ME555 Midterm

Problem 1

(a) False. If matrix A is defective, then the set couldn't be a basis. For example,

$$A=\begin{bmatrix}0&0\\0&0\end{bmatrix},\quad A^2=A^3=\begin{bmatrix}0&0\\0&0\end{bmatrix},\quad k=2^2-1=3.$$

(b) False. From transfer function to state-space representation is always not unique, different realizations can lead to different results. (e.g. CCF & OCF)

(c) False. Linearizing a system can only guarantee it would be a linear system, but it may be time-invariant or time-varying.

(d) False. In LTI system, Lyuapunov stability only consider A matrix, but we should also consider B and C matrices when talking about BIBO stable.

(e) True. From state-space to transfer function is always unique.

$$(sI-A)=\begin{bmatrix}s-2&-1\\0&s-3\end{bmatrix},\quad (sI-A)^{-1}=\frac{1}{(s-2)(s-3)}\begin{bmatrix}s-3&1\\0&s-2\end{bmatrix}=\begin{bmatrix}\frac{1}{s-2}&\frac{1}{(s-2)(s-3)}\\0&\frac{1}{s-3}\end{bmatrix}$$

$$C(sI-A)^{-1}B+D=\begin{bmatrix}1&0\end{bmatrix}\begin{bmatrix}\frac{1}{s-2}&\frac{1}{(s-2)(s-3)}\\0&\frac{1}{s-3}\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}+0=\frac{1}{(s-2)(s-3)}=\frac{1}{s^2-5s+6}$$

Problem 2

(a) $\{x_1, x_2, x_3\}$ is a basis for $\mathbb{R}^3$, then

$$a x_1 + b x_2 + c x_3 = 0 \quad \text{iff} \quad a=b=c=0$$

Now apply this to new basis $\{\hat{x}_1, \hat{x}_2, \hat{x}_3\}$,

$$a\hat{x}_1 + b\hat{x}_2 + c\hat{x}_3 = a x_1 + b(x_1 + 2x_2) + c(3x_2 + 2x_3)$$
$$= (a+b)x_1 + (2b+3c)x_2 + 2c\,x_3 = 0$$

$$\Rightarrow \begin{cases} a+b=0 \\ 2b+3c=0 \\ 2c=0 \end{cases} \Rightarrow \begin{cases} a=0 \\ b=0 \\ c=0 \end{cases}$$

Therefore, the new ~~basis~~ set $\{\hat{x}_1, \hat{x}_2, \hat{x}_3\}$ is also basis for $\mathbb{R}^3$.

(b) We have $\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 3 & 2 \end{bmatrix}^{-1} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix}$

$$b = \begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\hat{b} = \underbrace{\begin{bmatrix} b_1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & b_3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 3 & 2 \end{bmatrix}^{-1}}_{\text{representation in } \{\hat{x}_1, \hat{x}_2, \hat{x}_3\}} \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix}$$

(c) Denote the transition matrix from $\{x_1, x_2, x_3\}$ to $\{\hat{x}_1, \hat{x}_2, \hat{x}_3\}$ as P, i.e.

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 3 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = P \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Suppose we have vectors a, b in basis $\{x_1, x_2, x_3\}$, which C could transform b to a

$$a = Cb$$

Now, in new basis $\{\hat{x}_1, \hat{x}_2, \hat{x}_3\}$,

$$P^{-1}\hat{a} = CP^{-1}\hat{b}$$

$$\hat{a} = PCP^{-1}\hat{b}$$

Therefore, its representation in the $\{\hat{x}_1, \hat{x}_2, \hat{x}_3\}$ basis is PCP^{-1}.

Problem 3

(a) Using the property $\frac{d\Phi(t, t_0)}{dt} = A \cdot \Phi(t, t_0)$

$$\frac{d}{dt}\Phi(t, t_0) = \begin{bmatrix} 2t \cdot e^{t^2 - t_0^2} & 0 \\ 0 & 2t \cdot e^{t^2 - t_0^2} \end{bmatrix} = \begin{bmatrix} 2t & 0 \\ 0 & 2t \end{bmatrix} \begin{bmatrix} e^{t^2 - t_0^2} & 0 \\ 0 & e^{t^2 - t_0^2} \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 2t & 0 \\ 0 & 2t \end{bmatrix}$$

