Introduction to Binary Decision Diagram

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Outlines

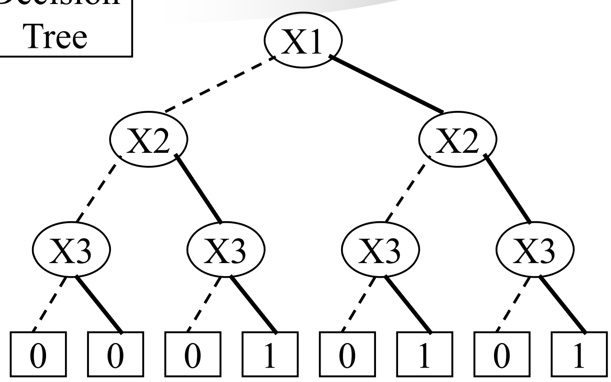
- Representing Boolean Functions 表示式
  - Decision graph structure
  - Reduction to canonical form
  - Effect of variable ordering
  - Variants to reduce storage
- Algorithms 儲存方式
  - General framework
  - Basic operations
    - » Restriction (Cofactor)
    - » If-Then-Else
  - Derived operations
  - Computing functional properties

Decision Structures

Truth Table

X1	X2	X3	f
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

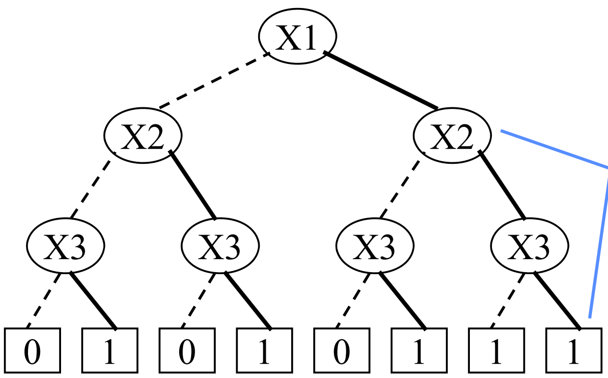
Decision Tree



- Vertex represents decision
- Follow dashed line for value 0
- Follow solid line for value 1
- Function value determined by leaf value

Binary Decision Diagram (BDD)

$f = x_1x_2+x_3$



---- : 0

—— : 1

leaf

terminal node :

- attribute
  - value(v) = 0
  - value(v) = 1

nonterminal node :

- index(v) = i
- two children
  - low(v)
  - high(v)

BDD Binary Decision Diagram

A BDD graph which has a vertex v as root corresponds to the function F_v :

(1) If v is a terminal node :

a) if value(v) is 1, then $F_v = 1$

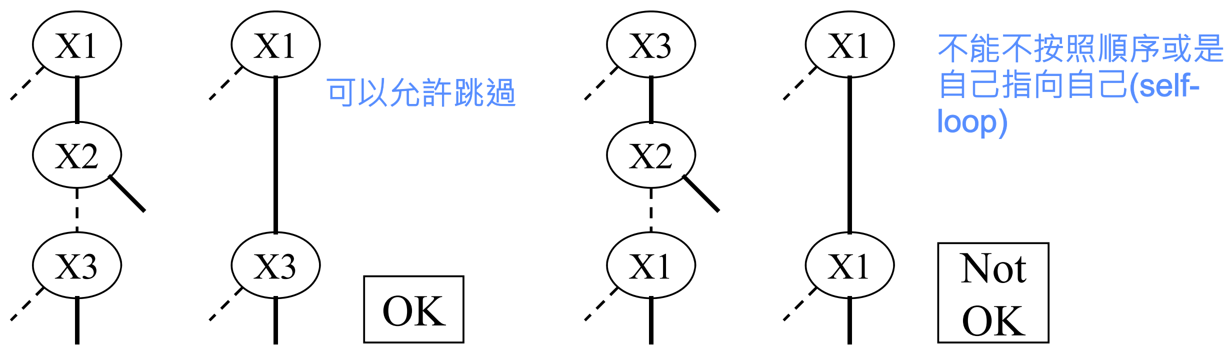
b) if value(v) is 0, then $F_v = 0$

(2) If F is a nonterminal node (with index(v) = i)

$$F_v(x_1, \dots, x_n) = x_i' F_{low(v)}(x_{i+1}, \dots, x_n) + x_i F_{high(v)}(x_{i+1}, \dots, x_n)$$

Variable Ordering

- Assign arbitrary total ordering to variable
  e.g. X1 < X2 < X3 需要遵循一定的order
- Variable must appear in ascending order along all paths

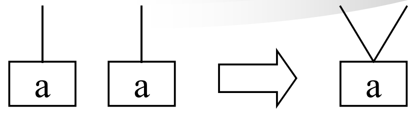


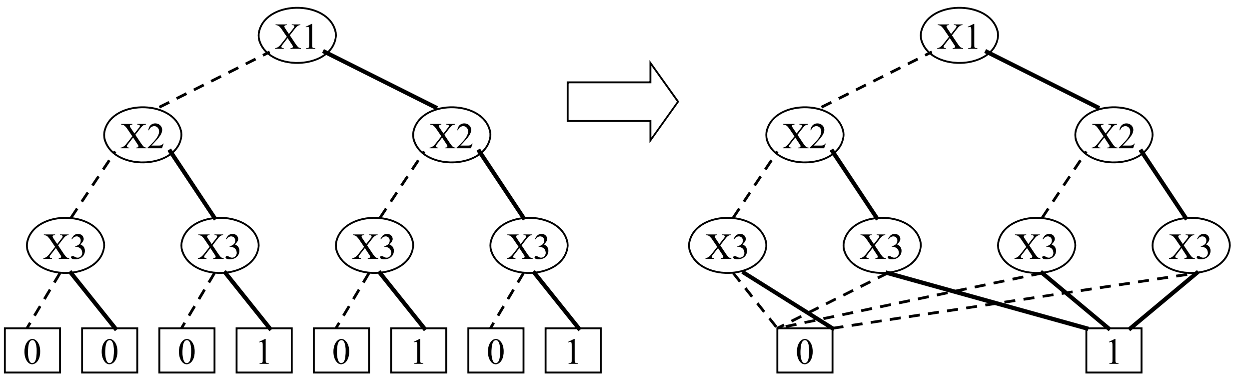
- Properties
  – No conflicting variable assignments along path
  – Simplifies manipulation

