frequency, ω_0. (Note: The frequency of the n^{th} sine and cosine is $n \times \omega_0$).

The fundamental frequency of $x(t)$ is $\omega_0 = 2\pi f_0\ rad/s$ where $f_0 = 1/T\ Hz$ and T is the period of $x(t)$, (i.e., $x(t+T) = x(t)$).

Algebraic work on the Fourier series is made much simpler by using complex exponentials to represent the sine and cosine terms. i.e.,

$$\cos x = \frac{e^{jx}+e^{-jx}}{2}\ ,\ \sin x = \frac{e^{jx}-e^{-jx}}{2j}$$

This reduces the trigonometric series to:

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

In this form, the Fourier series coefficients, c_n, are *complex* numbers and satisfy the property.

$$c_{-n} = \overline{c_n}$$

where $\bar{z}$ represents the complex conjugate of z.

Exercise 1.2:

Expanding the trigonometric series as complex exponentials to obtain the complex form of the Fourier series is left as an exercise. In this process verify that:

$$c_n = \frac{A_n - jB_n}{2}\ and\ c_{-n} = \frac{A_n + jB_n}{2}, \forall\, n \geq 1$$

Note: The components (complex exponentials or sinusoids, based on which form of the Fourier series is used) of different frequencies (multiples of the fundamental frequency) are commonly referred to as harmonics. i.e., the component of frequency $\frac{2}{T}Hz$ is referred to as the second harmonic; the component of frequency $\frac{3}{T}Hz$ is referred to as the third harmonic and so on.

Note: In general, the Fourier series expansion of a periodic signal will have infinitely many components (harmonics). However in some cases, depending on the signal, it might turn out that some of the coefficients are zero. It may also turn out that $c_n = 0, \forall n > N$ in which case the expansion has a finite number (N) of components (harmonics).

In order to calculate the values of the Fourier coefficients, consider that the Fourier series expands any periodic function, $x(t)$, as

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

and multiply on both sides by $e^{-jk\omega_0 t}$, for some fixed k

$$\begin{aligned} e^{-jk\omega_0 t} x(t) &= e^{-jk\omega_0 t} \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t} \\ &= \cdots + e^{-jk\omega_0 t} c_k e^{jk\omega_0 t} + \cdots \\ &= \cdots + c_k + \cdots \end{aligned} \tag{2.1}$$

Thus,

$$\begin{aligned} c_k &= e^{-jk\omega_0 t} x(t) - \sum_{n=-\infty, n\neq k}^{\infty} c_n e^{-jk\omega_0 t} e^{jn\omega_0 t} \\ &= e^{-jk\omega_0 t} x(t) - \sum_{n=-\infty, n\neq k}^{\infty} c_n e^{j\omega_0 (n-k) t} \end{aligned}$$

Integrating on both sides with respect to t in any interval of length T, $[a, a+T]$, where T is the period of $x(t)$.

$$\int_a^{a+T} c_k dt = \int_a^{a+T} x(t) e^{-jk\omega_0 t} dt - \int_a^{a+T} \sum_{n=-\infty, n\neq k}^{\infty} c_n e^{j\omega_0 (n-k) t} dt$$

$$c_k \int_a^{a+T} dt = \int_a^{a+T} x(t)\, e^{-jk\omega_0 t} dt - \sum_{n=-\infty, n\neq k}^{\infty} c_n \int_a^{a+T} e^{j\omega_0 (n-k) t} dt$$

It can be seen that the integral inside the summation in the second term on the right is always zero

$$\int_{a}^{a+T} e^{j\omega_0(n-k)t}dt = \frac{1}{j\omega_0(n-k)} e^{j\omega_0(n-k)t}\Big]_{a}^{a+T}$$

$$= \frac{1}{j\omega_0(n-k)} e^{j\omega_0(n-k)a}\left(e^{j\frac{2\pi}{T}(n-k)T} - 1\right)$$

$$= \frac{1}{j\omega_0(n-k)} e^{j\omega_0(n-k)a}\left(e^{j2\pi\,(n-k)} - 1\right)$$

