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Pricing options on realized variance

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Abstract. Models which hypothesize that returns are pure jump processes with independent increments have been shown to be capable of capturing the observed variation of market prices of vanilla stock options across strike and maturity. In this paper, these models are employed to derive in closed form the prices of derivatives written on future realized quadratic variation. Alternative work on pricing derivatives on quadratic variation has alternatively assumed that the underlying returns process is continuous over time. We compare the model values of derivatives on quadratic variation for the two types of models and find substantial differences.

Key words: Options on variance swaps, options on time changes, self decomposability and its hierarchy

JEL Classification: G10, G12, G13

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1 Introduction

Risk neutral models for the prices of underlying assets are fundamentally models of martingales after adjusting for cost of carry considerations. Ocone (1993) shows that when symmetric martingales are conditioned on their quadratic variation, they become processes of conditionally independent increments. As these latter processes are well understood, one may view the modeling of quadratic variation as fundamental to the study of the risk neutral law for the price of a financial asset.

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Undoubtedly, this law is not symmetric and we shall allow for a prespecified degree of asymmetry in constructing models for the stock price from its quadratic variation. We do however wish to focus attention here on quadratic variation as the primary object to be modeled.

In a related market development, swap contracts on realized variance have now been trading over the counter for some years with a fair degree of liquidity. The floating leg for these so called variance swaps is just the sum of squared daily log price relatives, which differs from the quadratic variation of the log price only by the sampling frequency. The development of these markets for speculating and hedging in quadratic variation suggests that uncertainty in future quadratic variation is clearly perceived. More recently, derivatives whose payoffs are nonlinear functions of realized quadratic variation have also begun to trade over the counter. In particular, a natural outgrowth of the variance swap market is an interest in volatility swaps, which are essentially forward contracts written on the square root of realized quadratic variation. Furthermore, several firms are now making markets in options on realized variance. To price these contracts, it is clearly useful to get a better understanding of the risk-neutral probability law of the prospective outcomes.

These considerations lead us to investigate approaches which directly model the quadratic variation of log returns. Given the large literature on pricing vanilla options and the extent of liquidity in this market, relative to any market for quadratic variation or its derivatives, it is reasonable to restrict attention to models for quadratic variation that are implied by models capable of synthesizing the prices of vanilla options across the strike and maturity spectrum. We will refer to such models as smile-consistent. As is well known, the Black and Scholes (1973) and Merton (1973) model for stock returns is not smile-consistent as no options market has displayed flat smiles for quite some time. Furthermore, since Black/Merton/Scholes model the stock price as geometric Brownian motion, the realized quadratic variation for log returns has no uncertainty and hence is not appropriate for pricing options on realized variance.

In the (time-dependent) Black/Merton/Scholes model, the log price process employed can be characterized as the only continuous time process which has both independent increments and sample paths which are continuous over time. Hence, in generalizing the model, it is natural to consider relaxing either the independent increments assumption or the path continuity assumption (or both). For example, one can relax the independent increments assumption while retaining continuity by using a calibrated local volatility model that relates the instantaneous volatility functionally to the underlying spot price and to calendar time. Such models calibrate to the surface of option prices by design, but the probability laws of realized quadratic variation would be nearly impossible to decipher analytically. Stochastic volatility models also relax the independent increments assumption while retaining continuity. The prototypical model in this class is Heston (1993), which Dufresne (2001) suggests may be used to price options on realized variance. Moreover, Heston and Nandi (2000) consider a special case of the Heston stochastic volatility model (also studied by Janicki and Krajna 2001) for pricing options on realized variance. In contrast to the local volatility models, these models lend themselves to an analytic description of the distribution of realized variance. Moreover, one can further relax path continuity and introduce jumps in either the returns process as shown in Carr et al. (2003) or in the variance process as shown in Nicolato and Venardos (2003). However, these authors do not consider the pricing of derivatives on quadratic variation per se.

Whether jumps are present or not, stochastic volatility models all increase the Markov dimension of the system from one to two, by augmenting the stock price with the level of the instantaneous variance rate. There are many complexities associated with the introduction of this second Markov dimension. In a first attempt at synthesizing the distribution of realized variance, it seems prudent to first consider relaxing the continuity assumption instead of the independent increments assumption, while keeping the Markov dimension equal to one. We have argued elsewhere, both theoretically and empirically, in favor of using processes where the jump component is not only present, but of such high activity that no continuous martingale component is necessary (Geman et al. 2001; Carr et al. 2002). As we intend to keep the independent increments assumption for tractability reasons, we shall be working with the class of pure jump additive processes. To narrow the focus slightly, we study pure jump additive processes which have the additional property that the process at unit time has a distribution which is self-decomposable. One such class of processes which we will study is the set of pure jump Lévy processes, which supplement the independent increments assumption with a stationarity criterion. Another class of pure jump additive processes which we will also study are the time-inhomogeneous processes introduced by Sato (1991). For ease of exposition, we will henceforth drop the modifier "pure jump", when describing the processes studied, it being understood that the only stochastic processes studied in this paper have no continuous martingale component. We will also refer to the time inhomogeneous additive processes introduced by Sato as Sato processes.

To summarize to this point, realistic modelling of quadratic variation requires either the relaxation of independent increments or the relaxation of continuity of returns (or both). The present literature on pricing derivatives on quadratic variation has relaxed the independent increments assumption, while retaining continuity of the sample paths of returns. In contrast, this paper focusses on pricing derivatives on quadratic variation by relaxing the continuity of returns, while retaining independent increments. There is another interesting distinction between the two modelling approaches. A theorem of Monroe (1978) implies that every martingale can be represented as a stochastic time change of a Brownian motion. If the martingale is continuous, then by the well known result of Dambis (1965), Dubins and Schwarz (1965), the stochastic clock used to time change the Brownian motion is just the quadratic variation of the continuous martingale. Hence, a derivative security written on quadratic variation in the continuous context is the same entity as a derivative security written on the stochastic clock used to time change the Brownian motion. However, if the martingale has jump components, then its quadratic variation is distinct from the time change of Brownian motion used to generate it. Building on an insight in Carr and Lee (2004), we show that so long as the stochastic clock is independent of the Brownian motion that it time changes, then one can always price derivatives on this stochastic clock by referring to the market prices of standard options written on the time-changed Brownian motion.

