HW4

Problem 1

(a) Compute controllability matrix

$$P = [B \,\vdots\, AB \,\vdots\, A^2B \,\vdots\, A^3B]$$

$$= \left[\begin{array}{cc:cc:cc:cc} 0 & 0 & 1 & 0 & 0 & 2\omega & -\omega^2 & 2\omega \\ 1 & 0 & 0 & 2\omega & -\omega^2 & 2\omega & -4\omega^2 & -2\omega^3+2\omega \\ 0 & 0 & 0 & 1 & -2\omega & 1 & -2\omega & 1-4\omega^2 \\ 0 & 1 & -2\omega & 1 & -2\omega & 1-4\omega^2 & 2\omega^3-2\omega & 1-8\omega^2 \end{array}\right]$$

Since $\text{rank}(P) = 4 = n$, so it's controllable from u.

(b) When the radial thruster fails, $u = [\cancel{u_1}, u_2]^T$, B matrix becomes to $b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, we can compute controllability matrix again:

$$P = [b \,\vdots\, Ab \,\vdots\, A^2b \,\vdots\, A^3b]$$

$$= \left[\begin{array}{c:c:c:c} 0 & 0 & 2\omega & 2\omega \\ 0 & 2\omega & 2\omega & -2\omega^3+2\omega \\ 0 & 1 & 1 & 1-4\omega^2 \\ 1 & 1 & 1-4\omega^2 & 1-8\omega^2 \end{array}\right], \quad \text{rank}(P) = 4 \Rightarrow \text{controllable}$$

When the tangential thruster fails, $u = [u_1, \cancel{u_2}]^T$, B matrix becomes to $b = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, we compute controllability matrix again:

$$P = [b \,\vdots\, Ab \,\vdots\, A^2b \,\vdots\, A^3b]$$

$$= \left[\begin{array}{c:c:c:c} 0 & 1 & 0 & -\omega^2 \\ 1 & 0 & -\omega^2 & -4\omega^2 \\ 0 & 0 & -2\omega & -2\omega \\ 0 & -2\omega & -2\omega & 2\omega^3-2\omega \end{array}\right], \quad \text{rank}(P) = 3 < n \Rightarrow \text{uncontrollable}$$

Problem 2

① Check controllability

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ e^{-t} \end{bmatrix} u$$

$$\begin{cases} \dot{x}_1 = 0 \\ \dot{x}_2 = -x_2 \end{cases} \Rightarrow \begin{cases} x_1 = x_1(0) \\ x_2 = x_2(0) \cdot e^{-t} \end{cases}$$

Pick $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as initial conditions, solutions are

$$X(t) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-t} \end{bmatrix}, \quad X^{-1}(t) = \begin{bmatrix} 1 & 0 \\ 0 & e^{t} \end{bmatrix}$$

$\Rightarrow$ state transition matrix

$$\Phi(t_0, \tau) = X(t_0) X^{-1}(\tau) = \begin{bmatrix} 1 & 0 \\ 0 & e^{-t_0} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{\tau} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{\tau - t_0} \end{bmatrix}$$

Then, compute the controllability Gramian

$$G_c(t_0, t_1) = \int_{t_0}^{t_1} \Phi(t_0, \tau) B(\tau) B^T(\tau) \Phi^T(t_0, \tau) \, d\tau$$

$$= \int_{t_0}^{t_1} \begin{bmatrix} 1 & 0 \\ 0 & e^{\tau - t_0} \end{bmatrix} \begin{bmatrix} 1 \\ e^{-\tau} \end{bmatrix} \begin{bmatrix} 1 & e^{-\tau} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{\tau - t_0} \end{bmatrix} d\tau$$

$$= \int_{t_0}^{t_1} \begin{bmatrix} 1 \\ e^{-t_0} \end{bmatrix} \begin{bmatrix} 1 & e^{-\tau} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{\tau - t_0} \end{bmatrix} d\tau = \int_{t_0}^{t_1} \begin{bmatrix} 1 & e^{-\tau} \\ e^{-t_0} & e^{-\tau - t_0} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{\tau - t_0} \end{bmatrix} d\tau$$

$$= \int_{t_0}^{t_1} \begin{bmatrix} 1 & e^{-t_0} \\ e^{-t_0} & e^{-2t_0} \end{bmatrix} d\tau = (t_1 - t_0) \cdot \begin{bmatrix} 1 & e^{-t_0} \\ e^{-t_0} & e^{-2t_0} \end{bmatrix}$$

$\det(G_c) = (t_1 - t_0) \cdot (e^{-2t_0} - e^{-2t_0}) = 0$, thus $\operatorname{rank}(G_c) = 1 < 2$

$\Rightarrow$ not controllable

② Check observability

Similarly, compute the observability Gramian

$$G_o(t_0, t_1) = \int_{t_0}^{t_1} \Phi^T(\tau, t_0) C^T(\tau) C(\tau) \Phi(\tau, t_0)\, d\tau$$

$$= \int_{t_0}^{t_1} \begin{bmatrix} 1 & 0 \\ 0 & e^{t_0-\tau} \end{bmatrix} \begin{bmatrix} 0 \\ e^{-\tau} \end{bmatrix} \begin{bmatrix} 0 & e^{-\tau} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{t_0-\tau} \end{bmatrix} d\tau$$

$$= \int_{t_0}^{t_1} \begin{bmatrix} 0 \\ e^{t_0-2\tau} \end{bmatrix} \begin{bmatrix} 0 & e^{-\tau} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{t_0-\tau} \end{bmatrix} d\tau = \int_{t_0}^{t_1} \begin{bmatrix} 0 & 0 \\ 0 & e^{t_0-3\tau} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{t_0-\tau} \end{bmatrix} d\tau$$

$$= \int_{t_0}^{t_1} \begin{bmatrix} 0 & 0 \\ 0 & e^{2t_0-4\tau} \end{bmatrix} d\tau = \begin{bmatrix} 0 & 0 \\ 0 & e^{2t_0}\left(-\frac{1}{4}\right)\left(e^{-4t_1} - e^{-4t_0}\right) \end{bmatrix}$$