Since, n and k are integers, $n-k$ is an integer and hence $e^{j2\pi(n-k)} = 1$. Thus,

$$\int_{a}^{a+T} e^{j\omega_0(n-k)t}dt = \frac{1}{j\omega_0(n-k)} e^{j\omega_0(n-k)a}(1-1)$$

$$= 0$$

This reduces the summation to zero and hence,

$$c_k T = \int_{a}^{a+T} x(t)\, e^{-jk\omega_0 t}dt$$

$$c_k = \frac{1}{T}\int_{a}^{a+T} x(t)\, e^{-jk\omega_0 t}dt$$

This is valid for any value of a and typically it is chosen as $-\frac{T}{2}$ to give

$$c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\omega_0 t}dt$$

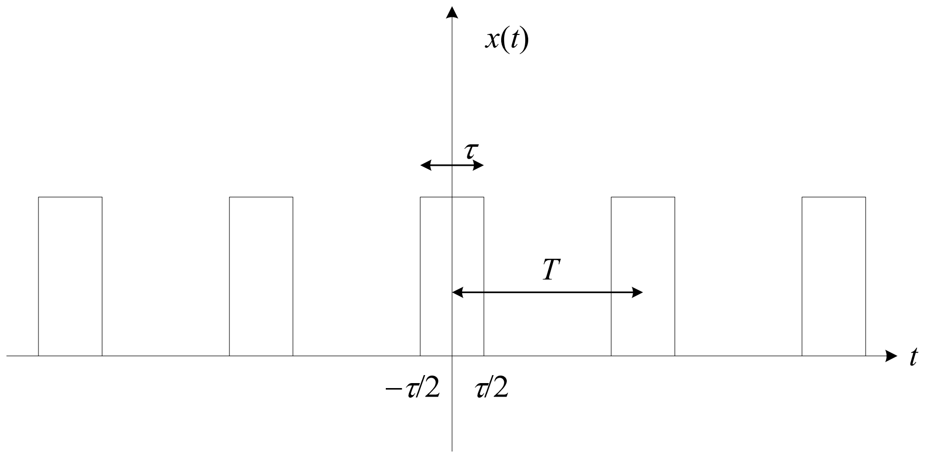
$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

Note: It is also common to take $a = 0$ resulting in

$$c_k = \frac{1}{T}\int_{0}^{T} x(t)e^{-jk\omega_0 t}dt$$

Example 1.1:

Consider the following periodic signal, $x(t)$



To express this signal as a Fourier series, the Fourier coefficients are computed as

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jn\omega_0 t}dt$$

$$= \frac{1}{T}\int_{\frac{\tau}{2}}^{\frac{\tau}{2}} 1 \cdot e^{-jn\omega_0 t}dt$$

$$= \frac{1}{Tn\omega_0}\left[-e^{-jn\omega_0 t}\right]_{-\tau/2}^{\tau/2}$$

$$= 2\frac{\sin\frac{n\omega_0\tau}{2}}{n\omega_0 T}$$

$$= \frac{\tau}{T}\frac{\sin\frac{n\omega_0\tau}{2}}{\frac{n\omega_0\tau}{2}}$$

$$c_n = \frac{\tau}{T} sinc\left(\frac{n\omega_0\tau}{2}\right)$$

To illustrate the idea of the Fourier series, the following series of plots show the signal obtained by summing the first few harmonics of the periodic pulse train given above using the computed Fourier coefficients, c_n.