Carr and Lee further assume continuity of returns and use the DDS result to price derivatives on quadratic variation. In a jump context, the quadratic variation of the martingale is distinct from the stochastic clock used to generate it. However, it is an open numerical question as to whether the pricing of derivatives on the clock is at least close to the pricing of derivatives on quadratic variation. Using the CGMY model, we answer this question in the negative by showing that there are large numerical differences between the price of a claim paying the square root of the stochastic clock and the price of a volatility swap.

The remainder of the paper is structured as follows. In the next section, we first study which properties of the return process are inherited by the quadratic variation process. We restrict our analysis to Lévy and Sato processes for returns. The properties studied include infinite activity, variation, complete monotonicity, self decomposability, and membership in the hierarchy of higher orders of decomposability for forward returns and realized variations. We also consider how one may reverse engineer a price process with a pre-specified skewness so that it is consistent with a given quadratic variation process. Details for the specific parametric class of the $CGMY$ model are presented in Sect. 3. In Sect. 4, we provide explicit formulae for the Laplace transform of quadratic variation for the particular Lévy or Sato processes introduced in Sect. 3. Section 5 shows how these transforms may be employed to price options on realized variance and volatility. Section 6 describes how one may synthesize the Laplace transform of implied time changes from the characteristic function for the log price. Section 7 reports on a study comparing the price of a contract paying the square root of the stochastic clock with the price of a contract paying the volatility, as measured by the square root of realized quadratic variation.

2 Quadratic variation processes

We restrict attention to the class of Lévy processes and Sato processes which are consistent with a given self decomposable law when evaluated at unit time. We briefly describe the structure of the Sato process.

2.1 The Sato process

A self decomposable random variable X has the property that for every $c, 0 < c < 1$, there exists an independent random variable $X^{(c)}$ satisfying

$$X \overset{(d)}{=} cX + X^{(c)}. \tag{1}$$

These random variables are infinitely divisible with a Lévy density $k(x)$ of the special form

$$k(x) = \frac{h(x)}{|x|}$$

where $h(x)$ is decreasing for positive x, and increasing for negative x. We define

$$h_p(x) = h(x),\; x > 0$$
$$h_n(-x) = h(x),\; x < 0$$

as the pair of self decomposability characteristics of the self decomposable random variable X. These are both nonincreasing functions defined on the positive half line. We assume that these functions are both differentiable.

One may always also associate with this Lévy density a Lévy process that has the self decomposable law as its unit time distribution. Sato (1991) considered γ self similar processes defined by the property that

$$(X_{ct}, t \geq 0) \stackrel{(d)}{=} (c^{\gamma} X_t, t \geq 0)\,.$$

Sato (1991) proves that for every self decomposable law and every $\gamma > 0$ there exists an additive self similar process with this law at unit time. Carr et al. (2003) identify the Lévy system density of this process $g(x,t)$ as

$$g(x,t) = \begin{cases} \dfrac{\gamma h'\left(\frac{x}{t^{\gamma}}\right)}{t^{1+\gamma}}, & x < 0 \\ -\dfrac{\gamma h'\left(\frac{x}{t^{\gamma}}\right)}{t^{1+\gamma}}, & x > 0 \end{cases}. \tag{2}$$

We shall be concerned here with the process for quadratic variation implied by the Lévy and Sato processes for the underlying stock price associated with a particular self decomposable law at unit time.

2.2 *Lévy and Sato implied quadratic variation*

Consider now the process for the quadratic variation $Q(t)$ of an additive Sato process with the system of Lévy densities $g(x,t)$. The analysis for the Lévy process follows easily on dropping the dependence of $g(x,t)$ on t. The process for quadratic variation is defined in terms of the Lévy or Sato process by

$$Q(t) = \sum_{s \leq t} (\Delta X_s)^2.$$

The following result identifies the Lévy system for the quadratic variation as an increasing additive process.

Theorem 1 *The process $Q(t)$ of quadratic variation associated with the additive process with Lévy system $g(x,t)$ admits as its Lévy system density $q(y,t)$ where*

$$q(y,t) = \frac{g(\sqrt{y},t)}{2\sqrt{y}} + \frac{g(-\sqrt{y},t)}{2\sqrt{y}}. \tag{3}$$

The Lévy case is covered by suppressing the dependence on t in both q and g.

Proof Let $f(x)$ be a test function and consider the evaluation of the expectation

$$E\left[\sum_{s\le t} H_s f\big((\Delta X_s)^2\big)\right]$$

for a bounded predictable process H_s, where $X(t)$ is the given additive process. Since the process

$$M(t)=\sum_{s\le t} H_s f\big((\Delta X_s)^2\big)-\int_0^t\int_{-\infty}^{\infty} H_s f(x^2)g(x,s)dxds$$

is a compensated jump martingale, the required expectation is given by

$$\begin{aligned}
&E\left[\int_0^t\int_{-\infty}^{\infty} H_s f(x^2)g(x,s)dxds\right]\\
&=E\left[\int_0^t\int_0^{\infty} H_s f(y)\left(\frac{g(\sqrt{y},s)}{2\sqrt{y}}+\frac{g(-\sqrt{y},s)}{2\sqrt{y}}\right)dyds\right]\\
&=E\left[\int_0^t\int_0^{\infty} H_s f(y)q(y,s)dyds\right]
\end{aligned}$$

and hence the Lévy system for the quadratic variation is identified by (3). □

As observed earlier, the quadratic variation of a martingale describes an important part of the martingale. Ocone (1993) shows how a symmetric martingale conditional on its quadratic variation is constructed as a process of conditionally independent increments. Given the quadratic variation one has the size and absolute value of all the jumps and conditional on this information, under symmetry, the process is a fair coin toss between the positive and the negative moves. We may generalize somewhat from symmetry in the interests of working with risk neutral processes that are asymmetric.

