ELEC3104: Digital Signal Processing

Chapter 3: Discrete Time Systems

3.1 Introduction

A discrete - time system is a device or algorithm that operates on a discrete-time signal called the input or excitation according to some well-defined rule, to produce another discrete-time signal called the output or response.

We say that the input signal, $x[n]$, is transformed by the system into a signal, $y[n]$, and express the general relationship between $x[n]$ and $y[n]$ as:

$$y[n] = H\{x[n]\}$$

where the symbol H denotes the transformation or processing performed by the system on $x[n]$ to produce $y[n]$ (see Figure 3.1).

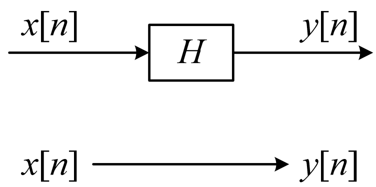


Figure 3.1: Block diagram representation of a discrete-time system

Note: It is possible for the output, $y[n]$ to depend on other values of itself as well as the input $x[n]$. For instance the relationship, $y[n] = 2x[n] + 3x[n-1] + 0.5y[n+2]$ is a system of the form $y[n] = H\{x[n]\}$.

3.2 Classification of Discrete-time systems

In the analysis as well as in the design of systems, it is desirable to classify the systems according to the general properties that they satisfy. For a system to possess a given property, the property must hold for every possible input signal to the system. If a property holds for some input signals but for others, the system does not possess the property.

General Categories are:

- Static systems
- Linear systems
- Time - invariant systems
- Causal systems
- Stable systems

3.2.1 Static Systems

A discrete-time system is called static or memoryless if its output at any instant 'n' depends at most on the input sample at the same time, but not on past or future samples of the input.

Example 3.1

$$y[n] = ax[n]$$
$$y[n] = nx[n] + bx^3[n]$$

Both are static or memoryless

On the other hand, the systems described by the following input-output relations

$$y[n] = x[n] + 3x[n-1]$$
$$y[n] = \sum_{k=0}^{N} x[n-k]$$

are dynamic systems or system with memory.

3.2.2 Linear Systems

A linear system is defined as follows:

$$H\{a_1x_1[n] + a_2x_2[n]\} = a_1y_1[n] + a_2y_2[n]$$

where a_1 and a_2 are arbitrary constants.

Example 3.2:

Three sample averager

$$y[n] = H\{x[n]\} = \frac{1}{3}(x[n+1] + x[n] + x[n-1])$$

$$H\{ax_1[n] + bx_2[n]\}$$
$$= \frac{1}{3}(ax_1[n+1] + bx_2[n+1] + ax_1[n] + bx_2[n]$$
$$+ \ ax_1[n-1] + bx_2[n-1])$$

$$H\{ax_1[n] + bx_2[n]\} = (ay_1[n] + by_2[n])$$

The 3-sample averager is a linear system

Example 3.3

$$y[n] = H\{x[n]\} = x^2[n]$$
$$H\{ax_1[n] + bx_2[n]\} = \{ax_1[n] + bx_2[n]\}^2$$
$$= a^2x_1^2[n] + b^2x_2^2[n] + 2abx_1[n]x_2[n]$$

which is not equal to $ax_1^2[n] + bx_2^2[n]$. This system is nonlinear.

Example 3.4

$$y[n] = H\{x[n]\} = nx[n]$$
$$H\{ax_1[n] + bx_2[n]\} = n(ax_1[n] + bx_2[n])$$
$$= anx_1[n] + bnx_2[n]$$
$$= ay_1[n] + by_2[n]$$

The system is linear.

3.2.3 Time-Invariant systems

A time-invariant system is defined as follows:

$$H\{x[n-n_0]\} = y[n-n_0]$$

where, $y[n] = H\{x[n]\}$.

Specifically, a system is time invariant if a time shift in the input signal results in an identical time shift in the output signal.

Example 3.5

Determine if the system is time variant or time invariant.

$$y[n] = H\{x[n]\} = nx[n]$$

The response of this system to $x[n-k]$ is

$$w[n] = nx[n-k]$$

But,

$$\begin{aligned} y[n-k] &= (n-k)x[n-k] \\ &\neq w[n] \end{aligned}$$

Therefore this system is time variant.