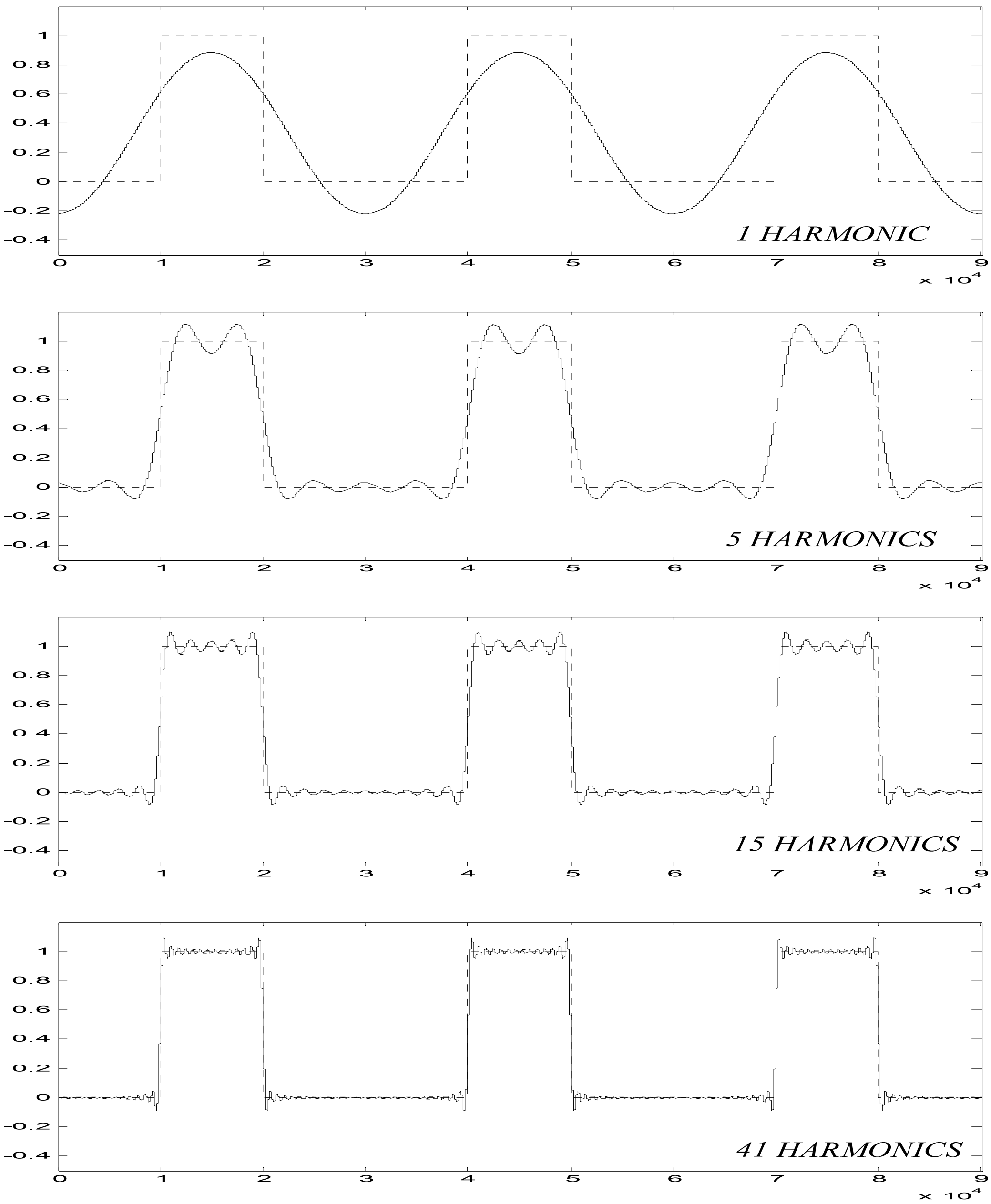


Figure 1.3: Sum of first *N* harmonics of the Fourier series (solid line) for a periodic pulse train (dotted line)

1.3.3 The Spectrum

A periodic signal, $x(t)$, of period T repeats itself according to $x(t+T) = x(t)$ and so do all the individual terms of its Fourier series. (Although the n^{th} term has a fundamental period of T/n, it also repeats at a period of T). However, it is not natural to talk of the signal $x(t)$ having a frequency $1/T\ Hz$ (or any other value for that matter). Rather, the Fourier series suggests that it is composed of many harmonics, each of a different frequency (both positive and negative and perhaps an infinite number of them). The set of frequencies present in a given periodic signal is the spectrum of the signal.

Note: Because of the conjugate symmetry relationship, $c_{-n} = \overline{c_n}$, the coefficients c_n and c_{-n} are either both zero or both non-zero.