We define an additive process to be α asymmetric if its Lévy system density satisfies

$$g(-y,t)=e^{\alpha y}g(y,t) \tag{4}$$

where for risk neutral price processes we expect that α will generally be negative. We see from (3) and (4) that one may reverse engineer an α asymmetric process with a given additive quadratic variation with Lévy system $q(y,t)$ density by defining

$$\begin{aligned}
g(\sqrt{y},t)&=\frac{2\sqrt{y}q(y,t)}{\big(1+e^{\alpha\sqrt{y}}\big)},\\
g(-\sqrt{y},t)&=e^{\alpha\sqrt{y}}\frac{2\sqrt{y}q(y,t)}{\big(1+e^{\alpha\sqrt{y}}\big)}.
\end{aligned} \tag{5}$$

This process after a drift correction provides us with an α asymmetric martingale with the given quadratic variation.

2.3 Properties inherited across Lévy, Sato, and implied quadratic variation processes

We now ask what properties are shared by the original Lévy process, the Sato process and the quadratic variation process over forward intervals of time. In particular we are interested in the properties of infinite activity, infinite variation, complete monotonicity, and self decomposability at the initial and higher levels. We present in a summary subsection a statement of the various properties considered, followed in another subsection by a brief discussion of their financial relevance. An analysis of how these properties are shared across the original Lévy process, the Sato process and the implied process for quadratic variation is then taken up in separate subsections devoted to these issues. Apart from questions of self decomposability, these properties have been studied in Carr et al. (2002).

2.3.1 Definition of properties considered

A process of independent and inhomogeneous increments with in general a time inhomogeneous Lévy system density $k(x,t)$ is said to be of infinite activity (IA) if it has the property of infinitely many moves in any interval. This requires that

$$\int_{-\infty}^{\infty} k(x,t)dx = \infty, \text{ for all } t.$$

A process is of infinite variation (IV) if the sum of the absolute values of changes is infinite in any interval, or equivalently the process may not be written as the difference of two increasing processes. This requires that

$$\int_{|x|\leq 1} |x|\, k(x,t)dx = \infty \text{ for all } t.$$

We say that a process has the completely monotone (CM) property if large jumps in absolute value occur at a strictly smaller rate than jumps of a smaller size in absolute value. This property requires that the functions $k_p(x,t) = k(x,t)$ and $k_n(x,t) = k(-x,t), x > 0$, are completely monotone with k^{th} derivatives that have the sign of $(-1)^k$.

Self decomposability (SD) was defined earlier. It is a property of a random variable and we are also interested in its application to the increments $X(t) - X(s)$, $s < t$. We also refer to the (SD) class as the class L of random variables since they are the laws of limit random variables as studied by Lévy (1937) and Khintchine (1938).

To the extent one is interested in forward returns being of the class L, one is led to the subclasses of L identified by Urbanik (1972, 1973) and studied in detail by Sato (1980). The first subclass is L_1 and a self decomposable random variable Y is in L_1 if the residual $Y^{(c)}$ in the self decomposable decomposition is itself self decomposable. The variable is in L_2 if further the residual in the decomposition of $Y^{(c)}$ is itself self decomposable, and so on. The intersection of all the classes L_m over all m is the class L_∞. For a random variable to be in L_m, for fixed $m \geq 1$, it is necessary and sufficient that the functions $h_p(x) = h(x)$ and $h_n(x) = h(-x)$,

$x > 0$, have the property that the functions $a_p(s) = h_p(e^{-s})$, $a_n(s) = h_n(e^{-s})$ be monotone of order $m + 1$. This requires that all regular k^{th} order differences of all sizes δ for $k \leq m + 1$ are positive. The k^{th} order regular difference of size δ for an arbitrary function f is defined as

$$\Delta_\delta^k(f)(s) = \sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} f(s + \delta j).$$

Monotonicity of order m is equivalent to being $m - 2$ times continuously differentiable with all derivatives being nonnegative, nondecreasing and convex.

Jurek and Vervaat (1983) related the L_m property for a Lévy process X with a self decomposable law at unit time to the L_{m-1} property for the Background Driving Lévy Process (BDLP) associated with this process. These two characterizations are simply related to each other via a change of variable. We later use the Jurek and Vervaat characterization to study the hierarchy for the $CGMY$ model. For other related work on self decomposability we refer to Jurek (1983a,b), and Iksanov et al. (2004) for a related hierarchy.

The class L_∞ is particularly interesting; it contains the stable random variables and for a random variable to be in L_∞ the associated functions $a_p(s), a_n(s)$ must be of the form

$$a(s) = \int_0^2 e^{\alpha s} \Gamma(d\alpha)$$

for some positive measure on $(0, 2)$ satisfying

$$\int_0^2 \frac{\pi}{2} \csc\left(\frac{\pi\alpha}{2}\right) \Gamma(d\alpha) < \infty.$$

2.3.2 Financial relevance of properties

We comment briefly on the relevance of each of the properties introduced in the last subsection.

Infinite activity Most financial applications involve the study of prices of exchange traded assets with very high transaction volumes. It is reasonable to employ models with infinite activity to study such price processes and many models considered in the literature have this property. Finite activity models like various forms of the jump diffusion model appear only in conjunction with an infinite activity diffusion component. In the absence of such a component, it is even all the more appropriate to employ infinite activity processes. Carr et al. (2002) argue that in the presence of an infinite activity jump component synthesizing small and large moves, the use of a diffusion component is both theoretically and practically redundant.

Infinite variation Processes with infinite variation are fairly popular in the literature and include continuous diffusions, the normal inverse Gaussian process (Barndorff-Nielsen 1998) and the $CGMY$ model studied in Carr et al. (2002), for $Y > 1$. We observe later that increasing the value of Y does produce laws capable of belonging to the higher self decomposable classes and to the extent that this is a desired property, one may wish to accomodate infinite variation as these properties may be more easily delivered by such a class of processes.

Complete monotonicity The property of complete monotonicity is a structural property of the Lévy density and places the density by Bernstein's theorem in the class of Laplace transforms of positive measures on the half line. Hence we are in a sense considering mixtures of exponentials for the arrival rates of moves. The decay rates of the individual exponentials are an appealing property when we think of price responses to information shocks. For these responses to occur, in magnitude, the information event must reach a large number of people who act on it. As dissemination and actions have built in delay factors and resistances it is natural to speculate that small effects do in fact occur at a faster rate than larger effects. We take this as a good working hypothesis that economises the class of models to be investigated.

