HW3

Problem 1

(a) $\Phi_1(t) = e^{A_1 t} = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix}$

zero input response

$$x(t) = \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} x_0$$

(b) $(sI - A_2)^{-1} = \begin{bmatrix} s+1 & 0 \\ -2 & s+3 \end{bmatrix}^{-1} = \frac{1}{(s+1)(s+3)} \begin{bmatrix} s+3 & 0 \\ 2 & s+1 \end{bmatrix} = \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{2}{(s+1)(s+3)} & \frac{1}{s+3} \end{bmatrix}$

$$= \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{s+1} - \frac{1}{s+3} & \frac{1}{s+3} \end{bmatrix}$$

$$\mathcal{L}^{-1}\left((sI - A_2)^{-1}\right) = \begin{bmatrix} e^{-t} & 0 \\ e^{-t} - e^{-3t} & e^{-3t} \end{bmatrix}$$

zero input response

$$x(t) = \begin{bmatrix} e^{-t} & 0 \\ e^{-t} - e^{-3t} & e^{-3t} \end{bmatrix} x_0$$

(c) Both systems have eigenvalues $\lambda_1 = -1$, $\lambda_2 = -3 < 0$.

For LTI system, iff $Re(\lambda_i(A)) < 0$, the system is asymptotically stable.

Problem 2

(a) $\frac{\partial \Phi(t,\tau)}{\partial t} = A(t)\Phi(t,\tau)$

$\Phi(t_0, t) = \Phi^{-1}(t, t_0)$

$\Rightarrow \frac{\partial \Phi(t_0, t)}{\partial t} = \frac{\partial}{\partial t}\Phi^{-1}(t, t_0)$

In order to take derivative of an inverse matrix, we first prove that

$$\frac{d}{dt}M^{-1} = -M^{-1}\frac{dM}{dt}M^{-1}$$

$\because M(t)M^{-1}(t) = I$

$\therefore \frac{d}{dt}[M(t)M^{-1}(t)] = \frac{d}{dt}I$

$$M(t)\frac{d}{dt}M^{-1}(t) + \frac{dM(t)}{dt}M^{-1}(t) = 0$$

$$M(t)\frac{d}{dt}M^{-1}(t) = -\frac{dM(t)}{dt}M^{-1}(t)$$

$$\frac{dM^{-1}(t)}{dt} = -M^{-1}(t)\frac{dM(t)}{dt}M^{-1}(t)$$

$$\Rightarrow \frac{\partial\Phi(t_0,t)}{\partial t} = \frac{\partial}{\partial t}\Phi^{-1}(t,t_0) = -\Phi^{-1}(t,t_0)\cdot\frac{\partial\Phi(t,t_0)}{\partial t}\cdot\Phi^{-1}(t,t_0)$$

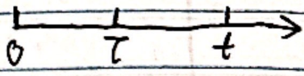
$$= -\Phi^{-1}(t,t_0)\cdot A(t)\Phi(t,t_0)\cdot\Phi^{-1}(t,t_0)$$

$$= -\Phi^{-1}(t,t_0)\cdot A(t)$$

Thus, $\frac{\partial\Phi(t_0,t)}{\partial t} = -\Phi(t_0,t)A(t)$

(b) According to the property of state transition matrix,

$$\Phi(t,0) = \Phi(t,\tau)\Phi(\tau,0)$$



$$\Rightarrow \Phi(t,\tau) = \Phi(t,0)\Phi^{-1}(\tau,0)$$

$$\Phi^{-1}(\tau,0) = \Phi^{T}(\tau,0) = \begin{bmatrix}\cos\tau^2 & -\sin\tau^2\\ \sin\tau^2 & \cos\tau^2\end{bmatrix}$$

$$\Phi(t,\tau) = \begin{bmatrix}\cos t^2 & \sin t^2\\ -\sin t^2 & \cos t^2\end{bmatrix}\begin{bmatrix}\cos\tau^2 & -\sin\tau^2\\ \sin\tau^2 & \cos\tau^2\end{bmatrix} = \begin{bmatrix}\cos(t^2-\tau^2) & \sin(t^2-\tau^2)\\ -\sin(t^2-\tau^2) & \cos(t^2-\tau^2)\end{bmatrix}$$

