HW #1 Solutions

1) a) Linear, time-invariant

b) Nonlinear $\left(f(x_1+x_2) \neq f(x_1)+f(x_2) \right.$

time-invariant $\quad a \neq a+a$

c) Linear, time-invariant

d) Nonlinear $\left(x_1+x_2+5 \neq x_1+5+x_2+5 \right)$

time-invariant

e) Linear, time-varying

f) Nonlinear $\left(\ddot{y}_1 + a_1 u_1 y_1 = 0 \right.$

$+$

time-invariant $\quad \ddot{y}_2 + a_2 u_2 y_2 = 0$

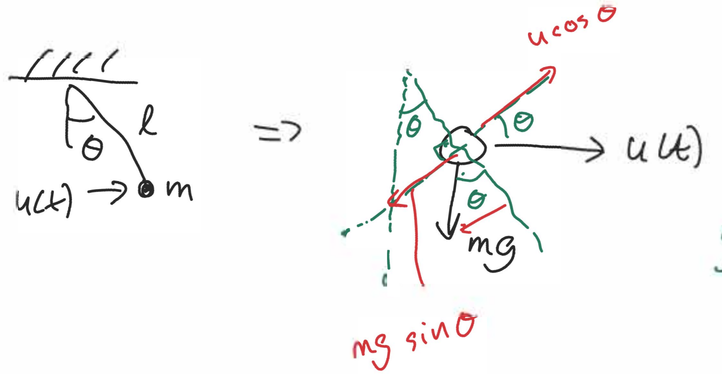
$$\ddot{y}_1 + \ddot{y}_2 + a_1 u_1 y_1 + a_2 u_2 y_2 = 0$$

is not the same as

$$\left. \ddot{y}_1 + \ddot{y}_2 + (a_1 u_1 + a_2 u_2)(y_1 + y_2) = 0 \right)$$

g) Linear, time-invariant

2) a)



$$\tau = I\ddot{\theta}$$

$$u(\cos\theta)\cdot l - mg\sin\theta \cdot l = ml^2\ddot{\theta}$$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta + \frac{u\cos\theta}{ml}$$

Let

$$x_1 = \theta \qquad x_2 = \dot{\theta}_2 \qquad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Then

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \ddot{\theta} = -\frac{g}{l}\sin x_1 + \frac{u\cos x_1}{ml}$$

small angle $\Rightarrow$ $\cos x_1 \simeq 1$, $\sin x_1 \simeq x_1$

$$\dot{x}_1 = x_2$$

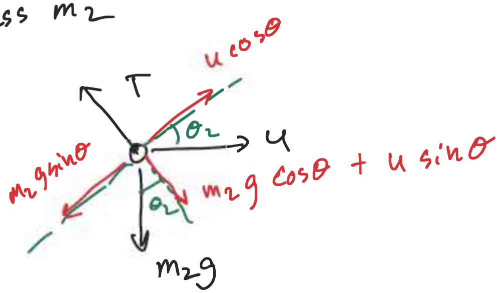
$$\dot{x}_2 = -\frac{g}{l} x_1 + \frac{u}{ml}$$

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -g/l & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1/ml \end{bmatrix} u$$

Linear.

b)

b) Mass m_2

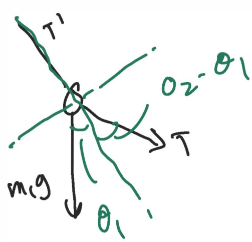


1) $u\cos\theta_2 l - m_2 g\sin\theta_2 l = m_2 l_2^2 \ddot{\theta}_2$

2) $T - m_2 g\cos\theta - u\sin\theta = 0$ ← not moving in this direction

$T = m_2 g\cos\theta_2 + u\sin\theta_2$

Mass M_1



1) $T\sin(\theta_2 - \theta_1) l_1 - m_1 g\sin\theta_1 l_1 = m_1 l_1^2 \ddot{\theta}_1$

(2) $T' = T\cos(\theta_2 - \theta_1)$ doesn't matter)

Let $x_1 = \theta_1$, $x_2 = \dot{\theta}_1$, $x_3 = \theta_2$, $x_4 = \dot{\theta}_2$

