Solving and Estimating Heterogeneous Agent Models Using Machine and Reinforcement Learning *

Yuanzhe Liu†

November 4, 2024

Abstract

In this paper, I introduce an innovative approach leveraging Deep Reinforcement Learning techniques to solve and estimate heterogeneous models. Contrasted with conventional solution methods, the deep learning approach offers a global solution while retaining the entirety of nonlinearity. Moreover, it exhibits remarkable scalability, making it suitable for handling models with hundreds or even thousands of state variables. I also explore the integration of this novel solution method with amortized likelihood-free Bayesian inference, opening up new possibilities for advanced probabilistic modeling and estimation.

Keywords: ML, NN, RL, Heterogeneous agent model

JEL Codes: C63, E21, C11

*I am grateful for continuous support and helpful comments from my advisor Peter Rupert, as well as from Kieran Walsh,and Yueyuan Ma. I am also grateful to all the participants of the UCSB Macro 290 group; special thanks to Xander Abajian and Yang Gao. All errors are my own.

†yliu793@ucsb.edu

1 Introduction

The heterogeneous agent model has been a workhorse in macroeconomics, widely used to study the distribution of wealth, income, consumption, inequality and the transmission of aggregate shock. Traditionally, dynamic programming has been the primary solution method for these models. However, it has several limitations. First, solving the model requires considerable manual effort. Second, scaling it to models with hundreds or even thousands of state variables is challenging. Third, it typically finds only local solutions rather than guaranteed global solutions.

Reinforcement learning (RL) is an unsupervised machine learning technique that has been successfully applied to various domains, including game playing, robotics, and autonomous driving. In many cases, such as Go and multiplayer online battle arena (MOBA) games, reinforcement learning has outperformed the best human players.

In this paper, I demonstrate how reinforcement learning can be adapted to solve heterogeneous agent models. I show that naively applying existing RL algorithms does not work effectively and propose a method to tailor commonly used RL algorithms for solving economic models. I demonstrate that this approach can reliably find the global solution without imposing additional assumptions and can easily scale to models with hundreds or thousands of state variables. Furthermore, I show how the learned policy function allows for the straightforward computation of the model's impulse response function (IRF) and enables counterfactual policy analysis, which is not feasible with traditional methods.

After obtaining the solution to the model, we may wish to estimate the model using data. Since we approximate the policy function with neural networks, we can't directly compute the likelihood function if we decide to take either the Bayesian or maximum likelihood approaches. Therefore, in this paper, I also introduce amortized likelihood-free Bayesian inference as a method to estimate the model.

Reinforcement learning (RL) is a type of machine learning used to train an agent to make decisions. The agent learns by interacting with an environment and receiving rewards

or penalties based on its actions. The goal is to develop a policy that maximizes the agent's discounted cumulative reward over time. RL agents learn through trial and error, exploring the environment and adjusting their actions based on the outcomes they experience. Most RL algorithms rely on estimating the value of different state-action pairs from past experiences to guide future decisions.

For example, consider the game of Snake, where the goal is to eat as many apples as possible without hitting the walls or the snake's own body. In this scenario, the agent is the snake, the environment is the game board, and the rewards are the number of apples the snake consumes. The snake learns how to navigate the board and collect apples by exploring its environment and adjusting its actions based on the rewards it receives. Its policy consists of the rules it follows to decide how to move. By trying different moves and observing the outcomes, the snake gradually improves its strategy.

Most problems in reinforcement learning (RL) can be formulated as a Markov decision process (MDP). An MDP provides a mathematical framework that describes the interaction between an agent and its environment. It consists of a set of states, a set of actions, a transition function that defines how the environment evolves in response to the agent's actions, and a reward function that assigns rewards to state-action pairs.

This framework is very similar to how we construct economic models. In economic models, the agent—such as a household or a firm—makes decisions based on the state of the economy and receives rewards or penalties (e.g., profits or utility) based on these decisions. The agent's objective is to maximize their expected discounted utility over time. Since many economic models can be expressed as MDPs, they can also be solved using reinforcement learning.

What is the difference between reinforcement learning and dynamic programming? The key distinction between reinforcement learning and dynamic programming lies in how the solution is obtained. In dynamic programming, the model is solved using techniques such as backward induction or value function iteration. This approach starts from the final (terminal)

period and works backward to determine the value function at each preceding period, by maximizing the expected value of the next period's value function. In reinforcement learning, the model is solved through trial and error. The agent begins with a random policy and gradually improves it by interacting with the environment and adjusting the policy based on the rewards it receives. The critical difference is that in dynamic programming, the transition function and reward function are assumed to be known in advance. In contrast, reinforcement learning requires the agent to learn these functions from simulated or observed data through interaction with the environment.

Traditional methods, such as dynamic programming, have several limitations when applied to models with large state spaces, like heterogeneous agent models. In these cases, the state space becomes too large to efficiently solve the model using dynamic programming. This challenge, known as the curse of dimensionality, makes it computationally infeasible to explore every possible state-action combination.

In contrast, reinforcement learning (RL) offers a promising alternative for handling large state spaces. RL algorithms can efficiently explore vast environments and, crucially, have the potential to find a global solution, while traditional methods typically only identify local solutions. The perturbation and projection method is one of the most commonly used techniques for solving heterogeneous agent models, but it also has limitations—it cannot guarantee a global solution and requires assumptions to simplify the model and make it tractable. In comparison, reinforcement learning can find global solutions without relying on these assumptions.