Generally, the spectrum is infinitely large since there are infinitely many terms in the Fourier series. However, If the coefficients are all zero from a certain point, i.e.,

$$c_n = 0, \qquad for\ |n| > N$$

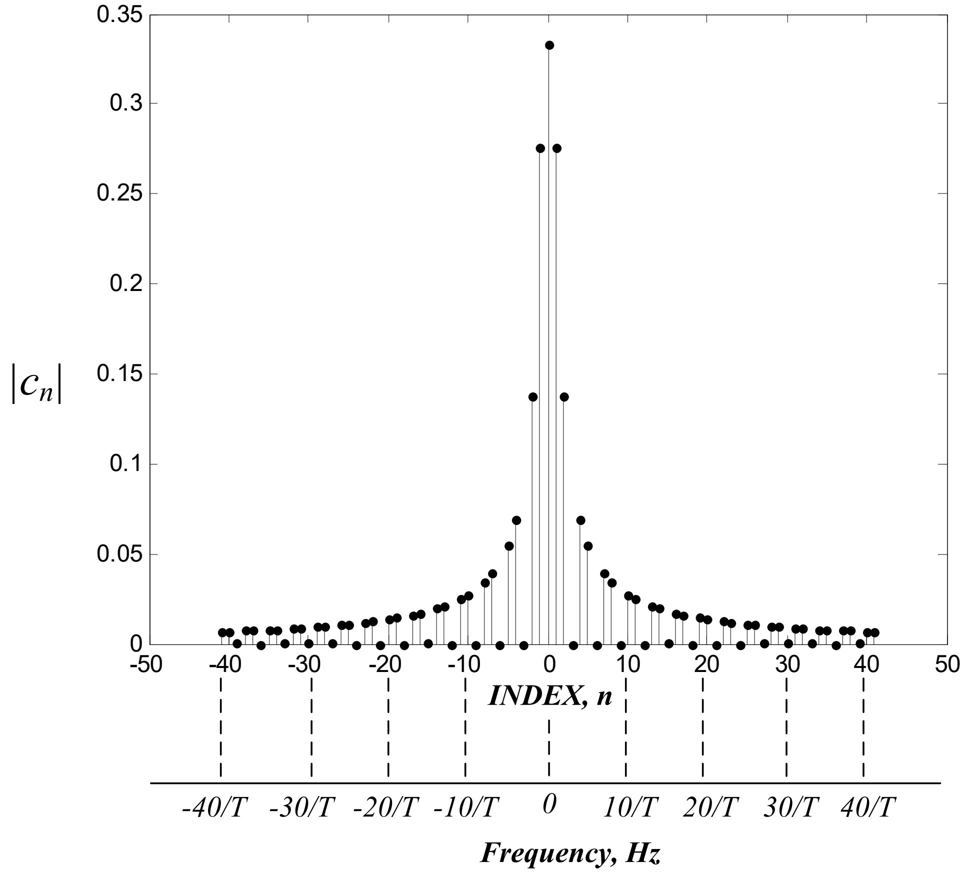
then the spectrum is limited between $-\frac{N}{T}\ Hz$ and $\frac{N}{T}\ Hz$, and the signal is said to be *bandlimited*.

The Fourier coefficients are generally complex numbers and can be represented by a magnitude and a phase.

$$c_n = |c_n| e^{j\phi_n}$$

Both the magnitude and the phase can be plotted as a function of discrete frequencies ($\pm n/T$) or alternately the index n that identifies the harmonics to give the magnitude and phase spectra respectively.