Reduction Rule #1

- Merge equivalent leaves

只會merge最底層的leaf



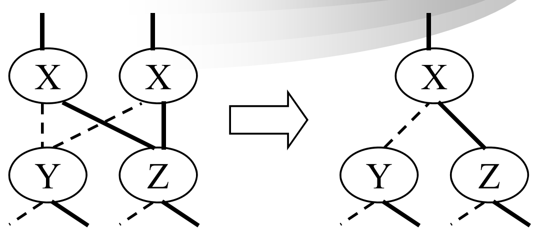


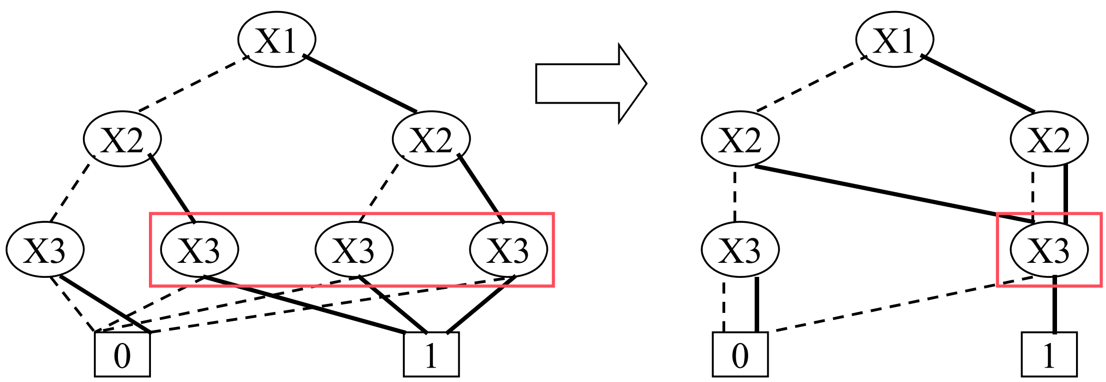
最後的output都是0與1，因此可以merge

Reduction Rule #2

- Merge isomorphic nodes

當兩個node的output完全一致，
稱他們為isomorphic



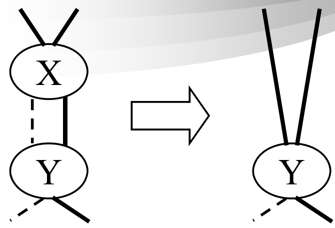


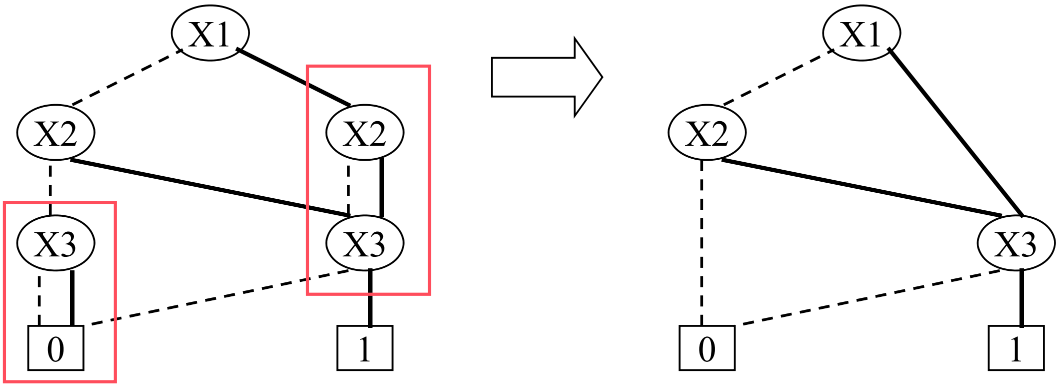
紅框處X3=0時output皆為0，X3=1時output皆為1 --> isomorphic

Reduction Rule #3

- Eliminate Redundant Tests

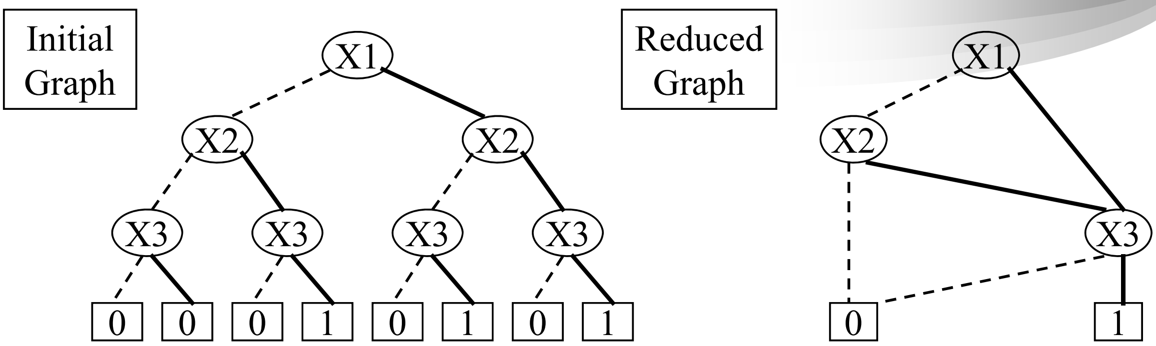
  當一個decision不管如何，其output都是同一個的話，代表該decision是一個多餘的判斷 --> 跳過(即移除該node)