While reinforcement learning is powerful, it is not without its challenges. RL has its own set of limitations. Convergence Issues, RL is not always guaranteed to find the optimal policy. It can sometimes get stuck in local minima, failing to reach the global solution. Design and Initialization Sensitivity, the success of RL heavily depends on the careful design of the model, the algorithm, and the initialization. Poor design choices can hinder the agent's ability to learn effectively and find the global solution. Despite these challenges, with well-

designed models and algorithms, reinforcement learning remains a powerful tool for solving complex economic models that are otherwise intractable using traditional methods.

Literature Review

The most widely used method for solving heterogeneous agent models with aggregate shocks was introduced by Krusell and Smith (1998), followed by many extensions. A common feature of these approaches is their attempt to represent the cross-sectional distribution of agents using a small number of statistics, thereby reducing the dimensionality of the state space.

Reiter (2009) proposed a projection and perturbation method to solve heterogeneous agent models. However, this method requires linearization around the stationary steady state. When the model includes multiple aggregate shocks or a multidimensional cross-sectional distribution, the method becomes slower, and if the model is highly nonlinear in aggregate shocks, it becomes less accurate. Building on Reiter (2009), Auclert et al. (2021) introduced a new approach by perturbing the model to first order in aggregates. Rather than representing the equilibrium as a system of linear equations in state space, they express it as a system of linear equations in sequence space. However, their method still relies on linearization, meaning it cannot guarantee a global solution.

Han et al. (2021) proposed a novel approach to solving heterogeneous agent models using deep learning. In their method, a neural network approximates both the value function and the policy function. They employ fictitious play to iteratively update these functions. Their results demonstrate that the method can reliably find the global solution and scale efficiently to models with hundreds or thousands of state variables. Similarly, Azinovic et al. (2022) developed a deep learning-based method to solve heterogeneous agent models. Their approach uses a neural network to approximate the value function and applies Euler equations as constraints during the value function update process.

what exactly are we expecting from a good solution method? A good solution method

should meet the following criteria as discussed in Han et al. (2021):

- **Efficiency**: This is necessary in order to use the method for calibration, estimation, and further quantitative analysis.
- **Reliability**: In particular, it should be applicable beyond the local perturbation regime if nonlinearity matters, e.g. ZLB.
- **Generality**: The method should be applicable to a wide variety of different HA models. Multiple shocks and multidimensional cross-sectional distribution should be handled without much modification.

In this paper, I will show that reinforcement learning can meet all these criteria and provide a powerful alternative to traditional solution methods for heterogeneous agent models.

The rest of the paper is organized as follows. In Section 2, I provide a brief description of the heterogeneous agent model used as a benchmark. Section 3 introduces the reinforcement learning (RL) method applied to solve the model. In Section 4, I present the results obtained using the RL method and display the impulse response function (IRF). Section 5 explains how to use simulation-based inference to perform Bayesian estimation of the model. Finally, Section 6 concludes the paper and discusses directions for future research.

2 Model

Here is a brief review of Krusell and Smith (1998), rephrased using the language of reinforcement learning. In this model, we assume there is a continuum (infinite number) of households in the economy. Each household faces a decision-making problem that can be described by a Bellman equation, which represents the household's dynamic optimization problem.

$$v(k, l; \lambda, z) = \max_{c,k'} \{u(c) + \beta E \left[v \left(k', l'; \lambda', z'\right) \mid (\lambda, z)\right]\} \tag{1}$$

where the maximization is subject to

$$c + k' = \tilde{r}(K, N, z)k + w(K, N, z)l + (1 - \delta)k \tag{2}$$

$$\tilde{r} = \tilde{r}(K, N, z) = z\alpha \left(\frac{K}{N}\right)^{\alpha-1} \tag{3}$$

$$w = w(K, N, z) = z(1 - \alpha) \left(\frac{K}{N}\right)^{\alpha} \tag{4}$$

$$\lambda' = H(\lambda, z) \tag{5}$$

where (K, N) denotes aggregate capital and labor defined as:

$$K = \int k\lambda(k, l)dkdl \tag{6}$$

$$N = \int l\lambda(k, l)dkdl. \tag{7}$$

Here, $\lambda(k, l)$ represents the joint distribution of capital (k) and labor (l) across households, while $H(\cdot)$ in Equation 5 denotes the transition kernel. This distribution is a random function influenced by the aggregate shock (common noise) z. Typically, we assume that z follows an AR(1) process: $z' = (1 - \rho_z)\mu_z + \rho_z z + \epsilon_z$ where ϵ_z is drawn from a Gaussian distribution. Similarly, labor productivity s is also assumed to follow an AR(1) process.

Households choose their consumption c for the current period and savings k' for the next period, subject to the budget constraint in Equation 2. Here, $\tilde{r}$ denotes the interest rate, and w represents the wage. On the right-hand side of Equation 2, the budget constraint accounts for capital income, labor income, and the value of undepreciated capital from the previous period. The parameter δ represents the depreciation rate of capital.

The equilibrium is defined as follows:

1. Given the price functions, 3 and 4, and $H(\cdot)$, the value function solves the Bellman equation 1 and the optimal decision rule is $f(k, l, \lambda, z)$. ($f(k, l, \lambda, z)$ returns c and k')

2. The decision rule $f(k, l, \lambda, z)$ and the processes for l and z imply that today's distribution $\lambda(k, l)$ is mapped into tomorrow's $\lambda'(k, l)$ by H (I think is part is similar to the forward backward phrase in the mean field game literature.)

In the original paper by Krusell and Smith (1998), instead of tracking the full distribution λ, which has infinite dimensions, they approximate it by tracking only the first moment of λ. This simplification works remarkably well within their model. However, relying solely on the first moment may not be effective in other models where the entire distribution matters significantly. Important implications of the model may also be lost if we ignore higher-order moments or other features of the distribution.

