ELEC3104: Digital Signal Processing

Chapter 4: Z-Transform

4.1 Introduction

Consider two polynomials,

$$A(z) = a_0 + a_1 z^{-1} + a_2 z^{-2}$$
$$B(z) = b_0 + b_1 z^{-1} + b_2 z^{-2}$$

Let, $C(z) = A(z)B(z)$, then

$$\begin{aligned} C(z) &= (a_0 + a_1 z^{-1} + a_2 z^{-2})(b_0 + b_1 z^{-1} + b_2 z^{-2}) \\ &= \underline{a_0 b_0} + (\underline{a_1 b_0 + a_0 b_1}) z^{-1} + (\underline{a_2 b_0 + a_1 b_1 + a_0 b_2}) z^{-2} + (\underline{a_2 b_1 + a_1 b_2}) z^{-3} \\ &\quad + \underline{a_2 b_2} z^{-4} \end{aligned}$$

Now consider two discrete time signals,

$$x[n] = \begin{cases} a_0, & n = 0 \\ a_1, & n = 1 \\ a_2, & n = 2 \\ 0, & otherwise \end{cases}$$

$$y[n] = \begin{cases} b_0, & n = 0 \\ b_1, & n = 1 \\ b_2, & n = 2 \\ 0, & otherwise \end{cases}$$

And let, $\underline{p[n] = x[n] * y[n]} = \sum_k x[k] y[n-k]$,

For $n \leq -1$, $x[k]$ and $y[n-k]$ don't overlap, $\therefore p[n] = 0,\ n < 0$

For $n \geq 5$, $x[k]$ and $y[n-k]$ don't overlap, $\therefore p[n] = 0,\ n > 4$

And,

$$p[0] = x[0]y[0] = \underline{a_0 b_0}$$
$$p[1] = x[0]y[1] + x[1]y[0] = \underline{a_0 b_1 + a_1 b_0}$$
$$p[2] = x[0]y[2] + x[1]y[1] + x[2]y[0] = \underline{a_0 b_2 + a_1 b_1 + a_2 b_0}$$
$$p[3] = x[1]y[2] + x[2]y[1] = \underline{a_1 b_2 + a_2 b_1}$$
$$p[4] = x[2]y[2] = \underline{a_2 b_2}$$

The elements of $p[n]$ are exactly the coefficients of $C(z)$ and the elements of $x[n]$ and $y[n]$ are the coefficients of the polynomials $A(z)$ and $B(z)$ respectively. This suggests that convolution can be seen as the outcome of multiplying polynomials of a certain type. Specifically the polynomials obtained via the Z-transforms.

The discrete time Fourier transform (DTFT) transforms a discrete signal, $x[n]$, to a periodic continuous signal/function, $\hat{x}(\theta)$, which is defined for $\theta \in \mathbb{R}$. The Z-transform generalises the DTFT, extending its domain to a function defined on $z \in \mathbb{C}$, the complex plane.

The primary roles of the z-transform are the study of system characteristics and derivation of computational structures for implementing discrete-time systems on computers. The-transform is also used solve difference equations.

4.2 Definition

The z-transform is obtained by replacing the complex exponential ($e^{j\theta}$) in the DTFT definition with $z \in \mathbb{C}$. The z-transform of a discrete signal is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}, \qquad z \in \mathbb{C}$$

The is the two-sided z-transform (since n varies from $-\infty$ to ∞). When considering causal systems, often $x[n]$ will be zero for $n < 0$. In this case the z-transform is one sided and is given as

$$X(z) = \sum_{n=0}^{\infty} x[n]z^{-n}$$

Note: The independent variable, z, is a complex number and can be written in the form $z = re^{j\theta}$. When z is constrained to the unit circle, i.e., when $|z| = 1$, we get $r = 1$ and $z = e^{j\theta}$ and the z-transform reduces to the DTFT.