(b) The solution

$$x(t) = \Phi(t, 0)\, x_0 + \int_0^t \Phi(t, \tau)\, B(\tau)\, u(\tau)\, d\tau$$

For this case, $u = 0$,

$$x(1) = \Phi(1, 0)\, x_0 = \begin{bmatrix} e^1 & 0 \\ 0 & e^1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e \\ e \end{bmatrix}$$

(c) If $x(0)=0$, $u(t)=1$

$$x(t)=\int_0^t \Phi(t,\tau)B(\tau)\cdot 1\, d\tau$$

$$=\int_0^t \begin{bmatrix} e^{t^2-\tau^2} & 0 \\ 0 & e^{t^2-\tau^2} \end{bmatrix}\begin{bmatrix} e^{\tau^2} \\ e^{\tau^2} \end{bmatrix} d\tau$$

$$=\int_0^t \begin{bmatrix} e^{t^2} \\ e^{t^2} \end{bmatrix} d\tau = \begin{bmatrix} \tau e^{t^2} \\ \tau e^{t^2} \end{bmatrix}\Bigg|_0^t = \begin{bmatrix} te^{t^2} \\ te^{t^2} \end{bmatrix}$$

Problem 4

(a) The system is not stable.

$$|sI-A|=\begin{vmatrix} s-c & 0 & 0 \\ 0 & s+c & -2c \\ 0 & 0 & s-c \end{vmatrix}=(s-c)^2(s+c) \Rightarrow s_1=c,\ s_2=c,\ s_3=-c$$

Since $c\neq 0$, there is always positive eigen value(s) in this system.

(b) For $\lambda_1=-c$,

$$\lambda_1 v_1 = A v_1 \Rightarrow -c\begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix}=\begin{bmatrix} c & 0 & 0 \\ 0 & -c & 2c \\ 0 & 0 & c \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \\ v_{13} \end{bmatrix} \Rightarrow \begin{cases} -cv_{11}=cv_{11} \\ -cv_{12}=-cv_{12}+2cv_{13} \\ -cv_{13}=cv_{13} \end{cases}$$

$$\Rightarrow v_1=\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

For $\lambda_2 = c$,

$$[A-\lambda_2 I]v = 0 \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2c & 2c \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1' \\ v_2' \\ v_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow v_2' = v_3'$$

we can construct two eigen vectors $v_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Thus, $T = [v_2 \ \ v_3 \ \ v_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$\tilde{A} = T^{-1}AT = \begin{bmatrix} c & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & -c \end{bmatrix}$$

(c) Define the original system as $\dot{\tilde{x}} = A\tilde{x} + Bu$, $\tilde{x} = Tx$,
so the new system would be

$$T\dot{x} = ATx + Bu$$

$$\dot{x} = \underbrace{T^{-1}AT}_{\hat{A}}x + \underbrace{T^{-1}B}_{\hat{B}}u$$

$$\Rightarrow \hat{B} = T^{-1}B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ k \end{bmatrix} = \begin{bmatrix} 0 \\ k \\ -k \end{bmatrix}$$

(d) The homogeneous solution for LTI system is

$$x(t) = e^{At}x(0)$$

$$(sI-A)^{-1} = \begin{bmatrix} s-c & 0 & 0 \\ 0 & s+c & -2c \\ 0 & 0 & s-c \end{bmatrix}^{-1} = \frac{1}{(s-c)^2(s+c)}\begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix} = \begin{bmatrix} \frac{1}{s-c} & 0 & 0 \\ 0 & \frac{1}{s+c} & \frac{2c}{(s-c)(s+c)} \\ 0 & 0 & \frac{1}{s-c} \end{bmatrix}$$

$$\mathcal{L}^{-1}\left((sI-A)^{-1}\right) = \begin{bmatrix} e^{ct} & 0 & 0 \\ 0 & e^{-ct} & e^{ct}-e^{-ct} \\ 0 & 0 & e^{ct} \end{bmatrix}$$

Thus,

$$x(t) = \begin{bmatrix} e^{ct} & 0 & 0 \\ 0 & e^{-ct} & e^{ct}-e^{-ct} \\ 0 & 0 & e^{ct} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

I have neither given nor received outside aid during this exam, nor have I concealed any violation of the honor code. — Yuanzhu Zhan