Self decomposability and its subclasses Limit laws are probably the best explanation for the wide-spread use of the Gaussian law in the study of financial markets. The self decomposable laws are limit laws and this is also their appeal. They are provably unimodal and have realistic densities associated with them on this account.

Given the interest in forward returns, the considerations that drive us to having realistic densities for the holding period return for various terms suggest the same for forward returns. Hence, these should be unimodal limit laws for the same reasons. Since we now require differences to be selfdecomposable, the original law should at least be in the L_1 class. It may not be necessary to adopt laws from the higher L_m classes but it is interesting to try, as in this case one has limit laws at all levels, whether one models forward returns or forward return spreads and so on.

2.3.3 Results for infinite activity

The Sato process need not have infinite activity even if the initial Lévy process is one of infinite activity. For the Sato process to have infinite activity, say, on the positive side $\mathbb{R}^+$ we require that

$$\int_0^\infty g(x,t)dx = \infty.$$

Substituting for $g(x,t)$ from (2) we require that

$$\infty = -\int_0^\infty \frac{\gamma}{t^{1+\gamma}} h'\left(\frac{x}{t^\gamma}\right) dx = -\int_0^\infty \frac{\gamma}{t} h'(u)du = \frac{\gamma}{t} h(0)$$

Hence we must have the further condition that

$$h(0) = \lim_{x \to 0} xk(x) = \infty$$

or equivalently that $k(x)$ tends to infinity faster than $\frac{1}{x}$.

On the other hand if the Sato process has infinite activity and $g(x,t)$ integrates to infinity then a simple change of variable shows this is equivalent to the corresponding quadratic variation being a process of infinite activity.

2.3.4 Results for infinite variation

The Sato process has infinite variation just if

$$\int_{|x|<1} |x|\, g(x,t)dx = \infty.$$

Again, working on the positive side, we note that we now have

$$-\int_0^\infty \frac{\gamma x}{t^{1+\gamma}} h'\left(\frac{x}{t^\gamma}\right) dx = -\int_0^\infty \frac{\gamma}{t^{1-\gamma}} u h'(u) du = \frac{\gamma}{t^{1-\gamma}} \int_0^\infty h(u) du = \infty.$$

Since $xk(x) = h(x)$ we have that equivalently the original Lévy process must have infinite variation.

Quadratic variation on the other hand is always a process of finite variation as this is an increasing process.

2.3.5 Results on complete monotonicity

The process for quadratic variation for a Lévy system inherits completely monotonicity from this system. The reverse need not be true. Suppose for example that the Sato process has a Lévy system in the completely monotone class. We then have, on the positive side,

$$g(x,t) = \int_0^\infty e^{-ax} \kappa(da)$$

for some positive measure $\kappa(da)$.

The Lévy system for quadratic variation implied by this Lévy system is

$$q(y,t) = \frac{g(\sqrt{y},t)}{\sqrt{y}}$$

and we wish to see that q is completely monotone. For this we note that

$$\begin{aligned} g(\sqrt{y},t) &= \int_0^\infty e^{-a\sqrt{y}} \kappa(da) \\ &= \int_0^\infty \kappa(da) \int_0^\infty e^{-uy} \frac{a}{2\sqrt{\pi u^3}} e^{-\frac{a^2}{4u}} du \\ &= \int_0^\infty e^{-uy} \frac{1}{2\sqrt{\pi u^3}} \int_0^\infty a e^{-\frac{a^2}{4u}} \kappa(da). \end{aligned}$$

Hence $g(\sqrt{y},t)$ is completely monotone. As $\frac{1}{\sqrt{y}}$ is also completely monotone, it follows that the Lévy system for quadratic variation is completely monotone.

On the other hand if quadratic variation has the Lévy system $e^{-y}dy$ of an exponential, the process in the symmetric case would have a system, using (5) for $a = 0$, on the positive side of $2xe^{-x^2}dx$, and this is not completely monotone. Complete monotonicity also does not necessarily pass from the Lévy measure to the Sato process.

2.3.6 Results on self decomposability

Even though the Sato process associated with a self decomposable law need not have a self decomposable law, the process for quadratic variation is self decomposable. For the Sato process to be self decomposable it is necessary that the original Lévy process at time 1 has a law in the L_1 class, as we have commented earlier. The self decomposability for quadratic variation is established by the following theorem.

Theorem 2 *The quadratic variation at unit time of a self similar additive process with a self decomposable law at unit time is itself a self decomposable law.*

Proof By construction we have that

$$q(y,t) = -\frac{h'\left(\frac{\sqrt{y}}{t^\gamma}\right)\gamma}{t^{1+\gamma}2\sqrt{y}} + \frac{h'\left(-\frac{\sqrt{y}}{t^\gamma}\right)}{t^{1+\gamma}2\sqrt{y}}.$$

The characteristic exponent of quadratic variation at unit time is given by

$$\log\left(E\left[e^{iuQ(1)}\right]\right) = \int_0^1\int_0^\infty \left(e^{iuy}-1\right)q(y,s)dyds.$$

It follows that $Q(1)$ is infinitely divisible with Lévy density

$$\begin{aligned}
\int_0^1 q(y,s)ds &= \int_0^1\left(-\frac{h'\left(\frac{\sqrt{y}}{s^\gamma}\right)\gamma}{s^{1+\gamma}2\sqrt{y}} + \frac{h'\left(-\frac{\sqrt{y}}{s^\gamma}\right)}{s^{1+\gamma}2\sqrt{y}}\right)ds \\
&= \int_1^\infty -\frac{h'(\sqrt{y}u^\gamma)\gamma u^{\gamma-1}}{2\sqrt{y}}du + \int_1^\infty \frac{h'(-\sqrt{y}u^\gamma)\gamma u^{\gamma-1}}{2\sqrt{y}}du \\
&= \int_1^\infty -\frac{h'(\sqrt{y}w)}{2\sqrt{y}}dw + \int_1^\infty \frac{h'(-\sqrt{y}w)}{2\sqrt{y}}dw \\
&= \frac{h(\sqrt{y})}{2y} + \frac{h(-\sqrt{y})}{2y}.
\end{aligned}$$

Self decomposability of the original process at unit time then implies self decomposability of the quadratic variation at unit time. □

We also observe from this result that quadratic variation lies in the same L class as the original Lévy process. This is because the function $a(s)$ associated with quadratic variation is the one associated with the original Lévy process evaluated at $s/2$. Hence monotonicity of any order just passes through to quadratic variation.