Then

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{g}{l}\sin x_1 + \frac{m_2 g}{m_1 l_1}\cos x_3 \sin(x_3 - x_1) + \frac{u}{m_1 l_1}\sin x_3 \sin(x_3 - x_1)$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = -\frac{g}{l_2}\sin x_3 + \frac{\cos x_3}{m_2 l_2}u$$

small angle $\cos(\theta_1) \approx 1 \approx \cos(\theta_2)$ $\quad \sin\theta_2 \approx \theta_2$

$$\sin(\theta_1) \approx \theta_1$$

$$\theta_1 \theta_2 \approx 0$$

Thus:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{g}{l_1 m_1}(m_1 + m_2) & 0 & \frac{m_2 g}{m_1 l_1} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{g}{l_2} & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{1}{m_2 l_2} \end{bmatrix} u$$

Linear.

3) Equilibrium points

$$\theta_0 = \dot{\theta}_0 = \ddot{\theta}_0 = 0 \qquad F_0 = 0$$

$$x_0 = \dot{x}_0 = \ddot{x}_0 = 0$$

Let

$$\theta = \theta_0 + \delta\theta$$

$$x = x_0 + \delta x$$

$$F = F_0 + \delta F$$

Then the first equation becomes

$$(m+M)(\ddot{x}_0 + \delta\ddot{x}) + ml(\ddot{\theta}_0 + \delta\ddot{\theta})\cos(\theta_0 + \delta\theta) - ml(\dot{\theta}_0 + \delta\dot{\theta})^2 \sin(\theta_0 + \delta\theta) = F_0 + \delta F$$

↳ Applying operating points and neglecting higher order terms

$$(m+M)\,\delta\ddot{x} + ml\,\delta\ddot{\theta} = \delta F$$

↳ Similarly, the second equation becomes

$$(I+ml^2)\,\delta\ddot{\theta} + ml\,\delta\ddot{x} - mgl\,\delta\theta = 0$$

↳ Rewriting both

$$\delta\ddot{x} = -\frac{ml}{m+M}\,\delta\ddot{\theta} + \frac{1}{m+M}\,\delta F \quad (1)$$

$$\delta\ddot{\theta} = -\frac{ml}{1+ml^2}\,\delta\ddot{x} + \frac{mgl}{1+ml^2}\,\delta\theta \quad (2)$$

↳ Substitude $\delta\ddot{\theta}$ into (1) and $\delta\ddot{x}$ into (2)

$$\delta\ddot{x} = -\frac{m^2l^2g}{mI+mI+mMl^2}\,\delta\theta + \frac{I+ml^2}{mI+MI+mMl^2}\,\delta F$$

$$\delta\ddot{\theta} = \frac{(m+M)\,mgl}{mI+MI+mMl^2}\,\delta\theta - \frac{ml}{mI+MI+mMl^2}\,\delta F$$

Let

$$x = \begin{bmatrix} \delta x \\ \delta\dot{x} \\ \delta\theta \\ \delta\dot{\theta} \end{bmatrix}$$

Then $\dot{x} = Ax + B\,\delta F$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{m^2 l^2 g}{mI + mI + mMl^2} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{(m+M)\,mgl}{mI + MI + mMl^2} & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ \dfrac{I + ml^2}{mI + MI + mMl^2} \\ 0 \\ -\dfrac{ml}{mI + MI + mMl^2} \end{bmatrix}$$

4. a) $\dot{x}_1 = \dot{p}_x \cos\theta + p_x \dot{\theta} \sin\theta$
$+ \dot{p}_y \sin\theta + (p_y - 1)\dot{\theta}\cos\theta$

$= v\cos^2\theta + v\sin^2\theta +$
$[(p_y - 1)\cos\theta - p_x \sin\theta]\,\dot{\theta}$

$= v + x_2 \omega$

$= u_1 + x_2 u_2$

$\dot{x}_2 = -\dot{p}_x \sin\theta - p_x \dot{\theta}\cos\theta$
$+ \dot{p}_y \cos\theta - (p_y - 1)\dot{\theta}\sin\theta$

$= \cancel{-v\cos\theta} - p_x \dot{\theta}\cos\theta$
$+ \cancel{v\cos\theta} - (p_x - 1)\dot{\theta}\sin\theta$