Reduced Order Binary Decision Diagram

Example ROBDD



- Canonical representation of Boolean function for given variable ordering 當variable順序固定，一個boolean function只會對應唯一一個ROBDD
  - Two functions equivalent iff graphs isomorphic
    - » can be tested in linear time
  - Desirable property : The simplest form is canonical

Reduce

- Visit OBDD bottom up and label each vertex with an identifier
- Redundancy

  low: 條件為0
  high: 條件為1

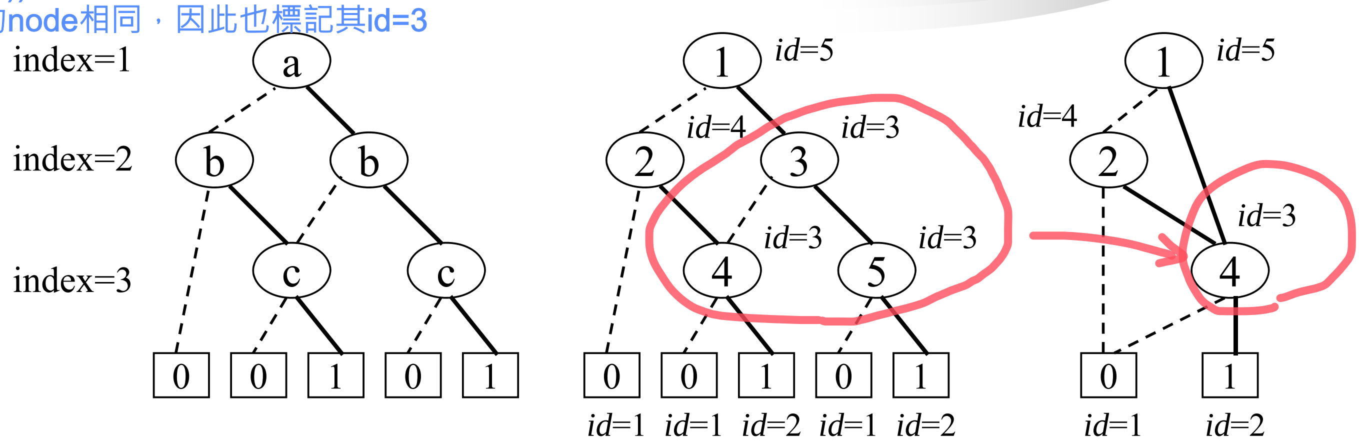
  – if id(low(v)) = id(high(v)), then vertex v is redundant

    $\Rightarrow$ set id(v) = id(low(v)) --> Rule 3

  – if id(low(v)) = id(low(u)) and id(high(v)) = id(high(u)), then set id(v) = id(u) --> Rule 2
- A different identifier is given to each vertex at level i
- Terminated when root is reached
- An ROBDD is identified by a subset of vertices with different identifiers

Reduce

1. Bottom-Up的去編號，因此leaf中的0、1會最先被編號(分別為id1, 2)
2. 接下來往上看，首先看到4，其中
id(low(4)) = 1
id(high(4)) = 2
是沒有出現過的點，因此標記其id=3
3. 然後看到5，
其中
id(low(5)) = 1
id(high(5)) = 2
與id=3的node相同，因此也標記其id=3
4. 接下來看2，其中
id(low(2)) = 0
id(high(2)) = 3
是沒有出現過的點，因此標記其id = 4
5. 接下來看3，其中
id(low(3)) = id(high(3)) = 3
因此其為一多餘的判斷條件
(Rule 2)，標記其id = 3



(a) (b) (c)

最後將所有id相同的點merge在一起

- - - - : 0

—— : 1

Construct ROBDD Directly

- Using a hash table called unique table
  - Contain a key for each vertex of an OBDD
  - Key : (variable, right children, left children)
  - Constructed bottom up
  - Each key uniquely identify the specific function
  - Look up the table can determine if another vertex in the table implements the same function

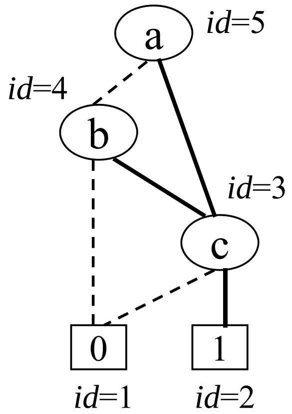
使用hash table去紀錄一個node是否有出現過(i.e. 其low, high是否與曾經的某個node一致)
value 就存對應的node*
詳細見17頁

通常會一邊建立一邊化簡，這樣比較省空間
如果是一開始先建立好再化簡，就會需要一個很大的初始空間

The Unique Table

- Represent an ROBDD
- A strong canonical form
- Check equivalence of two Boolean functions by comparing the corresponding identifiers
- Can represent multiple-output functions

Multi-Rooted ROBDD



Unique table

Key

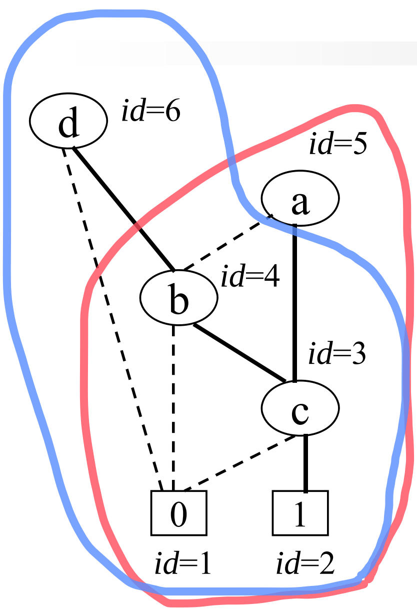
Identifier	Variable	Right child	Left child
5	a	3	4
4	b	3	1
3	c	2	1

f = (a+b) c

variable order (a, b, c)