We now present more direct arguments establishing that forward increments and differences of these increments are self decomposable, provided we begin in the classes L_1 and L_2 respectively. Later we shall identify specific models that lie in the L_1 and L_2 classes that we propose to employ for the purpose of pricing options on quadratic variation.

Theorem 3 *If the law at unit time is in L_1 then the increment of the additive self similar process over an arbitrary interval is a self decomposable law. Further the increments of the associated quadratic variation process also have this property.*

Proof Let X denote the random variable at unit time with an L_1 law. By construction we have that

$$X_t \stackrel{(d)}{=} t^\gamma X_1.$$

By the L_1 property we have that for all $c, 0 < c < 1,$

$$X_1 \stackrel{(d)}{=} c^\gamma X_1 + X^{(c)}$$

where $X^{(c)}$ is a self decomposable random variable independent of X_1. It follows that

$$t^\gamma X_1 \stackrel{(d)}{=} (ct)^\gamma X_1 + t^\gamma X^{(c)} \stackrel{(d)}{=} X_{ct} + t^\gamma X^{(c)}.$$

Hence we see that

$$X_t \stackrel{(d)}{=} X_{ct} + t^\gamma X^{(c)}.$$

As the process for X_t is an additive process the characteristic function of the increment $X_t - X_{ct}$ is that of $t^\gamma X^{(c)}$, a self decomposable random variable.

By an integration similar to the one conducted in Theorem 2 for the quadratic variation we observe that the Lévy density for the increment $X_t - X_u$ is

$$k_{(X_t - X_u)}(x) = \frac{h(\frac{x}{t^\gamma}) - h(\frac{x}{u^\gamma})}{x},$$

and if $X \in L_1$ we have that h is monotone of order 2 so that the first differences of the functions a_p, a_n are positive and nondecreasing. This is equivalent to $h(\frac{x}{t^\gamma}) - h(\frac{x}{u^\gamma})$ being nonincreasing or $X_t - X_u$ being self decomposable.

For the increment $Q_t - Q_u$ in quadratic variation we note that the Lévy density is given by the change of variable $x^2 = y$ applied to the Lévy density for the increment and so

$$k_{(Q_t - Q_u)}(y) = \frac{h(\frac{\sqrt{y}}{t^\gamma}) - h(\frac{\sqrt{y}}{u^\gamma})}{2y}$$

and we require that $a_p(s/2), a_n(s/2)$ be monotone of order 2. $\square$

We note that if one is pricing options on forward returns as for example in forward start options then it would be desirable to employ processes that calibrate the surface of option prices using self similar additive processes associated with a unit time law that is in L_1. The conditions provided by Sato (1980) are useful in this regard.

The class L_2 is also of potential financial interest when pricing return spreads. Here we have a self decomposability result for spreads over scaled intervals. Let $t_1 < t_2$ and consider the interval (ct_1, ct_2) for some $c < 1$. In general we would be interested in cases where $ct_2 < t_1$. The return spread is given by

$$Z = X_{t_2} - X_{t_1} - (X_{ct_2} - X_{ct_1}) .$$

Theorem 4 *For $X \in L_2$ the return spread Z is a self decomposable random variable.*

Proof Since $X \in L_2$ the decomposition of X used in the proof of Theorem 3 produces a residual $X^{(c)}$ that in fact is in L_1. It follows that $X_t - X_u$ belongs to L_1. Therefore for any $c, 0 < c < 1$, there exists a residual that we denote $(X_t - X_u)^{(c)}$ such that

$$X_t - X_u \overset{(d)}{=} c^{\gamma} (X_t - X_u) + (X_t - X_u)^{(c)} .$$

By the scaling property

$$c^{\gamma} (X_t - X_u) \overset{(d)}{=} X_{ct} - X_{cu}$$

and so we may write that

$$X_t - X_u \overset{(d)}{=} X_{ct} - X_{cu} + (X_t - X_u)^{(c)} .$$

Now provided as we suppose that $cu < t$, we may use the additivity of the process to conclude that

$$X_t - X_u - (X_{ct} - X_{cu}) \overset{(d)}{=} (X_t - X_u)^{(c)} .$$

The self decomposability of the spread follows from that of $(X_t - X_u)^{(c)}$ as $X_t - X_u$ is in L_1. □

The higher classes go on to establish the self decomposability of higher order differences, but it is unlikely that contracts would be priced at this level of complexity in differencing. If one wishes to be guaranteed of self decomposability at all levels then processes in L_∞ may be entertained.

2.3.7 The Sato process of the unit time quadratic variation of a Sato process and the quadratic variation of the original Sato process

The quadratic variation of a Sato process has the Lévy system identified by (3) in terms of the Lévy system of the Sato process of a self decomposable law identified by (2). As we observed in the last subsection the unit time quadratic variation has a self decomposable law with the Lévy measure identified by the following Lévy Khintchine decomposition for $Q(1)$:

$$E\left[\exp\left(iuQ(1)\right)\right] = \exp\left(-\int_0^\infty \left(1-e^{iuy}\right)\left[\frac{h(\sqrt{y})}{2y} + \frac{h(-\sqrt{y})}{2\sqrt{y}}\right] dy\right).$$

Theorem 5 *The Sato process associated with $Q(1)$ scaled at 2γ is the same as quadratic variation of the original Sato process with Lévy system (3).*

Proof The required Lévy system for the Sato process associated with $Q(1)$ at 2γ scaling is given on applying (2) by

$$\begin{aligned} & -\frac{2\gamma}{t^{1+2\gamma}}\left[h'\left(\sqrt{\frac{y}{t^{2\gamma}}}\right)\frac{1}{4\sqrt{y/t^{2\gamma}}} - h'\left(-\sqrt{\frac{y}{t^{2\gamma}}}\right)\frac{1}{4\sqrt{y/t^{2\gamma}}}\right] \\ &= -\frac{\gamma}{t^{1+\gamma}}\left[h'\left(\frac{\sqrt{y}}{t^{\gamma}}\right)\frac{1}{2\sqrt{y}} - h'\left(-\frac{\sqrt{y}}{t^{\gamma}}\right)\frac{1}{2\sqrt{y}}\right]. \end{aligned}$$

□

3 Candidate processes for use in pricing options on realized quadratic variation

A number of processes are known to fall in the self decomposable class. These include the Normal Inverse Gaussian (NIG) (Barndorff-Nielsen 1998) and Meixner (Schoutens 2002) models. Unfortunately, except for the VG process the Lévy densities involved are quite complex. The VG process though self decomposable is not in L_1 as the associated function is

$$a(s) = e^{-e^{-s}}$$

and this function is not convex. This fact is also easily seen from the characterization provided by Jurek and Vervaat (1983) as the $BDLP$ for the VG has an exponential Lévy measure which is not a self decomposable Lévy measure.