3.2.4 Causal systems

A system is said to be *causal* if the output of the system at any time 'n' depends only on present and past inputs, but does not depend on future inputs. If a system does not satisfy this definition, it is called *noncausal*. Such a system has an output that depends not only on present and past inputs but also on future inputs.

Example 3.6

$$\begin{aligned} y[n] &= x[n] - x[n-1] && \rightarrow Causal \\ y[n] &= ax[n] && \rightarrow Causal \\ y[n] &= x[n] + 3x[n+4] && \rightarrow Noncausal \\ y[n] &= x[-n] && \rightarrow Noncausal \end{aligned}$$

Note: In the last example, let $n = -1$, then $y[-1] = x[1]$ hence non-causal.

3.2.5 Stable systems

- A discrete signal $x[n]$ is bounded if there exists a finite M such that $|x[n]| < M$ for all n.
- A discrete - time system in Bounded Input-Bounded Output (**BIBO**) stable if every bounded input sequence $x[n]$ produced a bounded output sequence.

$$If\ |x[n]|_{max} \leq A, \qquad then\ |y[n]|_{max} \leq B$$

Example 3.7

The discrete - time system

$$y[n] = ny[n-1] + x[n], \qquad n > 0$$

is at rest. (i.e., $y[-1] = 0$). Check if the system is BIBO stable.

If

$$x[n] = u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

then $|x[n]| \leq 1$.

But for this bounded input, the output is

$$n = 0 \rightarrow y[0] = x[0] = 1$$
$$n = 1 \rightarrow y[1] = 1y[0] + x[1] = 2$$
$$n = 2 \rightarrow y[2] = 1y[1] + x[2] = 5$$
$$\vdots \qquad\qquad \vdots$$

which is unbounded. Hence the system is unstable.

3.3 Convolution

Convolution for discrete time signals is analogous to that of the continuous time signals and is given as

$$x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k]$$

Commutative Law:

$$x[n] * y[n] = y[n] * x[n]$$

Associative Law:

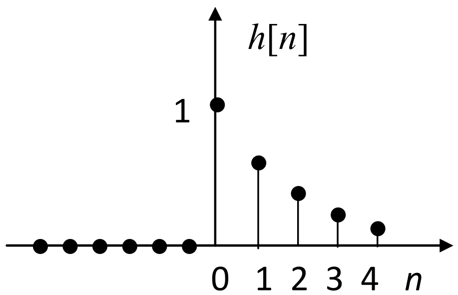
$$(x[n] * y[n]) * z[n] = x[n] * (y[n] * z[n])$$

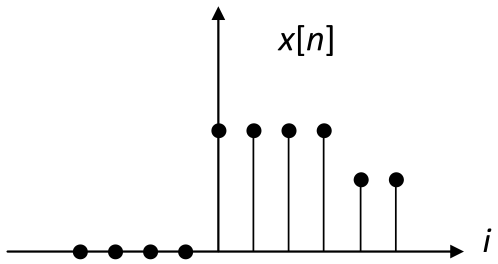
Distributive Law:

$$x[n] * (y[n] + z[n]) = x[n] * y[n] + x[n] * z[n]$$

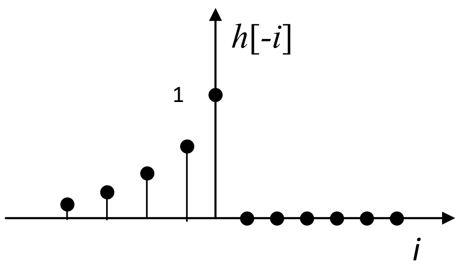
3.4 Graphical interpretation of convolution

The convolution of two signals $x[n]$ and $h[n]$ is shown in steps in the diagram below.

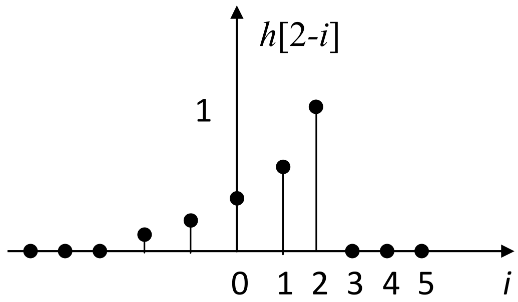




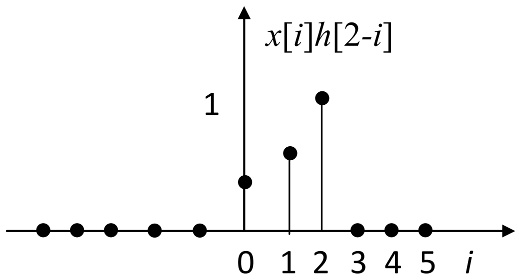
Step 1: Fold $h[i]$ over in time: this gives $h[-i]$



Step 2: Shift $h[-i]$ through a distance n. This gives $h[n-i]$. We have chosen $n = 2$ in the diagram.

