HW #2 Solutions

1) a)

(i) $a\begin{bmatrix}1\\0\\3\end{bmatrix} + b\begin{bmatrix}2\\1\\1\end{bmatrix} + c\begin{bmatrix}0\\5\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$

$$a + 2b = 0 \Rightarrow a = 2b$$
$$b + 5c = 0 \Rightarrow c = 1/5\, b$$
$$3a + b + c = 0 \Rightarrow 6b + b + 1/5\, b = 0$$
$$\hookrightarrow b = 0$$
$$\hookrightarrow c = 0,\ a = 0$$

The only solution is $a = b = c = 0$, so it is linearly independent.

(ii) Another way is to check determinant.

$$\begin{vmatrix} 4 & 3 & 2 \\ 5 & 1 & 8 \\ 1 & 2 & -2 \end{vmatrix} = 0 \Rightarrow \text{lin dependant}$$

Double check: $a = 2$ $b = -2$ $c = -1$

b) $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $A = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ $A^2 = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$

$$2 \cdot I + A - A^2 = 0$$

↳ lin dependant

The C-H Theorem say

$$\alpha_n I + \alpha_{n-1} A + \alpha_{n-2} A^2 = 0$$

so $\{I, A, A^2, \dots\}$ are always lin dependent

2) a)

$$\|x_1\|_1 = |2| + |-3| + |1| = 6$$

$$\|x_1\|_2 = \sqrt{2^2 + (-3)^2 + 1^2} = \sqrt{14}$$

$$\|x_1\|_\infty = \max(|2|, |-3|, |1|) = 3$$

$$\|x_2\|_1 = 1+1+1 = 3$$

$$\|x_2\|_2 = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

$$\|x_2\|_\infty = \max(1,1,1) = 1$$

b) Use Gram-Schmidt Normalization:

$$u_1 = x_1 \qquad q_1 = \frac{u_1}{\|u\|_2} = \frac{1}{\sqrt{14}}\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

$$u_2 = x_2 - (q_1^T x_2)\, q_1$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \left(\frac{1}{\sqrt{14}}\begin{bmatrix} 2 & -3 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right)\frac{1}{\sqrt{14}}\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$q_2 = \frac{u_2}{\|u_2\|} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Orthonormal basis:

$$q_1 = \frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \qquad q_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

3)

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 4 & 1 & -1 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

a) $\text{Rank}(A_1) = 2$ (# of lin ind. columns)

$\text{Nullity}(A_1) = \dim - \text{Rank}(A_1) = 3 - 2 = 1$

$\text{Rank}(A_2) = 3$

$\text{Nullity}(A_2) = 3 - 3 = 0$

b) for A_1 columns 2 and 3 are lin. ind., so

Range space $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Null space $= \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$

for A_2, all columns are lin. ind., so:

Range space $= \left\{ \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} \right\}$

$$\text{Null space} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \quad \dim = 0$$

c) $\|A_1\|_1 = \max\left(1+0+0,\ 0+0+0,\ 0+0+1\right)$

$\|A_1\|_1 = 1$

$\|A_1\|_2 = \sqrt{\max(\lambda(A^T A))}$

$$A^T A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \lambda = 0, 1$$

$\|A_1\|_2 = \sqrt{1}$

$\|A_1\|_\infty = \max(0+0+0,\ 0+1+0,\ 0+0+1)$

$\|A_1\|_\infty = 1$

· Same process for A_2:

$\|A_2\|_1 = 8 \quad \|A_2\|_2 \approx \sqrt{31.4} \approx 5.6 \quad \|A_2\|_\infty = 6$

4) a) $B_1 = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

e-values are $1, 2, 3$ since this is upper triangular. So

$$A_J = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

- determine e-vectors

1) $(1I - A)v_1 = 0 \implies v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

2) $(2I - A)v_2 = 0 \implies v_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$

3) $(3I - A)v_3 = 0 \implies v_3 = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$

$$T = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

b)

$$B_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -12 & 6 \end{bmatrix} \rightarrow$$ control canonical form!