The intuition behind why this method works is that, when households make consumption and saving decisions, they need to forecast the next period's interest rate, $\tilde{r}'$. To make an accurate forecast, it is sufficient to know only the first moment of the distribution λ. While some researchers have attempted to incorporate higher moments into the method, the improvements in accuracy have been marginal.

This model provides a useful benchmark to explore whether RL can be effectively applied to this class of models. For more complex models, tracking only the first moment may no longer be sufficient, especially when the dynamics of aggregate capital K are highly nonlinear.

3 Method

Reinforcement Learning (RL) consists of four essential elements, as shown in Figure 1: Agents, Environment, Actions, and Rewards. In the context of the Krusell and Smith (1998) model, the agents are the households, the environment corresponds to the transition function that governs how the state variables evolve over time, the actions are the households' consumption and saving decisions, and the rewards is the utility derived from consumption. The agent's objective is to learn a policy that maximizes its expected cumulative reward over time.

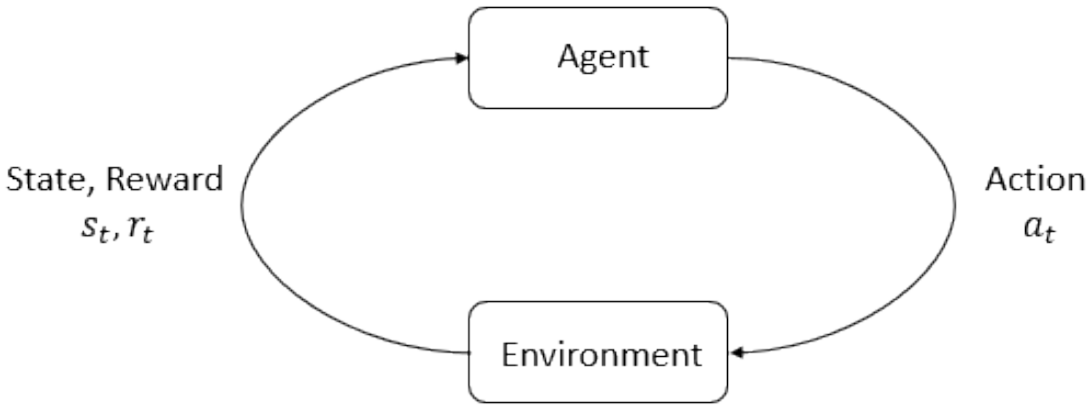

Figure 1: Reinforcement Learning Framework

3.1 Policy Gradient Methods

Policy gradient methods are a class of reinforcement learning techniques that update the parameters of a function approximator (in our case, a neural network) to increase the value function. The policy gradient theorem states that the gradient of the expected return with respect to the policy parameters is proportional to the sum of rewards, weighted by the gradient of the log probability of the chosen action. Mathematically, the policy gradient theorem can be expressed as:

$\nabla_\theta J(\boldsymbol{\theta}) \propto \sum_s \mu(s) \sum_a q_\pi(s,a) \nabla \pi(a \mid s, \boldsymbol{\theta})$

Here, $q_\pi(s,a)$ represents the action-value function, which denotes the expected cumulative future rewards an agent can obtain by starting from state s, taking action a, and then following a specific policy π_θ in this context. More details and the proof of the policy gradient theorem can be found in Sutton and Barto (2018).

The distribution μ here is the on-policy distribution under π. All parts of $\nabla_\theta J(\boldsymbol{\theta})$ can be estimated from simulated data. Let $J(\boldsymbol{\theta}) = v_{\pi_{\boldsymbol{\theta}}}(s)$ be the performance measure, where v_{π_θ} represents the true value function for the policy function π_θ. Here, θ denotes the parameter of the policy function. For instance, when approximating the policy function using a neural network (NN), θ corresponds to the weights and biases of the network. The task is to update

θ to maximize the value function.

There are various policy gradient methods, such as REINFORCE, PPO, DDPG, and TRPO. In this paper, I use the Deep Deterministic Policy Gradient (DDPG) algorithm (Algorithm 1) to solve the heterogeneous agent model. DDPG is an off-policy actor-critic algorithm that uses a deterministic policy to select actions. It is well-suited for continuous action spaces and has been shown to be effective across many problems. Off-policy means that the policy used to generate the data differs from the policy used to update the value function. This feature is useful in practice, as it allows us to reuse data collected from previous policies.

The actor-critic structure of DDPG means that the algorithm learns both a policy (actor) and a value function (critic) simultaneously. The actor updates the policy using the gradient of the value function, which provides a more stable estimate of the policy gradient.

An example of an on-policy method is the REINFORCE algorithm. In REINFORCE, the policy is updated using the gradient of the expected return with respect to the policy parameters. The expected return is estimated by running a simulation over many periods and computing the sum of rewards. The policy is then updated using the gradient of the log probability of the action, multiplied by the return. In contrast, for off-policy methods, we simulate the environment to collect data and then sample from this data to estimate the gradient.