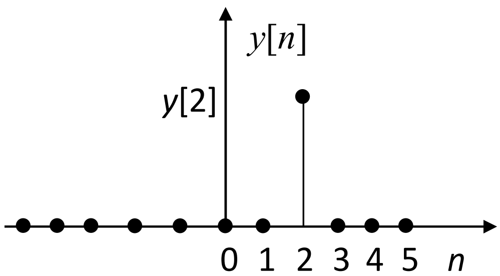


Step 3: Multiply $x[i]$ by $h[n-i]$

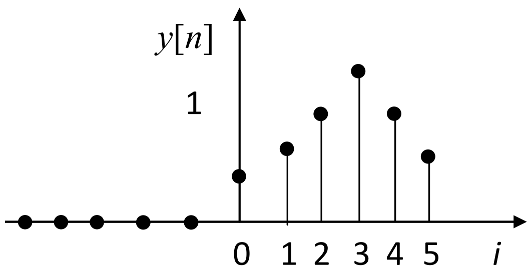


Step 4: Sum this product over all i. This gives the signal value $y[2]$.

$$y[2] = \sum_{i=-\infty}^{\infty} x[i]h[2-i]$$



Step 5: Vary n from $-\infty$ to ∞. This gives $y[n]$.



3.5 Linear Time-Invariant Systems

A discrete-time signal, $x[n]$ may be shifted in time (delayed or advanced) by replacing the variables n with $n-k$ where $k>0$ is an integer

$$x[n-k] => x[n]\ is\ delayed\ by\ k\ samples$$

$$x[n+k] => x[n] is\ advanced\ by\ k\ samples$$

For example consider a shifted version of the unit impulse function (see Figure 3.2). If we multiply an arbitrary signal $x[n]$ by this function, we obtain a signal that is zero everywhere, except at $n = k$. i.e.,

$$y[n] = x[n] \cdot \delta[n-k] = x_k \delta[n-k]$$

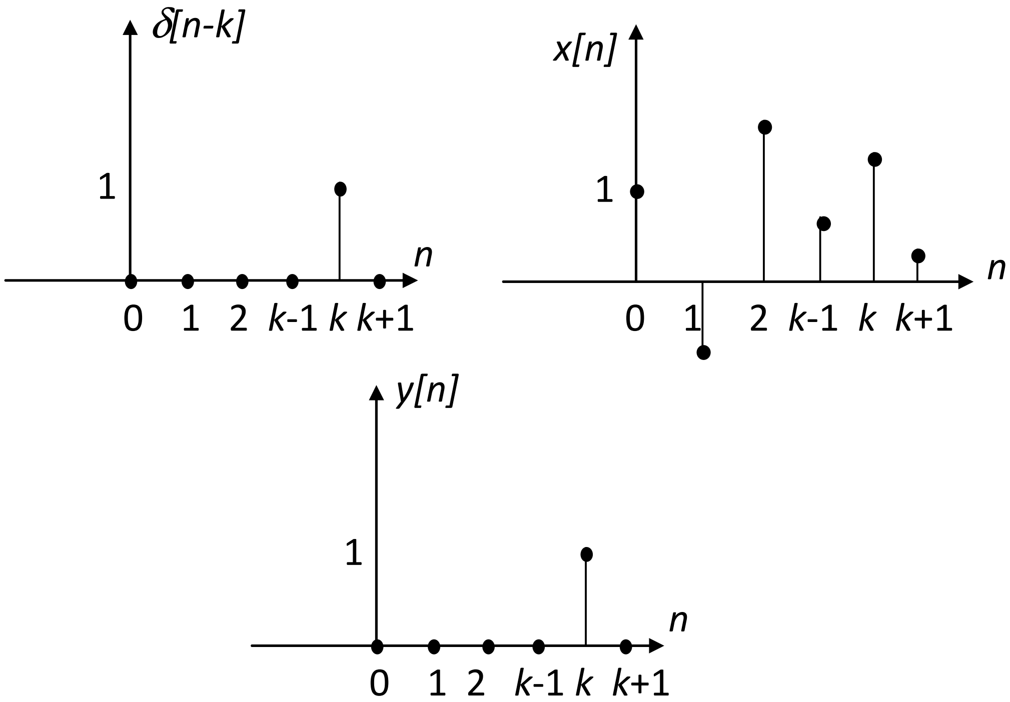
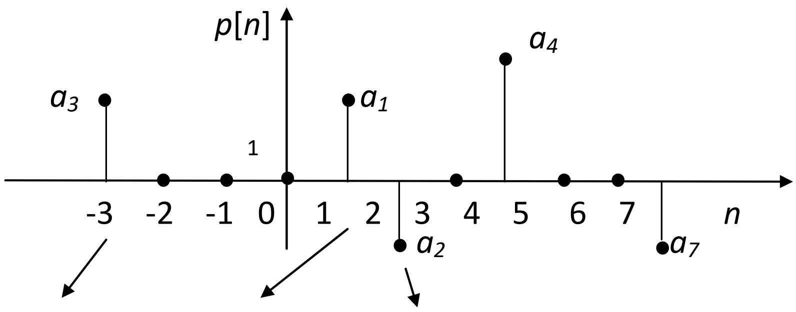


