HW2

Problem 1

(a) Since there are 6 vectors in $\mathbb{R}^3$ (can be represented by 3 linear independent basis), they must be linearly dependent.

(b) $\{I, A, A^2\}$ is not a linearly independent set in $\mathbb{R}^{2\times 2}$.

Cayley-Hamilton Theorem says that a matrix satisfies its own C.E.

$$\Delta s = \det(sI - A) = \begin{vmatrix} s & -2 \\ -1 & s-1 \end{vmatrix} = s(s-1)-2 = s^2-s-2=0$$

$$\Rightarrow A^2 - A - 2I = 0 \quad , \text{ i.e. } A^2 = A+2I$$

which shows A^2 can be written as a linear combination of A and I, so they are not linearly independent.

Problem 2

(a) 1-norm:

$$\|x_1\|_1 = \sum_{i=1}^{n} |x_{1i}| = 2+3+1 = 6 \qquad \|x_2\|_1 = \sum_{i=1}^{n} |x_{2i}| = 1+1+1 = 3$$

2-norm:

$$\|x_1\|_2 = \sqrt{2^2+(-3)^2+1^2} = \sqrt{14} \qquad \|x_2\|_2 = \sqrt{1^2+1^2+1^2} = \sqrt{3}$$

∞-norm:

$$\|x_1\|_\infty = \max_i |x_{1i}| = 3 \qquad \|x_2\|_\infty = \max_i |x_{2i}| = 1$$

(b) Applying the Gram-Schmidt process

① Normalize x_1

$$x_1 := \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} \Rightarrow q_1 := \frac{x_1}{\|x_1\|_2} = \frac{1}{\sqrt{14}} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

② Project x_2 onto x_1 and subtract

$$x_2 := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow u_2 := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \left(\left[\frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}, \frac{1}{\sqrt{14}}\right] \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) \cdot \begin{bmatrix} \frac{2}{\sqrt{14}} \\ \frac{-3}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow q_2 := \frac{u_2}{\|u_2\|_2} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Therefore, we find an orthonormal basis for the space spanned by x_1 and x_2

$$q_1 := \frac{1}{\sqrt{14}}\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \quad q_2 := \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

Problem 3

(a) Since the second row of A_1 is all zeros, $\text{rank}(A_1) = 2$.

Nullity of A_1 is $\nu(A_1) = 3 - 2 = 1$.

Perform row reduction to A_2

$$A_2 = \begin{bmatrix} 4 & 1 & -1 \\ 3 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

All three rows are linearly independent, $\text{rank}(A_2) = 3$.

$\nu(A_2) = 3 - 3 = 0$.

(b) For A_1:

Basis for range space: $a_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $a_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Basis for null space: Solve for $A_1 x = 0$

$\Rightarrow n_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

For A_2:

Basis for range space: $a_1 = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$, $a_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $a_3 = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$

Since $\nu(A_2) = 0$, we can't find null space (or $n_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$)

(c) For A_1:

$\|A_1\|_1 = |1| + |0| + |0| = 1$

$$|\lambda I - A_1^T A_1| = \begin{vmatrix} \lambda & 0 & 0 \\ 0 & \lambda-1 & 0 \\ 0 & 0 & \lambda-1 \end{vmatrix} = \lambda(\lambda-1)^2 \Rightarrow \lambda_{max} = 1 \quad \text{Thus, } \|A_1\|_2 = \sqrt{1} = 1$$

$\|A_1\|_\infty = |0| + |1| + |0| = 1$

For A_2:

$\|A_2\|_1 = |4| + |3| + |1| = 8$

$$|\lambda I - A_2^T A_2| = \begin{vmatrix} \lambda-26 & -11 & 4 \\ -11 & \lambda-6 & 1 \\ 4 & 1 & \lambda-1 \end{vmatrix} = \lambda^3 - 33\lambda^2 + 50\lambda - 1 = 0 \Rightarrow \lambda_{max} = 31.409 \quad \text{Thus, } \|A_2\|_2 = \sqrt{31.409} \approx 5.604$$