Note: It is sometimes common to use $\mathcal{Z}\{\cdot\}$ to denote the z-transform operator. i.e., $X(z) = \mathcal{Z}\{x[n]\}$. It is also common to use $\overset{Z}{\leftrightarrow}$ to denote a z-transform pair. i.e., $X(z) \overset{Z}{\leftrightarrow} x[n]$.

The region where the z-transform converges is known as the **region of convergence** (ROC) and in this region the values of $X(z)$ are finite.

4.3 Properties

Linearity

$$ax[n] + by[n] \overset{Z}{\leftrightarrow} aX(z) + bY(z)$$

where,

$$x[n] \overset{Z}{\leftrightarrow} X(z)$$

$$y[n] \overset{Z}{\leftrightarrow} Y(z)$$

Shifting Property (Delay Theorem)

$$x[n-k] \overset{Z}{\leftrightarrow} z^{-k}X(z)$$

A very important property of the z-transform is the delay theorem.

$$\mathcal{Z}\{x[n-1]\} = z^{-1}X(z)$$

$$\mathcal{Z}\{x[n-2]\} = z^{-2}X(z)$$

Time reversal

$$x[-n] \overset{Z}{\leftrightarrow} X\left(\frac{1}{z}\right) \quad or \quad X(z^{-1})$$

Multiplication by exponential sequence

$$a^n x[n] \overset{Z}{\leftrightarrow} X(a^{-1}z)$$

In the special case of multiplication by $e^{jn\theta}$

$$e^{jn\theta}x[n] \overset{Z}{\leftrightarrow} X\left(e^{-j\theta}z\right)$$

Differentiation in the z-domain

$$nx[n] \overset{Z}{\leftrightarrow} -z\frac{d}{dz}X(z)$$

Discrete Convolution

$$x[n] * y[n] \overset{Z}{\leftrightarrow} X(z)Y(z)$$

$$\mathcal{Z}\{x[n] * y[n]\} = \mathcal{Z}\left\{\sum_{k=-\infty}^{\infty} x[k]y[n-k]\right\}$$
$$= \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x[k]y[n-k]\right)z^{-n}$$

Let $m = n - k$

$$\mathcal{Z}\{x[n] * y[n]\} = \sum_{m=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x[k]y[m]z^{-(m+k)}$$
$$= \left(\sum_{k=-\infty}^{\infty} x[k]z^{-k}\right)\left(\sum_{m=-\infty}^{\infty} y[m]z^{-m}\right)$$
$$= X(z)Y(z)$$

This result can also be proved for the single sided z-transform by considering causal sequences, $x[n] = x[n]u[n]$ and $y[n] = y[n]u[n]$. This is left as an **exercise**.