Figure 3.2: Multiplying a discrete-time signal, $x[n]$, with a shifted unit impulse function, $\delta[n-k]$, produces a discrete-time signal whose samples are zero except at $n = k$.

Note: The term $x[n] \cdot \delta[n-k]$ is a sample by sample product of two signals while the term $x_k \delta[n-k]$ is a product of a scalar (x_k is the scalar value taken by $x[n]$ at $n = k$) and a signal.

An arbitrary sequence can then be expressed as a sum of scaled and delayed unit impulses.



$$\boxed{p[n] = a_{-3}\delta[n+3] + a_1\delta[n-1] + a_2\delta[n-2] + a_4\delta[n-4] + a_7\delta[n-7]}$$

Figure 3.3: An example of expressing arbitrary discrete-time sequences as a sum of scaled and delayed unit impulses

More generally, any discrete-time sequence can be expressed according as

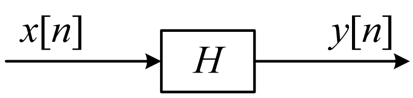
$$x[n] = \sum_{k=-\infty}^{\infty} x_k \delta[n-k]$$

Note: Once again $x[n]$ is a signal while x_k is a scalar.

The above expression can also be viewed as the convolution of $x[n]$ with $\delta[n]$ giving $x[n]$. i.e., convolution of a signal with $\delta[n]$ is itself.

3.5.1 The Impulse Response

Consider a LTI (Linear Time Invariant) system, $H\{\cdot\}$, operating on a signal $x[n]$ to produce an output $y[n]$.



$$x[n] \longrightarrow y[n]$$

i.e.,

$$\boxed{y[n] = H\{x[n]\}}$$

Now consider that the signal $x[n]$ can be written as

$$x[n] = \sum_{k=-\infty}^{\infty} x_k \delta[n-k]$$

Therefore,

$$y[n] = H\{x[n]\} = H\left\{\sum_{k=-\infty}^{\infty} x_k \delta[n-k]\right\}$$

Since the system, H, is linear; the system operating on a sum of signals is equal to the sum of the system operating on the individual signals. i.e.,

$$y[n] = \sum_{k=-\infty}^{\infty} H\{x_k \delta[n-k]\}$$

Since, x_k is a scalar and the H is linear

$$y[n] = \sum_{k=-\infty}^{\infty} x_k H\{\delta[n-k]\}$$

convolution

Let us define the **impulse response** $h[n]$ as

$$h[n] \stackrel{\text{def}}{=} H\{\delta[n]\}$$

Note: $h[n]$ is the output (response) of the system H, when the input is the unit impulse function, $\delta[n]$ and hence the name **impulse response**.

Since, H is time invariant,

$$H\{\delta[n-k]\} = h[n-k]$$

Thus,

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Recognising this as the convolution operation,

$$\boxed{y[n] = x[n] * h[n]}$$

The response to an arbitrary input signal, $x[n]$, is the convolution of $x[n]$ with the impulse response of the system.