$= -x_1 \omega$

$\dot{x}_3 = \dot{\theta}$
$= \omega$
$= u_2$

$$\dot{x} = \begin{bmatrix} u_1 + x_2 u_2 \\ -x_1 u_2 \\ u_2 \end{bmatrix}$$

b) $\cos\theta \, x_2 + \sin\theta \, x_1 = (p_Y - 1)\sin^2\theta + (p_Y - 1)\cos^2\theta$

$= p_Y - 1$

$\hookrightarrow p_Y = x_2 \cos\theta + x_1 \sin\theta + 1$

$\cos\theta \, x_1 - \sin\theta \, x_2 = p_x \cos^2\theta + p_x \sin^2\theta$

$= p_x$

$\hookrightarrow p_x = x_1 \cos\theta - x_2 \sin\theta$

$$y = \begin{bmatrix} x_1 \cos\theta - x_2 \sin\theta \\ x_2 \cos\theta + x_1 \sin\theta + 1 \\ x_3 \end{bmatrix}$$

c) Solution:

$$p_x = \sin t \qquad v = 1$$

$$p_y = 1 - \cos t \qquad \omega = 1$$

$$\theta = t$$

Plug in

$$x_1 = p_x \cos\theta + (p_y - 1)\sin\theta$$

$$= \sin t \cos t + (1 - \cos t - 1)\sin t$$

$$= 0$$

$$x_2 = -p_x \sin\theta + (p_y - 1)\cos\theta$$

$$= -\sin^2 t + (1 - \cos t - 1)\cos t$$

$$= -1$$

$$x_3 = \theta = t$$

$$x = \begin{bmatrix} 0 \\ -1 \\ t \end{bmatrix}, \quad u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

↳ Check this is a solution to

$$\dot{x} = \begin{bmatrix} u_1 + x_2 u_2 \\ -x_1 u_2 \\ u_2 \end{bmatrix}$$

$$\frac{d}{dt} \begin{bmatrix} 0 \\ -1 \\ t \end{bmatrix} = \begin{bmatrix} 1 + (-1)\, 1 \\ - 0 \cdot 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \checkmark$$

d) Now we can linearize

$$A = \begin{bmatrix} 0 & u_2 & 0 \\ -u_2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{u_2^{sol} = 1}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & x_2 \\ 0 & -x_1 \\ 0 & 1 \end{bmatrix} \quad x_1^{sol} = 0, \; x_2^{sol} = -1$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \begin{bmatrix} \cos x_3 & -\sin x_3 & -x_1 \sin x_3 - x_2 \cos x_3 \\ \sin x_3 & \cos x_3 & x_1 \cos x_3 - x_2 \sin x_3 \\ 0 & 0 & 1 \end{bmatrix} \Bigg|_{\substack{x_1 = 0 \\ x_2 = -1 \\ x_3 = t}}$$

$$= \begin{bmatrix} \cos t & -\sin t & -\cos t \\ \sin t & \cos t & \sin t \\ 0 & 0 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

5. $A = \begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$

$C = [0 \quad 1]$ for example

a) Transfer function:

$$C(sI-A)^{-1}B + \cancel{D}^{\,0}$$

$$(sI-A)^{-1} = \begin{bmatrix} s & -1 \\ 5 & s \end{bmatrix}^{-1}$$

$$= \frac{1}{s^2+5} \begin{bmatrix} s & 1 \\ -5 & s \end{bmatrix}$$

$$C(sI-A)^{-1}B = \frac{1}{s^2+5} [0 \quad 1] \begin{bmatrix} s & 1 \\ -5 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$$

$$= \frac{1}{s^2+5} [-5 \quad s] \begin{bmatrix} 0 \\ 1/2 \end{bmatrix}$$

$$= \frac{0.5s}{s^2+5}$$

Contents

Problem 5, Part b

```
A = [0 1; -5 0];
B = [0; 1/2];
C = [0 1];
D = 0;

[n d] = ss2tf(A,B,C,D);

transfer_function = tf(n,d)
```

```
transfer_function =

   0.5 s
  -------
  s^2 + 5

Continuous-time transfer function.
```

Problem 5, Part b

```
[A_c, B_c, C_c, D_c] = tf2ss(n,d)

% The states are just flipped, with x1 = angular velocity and x2 = angular
% displacement.
%
% Please note that the B_c and C_C matrices are also different, but
% represent the same system.
```

```
A_c =

         0   -5.0000
    1.0000         0


B_c =

     1
     0


C_c =

    0.5000         0


D_c =
```

```
0
```

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