Problem 3

(a) $\dot{x} = \begin{bmatrix} 0 & t \\ 0 & 1 \end{bmatrix} x$

$$\begin{cases} \dot{x}_1 = t x_2 \Rightarrow \dot{x}_1 = t x_{20} e^t \Rightarrow x_1 = x_{10} + x_{20} \int_0^t t e^t \, dt \\ \dot{x}_2 = x_2 \Rightarrow x_2 = x_{20} e^t \end{cases}$$

$$\because \int_0^t t e^t dt = \int_0^t t \, de^t = te^t\Big|_0^t - \int_0^t e^t dt = te^t - e^t\Big|_0^t = te^t - (e^t - 1) = (t-1)e^t + 1$$

$$\therefore x_1 = x_{10} + x_{20}[(t-1)e^t + 1]$$

To find fundamental matrix, pick $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as initial conditions;

solutions are $X(t) = \begin{bmatrix} 1 & (t-1)e^t + 1 \\ 0 & e^t \end{bmatrix}$, $X^{-1}(t) = \begin{bmatrix} 1 & -e^{-t} - (t-1) \\ 0 & e^{-t} \end{bmatrix}$

$$\Rightarrow \Phi(t, t_0) = X(t) X^{-1}(t_0) = \begin{bmatrix} 1 & (t-1)e^t + 1 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 1 & -e^{-t_0} - (t_0 - 1) \\ 0 & e^{-t_0} \end{bmatrix} = \begin{bmatrix} 1 & (1-t_0) + (t-1)e^{t-t_0} \\ 0 & e^{t-t_0} \end{bmatrix}$$

(b) $\dot{x} = \begin{bmatrix} -\sin t & 0 \\ 0 & -\cos t \end{bmatrix} x$

$$\begin{cases} \dot{x}_1 = -\sin t \, x_1 \\ \dot{x}_2 = -\cos t \, x_2 \end{cases} \Rightarrow \begin{cases} \frac{dx_1}{dt} = -\sin t \, x_1 \\ \frac{dx_2}{dt} = -\cos t \, x_2 \end{cases} \Rightarrow \begin{cases} \frac{dx_1}{x_1} = -\sin t \, dt \\ \frac{dx_2}{x_2} = -\cos t \, dt \end{cases} \Rightarrow \begin{cases} x_1 = e^{\cos t}\Big|_0^t \cdot x_{10} \\ x_2 = e^{-\sin t}\Big|_0^t \cdot x_{20} \end{cases}$$

$$\Rightarrow \begin{cases} \ln x_1 = \cos t\Big|_0^t + C_1 \\ \ln x_2 = -\sin t\Big|_0^t + C_2 \end{cases} \Rightarrow \begin{cases} \ln x_1 = \cos t - 1 + C_1 \\ \ln x_2 = -\sin t + C_2 \end{cases} \Rightarrow \begin{cases} x_1 = x_{10} e^{\cos t - 1} \\ x_2 = x_{20} e^{-\sin t} \end{cases}$$

To find fundamental matrix, pick $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as initial conditions,

solutions are $X(t) = \begin{bmatrix} e^{\cos t - 1} & 0 \\ 0 & e^{-\sin t} \end{bmatrix}$, $X^{-1}(t) = \begin{bmatrix} e^{1-\cos t} & 0 \\ 0 & e^{\sin t} \end{bmatrix}$

$$\Rightarrow \Phi(t, t_0) = X(t) X^{-1}(t_0) = \begin{bmatrix} e^{\cos t - \cos t_0} & 0 \\ 0 & e^{-\sin t + \sin t_0} \end{bmatrix}$$

Problem 4

(a) $|\lambda I - A| = \begin{vmatrix} \lambda+1 & 0 & -2 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{vmatrix} = (\lambda+1)\lambda^2 = 0 \Rightarrow \lambda_{1,2} = 0,\ \lambda_3 = -1$

For $\lambda_{1,2}=0$, there are two independent Jordan blocks.