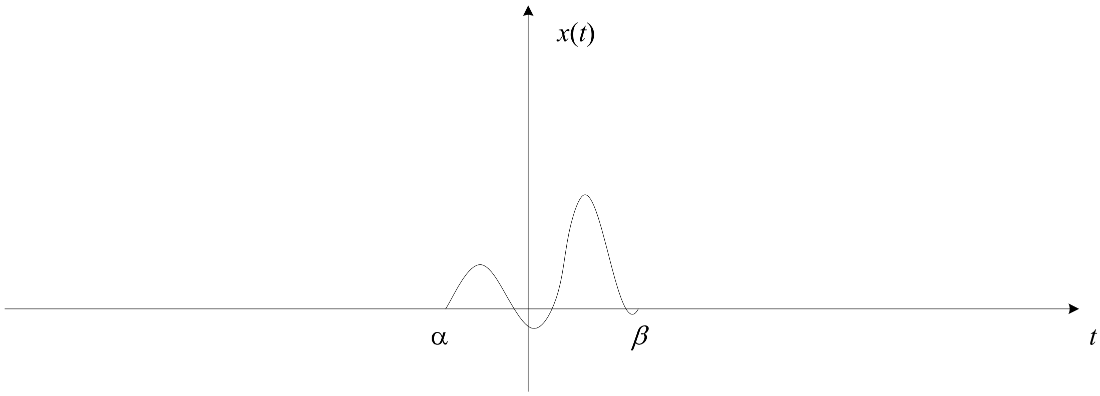
The magnitude spectrum of the periodic signal analysed in Example 1.1 is given below.



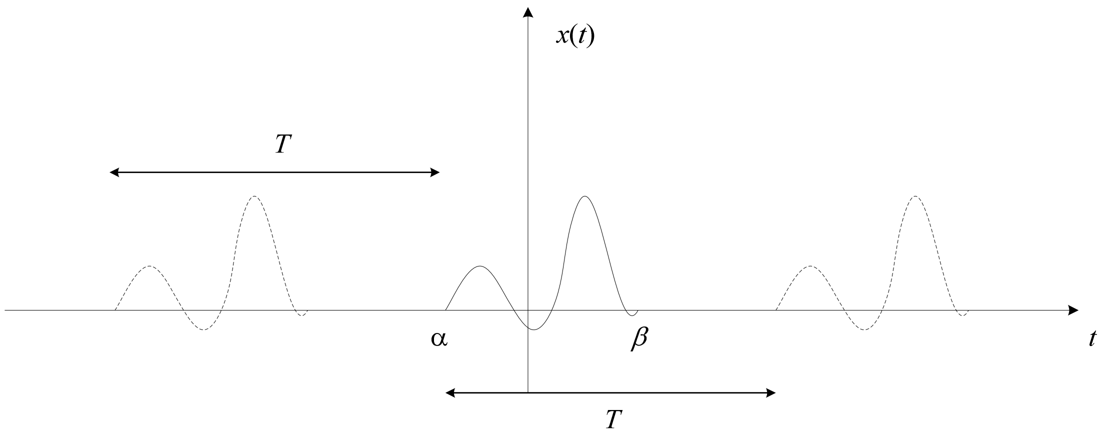
1.4 Fourier transform

The Fourier series allows any periodic signal to be written as a sum (an infinite sum in the general case) of sinusoids. However, a large number of signals are not periodic. If these signals have finite support, i.e., they are zero outside a finite interval:

$$x(t) = 0, \qquad t \notin [\alpha, \beta]$$



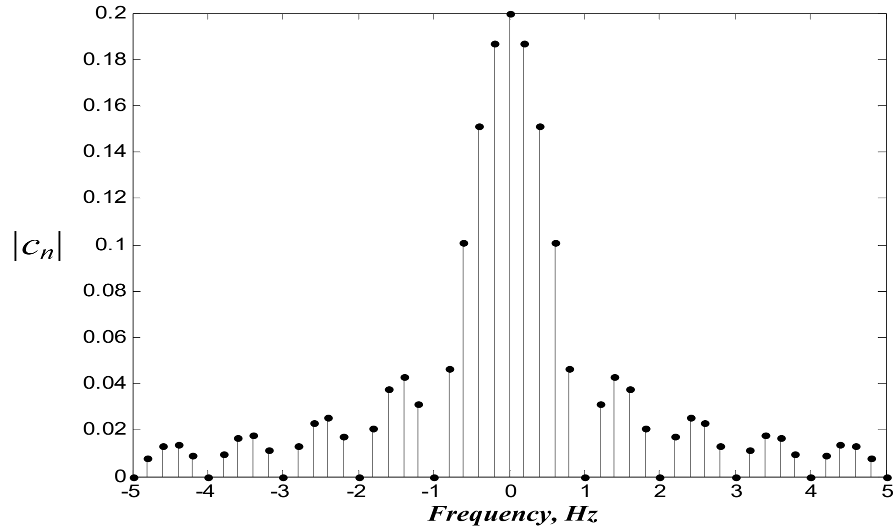
they can be made periodic by repeating them at a period larger than the interval in which they are non-zero. This is referred to as periodisation.