Example 4.1

Consider the unit step signal

$$x[n] = u[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0 \end{cases}$$

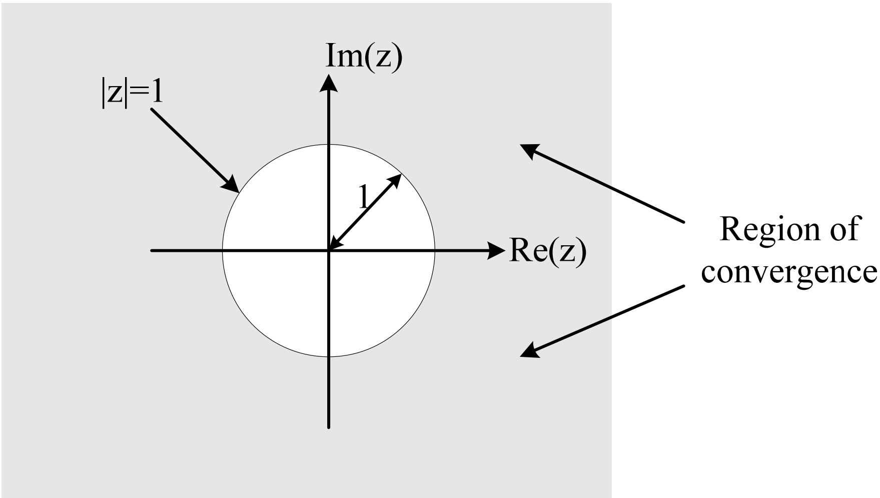
Its z-transform is given by

$$X(z) = \sum_{n=0}^{\infty} 1 \cdot z^{-n} = 1 + z^{-1} + z^{-2} + \cdots$$

This is a geometric series with a common ratio of z^{-1}. The series converges if $|z^{-1}| < 1$ or equivalently if $|z| > 1$.

$$X(z) = 1 + z^{-1} + z^{-2} + \cdots = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}$$

In this case, the z-transform is valid everywhere outside a circle of unit radius whose centre is at the origin (see below)



Note: $|z| = 1$ is a circle of unit radius and is commonly referred to as the 'unit circle'.

If $|z| > 1$, then $X(z)$ converges and if $|z| < 1$, then it diverges.

Let $z = 2$, then $X(z) = 1 + 2^{-1} + 2^{-2} + \cdots = \frac{1}{1-2^{-1}} = 2$

Let $z = 0.5$, then $X(z) = 1 + \left(\frac{1}{2}\right)^{-1} + \cdots = 1 + 2 + 4 + \cdots$

Example 4.2

The z-transform of the delta sequence, $\delta[n]$ is given by

$$\mathcal{Z}\{\delta[n]\} = \sum_{k=-\infty}^{\infty} \delta[k]z^{-k} = \delta[0]z^{-0} = 1$$

Example 4.3

Consider the causal geometric sequence, $x[n] = a^n u[n]$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \frac{1}{1-\frac{a}{z}} = \frac{z}{z-a}, \quad for \left|\frac{a}{z}\right| < 1$$

sum of the polynomial sequence

Or equivalently, $|z| > |a|$.

Note: When $a = 1$, the sequence $x[n] = a^n u[n]$ reduces to the unit step sequence $x[n] = u[n]$. Substituting $a = 1$ in $X(z)$, reduces it to $X(z) = \frac{z}{z-1}$; $ROC: |z| > 1$, which is the z-transform of the unit step sequence.

Example 4.4

Consider the complex exponential sequence, $x[n] = e^{jn\theta},\ n \geq 0$.

$$\mathcal{Z}\{e^{jn\theta}\} = \sum_{n=0}^{\infty} e^{jn\theta} z^{-n} = \frac{1}{1 - \frac{e^{j\theta}}{z}} = \frac{z}{z - e^{j\theta}}$$

$$= \frac{z}{z - e^{j\theta}} \times \frac{z - e^{-j\theta}}{z - e^{-j\theta}} = \frac{z\left(z - e^{-j\theta}\right)}{z^2 - (e^{j\theta} + e^{-j\theta})z + 1}$$

$$= \frac{z(z - \cos\theta + j\sin\theta)}{z^2 - 2z\cos\theta + 1}$$

$$= \frac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1} + j\frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

But,

$$\mathcal{Z}\{e^{jn\theta}\} = \mathcal{Z}\{\cos n\theta + j\sin n\theta\}$$

$$= \mathcal{Z}\{\cos n\theta\} + j\mathcal{Z}\{\sin n\theta\}$$

due to the linearity property. Hence, comparing real and imaginary parts

$$\mathcal{Z}\{\cos n\theta\} = \frac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1}$$

$$\mathcal{Z}\{\sin n\theta\} = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

Extremely useful in digital oscillator design!!!!!!

