HW#3 Solutions

1) a) $A = \begin{bmatrix} -1 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$$e^{At} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$$

$$\Phi(t,\tau) = \begin{bmatrix} e^{-(t-\tau)} & 0 \\ 0 & e^{-3(t-\tau)} \end{bmatrix}$$

$$x(t) = e^{A(t-t_0)} x_0$$

$$= \begin{bmatrix} e^{-(t-t_0)} & 0 \\ 0 & e^{-3(t-t_0)} \end{bmatrix} x_0$$

b) $A = \begin{bmatrix} -1 & 0 \\ 2 & -3 \end{bmatrix} = MJM^{-1}$

$$= \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -3 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-3t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t} & 0 \\ e^{-t} - e^{-3t} & e^{-3t} \end{bmatrix}$$

$$\Phi(t, \tau) = \begin{bmatrix} e^{-(t-\tau)} & 0 \\ e^{-(t-\tau)} - e^{-3(t-\tau)} & e^{-3(t-\tau)} \end{bmatrix}$$

$$x(t) = e^{A(t-t_0)} x_0$$

$$x(t) = \begin{bmatrix} e^{-(t-t_0)} & 0 \\ e^{-(t-t_0)} - e^{-3(t-t_0)} & e^{-3(t-t_0} \end{bmatrix} x_0$$

c) $\lambda_{1,2} = -3, -1$ and $Re(\lambda_i) < 0$, so both systems are stable (asymptotically).

2) Show $\frac{\partial \Phi(t_0,t)}{\partial t} = -\Phi(t_0,t)\cdot A(t)$.

$$\frac{\partial}{\partial t}\left(\Phi(t_0,t)\right) = \frac{d}{dt}\left(X(t_0)\, X^{-1}(t)\right)$$

$$= \cancel{\dot{X}(t_0)}^{\to 0}\, X^{-1}(t) + X(t_0)\frac{d}{dt}\left(X^{-1}(t)\right)$$

1 $$\frac{\partial}{\partial t}\left(\Phi(t_0,t)\right) = X(t_0)\frac{d}{dt}\left(X^{-1}(t)\right)$$

$$\frac{d}{dt}(I) = 0 = \frac{d}{dt}\left(X(t)\, X^{-1}(t)\right)$$

$$= \dot{X}(t)\, X^{-1}(t) + X(t)\, \dot{X}^{-1}(t)$$

$$X(t)\,\dot{X}^{-1}(t) = -\dot{X}(t)\, X^{-1}(t)$$

$$\dot{X}^{-1}(t) = -X^{-1}(t)\,\dot{X}(t)\, X^{-1}(t)$$

$$\dot{X}^{-1}(t) = -X^{-1}(t)\cdot A(t)\, \underbrace{\cancel{X(t)\, X^{-1}(t)}}_{I}$$

2 $$\dot{X}^{-1}(t) = -X^{-1}(t)\, A(t)$$

③ → ①

$$\frac{\partial}{\partial t}\left(\Phi(t_0,t)\right) = \underbrace{X(t_0)\left(-X^{-1}(t)\right.}_{-\Phi(t_0,t)}\left. A(t)\right)$$

$$= -\Phi(t_0,t)\cdot A(t)$$

b) $$\Phi(t,0) = \begin{bmatrix} \cos t^2 & \sin t^2 \\ -\sin t^2 & \cos t^2 \end{bmatrix}$$

$$\Phi(0,\tau) = \Phi^{-1}(\tau,0)$$
$$= \begin{bmatrix} \cos \tau^2 & -\sin \tau^2 \\ \sin \tau^2 & \cos \tau^2 \end{bmatrix}$$

$$\Phi(t,\tau) = \Phi(t,0)\,\Phi(0,\tau) =$$

$$= \begin{bmatrix} \cos t^2 \cos \tau^2 + \sin t^2 \sin \tau^2 & -\cos t^2 \sin \tau^2 + \sin t^2 \cos \tau^2 \\ -\sin t^2 \sin \tau^2 + \cos t^2 \cos \tau^2 & \sin t^2 \sin \tau^2 + \cos t^2 \cos \tau^2 \end{bmatrix}$$

$$= \begin{bmatrix} \cos(t^2-\tau^2) & \sin(t^2-\tau^2) \\ -\sin(t^2-\tau^2) & \cos(t^2-\tau^2) \end{bmatrix}$$

3) a) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & t \\ 0 & 1 \end{bmatrix} x$

① $\dot{x}_1 = t x_2$

② $\dot{x}_2 = x_2$

② $\int \frac{1}{x_2} dx_2 = \int dt \Rightarrow \ln x_2 = t + c$

$x_2 = x_2(0) e^t$

① $\dot{x}_1 = t x_2$

$\int \dot{x}_1 = \int t \, x_2(0) e^t$

$x_1 = x_2(0)(t e^t - e^t) + C$

$x_1(0) = x_2(0)(-1) + C \Rightarrow C = x_1(0) + x_2(0)$

$x_1 = x_2(0)(te^t - e^t) + x_1(0) + x_2(0)$

- Pick $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as I.C.