建立新的ROBDD時候，與前一張圖共用一張unique table，如果在table中有查到一樣的就共用，沒有就正常new 一個node

Multi-Rooted ROBDD



f is constructed first and is associated with *id*=5

g : *id*=6

Unique table

Key

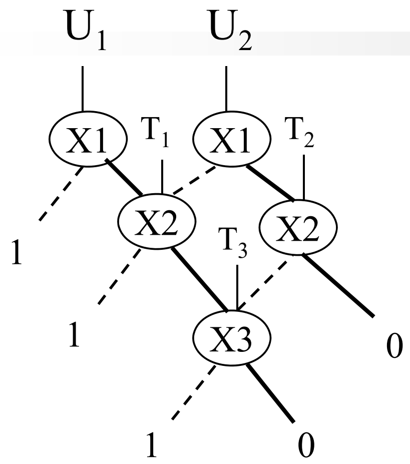
Identifier	Variable	Right child	Left child
6	d	4	1
5	a	3	4
4	b	3	1
3	c	2	1

f = (a+b) c

g = b c d

variable order (d, a, b, c)

The Unique Table

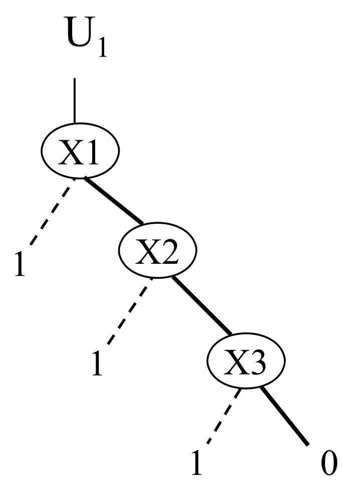


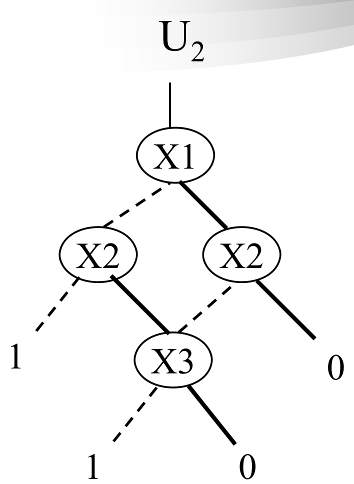
Hash Table Mapping
(X1, T1, 1) --> U1
(X1, T2, T1) --> U2
(X2, T3, 1) --> T1
(X2, 0 , T3) --> T2
(X3, 0 , 1) --> T3

true false

- Unique table : hash table mapping (Xi, G, H) into a node in the DAG
  - before adding a node to the DAG, check to see if it already exists
  - avoids creating two nodes with the same function
  - strong canonical form : pointer equality determines function equality

Non-Shared ROBDD

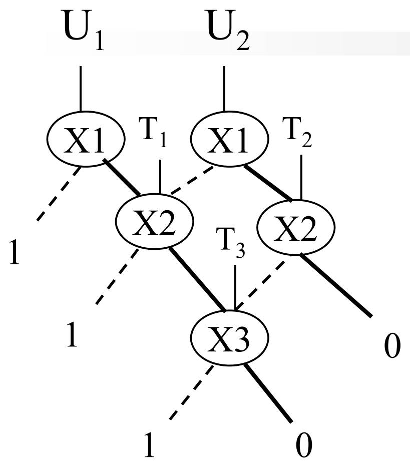




U1 = X1’ + X2’ + X3’

U2 = X1’X2’ + X1’X3’

Multi-Rooted (Shared) ROBDD



$U_1 = X_1'+X_2'+X_3' = (X_1, T_1, 1)$
$U_2 = X_1'X_2'+X_1'X_3' = (X_1, T_2, T_1)$
External functions
User functions

$T_1 = X_2'+X_3' = (X_2, T_3, 1)$
$T_2 = X_2'X_3' = (X_2, 0, T_3)$
$T_3 = X_3' = (X_3, 0, 1)$
$0 = (X_\infty, 0, 0)$
$1 = (X_\infty, 1, 1)$
Internal functions

- A DAG node F is represented by a tuple (Xi, G, H)
  - Xi is called the top variable of F
  - node (Xi, G, H) represents the function ite(Xi, G, H) = XiG + Xi'H
- DAG contains both external and internal functions

Separated vs. Shared

- Separated
  - 51 nodes for 4-bit adder
  - 12481 nodes for 64-bit adder
  - Quadratic growth
- Shared
  - 31 nodes for 4-bit adder
  - 571 nodes for 64-bit adder
  - Linear growth

Maintaining Shared ROBDD

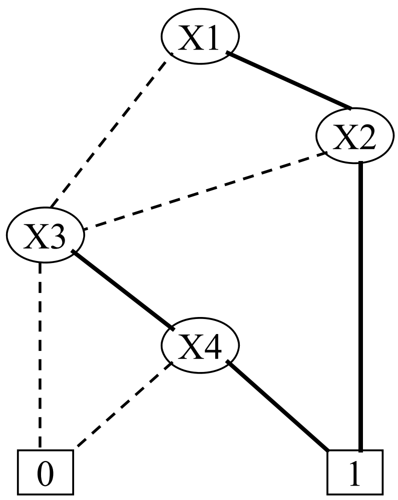
- Storage Model
  - Single, multiple-rooted DAG
  - Function represented by pointer to node in DAG
  - Maintain Unique (hash) table to keep canonical
- Storage Management 不能亂刪node，因為有些node是與其他ROBDD共用的
  - User cannot know when storage for node can be freed
  - Must implement automatic garbage collection
- Algorithmic Efficiency 當一個function(node)不用時，將其標記(還沒刪除)，等到一段時間過後檢查發現還是沒被用再刪除
  - Functions equivalent iff pointer equal
    - » if (p1 == p2) ...
  - Can test in constant time