4.4 Transfer Function

Consider a system, $y[n] = H\{x[n]\}$, given by

$$y[n] = a_0 x[n] + a_1 x[n-1] + a_2 x[n-2] + b_1 y[n-1] + b_2 y[n-2]$$

Taking the z-transform

$$Y(z) = a_0 X(z) + a_1 z^{-1} X(z) + a_2 z^{-2} X(z) + b_1 z^{-1} Y(z) + b_2 z^{-2} Y(z)$$

Grouping terms with $Y(z)$ and $X(z)$,

$$Y(z)[1 - b_1 z^{-1} - b_2 z^{-2}] = X(z)[a_0 + a_1 z^{-1} + a_2 z^{-2}]$$

The transfer function, $H(z)$, is then defined as

$$H(z) = \frac{Y(z)}{X(z)} = \frac{a_0 + a_1 z^{-1} + a_2 z^{-2}}{1 - b_1 z^{-1} - b_2 z^{-2}}$$

Note: Based on the definition of the transfer function,

$$Y(z) = H(z)X(z)$$

Using the convolution in time property of z-transforms,

$$y[n] = h[n] * x[n]$$

However, for an LTI system we know that the output, $y[n]$ is obtained by convolving the input, $x[n]$ with the impulse response. Hence, $h[n]$ is the impulse response of the system. i.e., the transfer function, $H(z)$ is the z-transform of the impulse response, $h[n]$ for an LTI system.

Note: The transfer function is also referred to as the system function.

Example 4.5

Find the difference-equation of the following transfer function

$$H(z) = \frac{5z + 2}{z^2 + 3z + 2}$$

First rewrite $H(z)$ as a ratio of polynomials in z^{-1}

$$\frac{Y(z)}{X(z)} = H(z) = \frac{5z^{-1} + 2z^{-2}}{1 + 3z^{-1} + 2z^{-2}}$$

$$Y(z) + 3z^{-1}Y(z) + 2z^{-2}Y(z) = 5z^{-1}X(z) + 2z^{-2}X(z)$$

Taking the inverse z-transform

$$y[n] + 3y[n-1] + 2y[n-2] = 5x[n-1] + 2x[n-2]$$

Thus giving the difference equation of the system

$$\boxed{y[n] = 5x[n-1] + 2x[n-2] - 3y[n-1] - 2y[n-2]}$$

Example 4.6

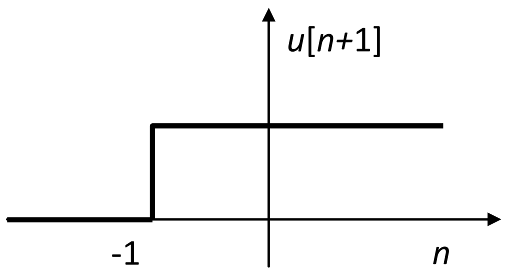
If $x[n] = u[n] - u[n-10]$, find $X(z)$.

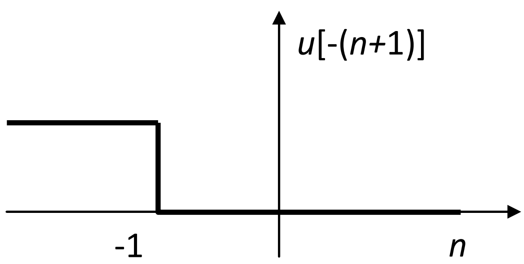
$$X(z) = \sum_{n=0}^{9} (1) z^{-n} = \frac{1 - z^{-10}}{1 - z^{-1}}$$

Example 4.7

If $y[n] = -\alpha^n u[-n-1]$, fund $Y(z)$.