3.5.2 Finite Impulse Response (FIR) System

If the impulse response of a LTI system is of finite duration, the system is said to be Finite Impulse Response (FIR).

Example 3.8

Consider the system

$$y[n] = H\{x[n]\} = 2x[n] - 0.5x[n-1]$$

To find the impulse response, let the input be the unit impulse. i.e., $x[n] = \delta[n]$. Then the impulse response, $h[n]$ is

$$h[n] = 2\delta[n] - 0.5\delta[n-1]$$

$$\begin{aligned}
n &= 0, \quad & y[0] &= h[0] = 2\delta[0] - 0.5\delta[-1] = 2 \\
n &= 1, \quad & y[1] &= h[1] = 2\delta[1] - 0.5\delta[0] = -0.5 \\
n &= 2, \quad & y[2] &= h[2] = 2\delta[2] - 0.5\delta[1] = 0 \\
&\vdots & &\vdots
\end{aligned}$$

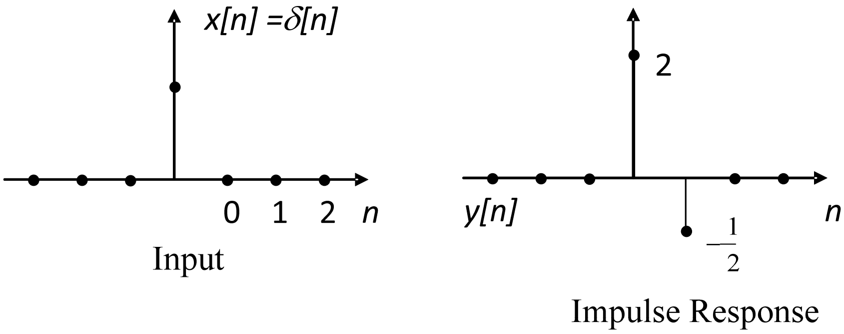


Figure 3.4: An example of a finite impulse response

3.5.3 Infinite Impulse Response

If the impulse response of a linear time-invariant system is of infinite duration, the system is said to be an Infinite Impulse Response (IIR) system.

Example 3.9

Consider the system,

$$y[n] = x[n] + ay[n-1]$$

to find impulse response we assume input to be δ[n]

To find the impulse response, $h[n]$, let $x[n] = \delta[n]$.

h[n]

$$\begin{aligned}
n = 0, &\quad y[0] = h[0] = \delta[0] + ay[-1] = 1 \\
n = 1, &\quad y[1] = h[1] = \delta[1] + ay[0] = a \\
n = 2, &\quad y[2] = h[2] = \delta[2] + ay[1] = a^2 \\
&\quad \vdots \\
n = n, &\quad y[n] = h[n] = \delta[n] + ay[n-1] = a^n
\end{aligned}$$

$y[n] = h[n] = 0$ for $n < 0$, because $\delta[n] = 0$ for $n < 0$ and the system is assumed to be as rest and not have an output before an input is given to it. Hence, $h[n] = a^n u[n]$ for all n. Where, $u[n]$ is the unit step function defined as

$$u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

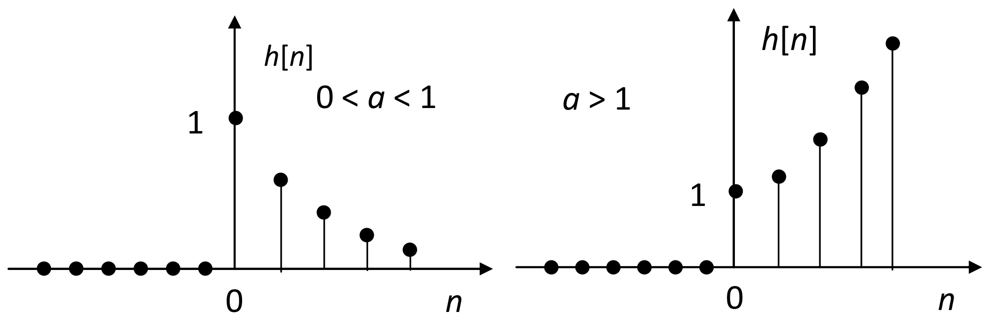


Figure 3.5: The infinite impulse response for $0 < a < 1$ and $a > 1$

∵ y[n] = a^n · u[n];
y[n] = x[n]*h[n];
x[n] = δ(n);
∴ y[n] = h[n];
∴ h[n] = a^n · u[n];

3.5.4 Stability of Linear Time-Invariant Systems

An LTI system is stable if, and only if, the stability factor denoted by S, and defined by

$$S = \sum_{k=-\infty}^{\infty} |h[k]|$$

is finite. i.e., $S < \infty$.