Ordering Effects

- The size of ROBDD depends on the ordering of variables

ex : $x_1x_2 + x_3x_4$

第一種順序 $x_1 < x_2 < x_3 < x_4$

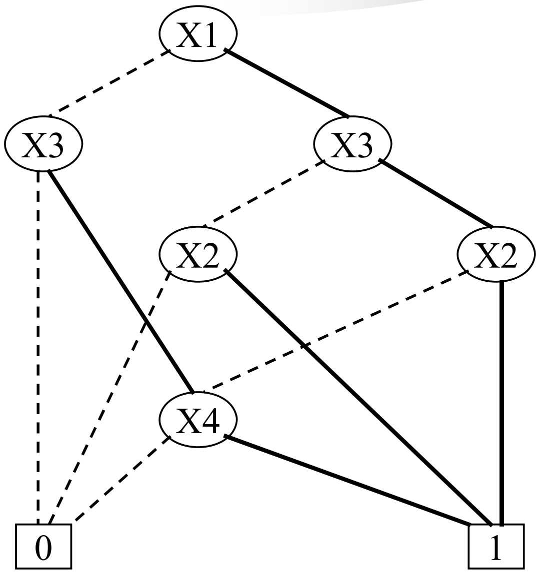


Ordering Effects (cont'd)

第二種順序 $x_1 < x_3 < x_2 < x_4$

可以發現此種順序之ROBDD較第一種順序的ROBDD複雜

因此選擇好的順序也很重要，但是我們並無法預先得知何種順序效果較好。因此可以嘗試在建立到一半時去檢驗該處使用哪個variable會比較好



Sample Function Classes

Function Class	Best	Worst	Ordering Sensitivity
ALU (Add/Sub)	Linear	Exponential	High
Symmetric	Linear	Quadratic	None
Multipication	Exponential	Exponential	Low

乘法使用ROBDD建立是非常沒有效率的

- General Experience
  - Many tasks have reasonable ROBDD representations
  - Algorithms remain practical for up to 100,000 vertex ROBDD node更多的話就會很慢
  - Heuristic ordering methods generally satisfactory

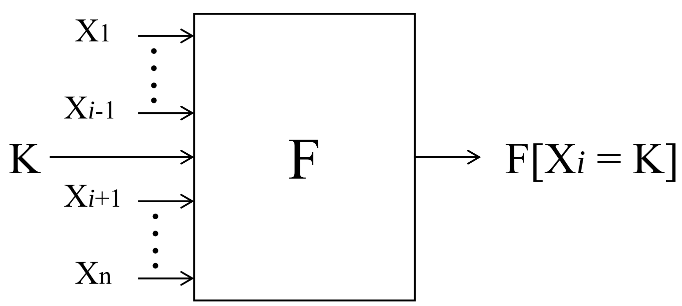
Symbolic Manipulation

- Strategy
  - Represent data as set of ROBDDs
    - » with identical variable orderings
  - Express solution method as sequence of symbolic operations
  - Implement each operation by ROBDD manipulation
- Algorithmic Properties
  - Arguments are ROBDDs with identical variable orderings
  - Result is ROBDD with same ordering
  - "Closure Property"
- Two Basic Operations
  - Restriction 限制某一variable的值固定為0或1
  - If-Then-Else

兩個ROBDD在進行運算的前提是他們具有相同的order，在此前提下進行運算的output就是"具有相同order的ROBDD" --> 封閉性(closure property)

Restriction Operation

- Concept
  - Effect of setting function argument Xi to constant K(0,1)
  - Also called Cofactor operation

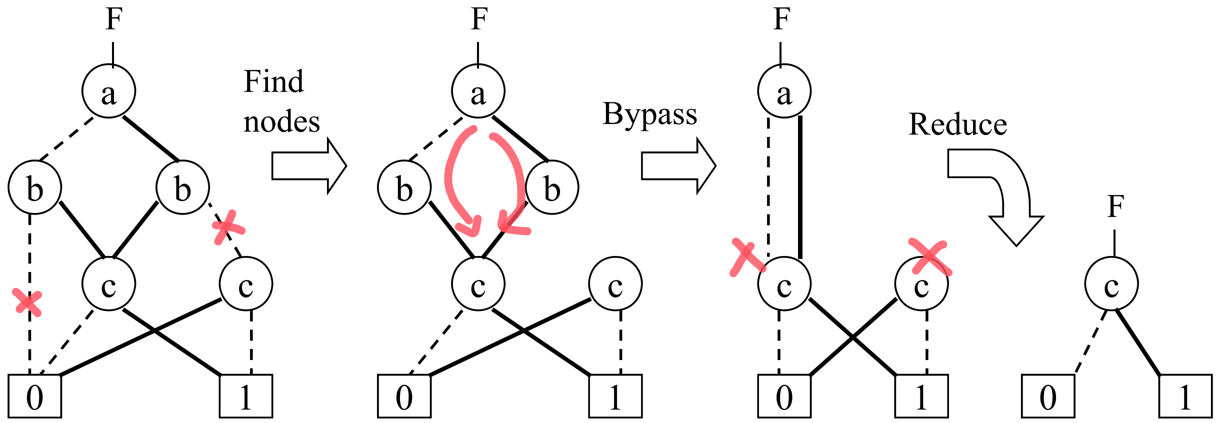


- Implementation
  - Depth-first traversal
  - Complexity near-linear in argument graph size

Restriction Algorithm

Restrict (F, x, k)
  Bypass any nodes for variable x
    Choose Hi child for k = 1
    Choose Lo child for k = 0
  Reduce result

e.g. Restrict variable b to 1



因為b=1，因此移除b=0的路徑

b為多餘的判斷條件，bypass

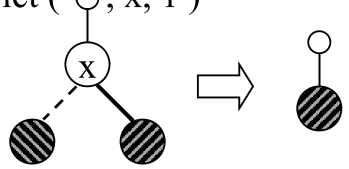
a為多餘的判斷條件 --> 移除
沒有條件會通往右側的c --> 移除

Special cases of Restriction

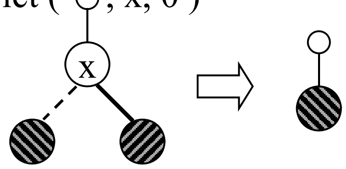
- Case 1 : Restrict on root node variable