$\|A_2\|_\infty = |4| + |1| + |-1| = 6$

Problem 4

(a) $$|\lambda I - B_1| = \begin{vmatrix} \lambda-1 & -4 & -10 \\ 0 & \lambda-2 & 0 \\ 0 & 0 & \lambda-3 \end{vmatrix} = (\lambda-1)(\lambda-2)(\lambda-3) \Rightarrow \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$$

for $\lambda_1 = 1$, $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$; for $\lambda_2 = 2$, $v_2 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$; for $\lambda_3 = 3$, $v_3 = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$

Thus, $T = [v_1 \ v_2 \ v_3] = \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\tilde{B}_1 = T^{-1} B_1 T = \begin{bmatrix} 1 & -4 & -5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

(b) $|\lambda I - B_2| = \begin{vmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -8 & 12 & \lambda-6 \end{vmatrix} = \lambda^3 - 6\lambda^2 + 12\lambda - 8 = (\lambda-2)^3$

$\Rightarrow \lambda = 2$, multiplicity 3

for $\lambda_1 = 2$,

$$[B_2 - \lambda_1 I]v_1 = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 8 & -12 & 4 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_1 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

for $\lambda_2 = 2$,

$$[B_2 - \lambda_2 I]v_2 = v_1 \Rightarrow \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 8 & -12 & 4 \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = \begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \Rightarrow v_2 = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix}$$

for $\lambda_3 = 2$,

$$[B_2 - \lambda_3 I]v_3 = v_2 \Rightarrow \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 8 & -12 & 4 \end{bmatrix} \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix} = \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 4 \end{bmatrix} \Rightarrow v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Thus, $T = [v_1 \ \ v_2 \ \ v_3] = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 4 & 1 \end{bmatrix}$

$$\tilde{B}_2 = T^{-1} B_2 T = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -4 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -12 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

Problem 5

(a) C.E. $|sI - C| = \begin{vmatrix} s-1 & -1 & 0 \\ 0 & s & -1 \\ 0 & 0 & s-1 \end{vmatrix} = s(s-1)^2$, and $f(\lambda) = \lambda^{10}$

$\Rightarrow s_1 = 0$, $s_2 = 1$ of multiplicity 2. Set $h(\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2$

for $s_1 = 0$, we have $f(0) = h(0)$

$\Rightarrow 0^{10} = \beta_0 \Rightarrow \beta_0 = 0$

for $s_2 = 1$, we have $\begin{cases} f(1) = h(1) \\ f'(1) = h'(1) \end{cases} \Rightarrow \begin{cases} 1^{10} = \beta_0 + \beta_1 + \beta_2 \\ 10 \cdot 1^9 = \beta_1 + 2\beta_2 \cdot 1 \end{cases} \Rightarrow \begin{cases} \beta_1 = -8 \\ \beta_2 = 9 \end{cases}$

Thus, $h(\lambda) = -8\lambda + 9\lambda^2$

$$C^{10} = -8C + 9C^2 = -8\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + 9\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) Set $f(\lambda) = \lambda^{100}$

for $s_1 = 0$, we have $f(0) = h(0) \Rightarrow 0^{100} = \beta_0 \Rightarrow \beta_0 = 0$

for $s_2 = 1$, we have $\begin{cases} f(1) = h(1) \\ f'(1) = h'(1) \end{cases} \Rightarrow \begin{cases} 1^{100} = \beta_0 + \beta_1 + \beta_2 \\ 100 \cdot 1^{99} = \beta_1 + 2\beta_2 \cdot 1 \end{cases} \Rightarrow \begin{cases} \beta_1 = -98 \\ \beta_2 = 99 \end{cases}$