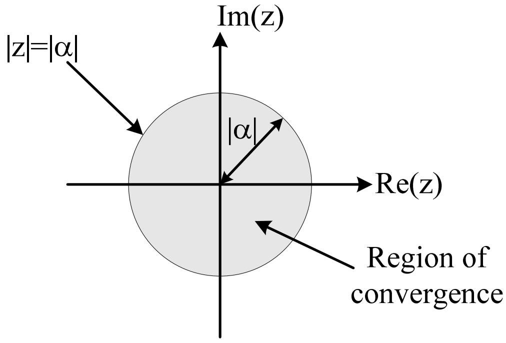
$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} -\alpha^n u[-n-1] z^{-n} \\ &= \sum_{n=-\infty}^{-1} \left(\frac{\alpha}{z}\right)^n \\ &= \sum_{k=1}^{\infty} \left(\frac{z}{\alpha}\right)^k \\ &= 1 - \sum_{k=0}^{\infty} \left(\frac{z}{\alpha}\right)^k \end{aligned}$$





The sum converges provided $\left|\frac{z}{\alpha}\right| < 1$. i.e., $|z| < |\alpha|$.

$$\begin{aligned} Y(z) &= 1 - \frac{1}{1 - z\alpha^{-1}}, \qquad |z| < |\alpha| \\ &= \frac{z}{z - \alpha}, \qquad ROC: |z| < |\alpha| \end{aligned}$$



Example 4.8

Determine the system function $H(z)$ of the system described by

$$y[n] = x[n] + ay[n-1]$$

Taking the z-transform

$$Y(z) = X(z) + az^{-1}Y(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - az^{-1}}$$

4.5 Inverse Z-Transform

The inverse z-transform allows for the computation of a discrete time signal $x[n]$ from its z-transform, $X(z)$.

$$x[n] = \mathcal{Z}^{-1}\{X(z)\}$$

There are three approaches to inverting the z-transform

- Evaluating a contour integral in the z-plane.
- Expanding $X(z)$ as a power series (Laurent series). Then the coefficients of z^{-n} are the values of the discrete time signal.
- By algebraic manipulation (using partial fractions) to split $X(z)$ in parts which can be recognised as z-transforms of known functions.

Contour Integral

The contour integral relating $x[n]$ and $X(z)$ is

$$x[n] = \frac{1}{2\pi j}\oint_C X(z)z^{n-1}dz$$

where, C is any simple, counter clockwise, closed contour of the complex plane in the ROC ($C \subseteq ROC_X$) and with the origin in the interior of C.

Power Series Expansion

If the z-transform, $X(z)$, of a discrete-time signal, $x[n]$, can be expanded in a power series of the form

$$\underline{X(z) = \cdots + a_{-2}z^{-(-2)} + a_{-1}z^{-(-1)} + a_0 + a_1z^{-1} + a_2z^{-2} + \cdots}$$

The values of the discrete time signal $x[n]$ are the coefficients of z^{-n}. i.e.,

$$\vdots$$
$$x[-2] = a_{-2}$$
$$x[-1] = a_{-1}$$
$$x[0] = a_0$$
$$x[1] = a_1$$
$$x[2] = a_2$$
$$\vdots$$

In general, $x[n] = a_n$.

Note: This method is obvious to see by comparing the power series

$$X(z) = \sum_{n=-\infty}^{\infty} a_n z^{-n}$$

and the z-transform formula

an = x[n]

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n}$$

Partial Fraction Method

Since the z-transform is a linear operation, by splitting $X(z)$ into a linear combination of expressions which are known z-transforms, the inverse transform can be found as the same linear combination of the corresponding discrete time signals. i.e., if $X(z)$ can be written as

$$X(z) = p_1 X_1(z) + p_2 X_2(Z) + \cdots$$

And we know that

$$\mathcal{Z}^{-1}\{X_1(z)\} = x_1[n]$$
$$\mathcal{Z}^{-1}\{X_2(z)\} = x_2[n]$$
$$\vdots$$