Initial policy and action-value function parameters θ and ϕ, and a reply buffer $\mathcal{D}$;

while *Not Converge* **do**

Observe state s and select action $a = \text{clip}\left(\pi_\theta(s) + \epsilon, a_{\text{Low}}, a_{\text{High}}\right)$, where $\epsilon \sim \mathcal{N}$;

Execute a ;

Observe next state s', utility r ;

Store (s, a, r, s') in reply buffer $\mathcal{D}$;

if *it is time to update* **then**

Randomly sample a batch $B = \{(s, a, r, s')\}$ from $\mathcal{D}$;

$y\left(s'\right) = r + \beta Q_{\phi_{\text{targ}}}\left(s', \pi_{\theta_{\text{targ}}}\left(s'\right)\right)$;

$\phi = \phi - \nabla_\phi \frac{1}{|B|} \sum_{(s,a,r,s')\in B} \left(Q_\phi(s, a) - y\left(s'\right)\right)^2$;

$\theta = \theta + \nabla_\theta \frac{1}{|B|} \sum_{s\in B} Q_\phi\left(s, \pi_\theta(s)\right)$;

$\phi_{\text{targ}} \leftarrow \rho\phi_{\text{targ}} + (1-\rho)\phi$

$\theta_{\text{targ}} \leftarrow \rho\theta_{\text{targ}} + (1-\rho)\theta$

end

end

Algorithm 1: Deep Deterministic Policy Gradient

The key idea of Deep Deterministic Policy Gradient (DDPG) is to use a deterministic policy to select actions and learn a value function that estimates the expected return of the policy. The policy is updated using the gradient of the value function, which provides a more stable estimate of the policy gradient. The value function is updated according to the Bellman equation, similar to how it is done in Q-learning.

Since DDPG trains a deterministic policy in an off-policy fashion, it may initially fail to explore a wide enough variety of actions, potentially getting stuck in a local solution. To address this, exploration noise is added to the action selection process. This noise encourages the policy to explore more diverse actions. As the policy becomes more confident over time, the noise is annealed (gradually reduced) to focus more on exploitation. More details can

be found in the original papers Silver et al. (2014) and Lillicrap et al. (2015).

3.2 Implementation details

In the Krusell-Smith (KS) economy, each agent observes the state of the economy in each period, including the interest rate (r), wage (w), total factor productivity (TFP), and the cross-sectional distribution of asset holdings and labor endowments. Based on this information, they make consumption and saving decisions and then transition to the next period.

We simulate 100 KS agents. For $i \in \{1, 2, 3...100\}$

$s_{t,i} = \{k_{t,1}, k_{t,2}..., k_{t,100}; l_{t,i}, z_t\}$

$a_{t,i} = \pi_\theta(s_{t,i})$

$u_{t,i} = log(a_{t,i})$

$k_{t+1,i} = (1 + r_t) * k_{t,i} + w_t * l_{t,i} - a_{t,i}$

Once every agent has made their decision, we draw a new TFP shock, z_{t+1}, and transition to the next period.

$s_{t+1,i} = \{k_{t+1,1}, k_{t+1,2}..., k_{t+1,100}; l_{t+1,i}, z_{t+1}\}$

Store $\{s_{t,i}, a_{t,i}, u_{t,i}, s_{t+1,i}\}$ in the memory reply buffer and update policy and action-value function according to Algorithm 1.

To make the training process more stable and ensure that the RL algorithm converges to the global solution, I made the following modifications to the original DDPG algorithm.

First, we pre-train the policy function using a supervised learning algorithm to provide a better starting point for the RL algorithm. This stabilizes the training process by reducing the chances of poor initial performance. The initial policy function is trained under the assumption that each agent consumes only their labor income and interest income, meaning their capital stock remains unchanged over time. Although this is not the optimal policy, it serves as a reasonable baseline from which the RL algorithm can make further improvements.

The second modification is instead of directly inputting the entire joint distribution of capital and labor into the policy function, we use a summary network to condense the distri-

bution into a few key statistics. This summary network is shared between the policy function and the value function, promoting more effective learning and stability during training. The intuition behind this modification is that the information from the cross-sectional distribution should be consistent for both the policy and value functions. Additionally, this approach reduces the dimensionality of the state space, making the training process more efficient. It also make the algorithm permutation invariant, which is important in the context of the heterogeneous agent model since the order of agents should not affect the policy function. We could have input the entire distribution into the policy function, however, it would have made the training process much slower and less stable because the policy function would have to learn the fact that the order does not matter.

I use Φ_{θ_s} to denote the summary network. The input of the summary network is the individual asset holding and labor endowment, and the output is the summary statistics. The *mean* of the summary statistics,S_t, are then input into the policy function and the value function.

$$\Phi_{\theta_s}(k_{t,i}, l_{t,i}) = s_{t,i} \tag{8}$$

$$S_t = \frac{1}{N} \sum_{i=1}^{N} \Phi_{\theta_s}(k_{t,i}, l_{t,i}) \tag{9}$$

Han et al. (2021) also employed a similar approach in their work. They used a neural network to represent a generalized moment in contrast to Krusell and Smith (1998) which only keep track of the first moment. The output of the summary network is not necessarily one dimension but can be multiple dimensions. The summary network is meant to capture the key information of the cross-sectional distribution. In practice, I found that setting the output dimension to be 4 works well for the Krusell-Smith model.

Table 1 shows the parameter values used in the model, while Table 2 shows the parameters of the learning algorithm. Some Parameters are then estimated using Bayesian estimation.

The details of the estimation process are described in Section 5.

Table 1: Model Parameter Values

Parameter	**Description**	**Value**
β	Discount Factor	0.98
μ_z	Mean of TFP	1
ρ_z	Persistence of TFP process	0.95
ϵ_z	st of TFP process	0.02
μ_s	Mean of labor	0.117
ρ_s	Persistence of labor process	0.95
ϵ_s	st of labor shock	0.05
α	Weight of capital in production function	0.33
δ	Depreciation rate of Capital	0.025
σ	utility	1

Table 2: DDPG Parameter Values

Parameter	**Description**	**Value**
lr_{actor}	Learning rate of policy function	$5e^{-4}$
lr_{critic}	Learning rate of action-value function	$1e^{-3}$
B	Batch size	64
S_θ	Structure of NN_{policy}	$[256, 128, 32]$
S_ϕ	Structure of NN_{value}	$[256, 128, 32]$
τ	soft update	$1e^{-3}$
ϵ	Exploration noise	0.001

Equilibrium conditions discussed in Section 2

4 RL Result

The learning rate for the policy function $5e^{-4}$, the learning rate for the action-value function $1e^{-3}$, the batch size $|B|= 64$, the target update rate $\tau = 1e^{-3}$, and the exploration noise

$\epsilon = 0.01$. I use a neural network with three hidden layers of 128 units each and Tanh activation functions to approximate the policy function and the action-value function.

