HW1

Problem 1

(a) linear, time-invariant

(b) $y(t_1+t_2)=K$, $y(t_1)+y(t_2)=K+K=2K$

$\because y(t_1+t_2) \neq y(t_1)+y(t_2)$

$\therefore$ nonlinear, time-invariant

(c) linear, time-invariant

(d) $\because x_1+5+x_2+5 \neq (x_1+x_2)+5$

$\therefore$ nonlinear, time-invariant

(e) $\because t^2 x_1 + t^2 x_2 = t^2(x_1+x_2)$

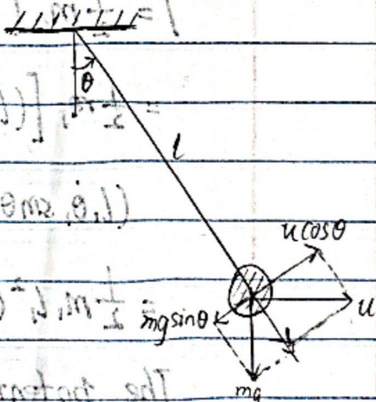
$\therefore$ linear, time-varying

(f) nonlinear, time-invariant

(g) linear, time-invariant

Problem 2

(a) $\begin{cases} (u\cos\theta - mg\sin\theta)l = I\ddot{\theta} \\ I = ml^2 \end{cases}$



$\Rightarrow (u\cos\theta - mg\sin\theta)l = ml^2\ddot{\theta}$

$\ddot{\theta} = -\frac{g}{l}\sin\theta + \frac{u}{ml}\cos\theta$

Define the states $X = \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix}$, then

$$\dot{X} = \begin{bmatrix} \dot{\theta} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} \dot{\theta} \\ -\frac{g}{l}\sin\theta + \frac{u}{ml}\cos\theta \end{bmatrix}$$

Define the output $y=\theta$, then $y = [1 \quad 0]X$

If θ is very small, we can linearize the model

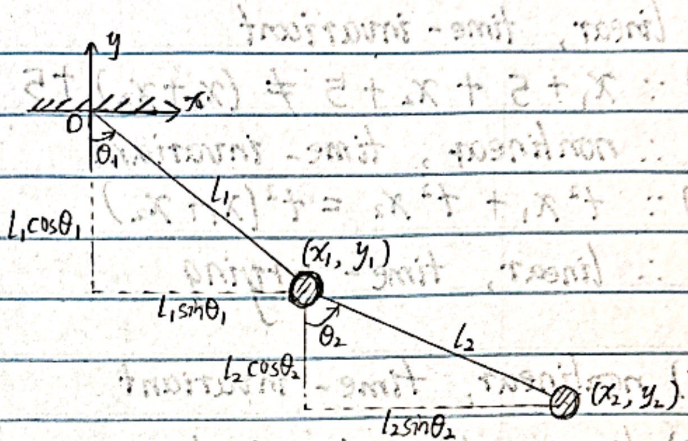
$$A = \frac{\partial f}{\partial X}\bigg|_{\theta=0} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\sin\theta - \frac{u}{ml}\sin\theta & 0 \end{bmatrix}\bigg|_{\theta=0} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u}\bigg|_{\theta=0} = \begin{bmatrix} 0 \\ \frac{1}{ml}\cos\theta \end{bmatrix}\bigg|_{\theta=0} = \begin{bmatrix} 0 \\ \frac{1}{ml} \end{bmatrix}$$

The linearized model is

$$\dot{X} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ \frac{1}{ml} \end{bmatrix} u$$

(b) Define the coordinate system as shown in the diagram, then



$$\begin{cases} x_1 = l_1\sin\theta_1 \\ y_1 = -l_1\cos\theta_1 \end{cases}$$

$$\begin{cases} x_2 = l_1\sin\theta_1 + l_2\sin\theta_2 \\ y_2 = -l_1\cos\theta_1 - l_2\cos\theta_2 \end{cases}$$