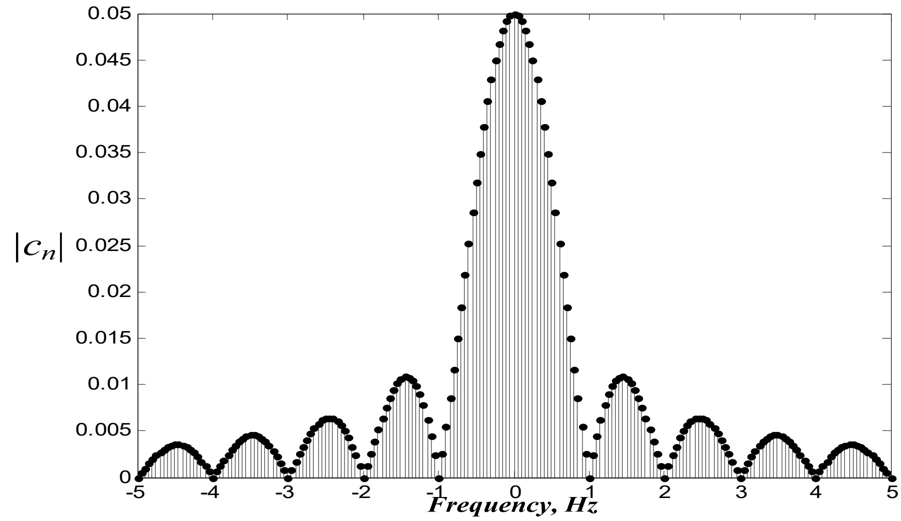


$$T > |\beta - \alpha|$$

Note: Periodisation allows for the analysis of non-periodic signals with finite support using the Fourier series. As the interval of support, $[\alpha, \beta]$, becomes larger, the period, T, also increases and the spacing between the frequency components in the spectrum decreases (since the frequencies present are $\pm n/T\ Hz$). The following plots show the magnitude spectrum of a pulse train (as in Example 1.1) with periods $T = 5\ secs$ and $T = 20\ secs$.

T





The limiting case of $T \to \infty$ can then allows for any non-periodic signal (since the interval of support, $[\alpha, \beta]$ can also be infinite).

Using the term $X(\omega_n)$ to represent the Fourier series coefficient, c_n scaled by the period, T:

$$X(\omega_n) = Tc_n = \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jn\omega_0 t}dt$$

Using the term $\omega_n = n\omega_0 = 2\pi n/T$ to represent the frequency,

$$X(\omega_n) = \int_{-T/2}^{T/2} x(t)e^{-j\omega_n t}dt$$

Allowing $T \to \infty$ reduces the spacing between frequency components to zeros and makes it a continuous variable ω.

$$\omega_n \to \omega, \qquad as\, T \to \infty$$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

Thus, we define the Fourier Transform, $X(\omega)$, of a signal, $x(t)$, as above.

Now consider the Fourier series expansion,

$$x(t) = \sum_{-\infty}^{\infty} c_n e^{jn\omega_0 t}$$

Since the Fourier series coefficients can be written as

$$c_n = \frac{1}{T}X(\omega_n)$$

Thus,

$$x(t) = \sum_{-\infty}^{\infty} \frac{1}{T}X(\omega_n)\, e^{j\omega_n t}$$

The points, $\omega_n = 2\pi n/T$ are spaced $2\pi/T$ apart, so we can think of $1/T$ as $\Delta\omega/2\pi$, where $\Delta\omega$ is the spacing between consecutive ω_n.

$$x(t) = \frac{1}{2\pi}\sum_{-\infty}^{\infty} X(\omega_n)e^{j\omega_n t}\Delta\omega$$