The following figures show the results of the RL method. Figure 2 shows the distribution of wealth in the economy after half a million iterations.

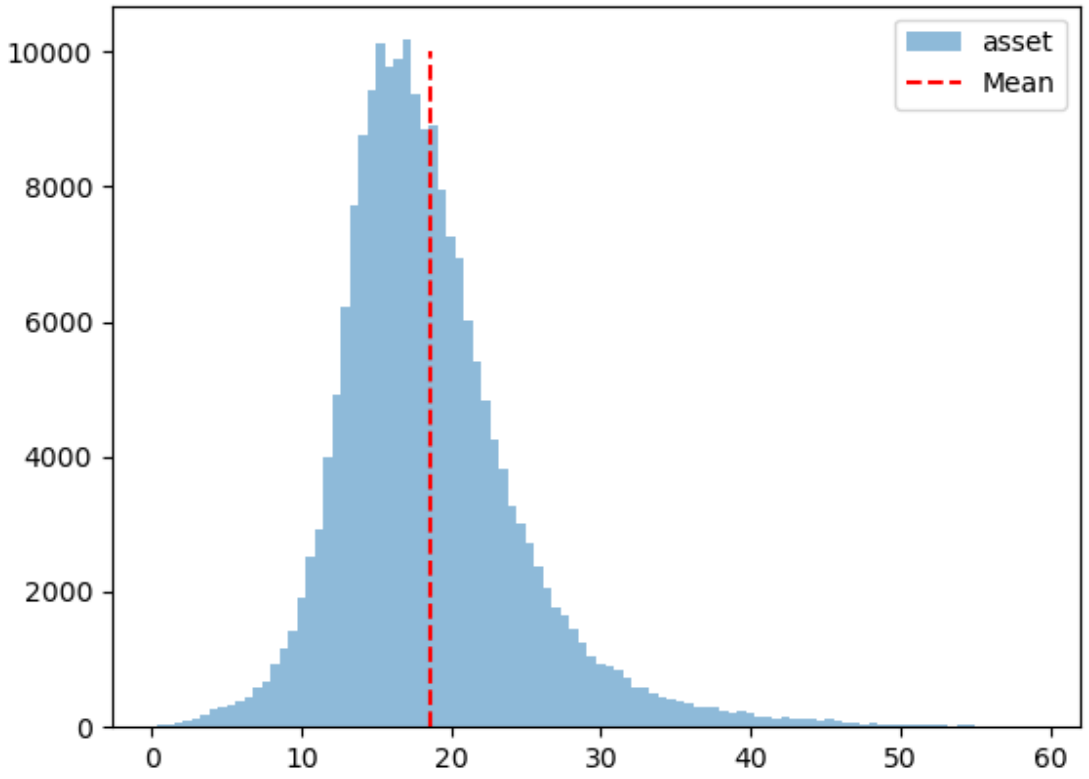

Figure 2: Wealth distribution after 500,000 iterations.

The following figures illustrate the consumption and saving policy functions obtained using the RL algorithm. Figure 3 shows the consumption function, while Figure 4 presents the saving function. Both figures closely resemble the policy functions obtained through the traditional method. As noted in the original paper, the policy functions are nearly linear because the marginal propensity to consume (MPC) remains almost constant across the wealth distribution.

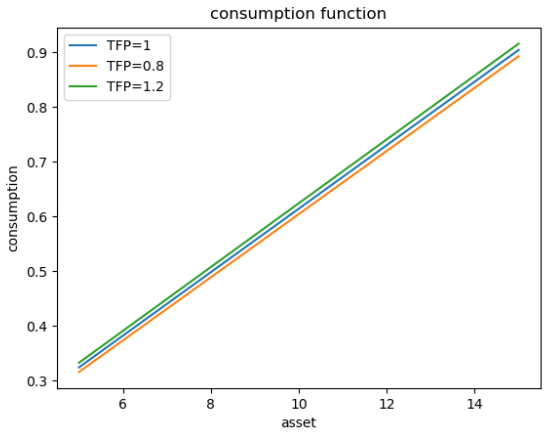

Figure 3: consumption function.

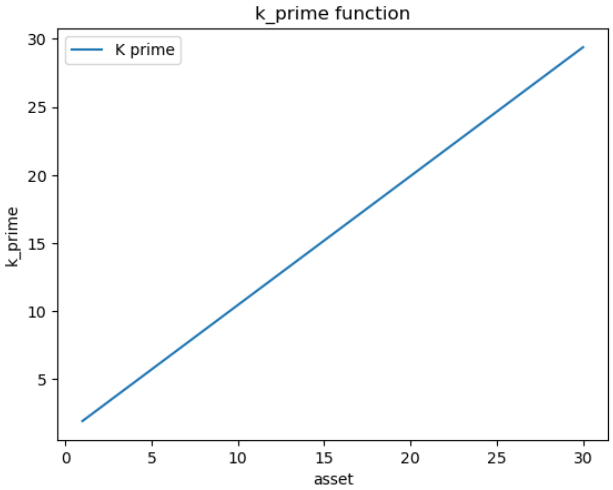

Figure 4: k'

Figure 5 shows the impulse response function of a transfer policy. The transfer policy is a policy that transfers a fixed amount of wealth from the rich to the poor. The social planner take one unit of wealth from agents whose wealth is at the top one third and give it to agents whose wealth is at the lower one third. The figure 5 shows the effect of this transfer policy on middle class who are in the middle one third of the wealth distribution. The original method proposed by Krusell and Smith (1998) states that only the mean of the distribution matters and the MPC is almost the same across the wealth distribution, therefore, the transfer policy should not have any impact on the middle class.