The kinetic energy is

$$T = \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2)$$

$$= \frac{1}{2}m_1\left[(l_1\dot{\theta}_1\cos\theta_1)^2 + (l_1\dot{\theta}_1\sin\theta_1)^2\right] + \frac{1}{2}m_2\left[(l_1\dot{\theta}_1\cos\theta_1 + l_2\dot{\theta}_2\cos\theta_2)^2 + (l_1\dot{\theta}_1\sin\theta_1 + l_2\dot{\theta}_2\sin\theta_2)^2\right]$$

$$= \frac{1}{2}m_1 l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left[l_1^2\dot{\theta}_1^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right]$$

The potential energy is:

$$V = -m_1gl_1\cos\theta_1 - m_2g(l_1\cos\theta_1 + l_2\cos\theta_2)$$

We can write down the Lagrangian

$$L = T - V$$

$$= \frac{1}{2}m_1l_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\left[l_1^2\dot{\theta}_1^2 + 2l_1l_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)\right] + m_1gl_1\cos\theta_1 + m_2g(l_1\cos\theta_1 + l_2\cos\theta_2)$$

According to Lagrange equation,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = Q \qquad q = \{\theta_1, \theta_2\}$$

$$\Rightarrow (m_1+m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 \cos(\theta_1-\theta_2) + m_2 l_1 l_2 \dot{\theta}_2^2 \sin(\theta_1-\theta_2) + (m_1+m_2) g l_1 \sin\theta_1 = 0$$

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 \cos(\theta_1-\theta_2) - m_2 l_1 l_2 \dot{\theta}_1^2 \sin(\theta_1-\theta_2) + m_2 g l_2 \sin\theta_2 = u l_2 \cos\theta_2$$

When θ_1 and θ_2 are very small,

$\sin\theta_1 \approx \theta_1$, $\sin\theta_2 \approx \theta_2$, $\cos(\theta_1-\theta_2) \approx 1$

we can linearize the equations above

$$\begin{cases} (m_1+m_2)l_1^2\ddot{\theta}_1 + m_2 l_1 l_2 \ddot{\theta}_2 + (m_1+m_2) g l_1 \theta_1 = 0 \\ m_2 l_2^2 \ddot{\theta}_2 + m_2 l_1 l_2 \ddot{\theta}_1 + m_2 g l_2 \theta_2 = u l_2 \end{cases}$$

$$\Rightarrow \begin{cases} \ddot{\theta}_1 = -\dfrac{(m_1+m_2)g}{m_1 l_1}\theta_1 + \dfrac{m_2 g}{m_1 l_1}\theta_2 - \dfrac{1}{m_1 l_1}u \\ \ddot{\theta}_2 = \dfrac{(m_1+m_2)g}{m_1 l_2}\theta_1 - \dfrac{(m_1+m_2)g}{m_1 l_2}\theta_2 + \dfrac{m_1+m_2}{m_1 m_2 l_2}u \end{cases}$$

Define the states $X = [\theta_1 \ \ \dot{\theta}_1 \ \ \theta_2 \ \ \dot{\theta}_2]^T$

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{(m_1+m_2)g}{m_1 l_1} & 0 & \dfrac{m_2 g}{m_1 l_1} & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{(m_1+m_2)g}{m_1 l_2} & 0 & -\dfrac{(m_1+m_2)g}{m_1 l_2} & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ -\dfrac{1}{m_1 l_1} \\ 0 \\ \dfrac{m_1+m_2}{m_1 m_2 l_2} \end{bmatrix} u$$