We consider here the quadratic variation of the Lévy and Sato process with γ scaling, for the unit time distribution of the $CGMY$ Lévy process. This process was introduced in Mantegna and Stanley (1994) and signficantly expanded upon in Koponen (1995) and in Carr et al. (2002). We are interested in determining the levels of the parameter Y consistent with this process belonging to the higher L classes. We work with the Lévy density on the right. Similar calculations apply to negative moves with the parameter G replacing M.

The self decomposability characteristic for this law is given by the function

$$h(x) = C\frac{\exp(-Mx)}{x^Y}.$$

We have noted the VG case of $Y = 0$ that is clearly in L_0 and not in L_1. We now show that $CGMY$ enters L_1 and L_2 for sufficiently large Y values but it appears that $CGMY$ will not enter the class L_7.

We begin with the result of Jurek and Vervaat (1983) that makes the analysis somewhat simpler than relying on the monotonicity characterizations. This result states that a selfdecomposable law is in L_m if and only if the background driving Lévy process (BDLP) Z associated with X is in L_{m-1}. The relationship between X and Z is given by

$$X = \int_0^\infty e^{-s} dZ(s).$$

The relationship between the Lévy measures ν_X for X and ν_Z for Z is given by (e.g., Jeanblanc et al. 2002)

$$\nu_X(dz) = \begin{cases} \dfrac{\nu_Z[-\infty, z]}{|z|}, & z < 0 \\ \dfrac{\nu_Z[z, \infty]}{z}, & z > 0. \end{cases}$$

Hence we define the operator Q acting on f as

$$Q(f) = -\frac{d}{dz}(zf(z))$$

and we note that for positive z, with $\nu_X(z)$ the density of the measure $\nu_X(dz)$, we have the density of the Lévy measure for the process Z given by $\nu_Z = Q(\nu_X)$. For X to be in L_1 we must have that Z is self decomposable and hence for positive z we must have that $z\nu_Z(z)$ is decreasing in z. Alternatively we may write that

$$Q(\nu_Z) > 0.$$

It follows that $X \in L_m$ if and only if $Q^{m+1}(\nu_X) > 0$.

Define for $z > 0$ the function

$$q^0(z) = \frac{e^{-z}}{z^{1+Y}}.$$

We also define

$$q^{(n)}(z) = Q^n(q^0) = q^0(z)P_n(z)$$

where $P_n(z)$ is a polynomial in z. Differentiation yields the recursion

$$P_{n+1}(z) = P_n(z)(z+Y) - P_n'(z)z.$$

This recursion may be easily implemented using polynomial manipulations and we note that

$$P_2(z) = (z+Y)^2 - z$$

which is positive for $Y > 0.25$.

We implemented this recursion to order 20 for various values of Y and evaluated the zeros of the resulting polynomials to detect the presence of positive real roots. For a given Y the last polynomial P_n with no positive real roots defines the highest level $n(Y)$ in the L hierarchy that $CGMY$ attains with this particular value for Y. The table below shows the results, and we see that for these values of Y, $CGMY$ never enters L_7.

Y	0.25	0.5	0.75	1	1.25	1.5	1.75	1.999
$n(Y)$	1	1	2	3	4	5	6	6

This brings us to present the *conjecture*: The integer valued function : $Y \to n(Y), Y \in (0,2)$, is nondecreasing, and smaller than or equal to 6.

4 The Laplace transform for realized quadratic variation at time t

The Laplace transform of quadratic variation for the self similar additive process with unit time law given by the $CGMY$ model is given by

$$E\left[e^{-\lambda Q(t)}\right] = \exp\left(\int_0^t \int_0^\infty \left(e^{-\lambda y} - 1\right) q(y,s)dyds\right).$$

Performing the integration in the exponential with respect to the time variable as in the proof of Theorem 2 we get

$$E\left[e^{-\lambda Q(t)}\right] = \exp\left(\int_0^\infty \left(e^{-\lambda y} - 1\right)\left[\frac{h\left(\frac{\sqrt{y}}{t^\gamma}\right)}{2y} + \frac{h\left(-\frac{\sqrt{y}}{t^\gamma}\right)}{2y}\right] dy\right). \quad (6)$$

For the specific case of the $CGMY$ model we get for the Laplace transform

$$\begin{aligned} E\left[e^{-\lambda Q(t)}\right] &= \exp\left(-\int_0^\infty \left(1 - e^{-\lambda y}\right) Ct^{\gamma Y} \frac{e^{-\frac{M}{t^\gamma}\sqrt{y}} + e^{-\frac{G}{t^\gamma}\sqrt{y}}}{2y^{1+\frac{Y}{2}}} dy\right) \\ &= \exp\left(-Ct^{\gamma Y}\int_0^\infty \left(1 - e^{-\lambda x^2}\right)\frac{e^{-\frac{M}{t^\gamma}x} + e^{-\frac{G}{t^\gamma}x}}{x^{1+Y}} dx\right). \end{aligned}$$

We define

$$A = Ct^{\gamma Y},\ a_p = \frac{M}{t^\gamma},\ a_n = \frac{G}{t^\gamma}$$

and write that

$$E\left[e^{-\lambda Q(t)}\right] = \exp\left(-A\int_0^\infty (1 - e^{-\lambda x^2})[e^{-a_p x} + e^{-a_n x}]x^{-1-Y} dx\right).$$