It is very easy to obtain the IRF of the model using the policy function learned by the RL algorithm. We can simply simulate this transfer for many time and then compute the average of the outcome. If we were to use the traditional method, we would have to solve the model for each period and then compute the outcome. The RL method is much faster and more efficient than the traditional method when it comes to compute the IRF.

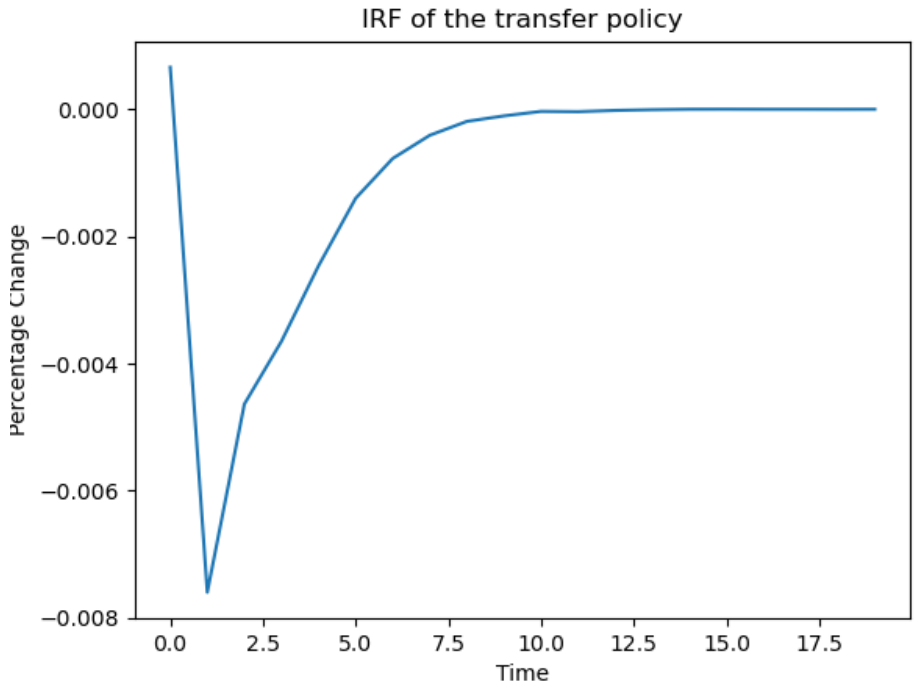

Figure 5: Impulse response function of a transfer policy.

5 Simulation Based inference

After solving the model, we may want to estimate some parameters with real data. However, because we use neural networks to approximate the policy function, we must rely on likelihood-free methods to estimate the model, as the likelihood function is not analytically available.

Traditionally, if the model is solved by linearizing around its steady state, we can follow Herbst and Schorfheide (2016) to express the model in state space, using the Kalman filter to compute the likelihood function and then apply sampling methods, such as Metropolis-Hastings, to obtain the posterior distribution of the parameters. In this case, however, this approach is not feasible because the likelihood function is not available. However, this approach only allows us to

Liu and Plagborg-Møller (2023) proposed a new method for incorporating the cross-

sectional distribution of variables, such as consumption, saving, and wealth, into Bayesian estimation. This approach requires solving for the conditional probability of the cross-sectional distribution given the state variables. Similarly, Kase et al. (2022) introduced a method that leverages information from the cross-sectional distribution to estimate the model. Their approach uses machine learning techniques to approximate the particle filter.

Another advantage of using the likelihood-free method is that, when combined with the RL solution method, it enables an end-to-end approach for solving and estimating our model. During estimation, we need to solve the model multiple times for different parameter values. With this approach, we can use each solved solution as a starting point for the next estimation, significantly reducing time and computational resources.

5.1 SBI method

When using RL or other methods involving neural networks to approximate the policy and value functions, an analytical expression for the likelihood function given observations is not available, rendering conventional Bayesian inference methods inapplicable. Instead, an alternative class of likelihood-free estimation methods has been developed in the literature, with Approximate Bayesian Computation (ABC) being a popular choice. ABC is a simulation-based method that approximates the posterior distribution of parameters by comparing simulated data with observed data.

The basic idea of ABC is as follows: for a given set of parameters drawn from the prior, we simulate the model over many periods and then compare the simulated data with the observed data. The distance between the simulated and observed data serves as a measure of likelihood for those parameters. If this distance exceeds a specified threshold, the parameters are rejected; otherwise, they are accepted. The accepted parameters are then used to construct an approximate posterior distribution.

Another way of seeing this is to think of this process as of obtaining the joint distribution. $p(X, \theta) = p(\theta) * p(X|\theta)$. The posterior distribution of the parameters is then given by

$p(\theta|X) = \frac{p(X,\theta)}{p(X)}$. The likelihood function $p(X|\theta)$ is not available analytically so does the joint distribution. If we draw many sets of parameters from the prior and simulate the model for many periods, we can obtain the joint distribution. The posterior distribution of the parameters is then given by the joint distribution divided by the marginal distribution of the data $p(X)$.

5.2 Normalizing Flow

Papamakarios and Murray (2018) proposed a new method to estimate the posterior distribution of the parameters using normalizing flow. Normalizing flow is a generative model that learns the distribution of the data by transforming a simple distribution into a complex distribution. The key idea of normalizing flow is to learn a series of invertible transformations that map a simple distribution to a complex distribution. The parameters of the transformations are learned using a neural network. The normalizing flow model can then be used to sample from the posterior distribution of the parameters. The advantage of normalizing flow is that it can capture complex dependencies in the data and can be trained efficiently using stochastic gradient descent. Normalizing flow has been shown to be effective in a wide range of problems, including density estimation, generative modeling, and variational inference.