Problem 3

Linearize around the operating point, $\cos\theta \approx 1$, $\sin\theta \approx \theta$

$$\Rightarrow \begin{cases} (m+M)\ddot{x} + ml\ddot{\theta} - \cancel{ml\dot{\theta}^2\theta} = F \\ (I+ml^2)\ddot{\theta} + ml\ddot{x} - mgl\theta = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \ddot{x} = \dfrac{m^2 l^2 g}{m^2 l^2 - (I+ml^2)(M+m)}\theta - \dfrac{I+ml^2}{m^2 l^2 - (I+ml^2)(M+m)}F \\ \ddot{\theta} = -\dfrac{(M+m)mgl}{m^2 l^2 - (I+ml^2)(M+m)}\theta + \dfrac{ml}{m^2 l^2 - (I+ml^2)(M+m)}F \end{cases}$$

Define the states $X = [x \;\; \dot{x} \;\; \theta \;\; \dot{\theta}]$

$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{m^2 l^2 g}{m^2 l^2 - (I + ml^2)(M+m)} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\dfrac{(M+m) mg l}{m^2 l^2 - (I+ml^2)(M+m)} & 0 \end{bmatrix} X + \begin{bmatrix} 0 \\ -\dfrac{I + ml^2}{m^2 l^2 - (I + ml^2)(M+m)} \\ 0 \\ \dfrac{ml}{m^2 l^2 - (I+ml^2)(M+m)} \end{bmatrix} F$$

Problem 4

(a) Given the coordinate transformation

$$x \triangleq \begin{bmatrix} p_x\cos\theta + (p_y-1)\sin\theta \\ -p_x\sin\theta + (p_y-1)\cos\theta \\ \theta \end{bmatrix}$$

we can derive $\dot{x}_1$, $\dot{x}_2$, and $\dot{x}_3$

$$\begin{cases} \dot{x}_1 = \dot{p}_x\cos\theta - p_x\dot{\theta}\sin\theta + \dot{p}_y\sin\theta + (p_y-1)\dot{\theta}\cos\theta \\ \dot{x}_2 = -\dot{p}_x\sin\theta - p_x\dot{\theta}\cos\theta + \dot{p}_y\cos\theta - (p_y-1)\dot{\theta}\sin\theta \\ \dot{x}_3 = \dot{\theta} \end{cases}$$

Substitute the model into these equations,

$$\begin{cases} \dot{x}_1 = v\cos^2\theta - p_x\dot{\theta}\sin\theta + v\sin^2\theta + (p_y-1)\dot{\theta}\cos\theta = v + x_2 w \\ \dot{x}_2 = -v\cos\theta\sin\theta - p_x\dot{\theta}\cos\theta + v\sin\theta\cos\theta - (p_y-1)\dot{\theta}\sin\theta = -x_1 w \\ \dot{x}_3 = \dot{\theta} = w \end{cases}$$

Therefore,

$$\dot{x} = f(x,u) = \begin{bmatrix} v + x_2 w \\ -x_1 w \\ w \end{bmatrix}$$

(b) Solving $y = [p_x, p_y, w]^T$ according to the coordinate transformation

$$p_x\cos^2\theta + (p_y-1)\sin\theta\cos\theta = x_1\cos\theta \quad ①$$

$$-p_x\sin^2\theta + (p_y-1)\sin\theta\cos\theta = x_2\sin\theta \quad ②$$

$$① - ② \Rightarrow p_x\cos^2\theta + p_x\sin^2\theta = x_1\cos\theta - x_2\sin\theta$$

$$p_x = x_1\cos x_3 - x_2\sin x_3$$

$$p_x\cos\theta\sin\theta + (p_y-1)\sin^2\theta = x_1\sin\theta \quad ③$$

$$-p_x\sin\theta\cos\theta + (p_y-1)\cos^2\theta = x_2\cos\theta \quad ④$$

$$③ + ④ \Rightarrow (p_y-1)\sin^2\theta + (p_y-1)\cos^2\theta = x_1\sin\theta + x_2\cos\theta$$

$$p_y = 1 + x_1\sin x_3 + x_2\cos x_3$$

Therefore, y in the new coordinates

$$y = \begin{bmatrix} x_1 \cos x_3 - x_2 \sin x_3 \\ x_1 \sin x_3 + x_2 \cos x_3 + 1 \\ x_3 \end{bmatrix}$$

