9. Proximal Gradient Methods

Proximal Gradient Methods

Proximal Gradient Methods

Proximal gradient methods are a class of iterative techniques used to solve non-differentiable convex optimization problems given by:

$$\min_{\boldsymbol{\theta}} f(\boldsymbol{\theta}) + \lambda \cdot g(\boldsymbol{\theta})$$

where $f: \mathbb{R}^n \to \mathbb{R}$ is a convex differentiable function and $g: \mathbb{R}^n \to \mathbb{R}$ is a continuous convex possibly non-smooth function.

Proximal gradient methods produce a sequence of parameter iterates given by:

$$\boldsymbol{\theta}^{(t+1)} = \mathrm{Prox}_{\frac{\lambda}{L} \cdot g}\left(\boldsymbol{\theta}^{(t)} - \frac{1}{L} \cdot \nabla f\left(\boldsymbol{\theta}^{(t)}\right)\right)$$

where $L > 0$ is an upper bound on the Lipschitz constant of ∇f and $\mathrm{Prox}_{\mu \cdot g}$ is the proximal operator given by

$$\mathrm{Prox}_{\mu \cdot g}(x) = \operatorname*{argmin}_{z} \{1/2 \cdot \|z - x\|^2 + \mu \cdot g(z)\}$$

NB: The proximal operator is a generalization of the projection operator onto a convex set.

Proximal Gradient Methods: Example

LASSO

The LASSO problem can be written as follows:

$$\min_{\boldsymbol{\theta}} \frac{1}{2} \cdot \|\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\theta}\|_2^2 + \lambda \cdot \|\boldsymbol{\theta}\|_1$$

where $\boldsymbol{X} \in \mathbb{R}^{n \times m}$ is the feature matrix, $\boldsymbol{y} \in \mathbb{R}^n$ is the predictions vector, and $\boldsymbol{\theta} \in \mathbb{R}^m$ is the unknown parameters vector

Note that the LASSO problem can also be written as follows:

$$\min_{\boldsymbol{\theta}} f(\boldsymbol{\theta}) + \lambda \cdot g(\boldsymbol{\theta})$$

where

$$f(\boldsymbol{\theta}) = 1/2 \cdot \|\boldsymbol{y} - \boldsymbol{X}\boldsymbol{\theta}\|_2^2$$

is a convex differentiable function and

$$\mathrm{g}(\boldsymbol{\theta}) = \lambda \cdot \|\boldsymbol{\theta}\|_1$$

is a continuous convex non-smooth function.

Proximal Gradient Methods: Example

LASSO Proximal Operator

The LASSO proximal operator is given by:

$$\mathrm{Prox}_{\mu \cdot g}(x) = \underset{z}{\mathrm{argmin}}\{1/2 \cdot \|z - x\|^2 + \mu \cdot \|z\|_1\}$$

where $\mu = \lambda / L$.

The calculation of the proximal operator involves finding the gradient or subgradient *f* and *g* resp.

Recall

Subderivative of $|x|$

Let

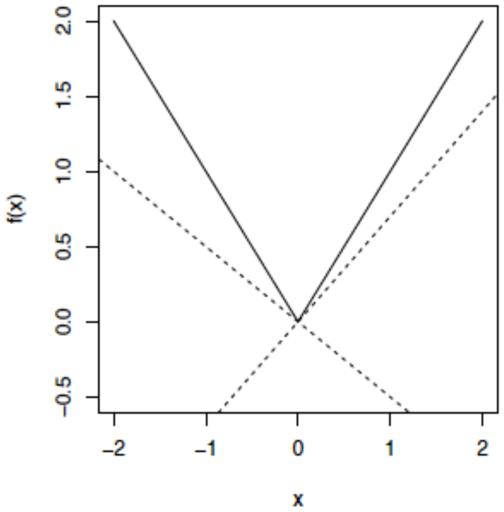
$$f(x) = |x| = \begin{cases} +x & x > 0 \\ -x & x < 0 \\ 0 & x = 0 \end{cases}$$