As $T \to \infty$, $\omega_n \to \omega$ and $\Delta\omega \to 0$ and the above approaches a Riemann sum for an integral. Thus giving the definition for the inverse Fourier transform,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

The Fourier Transform pair is given by

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

A Fourier transform pair may also be represented as

$$x(t) \overset{FT}{\leftrightarrow} X(\omega)$$

or

$$X(\omega) = \mathcal{F}\{x(t)\}$$

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\}$$

As in the case of the Fourier series, $X(\omega)$ is complex valued and be written as

$$X(\omega) = |X(\omega)|e^{j\phi(\omega)}$$

Thus giving both the magnitude spectrum, $|X(\omega)|$, and the phase spectrum, $\phi(\omega)$.

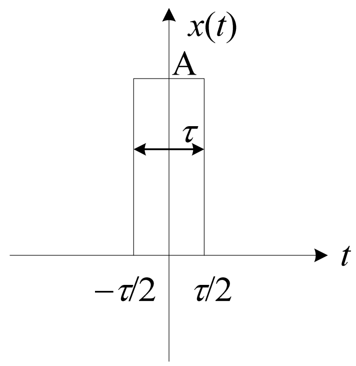
Note: As is the case of the Fourier series, the signal is said to be *bandlimited*, with a bandwidth of $\frac{B}{2\pi}$ Hz, if

$$X(\omega) = 0, \qquad |\omega| > B$$

and B is the lowest positive value for which it holds.

Example 1.2

Evaluate the Fourier transform of a rectangular pulse shown below:



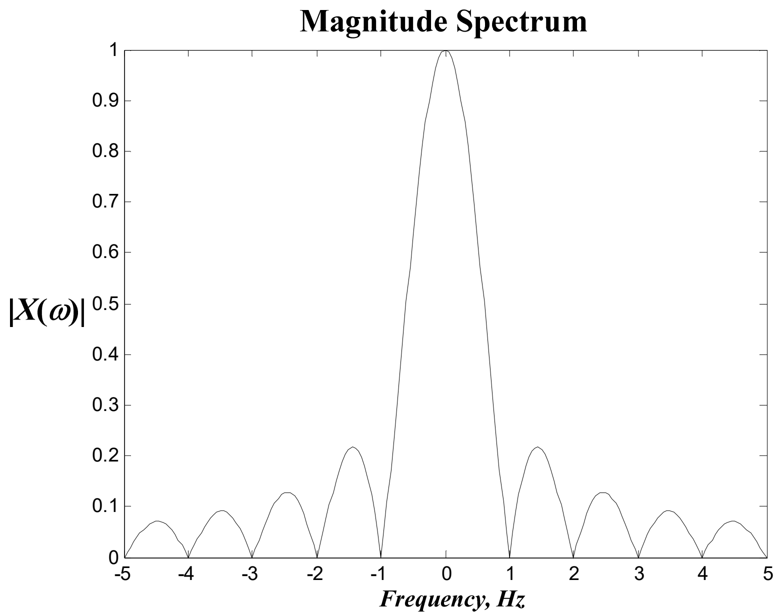
$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} Ae^{-j\omega t}dt$$

$$= \frac{A}{j\omega}\left[e^{-\frac{j\omega\tau}{2}} - e^{\frac{j\omega\tau}{2}}\right]$$

$$= A\tau\frac{\sin\frac{\omega\tau}{2}}{\frac{\omega\tau}{2}}$$

$$= A\tau \, sinc\left(\frac{\omega\tau}{2}\right)$$

Note: Here the $sinc$ function is defined as $sinc(x) = \frac{\sin x}{x}$. Alternatively, it can also be defined as $(x) = \frac{\sin \pi x}{x}$.



Note: The rectangular pulse is the limiting case of the pulse train (Example 1.1) when the period, $T \rightarrow \infty$. The spacing between the frequency components reduces to zero and the spectrum is continuous.