Radev et al. (2023) developed a more advanced method that amortized the estimation process. That is for any given observations, without the need to re-train the model, we can obtain the posterior distribution of the parameters simply by evaluating the trained NN. At its core, Radev et al. (2023) includes three neural networks: a summary network for encoding data into embeddings, a posterior network for amortized posterior inference, and a likelihood network to approximate likelihood functions. This combined approach allows for efficient estimation of marginal likelihoods and posterior predictive distributions, both of which are crucial for Bayesian model validation and comparison. Ultimately, Radev et al. (2023) represents a flexible and fully amortized approach that enhances the efficiency and practicality of Bayesian inference, making it suitable for complex probabilistic models and

computationally demanding simulations.

When we estimate heterogeneous agent models, we would like to include the cross section distributions such as the distribution of wealth, consumption, and saving. The traditional method proposed by Herbst and Schorfheide (2016) is to linearize the model around its steady state and then express the likelihood function. However, this method is not feasible in this case because the likelihood function is not available. The amortized inference method allows us to include the cross section distribution into the estimation. We first pass the data through a summary network to obtain the summary statistics. The summary statistics are then input into the posterior network to obtain the posterior distribution of the parameters.

However, in the original Radev et al. (2023) paper, they did not discuss identification issues. In the context of the economic models, the identification issue is very important. There could be two reasons that we may not be able to identify some parameters. First, the likelihood function could be very flat, therefore, we could not distinguish between different parameters. Second, the data we have may not be informative enough to identify the parameters. Identification is out of the scope of this paper, however, it is an important issue that needs to be addressed in the future research.

The following algorithm 2 shows the training process of the SBI method. The algorithm is based on the original paper by Radev et al. (2023).

Initialize the prior $p(\theta)$ and function $f_\Phi()$;

while *Not Converge* **do**

Sample $\{\theta_i\}_{i=1}^{i=N}$;

Simulate data $\{X_i\}_{i=1}^{i=N}$ from the model using $\{\theta_i\}_{i=1}^{i=N}$;

Pass $\{\theta_i, X_i\}_{i=1}^{i=N}$ through $f_\Phi()$ to get $\{z_i\}_{i=1}^{i=N}$;

Calculate the loss function Kullback-Leibler divergence between the empirical distribution of z and standard normal distribution;

Update Φ to minimize the loss function;

end

Algorithm 2: SBI: Training

The when inference, we can use the trained network to obtain the posterior distribution of the parameters. The following algorithm 3 shows the inference process of the SBI method.

Get the observed data x^o;

for *i in 1,2,3 ... N* **do**

Sample z_i from $\mathcal{N}_D(\mathbf{0}, \mathbb{I})$;

Pass $\{z_i; x^o\}$ through $f_\Phi^{-1}()$ to get θ_i;

end

Return $\{\theta_i\}_{i=1}^{i=N}$

Algorithm 3: SBI: Inference

5.3 Estimation Result

Figure 6 shows the posterior distribution of the parameters. The orange vertical line represents the true value that I used to generate the observation. The posterior distribution is centered around the true value, indicating that the estimation is accurate. The width of the posterior distribution reflects the uncertainty in the estimation. The narrower the distribution, the more certain we are about the parameter value. The posterior distribution can be used to make inferences about the parameters and to compute the posterior predictive distribution of the model.