Then, $x[n] = p_1 x_1[n] + p_2 x_2[n] + \cdots$

Example 4.9

Find the inverse z-transform for

$$X(z) = \frac{z^{-1}}{1 - 0.25z^{-1} - 0.375z^{-2}}$$

Here,

$$X(z) = \frac{z}{z^2 - 0.25z - 0.375} = \frac{z}{(z - 0.75)(z + 0.5)}$$

$$= z\left[\frac{A}{z - 0.75} + \frac{B}{z + 0.5}\right] = \frac{\left(\frac{4}{5}\right)z}{z - 0.75} - \frac{\left(\frac{4}{5}\right)z}{z + 0.5}$$

$$= \left(\frac{4}{5}\right)\left[\frac{1}{1 - 0.75z^{-1}}\right] - \left(\frac{4}{5}\right)\left[\frac{1}{1 + 0.5z^{-1}}\right]$$

Recognising that both terms are of the form $\frac{1}{1-\alpha z^{-1}}$ and identifying

$$\mathcal{Z}\{\alpha^n u[n]\} = \frac{1}{1 - \alpha z^{-1}}, \qquad ROC: |z| > |\alpha|$$

We can write,

$$\mathcal{Z}^{-1}\left\{\frac{1}{1 - \alpha z^{-1}}\right\} = \alpha^n u[n], \qquad |z| > |\alpha|$$

And hence,

$$x[n] = \mathcal{Z}^{-1}\{X(z)\} = \left(\frac{4}{5}\right)(0.75)^n u[n] - \left(\frac{4}{5}\right)(-0.5)^n u[n]$$

$$= \frac{4}{5}[(0.75)^n - (-0.5)^n]\, u[n]$$

$$= \frac{4}{5}[(0.75)^n - (-0.5)^n], \qquad n \geq 0$$

4.6 Relationship between the z-transform and the Laplace transform

The Laplace transform is defined on continuous time functions and serves as the analog world equivalent to the z-transform. Given a continuous time signal, $x(t)$, its Laplace transform is defined as

$$X(s) = \mathcal{L}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-st}dt, \qquad s \in \mathbb{C}$$

Similar to the z-transform, the independent variable in the Laplace transform, s, is complex valued. i.e., $s = \sigma + j\omega$ with $\sigma, \omega \in \mathbb{R}$.

The primary role of the Laplace transform in engineering is transient and stability analysis of causal LTI systems described by differential equations.

Note: Similar to the z-transform, the Laplace transform can be one-sided or two-sided. The above definition is the two-sided Laplace transform. The one-sided transform is identical but the integral is evaluated on the interval $[0, \infty]$ instead of $[-\infty, \infty]$.

Note: The region of convergence for a Laplace transform is the set of points, s, on the s-plane where the integral converges.

Recall that a discrete time signal, $x[n]$, obtained by sampling a continuous signal, $x(t)$, at regular intervals of T is defined as

$$x[n] \stackrel{\text{def}}{=} x(nT)$$

The "continuous" representation of the discrete signal is given by the aid of the Dirac comb as

$$
\begin{aligned}
x_d(t) &= x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT) \\
&= \sum_{n=-\infty}^{\infty} x(nT)\delta(t-nT) \\
&= \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT)
\end{aligned}
$$

Taking the Laplace transform of this function

$$
\begin{aligned}
X_d(s) = \mathcal{L}\{x_d(t)\} &= \int_{-\infty}^{\infty} x_d(t)e^{-st}dt \\
&= \int_{-\infty}^{\infty}\sum_{n=-\infty}^{\infty} x[n]\delta(t-nT)\,e^{-st}dt \\
&= \sum_{n=-\infty}^{\infty} x[n]\int_{-\infty}^{\infty}\delta(t-nT)e^{-st}dt \\
&= \sum_{n=-\infty}^{\infty} x[n]e^{-snT}
\end{aligned}
$$

This is equivalent to the z-transform of $x[n]$ when $z \leftarrow e^{sT}$

$$
X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n}
$$

i.e.,

$$
X_d(s) = X(z)\big|_{z=e^{sT}}
$$

Writing $s = \sigma + j\omega$,

$$
z = e^{sT} = e^{(\sigma+j\omega)T} = e^{\sigma T}\cdot e^{j\omega T}
$$

Thus, $|z| = e^{\sigma T}$ and $\angle z = \omega T = 2\pi\frac{f}{F_s} = \theta$. ($\theta$ - Digital Frequency)