如果restrict的node為root，則直接改變root即可

Restrict (○, x, 1)

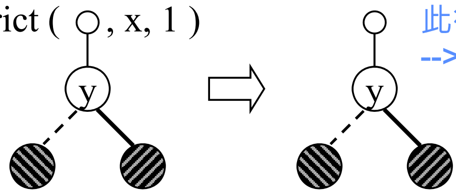


Restrict (○, x, 0)



- Case 2 : Restrict on variable less than root node
  - e.g. x < y

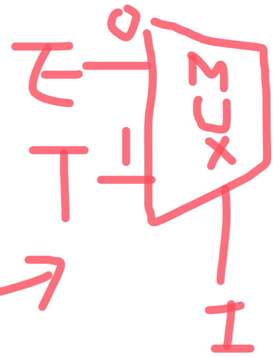
Restrict (○, x, 1)



restrict的條件為x，觀察當前root為y，且order為x<y，因此往下一定不會有x出現
-->直接return 即可

If-Then-Else Operation

- Concept
  - Basic technique for building ROBDD from network or formula
- Argument **I** (if), **T** (then), **E** (else)
  - Functions over variables X
  - Represented as ROBDDs
- Result
  - ROBDD representing composite function
  - IT + I’E
- Implementation
  - combination of depth-first traversal and dynamic programming
  - Worst case complexity : product of argument graph sizes



If-Then-Else Algorithm

- Recursive Formulation

F = x * F(x) + x' * F(x')
= x * F(1) + x' * F(0)

ITE (I, T, E) = x ITE(I[x=1], T[x=1], E[x=1]) +
x’ITE(I[x=0], T[x=0], E[x=0])

代入的方式就像前面的restriction一樣

- General Algorithm
  - Select top root variable x of I, T and E
  - Compute restrictions
    - » Guaranteed to be one of special cases
  - Apply recursively to get results Lo and Hi
  - Still remain canonical form
- Termination Conditions
  - I = 1 ==> Return T
  - I = 0 ==> Return E
  - T = 1, E = 0 ==> Return I
  - T = E ==> Return T

此兩種就是正常mux的功能

可以發現I = 1時return T = 1，I = 0時return E = 0，因此直接return I本身就好

不管I是啥結果都一樣，就隨便挑一個return 就好

An ITE Example

- Given f = ab + bc + ac, g = c under the order a < b < c

ITE (f, g, 0) 即I = f, T = g, E = 0

由於a < b < c，因此I先代a

= ITE[***a***, ITE(f(a=1), g(a=1), 0), ITE(f(a=0), g(a=0), 0)] T代ITE(a=1), E代ITE(a=0)

= ITE[***a***, ITE(b+bc+c, c, 0), ITE(bc, c, 0)]

b=1 *b=0*

= ITE{ ***a***, ITE[***b***, ITE(1, c, 0), ITE(c, c, 0)],

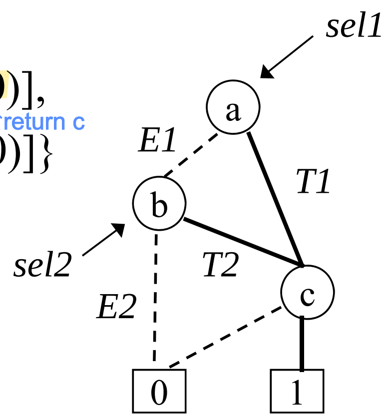
I=1 --> return T 等價於return c

ITE[***b***, ITE(c, c, 0), ITE(0, c, 0)]}

= ITE[***a***, ITE(b, c, c), ITE(b, c, 0)]

= ITE[***a***, c, ITE(b, c, 0)]

sel1 *T1* *E1=sel2* *T2*



Algorithmic Issues & Derived Operations

- Efficiency
  - Maintain computed table and unique table to increase efficiency
  - Worst case complexity product of graph sizes for I, T, E
- Derived operations
  - Express as combination of If-Then-Else and Restrict
  - Preserve closure property
    - » Result is a ROBDD with the same variable ordering

Detailed ITE Algorithm

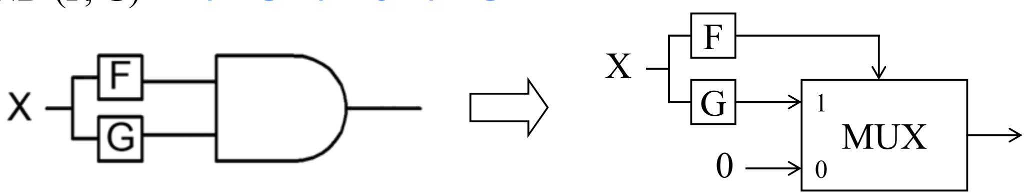
```
ITE(f, g, h) {
    if (terminal case)
        return (r = trivial result) ;
    else {                                                  /* exploit previous information */
        if (computed table has entry {(f, g, h), r} )
            return (r from computed table) ;
        else {
            x = top variable of f, g, h ;
            t = ITE(f_x, g_x, h_x) ;
            e = ITE(f_x', g_x', h_x') ;
            if (t == e)                                     /* children with isomorphic OBDDs */
                return (t) ;
            r = find_or_add_unique_table(x, t, e) ;         /* add r to unique table if not present */
            Update computed table with {(f, g, h), r} ;
            return (r) ;
        }
    }
}
```


Derived Algebraic Operations

- Other common operations can be expressed in terms of If-Then-Else

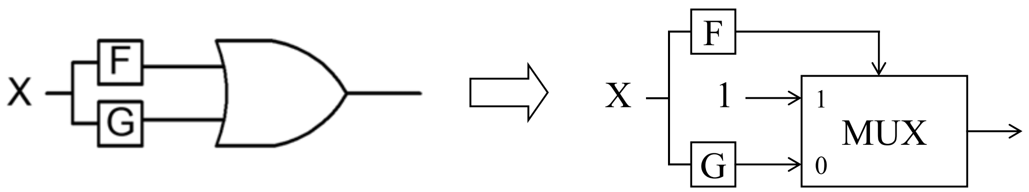
AND (F, G) $F * G + F' * 0 = F * G$

If-Then-Else (F, G, 0)