$\operatorname{rank}(G_o) = 1 < 2 \Rightarrow$ not observable

Problem 3

(a) (b)

For system 1

$$G_1(s) = C(sI-A)^{-1}B + D = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} s-2 & -1 & -2 \\ 0 & s-2 & -2 \\ 0 & 0 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 0 = \frac{1}{(s-2)^2}$$

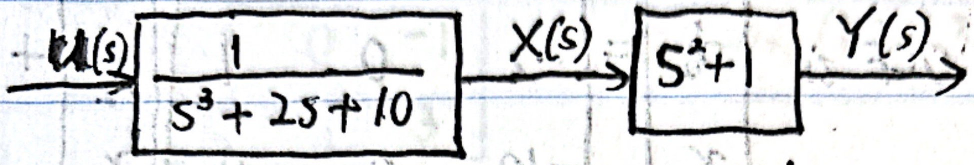
For system 2

$$G_2(s) = C(sI-A)^{-1}B + D = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} s-2 & -1 & -1 \\ 0 & s-2 & -1 \\ 0 & 0 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 0 = \frac{1}{(s-2)^2}$$

Therefore, (a) and (b) $\Rightarrow$ Yes!

Problem 4

(a) $G(s) = \dfrac{s^2+1}{s^3+2s+10} = \dfrac{Y}{U}$



$\hookrightarrow X = \dfrac{1}{s^3+2s+10} \cdot U$ $\qquad \hookrightarrow y = \ddot{x} + x$

$\Rightarrow x^{(3)} + 2\dot{x} + 10x = u$

$x^{(3)} = -10x - 2\dot{x} + u$

Set $x_1 = x$, $x_2 = \dot{x}$, $x_3 = \ddot{x}$

Then $\dot{x}_3 = -10x_1 - 2x_2 + u$

$y = x_1 + x_3$

The CCF will be

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

(b) $G(s) = \dfrac{s^2+1}{s^3+2s+10} \Rightarrow \dddot{y} + 2\dot{y} + 10y = \ddot{u} + u$

Set $y = x_1$

Then $\begin{cases} \dot{x}_1 = x_2 + u \\ \dot{x}_2 = -2x_1 + x_3 \\ \dot{x}_3 = -10x_1 + u \end{cases}$

Therefore, OCF is
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -2 & 0 & 1 \\ -10 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Problem 5

(a) $P = [b \,\vdots\, Ab \,\vdots\, A^2 b]$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix}, \quad \text{rank}(P) = 3 = n \Rightarrow \text{controllable}$$

$$Q = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix}, \quad \text{rank}(Q) = 1 < n \Rightarrow \text{not observable}$$

(b) Since the system is controllable, we don't need to do decomposition by controllability. Furthmore, since no uncontrollable part, we don't need to decompose that part based on observability either.

The only thing left is decomposing the controllable part by observability.

Construct similarity transformation matrix

$$T^{-1} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow T = \begin{bmatrix} 1 & -1 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Then,

$$\dot{\hat{x}} = T^{-1}AT\hat{x} + T^{-1}Bu = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -3 & -3 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ -1 & -2 & -1 \\ 0 & 1 & 0 \end{bmatrix} \hat{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \qquad y = [1 \;\; 2 \;\; 1] \begin{bmatrix} 1 & -1 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \hat{x} = [1 \;\; 0 \;\; 0]\,\hat{x}$$

i.e.

$$\begin{bmatrix} \dot{x}_{co} \\ \dot{x}_{c\bar{o}} \end{bmatrix} = \left[\begin{array}{c|cc} -1 & 0 & 0 \\ \hline -1 & -2 & -1 \\ 0 & 1 & 0 \end{array}\right] + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u \qquad y = [1 \,\vdots\, 0 \;\; 0] \begin{bmatrix} x_{co} \\ x_{c\bar{o}} \end{bmatrix}$$

Problem 6

(a) Check controllability:

$$P = [B \quad AB \quad A^2B \quad A^3B] = \begin{bmatrix} 0 & 1 & 4 & 12 \\ 1 & 2 & 4 & 8 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & 1 & -1 \end{bmatrix}, \quad rank(P) = 3 < 4 \Rightarrow \text{uncontrollable}$$

Eigenvalues:

$$|\lambda I - A| = \begin{vmatrix} \lambda-2 & -1 & 0 & 0 \\ 0 & \lambda-2 & 0 & 0 \\ 0 & 0 & \lambda+1 & 0 \\ 0 & 0 & 0 & \lambda+1 \end{vmatrix} = (\lambda-2)^2(\lambda+1)^2$$

$\Rightarrow \lambda_{1,2} = 2, \quad \lambda_{3,4} = -1$

The uncontrollable mode is $\lambda = 2$, it can not be shifted to -2 by state feedback. Therefore, we can't do anything to place the system to desired eigenvalues $(-2, -2, -1, -1)$, $(-2, -2, -2, -1)$ or $(-2, -2, -2, -2)$.

(b) Since $\lambda = 2$ is uncontrollable, the system is not stabilizable.