Thus, $h(\lambda) = -98\lambda + 99\lambda^2$

$$C^{100} = -98C + 99C^2 = -98\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + 99\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 99 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(c) Set $f(\lambda) = e^{\lambda t}$, $f'(\lambda) = te^{\lambda t}$

for $s_1 = 0$, we have $f(0) = h(0) \Rightarrow 1 = \beta_0$

for $s_2 = 1$, we have $\begin{cases} f(1) = h(1) \\ f'(1) = h'(1) \end{cases} \Rightarrow \begin{cases} e^t = \beta_0 + \beta_1 + \beta_2 \\ te^t = \beta_1 + 2\beta_2 \end{cases} \Rightarrow \begin{cases} \beta_1 = 2e^t - te^t - 2 \\ \beta_2 = te^t - e^t + 1 \end{cases}$

Thus, $h(\lambda) = 1 + (2e^t - te^t - 2)s + (te^t - e^t + 1)s^2$

$$e^{Ct} = \beta_0 I + \beta_1 C + \beta_2 C^2$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + (2e^t - te^t - 2)\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} + (te^t - e^t + 1)\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^t - te^t - 2 + 1 + te^t - e^t + 1 & 2e^t - te^t - 2 + te^t - e^t + 1 & te^t - e^t + 1 \\ 0 & 1 & 2e^t - te^t - 2 + te^t - e^t + 1 \\ 0 & 0 & 2e^t - te^t - 2 + 1 + te^t - e^t + 1 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & e^t - 1 & te^t - e^t + 1 \\ 0 & 1 & e^t - 1 \\ 0 & 0 & e^t \end{bmatrix}$$

Problem 6

(a) C.E.

$$|sI - A| = \begin{vmatrix} s+1 & -1 & 0 \\ 4 & s+2 & 5 \\ -5 & -2 & s-6 \end{vmatrix} = s^3 - 3s^2 - 2s - 1$$

Thus, we know $\alpha_0 = -1$, $\alpha_1 = -2$, $\alpha_2 = -3$

We can construct matrix $Q = [q_1\ q_2\ q_3]$,

which $q_3 = B = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$, $q_2 = AB + \alpha_2 B = \begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix} - 3\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$,

$$q_1 = A^2B + \alpha_2 AB + \alpha_1 B = \begin{bmatrix} -2 \\ -10 \\ 13 \end{bmatrix} - 3\begin{bmatrix} -1 \\ -3 \\ 4 \end{bmatrix} - 2\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow Q = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$T = Q^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

(b) $$\tilde{A} = TAT^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ -4 & -2 & -5 \\ 5 & 2 & 6 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

$$\tilde{B} = TB = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

(c) CCF:

$$\dot{\bar{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

Set $\bar{u} = k_1 \bar{x}_1 + k_2 \bar{x}_2 + k_3 \bar{x}_3$, plug it into CCF

$$\dot{\bar{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} \bar{x} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1+k_1 & 2+k_2 & 3+k_3 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix}$$

$$\det(sI - \tilde{A}) = \begin{vmatrix} s & -1 & 0 \\ 0 & s & -1 \\ -1-k_1 & -2-k_2 & s-3-k_3 \end{vmatrix} = s^3 - (3+k_3)s^2 - (2+k_2)s - (1+k_1) = 0$$

place all poles at -2

Since $(s+2)^3 = s^3 + 6s^2 + 12s + 8$

Compare $$\begin{cases} -(3+k_3) = 6 \\ -(2+k_2) = 12 \\ -(1+k_1) = 8 \end{cases} \Rightarrow \begin{cases} k_3 = -9 \\ k_2 = -14 \\ k_1 = -9 \end{cases}$$

Thus, $\bar{u} = \begin{bmatrix} -9 & -14 & -9 \end{bmatrix} \bar{x}$

(d) $\bar{x} = Tx$, $\bar{u} = K\bar{x}$

$$\Rightarrow u = KTx = \begin{bmatrix} -9 & -14 & -9 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} x = \begin{bmatrix} -18 & -23 & -32 \end{bmatrix} x$$

This is the control law for the original system