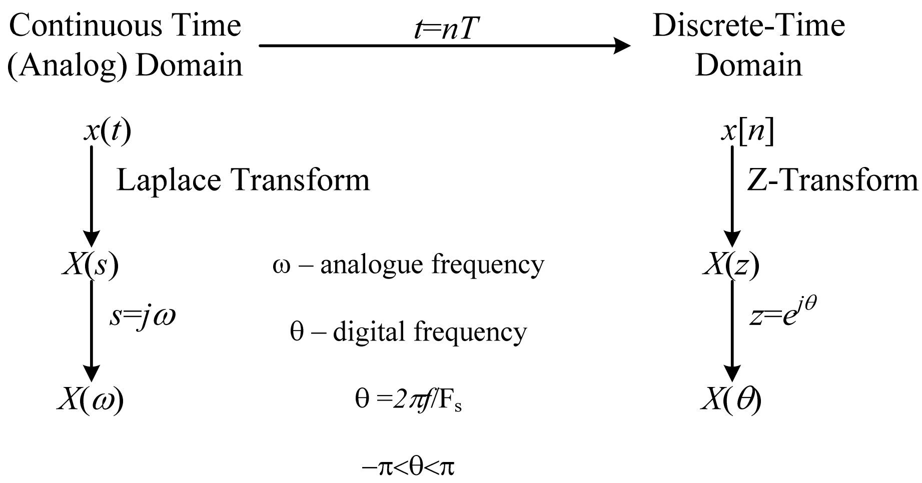
Thus,

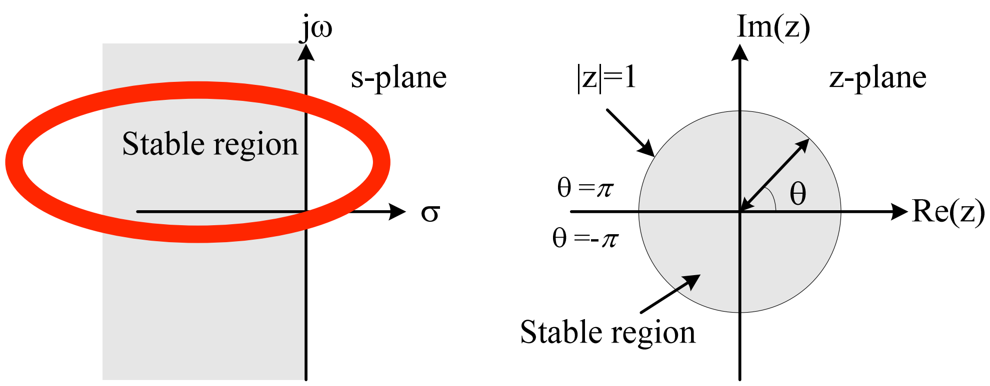
when $\sigma = 0$, $|z| = 1$. i.e., the $j\omega$ axis (vertical) of the s-plane is mapped onto the unit circle of the z-plane.

when $\sigma < 0$, $|z| = e^{\sigma T} < 1$. i.e., the left half of the s-plane is mapped onto the inside of the unit circle in the z-plane.

when $\sigma > 0, |z| = e^{\sigma T} > 1$. i.e., the right half of the s-plane is mapped onto the outside of the unit circle in the z-plane.

Note: Every point on the s-plane is mapped onto a point on the z-plane. However, all points, $s_k = \sigma + j(\omega_0 + 2\pi k)$, with $\omega_0 \in [-\pi, \pi]$ and $k \in \mathbb{Z}$ are mapped onto the same point $z = e^{\sigma T} e^{j\omega_0 T}$.