$\downarrow$

C.E. is

$s^3 - 6s^2 + 12s - 8 = 0$

E-values: $|A - \lambda I| = 0$

$\lambda = 2$

E-vector:

$$(B - \lambda I) v_1 = 0 \Rightarrow \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 8 & -12 & 4 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = 0$$

$-2v_{11} + v_{12} = 0 \Rightarrow v_{12} = 2v_{11}$

$-2v_{12} + v_{13} = 0 \Rightarrow v_{13} = 2v_{12} = 4v_{11}$

$8v_{11} - 12v_{12} + 4v_{13} = 0$

$v_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$ only e-vector

$$(B-\lambda I)\, v_2 = v_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

$$\left.\begin{aligned} -2v_{11} + v_{12} &= 1 \\ -2v_{12} + v_{13} &= 2 \\ 8v_{11} - 12v_{12} + 4v_{13} &= 4 \end{aligned}\right\} \quad V_2 = \begin{bmatrix} -1/2 \\ 0 \\ 2 \end{bmatrix}$$

$$(B-\lambda I)\, v_3 = V_2 = \begin{bmatrix} -1/2 \\ 0 \\ 2 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} 1/4 \\ 0 \\ 0 \end{bmatrix}$$

3x3 Jordan Block

$$T = \begin{bmatrix} 1 & -1/2 & 1/4 \\ 2 & 0 & 0 \\ 4 & 2 & 0 \end{bmatrix} \qquad J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

5)

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$|A - \lambda I| = 0 \Rightarrow \lambda = 0, 1, 1$$

a) $f(x) = x^{10}$

Let $h(\lambda) = B_0 + B_1 \lambda + B_2 \lambda^2$

$f(0) = h(0)$: $0 = \beta_0$

$f(1) = h(1)$: $1 = \beta_0 + \beta_1 + \beta_2$

$f'(1) = h'(1)$: $10 = \beta_1 + 2\beta_2$

$\beta_0 = 0$, $\beta_1 = -8$, $\beta_2 = 9$

$$C^{10} = \beta_0 I + \beta_1 C + \beta_2 C^2$$

$$= -8 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + 9 \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

b) Same process as above

$\beta_0 = 0 \quad \beta_1 = -98 \quad \beta_2 = 99$

$$C^{100} = \begin{bmatrix} 1 & 1 & 99 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

c) $f(x) = e^{xt}$

$f(0) = h(0) : \quad \beta_0 = 1$

$f(1) = h(1) : \quad e^t = \beta_0 + \beta_1 + \beta_2$

$f'(1) = h'(1) : \quad te^t = \beta_1 + 2\beta_2$

$\beta_0 = 1 \qquad \beta_2 = te^t - e^t$

$\beta_1 = 2e^t - te^t - 1$

$$e^{Ct} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \left(2e^t - te^t - 1\right) \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$+ \left(te^t - e^t\right) \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$e^{Ct} = \begin{bmatrix} e^t & e^t - 1 & te^t - e^t + 1 \\ 0 & 1 & e^t - 1 \\ 0 & 0 & e^t \end{bmatrix}$$

Contents

Part a

```
A = [-1,1,0; -4,-2,-5;5,2,6];
B = [0; -1;1];

%Find cofficients for CE
alpha = charpoly(A);
alpha2 = alpha(2);
alpha1 = alpha(3);

% Build Q
q_3 = B;
q_2 = A*q_3+alpha2*q_3;
q_1 = A*q_2+alpha1*B;
Q = horzcat(q_1,q_2,q_3);

%Determine transformation matrix
T = inv(Q)
```

```
T =

     1     1     1
     0     1     1
     1     0     1
```

Part b

```
A_con = T*A*inv(T)
B_con = T*B
```

```
A_con =

     0     1     0
     0     0     1
     1     2     3


B_con =

     0
     0
     1
```

Part c

```
%Desired CE
syms s k1 k2 k3
a=s+2;
out = expand(a^3)

% Calculate the closed loop system
charpoly(A_con + B_con*[k1 k2 k3])

% Calculate the gains
k_3_num = -3-6;
k_2_num = -12-2;
k_1_num = -1-8;

%Double check
K_con = [k_1_num k_2_num k_3_num]
charpoly(A_con + B_con*K_con)
```

```
out =

s^3 + 6*s^2 + 12*s + 8


ans =

[1, - k3 - 3, - k2 - 2, - k1 - 1]


K_con =

    -9   -14    -9


ans =

     1     6    12     8
```

Part d

```
%Get actual gains
K = K_con*T
```

```
K =

   -18   -23   -32
```

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