Integrating by parts gives

$$\int_0^\infty (1 - e^{-\lambda x^2}) e^{-ax} x^{-1-Y} dx$$
$$= -\int_0^\infty \frac{x^{-Y}}{-Y} \left[e^{-ax-\lambda x^2} 2\lambda x - a(1 - e^{-\lambda x^2}) e^{-ax} \right] dx.$$

We now write the result as the sum of two integrals to get

$$\frac{2\lambda}{Y} \int_0^\infty x^{(2-Y)-1} e^{-ax-\lambda x^2} dx - a \int_0^\infty \frac{x^{-Y}}{Y} (1 - e^{-\lambda x^2}) e^{-ax} dx.$$

Performing the second integral by parts we get

$$\frac{2\lambda}{Y} \int_0^\infty x^{(2-Y)-1} e^{-ax-\lambda x^2} dx + \frac{2\lambda a}{Y(1-Y)} \int_0^\infty x^{(3-Y)-1} e^{-ax-\lambda x^2} dx$$
$$-\frac{a^2}{Y(1-Y)} \frac{\Gamma(2-Y)}{a^{2-Y}} + \frac{a^2}{Y(1-Y)} \int_0^\infty x^{(2-Y)-1} e^{-ax-\lambda x^2} dx.$$

Now we use the fact that

$$I(\nu, a, \lambda) = \int_0^\infty x^{\nu-1} e^{-ax-\lambda x^2} dx = (2\lambda)^{-\nu/2} \Gamma(\nu) h_{-\nu} \left(\frac{a}{\sqrt{2\lambda}} \right)$$

where $h_\nu(z)$ is the Hermite function defined by

$$h_\nu(z) = \frac{1}{2\Gamma(-\nu)} \sum_{j=0}^\infty \Gamma\left(\frac{j-\nu}{2} \right) 2^{\left(\frac{j-\nu}{2}\right)} \frac{(-z)^j}{j!}.$$

The Hermite functions may be evaluated in terms of the hypergeometric functions U. Specifically,

$$h_\nu(z) = 2^{\left(\frac{\nu}{2}\right)} U\left(\frac{-\nu}{2}, \frac{1}{2}, \frac{z^2}{2} \right).$$

The hypergeometric U function can be obtained in terms of the confluent hypergeometric function ${}_1F_1$ using

$$U(a, b, z) = \frac{\pi}{\sin(\pi b)} \left\{ \frac{{}_1F_1(a, b, z)}{\Gamma(1 + a - b)\Gamma(b)} - z^{1-b} \frac{{}_1F_1(1 + a - b, 2 - b, z)}{\Gamma(a)\Gamma(2 - b)} \right\}.$$

We write the final result as

$$- \log E \left[\exp\left(-\lambda Q(t)\right) \right] / A \tag{7}$$
$$= \left[\frac{2\lambda}{Y} + \frac{a_p^2}{Y(1-Y)} \right] I(2 - Y, a_p, \lambda) + \left[\frac{2\lambda}{Y} + \frac{a_n^2}{Y(1-Y)} \right] I(2 - Y, a_n, \lambda)$$
$$+ \frac{2\lambda a_p}{Y(1-Y)} I(3 - Y, a_p, \lambda) + \frac{2\lambda a_n}{Y(1-Y)} I(3 - Y, a_n, \lambda)$$
$$- \frac{a_p^Y}{Y(1-Y)} \Gamma(2 - Y) - \frac{a_n^Y}{Y(1-Y)} \Gamma(2 - Y).$$

For the $CGMY$ Lévy process the result for the Laplace transform of quadratic variation is as in (7) with

$$A = Ct,\ a^p = M,\ a^n = G.$$

As the quadratic variation for Lévy process and Sato process in the $CGMY$ case have parametrically the same Laplace transforms we shall restrict all computations to the $CGMY$ case.

5 From the Laplace transform of quadratic variation to options on realized variance

An option on quadratic variation with strike K and maturity t pays at maturity the sum

$$(Q(t) - K)^+ .$$

The time zero value of this option using risk neutral valuation is given by

$$w(K,t) = E\left[e^{-rt}\left(Q(t) - K\right)^+\right].$$

Similar to the procedure in Carr and Madan (1998), we now define the Laplace transform (with respect to the strike) of the option value by

$$\gamma(\lambda,t) = \int_0^\infty e^{-\lambda K} w(K,t)dK,$$

and we observe on integration that

$$\gamma(\lambda,t) = e^{-rt}\left[\frac{\phi(\lambda,t)-1}{\lambda^2} - \frac{E[Q(t)]}{\lambda}\right],$$
$$\phi(\lambda,t) = E\left[e^{-\lambda Q(t)}\right].$$

Option prices may then be computed on inverting $\gamma(\lambda,t)$.

For the $CGMY$ Lévy process the expectation of quadratic variation may be explicitly computed on integrating x^2 against the Lévy density. The result is

$$E[Q(t)] = Ct\Gamma(2-Y)\left[\frac{1}{M^{2-Y}} + \frac{1}{G^{2-Y}}\right].$$

For options paying the positive part of the square root of $Q(t)$ minus the square root of the variance strike, K, we have the payoff

$$\left(\sqrt{Q(t)} - \sqrt{K}\right)^+ .$$

This contract may be valued as a portfolio of options on $Q(t)$ using standard methods for representing functions of an underlying risk as portfolios of options (see for example Carr and Madan 2001).

6 The Laplace transform of time changes

Suppose the uncertainty underlying the stock price is of the form of time changed Brownian motion with drift. Specifically we have

$$\begin{aligned} S(t) &= S(0)e^{(r-q+\omega)t+X(t)}, \\ \omega &= -\log E\left[\exp(X(t))\right] \end{aligned}$$

where $X(t)$ is of the form

$$X(t) = \gamma T(t) + W(T(t))$$

for a Brownian motion $W(t)$ and an increasing random time change process $T(t)$ independent of the Brownian motion. Following Carr and Lee (2004), who restricted attention to continuous time changes, one may develop the Laplace transform of the time change in terms of the characteristic function of log prices as described below.