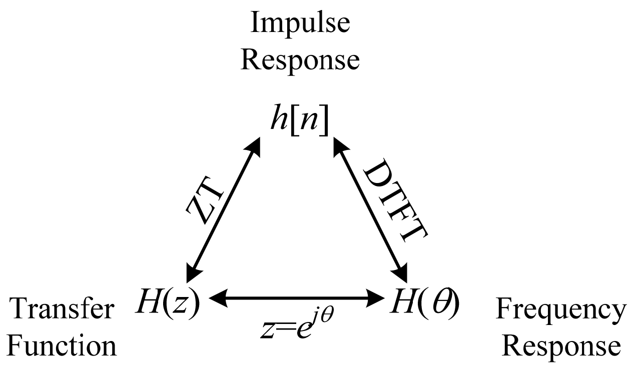


4.7 Relationship to Frequency Response

The frequency response of a system, $H(\theta)$, is defined as the transfer function evaluated on the unit circle. i.e.,

$$H(\theta) = H(z)\big|_{z=e^{j\theta}}$$

Note: This is equivalent to evaluating the DTFT of the impulse response.



Note: $H(\theta)$ is complex valued and can be written as $|H(\theta)|e^{j\phi(\theta)}$, where $|H(\theta)|$ is called the magnitude response and $\phi(\theta)$ is termed the phase response. Both are real valued functions.

<u>Example 4.10</u>

Find $H(\theta)$, the frequency response for a system described by the transfer function:

$$H(z) = \frac{1}{1 - az^{-1}}, \qquad 0 < a < 1, say\ a = 0.6$$

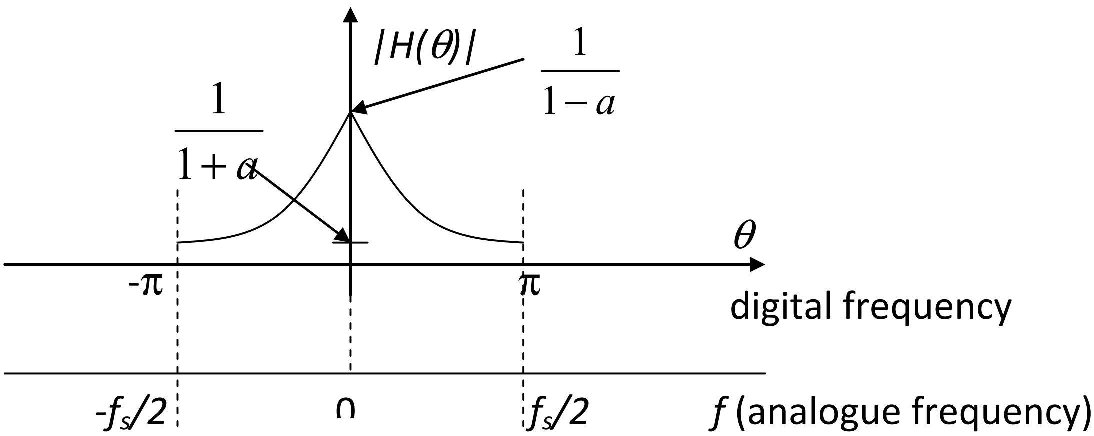
The frequency response is given by

$$H(\theta) = H(z)\big|_{z=e^{j\theta}}, \qquad -\pi \le \theta \le \pi$$

Thus,

$$H(\theta) = \frac{1}{1 - ae^{-j\theta}} = \frac{1}{(1 - a\cos\theta) + ja\sin\theta}$$

$$|H(\theta)| = \frac{1}{\sqrt{(1 - a\cos\theta)^2 + (a\sin\theta)^2}} = \frac{1}{\sqrt{1 - 2a\cos\theta + a^2}}$$



$\theta = \omega T$; $\theta = 2\pi f/f_s$; $\theta = \pi \Rightarrow f = f_s/2$

f_s – sampling frequency

Note: For a system with a real valued impulse response, the magnitude response is an even function and the phase response is an odd function.