Therefore, it's stable. However, it's not asymptotically stable since there exist 0 eigenvalues.

Furthermore, we can also solve the state-space equations directly.

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + 2x_3(t) \\ \dot{x}_2(t) = 0 \\ \dot{x}_3(t) = 0 \end{cases}$$

$\Rightarrow x_2(t) = x_2(0)$, $x_3(t) = x_3(0)$

$\dot{x}_1(t) = -x_1(t) + 2x_3(0) = -[x_1(t) - 2x_3(0)]$

$$\frac{dx_1(t)}{x_1(t) - 2x_3(0)} = -dt$$

$\ln[x_1(t) - 2x_3(0)] = -t + c'$

$x_1(t) - 2x_3(0) = ce^{-t}$

I.C. $\Rightarrow$ $x_1(0) = 2x_3(0) + C$

$C = x_1(0) - 2x_3(0)$

$\Rightarrow x_1(t) = 2x_3(0) + [x_1(0) - 2x_3(0)]e^{-t}$

$= x_1(0)e^{-t} + 2x_3(0)[1 - e^{-t}]$

Thus, $x_1(t)$ will converge to $2x_3(0)$. $x_2(t)$, $x_3(t)$ will stay at their own I.C.

(b) $|\lambda I - A| = \begin{vmatrix} \lambda+1 & 0 & -2 \\ 0 & \lambda & -1 \\ 0 & 0 & \lambda \end{vmatrix} = (\lambda+1)\lambda^2 = 0 \Rightarrow \lambda_{1,2} = 0,\ \lambda_3 = -1$

For $\lambda_{1,2}=0$, there is only one Jordan block (2×2), which means their eigenvectors are not linearly independent.

Thus, it's not stable and asymptotically stable.

Furthermore, we can also solve the state-space equations directly.

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + 2x_3(t) \\ \dot{x}_2(t) = x_3(t) \\ \dot{x}_3(t) = 0 \end{cases}$$

$\Rightarrow x_3(t) = x_3(0)$

$\dot{x}_2(t) = x_3(0), \quad x_2(t) = x_2(0) + x_3(0)t$

$\dot{x}_1(t) = -x_1(t) + 2x_3(0), \quad x_1(t) = x_1(0)e^{-t} + 2x_3(0)(1-e^{-t})$

Therefore, $\lim_{t\to\infty} x_2(t) = \infty$ (if $x_3(0) \neq 0$), diverge.

Problem 5

Transfer function

$G(s) = C(sI-A)^{-1}B + D, \qquad (sI-A)^{-1} = \begin{bmatrix} s+1 & -5 \\ 0 & s-2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s+1} & \frac{5}{(s+1)(s-2)} \\ 0 & \frac{1}{s-2} \end{bmatrix}$

$= [-2 \ \ 4] \begin{bmatrix} \frac{1}{s+1} & \frac{5}{(s+1)(s-2)} \\ 0 & \frac{1}{s-2} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + (-2) = \begin{bmatrix} \frac{-2}{s+1} & \frac{4s-6}{(s+1)(s-2)} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + (-2) = \frac{-4}{s+1} + (-2)$

pole $s_1 = -1$, left-hand plane

Therefore, the system is BIBO stable.

Problem 6

$$A^T P + PA - 2\lambda P = -Q$$

$$\Leftrightarrow (A-\lambda I)^T P + P(A-\lambda I) = -Q$$

Set $B = A-\lambda I$, then

$$B^T P + PB = -Q$$

Since P, Q are positive definite matrices, we can conclude

$$Re(\lambda_i(B)) \leq 0$$

So, $Re(\lambda_i(A-\lambda I)) = Re(\lambda_i(A)-\lambda) \leq 0$

Then we can say the real part of all eigenvalues of A are less than or equal to λ,

i.e. $Re(\lambda_i(A)) \leq \lambda$.