OR (F, G) $F * 1 + F' * G = F + F' * G$

If-Then-Else (F, 1, G)



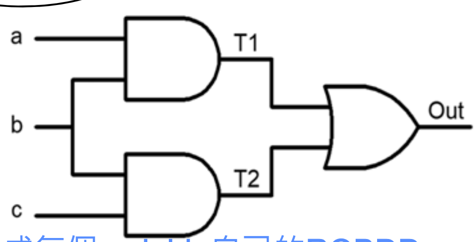
ITE Operators

Operator	Equivalent *ite* form
0	0
$f \cdot g$	$ite(f, g, 0)$
$f \cdot g'$	$ite(f, g', 0)$
f	f
$f' \cdot g$	$ite(f, 0, g)$
g	g
$f \oplus g$	$ite(f, g', g)$
$f + g$	$ite(f, 1, g)$
$(f + g)'$	$ite(f, 0, g')$
$(f \oplus g)'$	$ite(f, g, g')$
g'	$ite(g, 0, 1)$
$f + g'$	$ite(f, 1, g')$
f'	$ite(f, 0, 1)$
$f' + g$	$ite(f, g, 1)$
$(f \cdot g)'$	$ite(f, g', 1)$
1	1

Generating ROBDD from Network

- Task : Represent output functions of gate network as ROBDDs

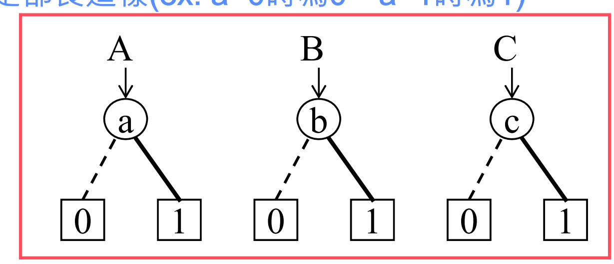
Network

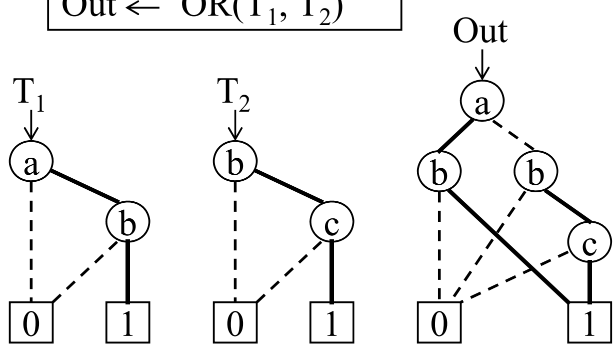


$A \leftarrow$ new_var("a")
$B \leftarrow$ new_var("b")
$C \leftarrow$ new_var("c")
$T_1 \leftarrow$ AND(A, B)
$T_2 \leftarrow$ AND(B, C)
Out $\leftarrow$ OR(T_1, T_2)

Evaluation

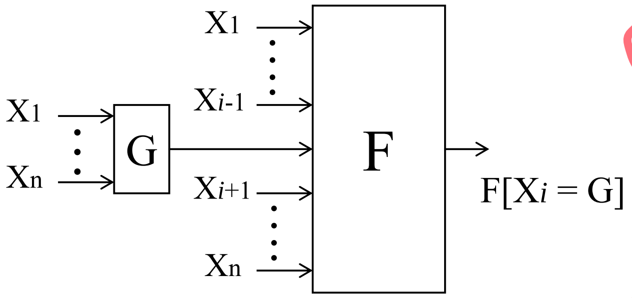
一開始先生成每個variable自己的ROBDD
固定都長這樣(ex: a=0時為0，a=1時為1)