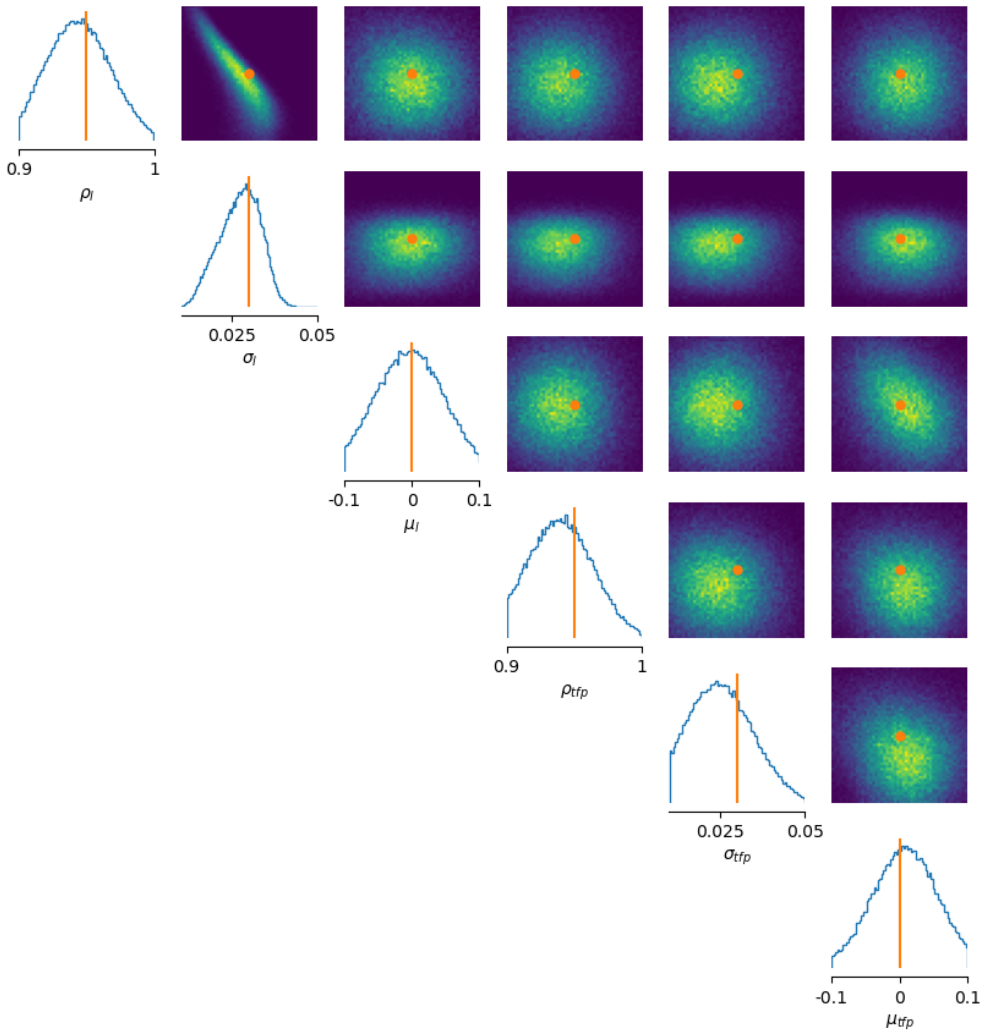

Figure 6: Posterior distribution of the parameters.

The orange vertical line represents the true parameter value that I used to generate the observation.

6 Conclusion

In this paper, I introduce a new pipeline for solving and estimating heterogeneous agent models. The pipeline combines the Reinforcement Learning (RL) method for solving the model and the Simulation-Based Inference (SBI) method for estimating it. The RL method is efficient, reliable, and flexible, capable of scaling up to models with hundreds or even

thousands of state variables. Moreover, it finds the global solution without additional assumptions. The SBI method is a powerful tool for estimating heterogeneous agent models by incorporating cross-sectional distributions of variables. Together, this pipeline provides a promising approach for solving and estimating heterogeneous agent models and can be applied to calibration, estimation, and further quantitative analysis.

The RL method has many potential applications. For example, it can be used to solve models with multiple assets and multiple shocks. Krusell and Smith (1997) extended the basic model to include a risk-free bond, where the mean return is zero. This scenario poses challenges for traditional methods like those proposed by Krusell and Smith (1998). If the model includes multiple, correlated asset classes, the complexity increases, even for the RL method, as the RL agent must learn the correlations between asset classes and additional elements such as bond prices. A potential solution to this complexity is multi-agent RL, where a portfolio manager agent manages multiple asset classes, and a broker agent determines the bond price.

When estimating the model, the first question to address is whether the parameters of interest can be identified. There are two primary reasons why some parameters may lack identification. First, the likelihood function may be very flat, making it difficult to distinguish between different parameter values. In other words, small changes in parameters may lead to minimal changes in the likelihood of observing the same data. Second, the data may not be informative enough to identify certain parameters. This could be due to noisy data or the absence of specific data points. Identification is a critical issue that should be further explored in future research.

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Appendix A. Benchmark

In this section, we use a simple RBC model to demonstrate that the performance of the RL approach. The RBC model is a standard model in macroeconomics that is used to study the business cycle. The model consists of a representative agent who maximizes utility subject to a budget constraint. The agent chooses consumption and saving to maximize utility subject to the budget constraint. The agent's utility function is given by $U(c) = \log(c)$, where c is consumption. The agent's budget constraint is given by $c + k' = (1 - \delta)k + y$, where k is capital, k' is next period's capital, δ is the depreciation rate of capital. The agent's production function is given by $y = zk^{\alpha}$, where y is output, z is total factor productivity which follows an $AR(1)$ process, and α is the capital share of output. The agent's Euler equation is given by $\frac{1}{c} = \beta E\left[\frac{1}{c'}(1 - \delta + \alpha z k^{\alpha-1})\right]$, where β is the discount factor, c' is next period's consumption, and E is the expectation operator. The agent's capital accumulation decision is given by $k' = (1 - \delta)k - c$.

The golden rule of solve this class of simple model is using value function iteration. The value function iteration is a dynamic programming method that solves for the value function of the agent by iterating on the Bellman equation. The Bellman equation is given by $V(k, z) = \max_{c,k'} \log(c) + \beta E[V(k', z')|z]$. We first discrete the sate variable k around its steady state ,and discrete z using method proposed by Rouwenhorst (1995). Then we iterate on the Bellman equation until convergence. The policy function is then obtained by taking the argmax of the Bellman equation. The policy function is the optimal decision rule of the agent. Value function iteration is a standard method in macroeconomics that is used to solve dynamic stochastic general equilibrium (DSGE) models. It is guaranteed to find the global solution to the model. However, it is computationally expensive and can be slow for large models.

We discrete k into 3000 grid points and z into 31 grid points. We measure the performance of the solution method using the mean squared error of the euler equation. The mean squared error of the euler equation is given by $\frac{1}{N}\sum_{i=1}^{N}\left(\frac{1}{c_i} - \beta E\left[\frac{1}{c'}(1 - \delta + \alpha z k^{\alpha-1})\right]\right)^2$, where N is the number of grid points, c_i is the consumption decision of the agent at grid point i, and c' is the next period's consumption decision. The mean squared error of the euler equation is a measure of how well the policy function satisfies the euler equation. The lower the mean squared error, the better the policy function. The MSE of the value function iteration is around 10^{-7}. The RL method I proposed in this paper has a MSE of around 10^{-}, which is in the same ballpark to the value function iteration method. Value function iteration is guaranteed to find the global solution to the model, therefore, in terms of accuracy the RL method is as good as the value function iteration method. However, the RL method is much faster and more efficient (less memory usage) than the value function iteration method.