1.4.1 Modifying the Spectrum

The spectrum, $\mathcal{F}\{x(t)\}$, of a signal, $x(t)$, can be modified by either:

- Adding the spectrum of another signal to it
- Multiplying it by a constant
- Multiplying it with the spectrum of another signal

If $y(t)$ is another signal, the signal corresponding to the sum of the two spectra, $\mathcal{F}\{x(t)\}$ and $\mathcal{F}\{y(t)\}$, is the sum of the signals.

$$\begin{aligned} h(t) &= \mathcal{F}^{-1}\{\mathcal{F}\{x(t)\} + \mathcal{F}\{y(t)\}\} \\ &= x(t) + y(t) \end{aligned}$$

The signal corresponding to the product of a spectrum, $\mathcal{F}\{x(t)\}$, and a constant, α, is the product of the signal, $x(t)$, and the constant.

$$\begin{aligned} h(t) &= \mathcal{F}^{-1}\{\alpha\mathcal{F}\{x(t)\}\} \\ &= \alpha x(t) \end{aligned}$$

These two results are due to the *linearity* of the Fourier transform.

The spectrum obtained as the product of the spectra of two signals, $x(t)$ and $y(t)$ is given by:

$$\begin{aligned} H(\omega) &= \mathcal{F}\{x(t)\} \cdot \mathcal{F}\{y(t)\} \\ &= \int_{-\infty}^{\infty} x(u)e^{-j\omega u}du \cdot \int_{-\infty}^{\infty} y(v)e^{-j\omega v}dv \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j\omega u}e^{-j\omega v}x(u)y(v)dudv \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j\omega(u+v)}x(u)y(v)dudv \end{aligned}$$

Let $t = u + v$, then $v = t - u$ and $dv = dt$

$$\begin{aligned} H(\omega) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j\omega t}x(u)y(t-u)du\,dt \\ &= \int_{-\infty}^{\infty} e^{-j\omega t}\left(\int_{-\infty}^{\infty} x(u)y(t-u)du\right)dt \\ &= \int_{-\infty}^{\infty} e^{-j\omega t}h(t)dt \\ &= \mathcal{F}\{h(t)\} \end{aligned}$$

where,

$$h(t) = \int_{-\infty}^{\infty} x(u)y(t-u)du$$

This is the signal corresponding to the spectrum obtained as the product of the spectra of two signals, $x(t)$ and $y(t)$. i.e.,

$$\underline{\mathcal{F}\{h(t)\} = \mathcal{F}\{x(t)\} \cdot \mathcal{F}\{y(t)\}}$$

The relationship between the signals, $h(t)$, $x(t)$ and $y(t)$ is given by

$$\underline{h(t) = \int_{-\infty}^{\infty} x(u)y(t-u)du}$$

This is the definition of the *convolution* operation and is usually denoted as

$$\underline{h(t) = x(t) * y(t)}$$

1.4.2 Properties of the Fourier Transform

Linearity	$\mathcal{F}\{ax(t) + by(t)\} = a\mathcal{F}\{x(t)\} + b\mathcal{F}\{y(t)\}$ $\mathcal{F}^{-1}\{aX(\omega) + bY(\omega)\}$ $= a\mathcal{F}^{-1}\{X(\omega)\} + b\mathcal{F}^{-1}\{Y(\omega)\}$
Frequency - shift	$e^{jkt}x(t) \overset{FT}{\leftrightarrow} X(\omega - k)$
Time – shift	$x(t - t_0) \overset{FT}{\leftrightarrow} e^{-j\omega t_0}X(\omega)$
Scaling	$x(at) \overset{FT}{\leftrightarrow} \frac{1}{\lvert a\rvert}X\left(\frac{\omega}{a}\right)$
Differentiation in time	$\frac{d}{dt}x(t) \overset{FT}{\leftrightarrow} j\omega X(\omega)$
Differentiation in frequency	$-jtx(t) \overset{FT}{\leftrightarrow} \frac{d}{d\omega}X(\omega)$
Convolution in time	$x(t) * y(t) \overset{FT}{\leftrightarrow} X(\omega)Y(\omega)$
Convolution in frequency	$2\pi x(t)y(t) \overset{FT}{\leftrightarrow} X(\omega) * Y(\omega)$