Where, $h[n]$ is the impulse response of the system.

Let $x[n]$ be a bounded input sequence (signal). i.e., $|x[n]| < M$ for all n, and M is a finite number.

We must show that the output is bounded when S is finite. To this end, we work again with the convolution formula.

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

If we take the absolute value of both sides of the above equation, we obtain

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right|$$

Now, the absolute value of the sum of terms is always less than or equal to the sum of the absolute values of the terms. i.e.,

$$|y[n]| \leq \sum_{k=-\infty}^{\infty} |h[k]||x[n-k]|$$

Since, $|x[n]| < M$,

$$|y[n]| \leq M \sum_{k=-\infty}^{\infty} |h[k]|$$

$$\leq MS$$

Hence, since both M and S are finite, the output is also bounded. i.e., a LTI system is stable if its impulse response is absolutely summable.

Example 3.10

Check the stability of the first-order recursive system shown below:

$$y[n] = ay[n-1] + x[n]$$

The impulse response of this system is

$$h[n] = a^n u[n], \qquad for\ all\ n$$

(This is worked out in Example 3.9)

Now,

$$S = \sum_{k=-\infty}^{\infty} |h[n]| = \sum_{k=-\infty}^{\infty} |a|^n$$

It is obvious that S is unbounded for $|a| > 1$, since then each term in the series is ≥ 1.

For $|a| < 1$, we can apply the infinite geometric sum formula, to find

$$S = \frac{1}{1-|a|}, \qquad for\ |a| < 1$$

Thus when $|a| < 1$, S is finite and the system is stable and when $|a| \geq 1$, the system is not stable.

3.6 Frequency Response

Consider that the output $y[n]$, of an LTI system with impulse response $h[n]$, to an input $x[n]$ is given by

$$y[n] = h[n] * x[n]$$

Taking the DTFT,

$$\begin{aligned}\hat{y}(\theta) &= \sum_{n=-\infty}^{\infty} y[n]e^{-j\theta n} \\ &= \sum_{n=-\infty}^{\infty} (x[n] * h[n])e^{-j\theta n} \\ &= \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]h[n-k]e^{-j\theta n} \\ &= \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} h[n-k]e^{-j\theta n}\end{aligned}$$

Noting,

$$\sum_{n=-\infty}^{\infty} h[n-k]e^{-j\theta n} = e^{-j\theta k}\hat{h}(\theta) \quad \text{phase shifting}$$

We get,

$$\begin{aligned}\hat{y}(\theta) &= \sum_{k=-\infty}^{\infty} x[k]e^{-j\theta k}\,\hat{h}(\theta) \\ &= \hat{x}(\theta)\hat{h}(\theta)\end{aligned}$$

Where, $\hat{h}(\theta)$ is referred to as the frequency response of the system.

Note: $\hat{h}(\theta)$ is complex valued and can be written as $|\hat{h}(\theta)|e^{j\angle\hat{h}(\theta)}$, where $|\hat{h}(\theta)|$ is called the magnitude response and $\angle\hat{h}(\theta)$ is termed the phase response. Both are real valued functions.

Writing the complex valued terms in the polar form

$$|\hat{y}(\theta)|e^{j\angle\hat{y}(\theta)} = |\hat{h}(\theta)|e^{j\angle\hat{h}(\theta)}|\hat{x}(\theta)|e^{j\angle\hat{x}(\theta)}$$

$$= |\hat{h}(\theta)||\hat{x}(\theta)|e^{j\left(\angle\hat{x}(\theta)+\angle\hat{h}(\theta)\right)}$$

Equating the magnitudes and phases,

$$|\hat{y}(\theta)| = |\hat{h}(\theta)||\hat{x}(\theta)|$$

$$\angle\hat{y}(\theta) = \angle\hat{x}(\theta) + \angle\hat{h}(\theta)$$

Thus the magnitude response gives the gain of the system at each frequency and the phase response gives the delay induced by the system at every frequency.