For $x \neq 0$, the derivative of $f(x) = |x|$ is given by:

$$\frac{df(x)}{dx} = sign(x)$$

For $x = 0$, the subderivative of $f(x) = |x|$ is given by:

$$[-1, +1]$$



$|\ x\ |$

Recall

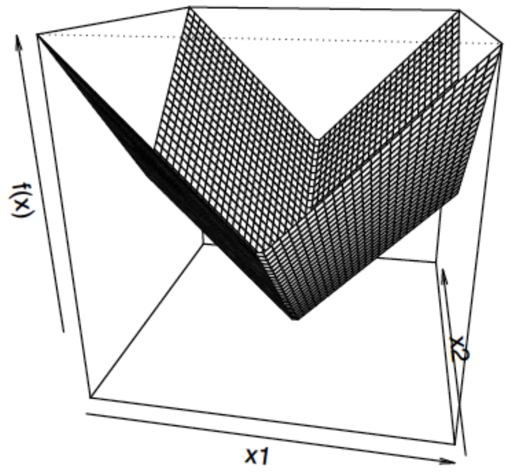
Subgradient of $\|x\|_1$

Let

$$f(\boldsymbol{x}) = \|\boldsymbol{x}\|_1 = |\boldsymbol{x}|_1 + |\boldsymbol{x}|_2 + \cdots + |\boldsymbol{x}|_n$$

For $x_i \neq 0$, the i-th component of the gradient equals $sign(x_i)$

For $x_i = 0$, the i-th component of the subgradient is any element of $[-1, +1]$



$\|x\|_1$

Proximal Gradient Methods: Example

LASSO Proximal Operator

The LASSO proximal operator is given by:

$$\mathrm{Prox}_{\mu \cdot g}(x) = \underset{z}{\mathrm{argmin}}\{1/2 \cdot \|z - x\|^2 + \mu \cdot \|z\|_1\}$$

where $\mu = \lambda / L$.

The proximal operator reduces to

$$\mathrm{Prox}_{\mu \cdot g}(x_i) = \begin{cases} x_i + \mu & x_i < -\mu \\ 0 & -\mu \le x_i \le +\mu \, , \\ x_i - \mu & x_i > +\mu \end{cases} \qquad i = 1, \dots, n$$

or equivalently to

$$\mathrm{Prox}_{\mu \cdot g}(x_i) = \max(0, 1 - \mu / |x_i|) \cdot x_i, \qquad i = 1, \dots, n$$

Iterative Soft Thresholding Algorithm (ISTA)

LASSO Problem

Recall the LASSO problem can also be written as follows:

$$\min_{\boldsymbol{\theta}} f(\boldsymbol{\theta}) + \lambda \cdot g(\boldsymbol{\theta})$$

where

$$f(\boldsymbol{\theta}) = 1/2 \cdot \|\boldsymbol{y} - \boldsymbol{X\theta}\|_2^2$$

is a convex differentiable function and

$$\mathrm{g}(\boldsymbol{\theta}) = \lambda \cdot \|\boldsymbol{\theta}\|_1$$

is a continuous convex non-smooth function.

ISTA Algorithm

Proximal gradient method produce a sequence of parameter iterates given by:

$$\boldsymbol{\theta}^{(t+1)} = \mathrm{Prox}_{\frac{\lambda}{L} \cdot g}\left(\boldsymbol{\theta}^{(t)} - \frac{1}{L} \cdot \nabla f\left(\boldsymbol{\theta}^{(t)}\right)\right)$$

Since$\nabla f(\boldsymbol{\theta}) = -\boldsymbol{X}^{\boldsymbol{t}} \cdot (\boldsymbol{y} - \boldsymbol{X\theta})$, the proximal gradient method also produces the sequence of parameter iterates given by:

$$\boldsymbol{\theta}^{(t+1)} = \mathrm{Prox}_{\frac{\lambda}{L} \cdot g}\left(\boldsymbol{\theta}^{(t)} + \frac{1}{L} \cdot \boldsymbol{X}^{\boldsymbol{t}} \cdot \left(\boldsymbol{y} - \boldsymbol{X\theta}^{(t)}\right)\right)$$