1) $x_1(0) = 1 \quad x_2(0) = 0$

$x_1 = 1$

$x_2 = 0$

2) $x_1(0) = 0 \quad x_2(0) = 1$

$x_1 = te^t - e^t + 1$

$x_2 = e^t$

· Fundamental Matrix

$$X(t) = \begin{bmatrix} 1 & te^t - e^t + 1 \\ 0 & e^t \end{bmatrix}$$

$$X^{-1}(t) = \frac{1}{e^t} \begin{bmatrix} e^t & -te^t + e^t - 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -t + 1 - e^{-t} \\ 0 & e^{-t} \end{bmatrix}$$

· State Transition Matrix

$$\Phi(t, t_0) = X(t)\, X^{-1}(t_0)$$

$$\Phi(t, t_0) = \begin{bmatrix} 1 & (te^t - e^t + 1)(-t_0 + 1 - e^{-t_0}) \\ 0 & e^t \cdot e^{-t_0} \end{bmatrix}$$

b.

$$\dot{x} = \begin{bmatrix} -\sin t & 0 \\ 0 & -\cos t \end{bmatrix} x$$

① $\dot{x}_1 = -\sin t \; x_1$

② $\dot{x}_2 = -\cos t \; x_2$

① $\dfrac{dx}{dt} = -\sin t \; x_1$

$$\int \frac{1}{x_1} dx = \int -\sin t \, dt$$

$$\ln x_1 = \cos t + c$$

$$x_1 = x_1(0) \, e^{\cos t}$$

② same reasoning as 1 $\quad x_2 = x_2(0) \, e^{-\sin(t)}$

Set I.C. to $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$.

$$x(0)=\begin{bmatrix}1\\0\end{bmatrix} \Rightarrow x(t)=\begin{bmatrix}e^{\cos t}\\0\end{bmatrix}$$

$$x(0)=\begin{bmatrix}0\\1\end{bmatrix} \Rightarrow x(t)=\begin{bmatrix}0\\e^{-\sin t}\end{bmatrix}$$

Fundamental matrix:

$$X(t)=\begin{bmatrix}e^{\cos t} & 0\\ 0 & e^{-\sin t}\end{bmatrix}$$

$$X^{-1}(t)=\begin{bmatrix}e^{-\cos t} & 0\\ 0 & e^{\sin t}\end{bmatrix}$$

state transition matrix:

$$\Phi(t, t_0) = X(t)\,X^{-1}(t_0)$$

$$\Phi(t, t_0)=\begin{bmatrix}e^{\cos t-\cos t_0} & 0\\ 0 & e^{\sin t_0-\sin t}\end{bmatrix}$$

4) c) $$\dot{x} = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x$$

$$|\lambda I - A| = \begin{vmatrix} \lambda+1 & 0 & -2 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} \Rightarrow \text{e-values are } -1, 0, 0$$

↳ NOT A.S.

E-vector for $\lambda = 0$

$$(\lambda I - A)v = 0$$

$$\begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} v = 0$$

Solution yields two independant e-vectors:

$$v_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

↳ stable is L

b)

$$\dot{x} = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$|\lambda I - A| = \begin{vmatrix} \lambda+1 & 0 & -2 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda \end{vmatrix} \Rightarrow \text{e-values are } -1, 0, 0$$

↳ NOT A.S.

For $\lambda = 0$:

$$(\lambda I - A)e = 0$$

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} v = 0$$

Solution yields only 1 e-vector:

$$v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \leftarrow \text{defective}$$

↳ NOT stable isL.

5)
$$\dot{x} = \begin{bmatrix} -1 & 5 \\ 0 & 2 \end{bmatrix} x + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u$$

$$y = [-2 \quad 4] x - 2u$$

$$G(s) = C(sI-A)^{-1} B + D$$

$$= [-2 \quad 4] \begin{bmatrix} s+1 & -5 \\ 0 & s-2 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} - 2$$

$$= [-2 \quad 4] \frac{1}{(s+1)(s-2)} \begin{bmatrix} s-2 & 5 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} - 2$$

$$= [-2 \quad 4] \begin{bmatrix} \frac{1}{s+1} & \frac{5}{(s+1)(s-2)} \\ 0 & \frac{1}{s-2} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} - 2$$

$$= -\frac{4}{s+1} - 2 = \frac{-2s-6}{s+1}$$

$\hookrightarrow$ e-value is $-1 < 0$, so BIBO stable.

6) $A^T P + PA - 2\lambda P = -Q$

$$A^T P - \lambda P + PA - \lambda P = -Q$$

$$(A^T - \lambda I) P + P(A - \lambda I) = -Q$$

$$(A - \lambda I)^T P + P(A - \lambda I) = -Q$$

Let $\tilde{A} = A - \lambda I$

$\tilde{A}^T P + P\tilde{A} = -Q$ satisfies lyapunov Equation

We know both A and $\tilde{A}$ have negative e-values

since e-values $(\tilde{A})$ = e-values $(A) - \lambda$:

$$Re(\lambda_i(\tilde{A})) = Re(\lambda_i(A)) - \lambda < 0$$

$\hookrightarrow$ $Re(\lambda_i(A)) < \lambda$.