The **phase delay** or **group delay** of the filter provides a useful measure of how the filter modified the phase characteristic of the signal. If we consider a signal that consists of several frequency components (eg. speech waveform) the phase delay of the filter is the amount of time delay each frequency component of the signal suffers in going through the filter.

$$phase\ delay, T_p = -\frac{\angle\hat{h}(\theta)}{\theta}$$

where $\angle\hat{h}(\theta)$ is the phase response of the system.

The group delay on the other hand is the average time delay the composite signal suffers at each frequency as it passes from the input to the output of the filter.

$$group\ delay, T_g = -\frac{d}{d\theta}\angle\hat{h}(\theta)$$

A *constant group delay* means that signal components at different frequencies receive the same delay in the filter.

A *linear phase filter* gives same time delay to all frequency components of the input signal. A filter with a nonlinear phase characteristic will cause a phase distortion in the signal that passes through it.

This is because the frequency components in the signal will each be delayed by an amount not proportional to frequency, thereby altering their harmonic relationship. Such a distortion is undesirable in many applications, for example music, video etc.

A filter is said to have a linear phase response if its phase response satisfies one of the following relationships:

$$\angle\hat{h}(\theta) = -a\theta$$
$$\angle\hat{h}(\theta) = b - a\theta$$

where a and b are constants.

Example 3.11

Calculate the magnitude and phase response of the 3-sample averager given by

$$h[n] = \begin{cases} \frac{1}{3}, & -1 \le n \le 1 \\ 0, & otherwise \end{cases}$$

The frequency response is given by

$$\begin{aligned} \hat{h}(\theta) &= \sum_{n=-\infty}^{\infty} h[n]e^{-j\theta n} \\ &= \sum_{n=-1}^{1} h[n]e^{-j\theta n} \\ &= \frac{1}{3}\left[e^{-j\theta} + 1 + e^{j\theta}\right] \\ &= \frac{1}{3}\left[1 + e^{j\theta} + e^{-j\theta}\right] \\ &= \frac{1}{3}[1 + 2\cos\theta] \end{aligned}$$

Note: When considering negative values it should be noted that -1 is equivalent to $e^{j\pi}$ and hence negative values result in an additional phase of π radians. Consider a transfer function of the form.

$$\hat{h}(\theta) = e^{j\psi(\theta)}B(\theta)$$

When $B(\theta)$ is positive, the phase response $\phi(\theta) = \psi(\theta)$ and when $B(\theta)$ is negative, it can be written as $B(\theta) = e^{\pm j\pi}|B(\theta)|$ and hence $\hat{h}(\theta) = e^{j\psi(\theta)}e^{\pm j\pi}|B(\theta)|$. The phase response in this case is then $\phi(\theta) = \psi(\theta) \pm \pi$.

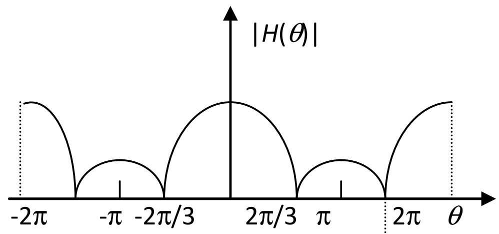
For the frequency response, $\hat{h}(\theta) = \frac{1}{3}[1 + 2\cos\theta]$, the magnitude response is given by

$$\left|\hat{h}(\theta)\right| = \left|\frac{1}{3}[1 + 2\cos\theta\right|$$

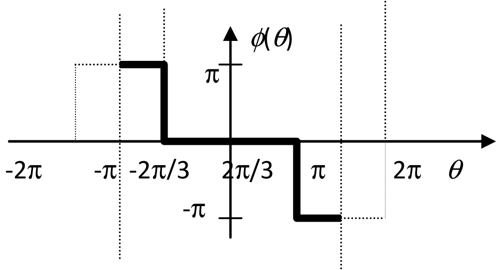
And the phase response is given by

$$\angle\hat{h}(\theta) = \begin{cases} 0, & \hat{h}(\theta) > 0, \quad -\frac{2}{3}\pi < \theta < \frac{2}{3}\pi \\ 0 \pm \pi, & \hat{h}(\theta) < 0, \quad -\frac{2}{3}\pi \leq |\theta| \leq \pi \end{cases}$$

The appropriate sign of π must be chosen to make $\phi(\theta)$ an odd function of frequency.



Even function



Odd function