(c) Given the trajectory,

$p_x = \sin t, \quad p_y = 1 - \cos t, \quad \theta = t, \quad v = 1, \quad w = 1$

Compute the new coordinate

$$\begin{cases} x_1 = \sin t \cos t - \cos t \sin t = 0 \\ x_2 = -\sin t \sin t - \cos t \cos t = -1 \\ x_3 = t \end{cases}$$

Plug into the dynamics in (a)

$$\dot{x} = f(x, u) = \begin{bmatrix} 1 - 1 \cdot 1 \\ -0 \cdot 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Then, compute the exact derivative of x

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

which gives the same answer with the dynamics.

Thus, the trajectory is still a solution in the new coordinates.

(d) In (a) and (b), we already have

$$\dot{x} = \begin{bmatrix} v + x_2 w \\ -x_1 w \\ w \end{bmatrix} \qquad y = \begin{bmatrix} x_1 \cos x_3 - x_2 \sin x_3 \\ x_1 \sin x_3 + x_2 \cos x_3 + 1 \\ x_3 \end{bmatrix}$$

$$A = \frac{\partial f}{\partial x}\bigg|_{x_e, u_e} = \begin{bmatrix} 0 & w & 0 \\ -w & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\Bigg|_{x_e, u_e} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B = \frac{\partial f}{\partial u}\bigg|_{x_e, u_e} = \begin{bmatrix} 1 & x_2 \\ 0 & -x_1 \\ 0 & 1 \end{bmatrix}\Bigg|_{x_e, u_e} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$C = \frac{\partial g}{\partial x}\bigg|_{x_e, u_e} = \begin{bmatrix} \cos x_3 & -\sin x_3 & -x_1 \sin x_3 - x_2 \cos x_3 \\ \sin x_3 & \cos x_3 & x_1 \cos x_3 - x_2 \sin x_3 \\ 0 & 0 & 1 \end{bmatrix}\Bigg|_{x_e, u_e} = \begin{bmatrix} \cos t & -\sin t & \cos t \\ \sin t & \cos t & \sin t \\ 0 & 0 & 1 \end{bmatrix}$$

$$D = \frac{\partial g}{\partial u}\bigg|_{x_e, u_e} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The linearized model is

$$\delta\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\delta x + \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}\delta u$$

$$\delta y = \begin{bmatrix} \cos t & -\sin t & \cos t \\ \sin t & \cos t & \sin t \\ 0 & 0 & 1 \end{bmatrix}\delta x + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\delta u$$

Problem 5

(a) Given

$$A = \begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}, \quad C = [1 \quad 0], \quad D = 0$$

The transfer function

$$G(s) = C(sI-A)^{-1}B + D$$

$$sI - A = \begin{bmatrix} s & -1 \\ 5 & s \end{bmatrix}, \quad (sI-A)^{-1} = \frac{1}{s^2+5}\begin{bmatrix} s & 1 \\ -5 & s \end{bmatrix}$$

$$\Rightarrow G(s) = [1 \quad 0] \frac{1}{s^2+5} \begin{bmatrix} s & 1 \\ -5 & s \end{bmatrix} \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} + 0 = \frac{1}{2(s^2+5)}$$

(b) Using ss2tf, MATLAB also returns

$$G(s) = \frac{1}{2(s^2+5)}$$ (see appendix code)

(c) MATLAB tf2ss gives $C' = [0 \quad \frac{1}{2}]$, which implies it used $\dot{\theta}$ as output. (see appendix code)

```
clear;clc;close;

A = [0, 1;
     -5, 0];
B = [0; 1/2];
C = [1, 0];
D = 0;

[b, a] = ss2tf(A, B, C, D)

[A1, B1, C1, D1] = tf2ss(b, a)
```