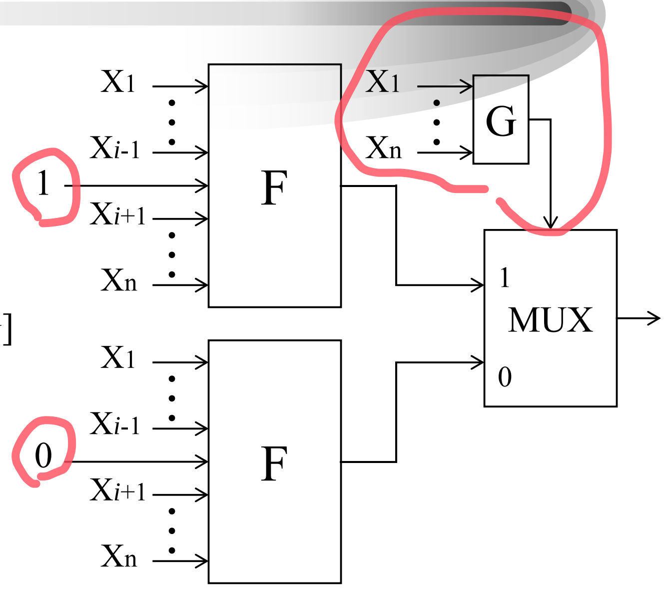




T1為AND，藉由前頁的
ITE可得出此ROBDD

Functional Composition

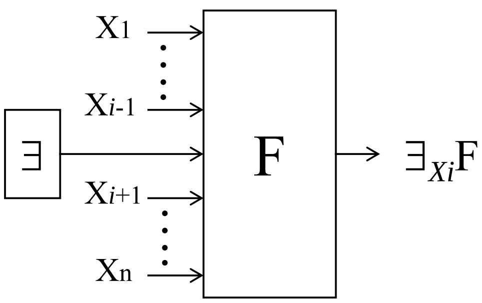


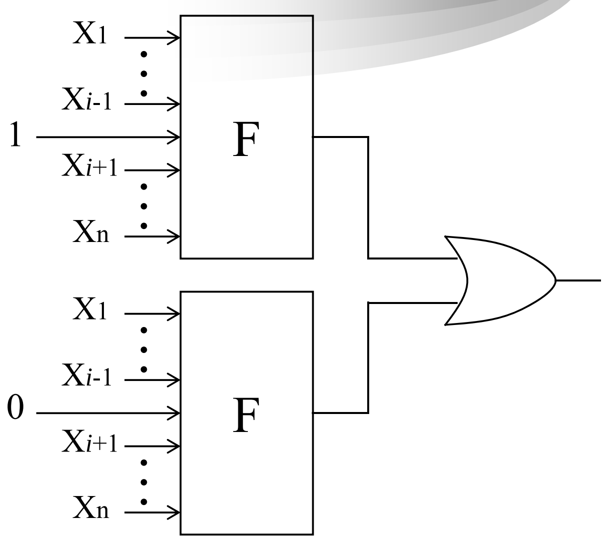


此種情況也可以使用ITE表示
--> ITE(G, ITE(G=1), ITE(G=0))

- Create new function by composing functions F and G
- Useful for composing hierarchical modules

Variable Qualification





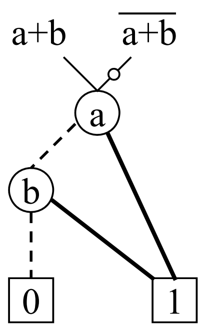
- Eliminate dependency on some argument through qualification

Variants & Optimizations

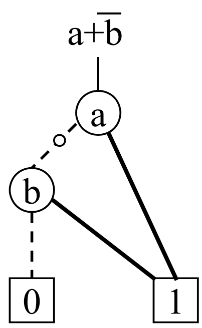
- Concept
  - Refinements to ROBDD representation
  - Do not change fundamental properties
- Objective
  - Reduce memory requirement
  - Improve algorithmic efficiency
  - Make commonly performed operations faster
- Common Optimizations
  - Share nodes among multiple functions
  - Negated arcs

Negation Arcs

- Concept
  - Dot on arc represents complement operator
    - » Invert function value
  - Can appear internal or external arc



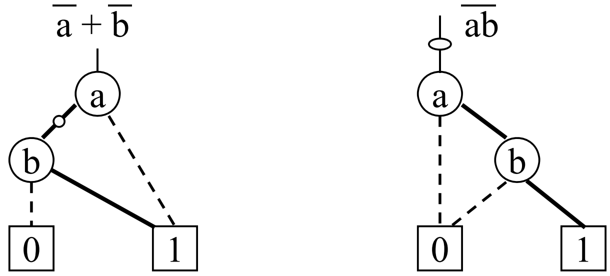
external



internal

Effect of Negation Arcs

- Storage Savings
  - At most 2X reduction in numbers of nodes
- Algorithmic Improvement
  - Can complement function in constant time
- Problem
  - Negation arc allow multiple representations of a function



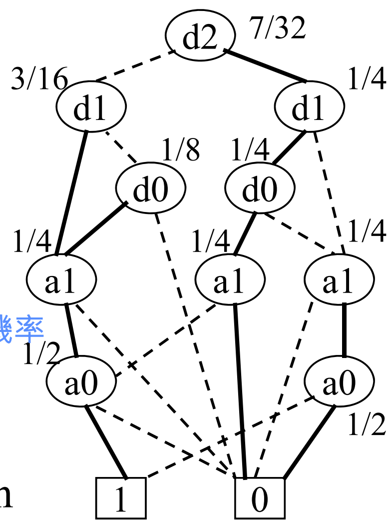
  - Modify algorithms with restricted conversions for use of negative arcs

Density Computation

- Definition
  - p(F) : fraction of variable assignments for which F = 1
- Applications
  - Testability measures
  - Probability computations
- Recursive Formulation
  - p(F) = [p(F[x=1]) + p(F[x=0])] / 2

可以bottom-up的去算每個節點為1的機率

- Computation
  - Compute bottom-up, starting at leaves
  - At each node, average density of children



Characteristic Function

Let E be a set and $A \subseteq E$

The characteristic function of A is the function

$X_A : E \rightarrow \{ 0, 1 \}$

$X_A(x) = 1$ if $x \in A$

$X_A(x) = 0$ if $x \notin A$

Ex :

$E = \{ 1, 2, 3, 4 \}$

$A = \{ 1, 2 \}$

$X_A(1) = 1$

$X_A(3) = 0$

Characteristic Function

Given a Boolean function

$f : B^n \rightarrow B^m$

the mapping relation denoted as $F \subseteq B^n \times B^m$ is defined as

$F(x, y) = \{ (x, y) \in B^n \times B^m \mid y = f(x) \}$

The characteristic function of a function f is defined for (x, y) s.t. $X_f(x, y) = 1$ iff $(x, y) \in F$

Characteristic Function

將input與output放在同一個BDD裡面，通常用在反向求解的時候。
ex: 知道x1與y的值，利用input與output在同一邊的characteristic function就可以反向推得x2的值

Ex : y = f(x1, x2) = x1 + x2

x1	x2	y
0	0	0
0	1	1
1	0	1
1	1	1

Fy(x1, x2, y) =

x1	x2	y	F
0	0	0	1
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

當x1=0, x2=0, 則y必為0，因此000的組合為true-->F=1
--->001的組合為false-->F=0

Summary

- ROBDD
  - Reduced graph representation of Boolean Function
  - Canonical for given variable ordering
  - Size sensitive to variable ordering
- Algorithmic Principles
  - Operations maintain closure property
    - » Result ROBDD with same ordering as arguments
    - » Can perform further operations on results
  - Limited set of basic operations to implement
    - » Restrict, If-Then-Else
    - » Other operations defined in terms of basic operations