We first note that many models fall in this category and include the Variance Gamma, the Normal Inverse Gaussian, and as we shall show here the $CGMY$. Evaluating for a possibly complex parameter p we observe that

$$S(t)^p = S(0)^p e^{p(r-q+\omega)t+p\gamma T(t)+pW(T(t))}.$$

Taking expectations first conditionally on the time change and then unconditionally we get that

$$E\left[S(t)^p\right] = S(0)^p e^{p(r-q+\omega)t} E\left[\exp\left(\left(\gamma p + \frac{p^2}{2}\right)T(t)\right)\right].$$

Solving for the inner exponent to be zero gives

$$p(\lambda) = -\left(\gamma + \sqrt{\gamma^2 - 2\lambda}\right)$$

and therefore

$$E\left[\exp\left(-\lambda T(t)\right)\right] = \phi_{\ln S}(-ip(\lambda))S(0)^{-p(\lambda)}\exp(-p(\lambda)(r-q+\omega)t)$$

where $\phi_{\ln S}(u)$ is the characteristic function of log prices. This characteristic function is available as soon as any option pricing model has been calibrated to option prices. For $CGMY$ the appropriate choice for γ is $(G-M)/2$.

Substitution of this γ in the explicit characteristic function for the $CGMY$ yields on simplification the Laplace transform of the subordinator $T(t)$ as

$$\begin{aligned} E[e^{-\lambda T(t)}] &= \exp\left(tC\Gamma(-Y)\left[2r^Y\cos(\eta Y) - M^Y - G^Y\right]\right), \\ r &= \sqrt{2\lambda + GM}, \\ \eta &= \arctan\left(\frac{\sqrt{2\lambda - \left(\frac{G-M}{2}\right)^2}}{\left(\frac{G+M}{2}\right)}\right). \end{aligned}$$

This Laplace transform may be employed to price options on the time change. For a continuous time change the result is tantamount to pricing options on quadratic variation. In the next section we implement both the pricing of options on quadratic variation for the $CGMY$ Lévy process and the pricing of options on the time change implicit in writing $CGMY$ as time changed Brownian motion with drift.

7 A numerical illustration

We calibrated the $CGMY$ model to prices of options on the $S\&P$ 500 index for the maturities 0.7968 and 1.0453 for June 2, 2003. The results of the calibration are presented in Table 1.

Table 1. CGMY on SPX 20030602

Maturity	C	G	M	Y	RMSE	AAE	APE	NOP
0.7968	0.3970	4.3120	19.5587	0.5839	0.0948	0.0794	0.0034	24
1.0453	0.3251	3.7103	18.4460	0.6029	0.1141	0.0889	0.0034	25

Apart from the parameter estimates, we present the root mean square error (RMSE), the average absolute error (AAE), the average percentage error (APE) and the number of options used (NOP). We observe the calibration is stable across the two maturities and the quality of the fit is good.

7.1 Quadratic variation and its square root

For reasons of numerical stability it is useful to work with the quadratic variation of the process $100X(t)$ as opposed to $X(t)$ and this has the effect of dividing G, M by 100 and scaling C by 100^Y. The resulting quadratic variations have to be scaled back by a factor of 10000 to get back to the quadratic variation of $X(t)$. It is customary to quote the associated entities in annualized volatility terms and this requires that we divide the realized variance by time to maturity and then take the square root. The expected payout on realized variance at the two maturities in annualized volatility terms was 22.28% and 22.61% respectively.

Prices of options on realized variance at these two maturities for strikes quoted in vol terms at $10\%, 20\%$ and 30% for the maturity 0.7968 were 19.68%, 15.38% and 11.89% respectively. The corresponding figures for the maturity 1.0453 were 20.57%, 16.73% and 13.40%.

To develop a quote on the square root of realized variance we first construct an optimal linear hedge that is attained by a bond position and a position in the realized variance contract that is tangent to the square root function at the level of the expected realized variance. We used 22% for the level of the tangent. The bond position is 0.0982 and the position in the realized variance contract is 2.5461 for the maturity 0.7968. For the maturity 1.0453 the corresponding values are 0.1125

and 2.2229. This linear hedge attains the Jensen's inequality upper bound that is the square root of 22%.

We may now cheapen the quote on the square root by writing options that bend the payoff of the hedge position to get down to the square root function. We just did this computing the slopes at a selected set of hedge strikes and writing options to the order of the change in the slope to the right of the tangency point while we wrote puts to the left of the tangency point, again to the order of the change in slope. We used the strikes of 100 to 900 in steps of 100 on the scale 10000 times realized variance. The resulting positions are provided in Table 2.

Table 2. Option positions for square root contract maturity

Strike	0.7968	1.0453
100	−1.4565,P	−1.4565,P
200	−0.6488,P	−0.6488,P
300	−0.3407,P	−0.3868,P
400	−0.2639,C	−0.2639,P
500	−0.1948,C	−0.1817,P
600	−0.1514,C	−0.1514,C
700	−0.1221,C	−0.1221,C
800	−0.1011,C	−0.1011,C
900	−0.0855,C	−0.0855,C

The resulting quotes on the square root of realized variance for the maturities 0.7968 and 1.0453 are respectively 17.47 and 20.93. The earlier maturity quote is considerably below the Jensen's inequality upper bound.

7.2 The CGMY time change and its square root

For the time change of Brownian that yields the $CGMY$ Lévy processes as calibrated at the two maturities we obtain the expected time change quoted here in volatility terms by numerically differentiating the Laplace transforms that are evaluated as described in Sect. 6. The resulting values are 16.64% and 16.43% at the 0.7968 and 1.0453 maturities respectively.

Prices of options on the time change at strikes in volatility terms of 8.66%, 15% and 20.61% at the 0.7968 maturity are 13.65%, 8.56% and 4.37%. The corresponding values at 1.0453 are 14.11%, 9.75% and 5.69%.

Constructing the tangent for the Jensen's inequality upper bound, the bond positions are 0.0714 and 0.0818 at maturities 0.7968 and 1.0453 respectively. The positions in the underlying time change are respectively 3.5 and 3.05. We used 13 strikes and constructed positions in a manner similar to the quadratic variation case. The resulting quotes for the square root of the time change are 14.21% and 16.04%.

We note that these quotes compare well with the quadratic variation quotes, with the exception that they are lower. This reflects the possibility that the time change is not the quadratic variation and is an underestimate of it.

8 Conclusion

This paper takes up directly the modeling of the evolution of quadratic variation for the logarithm of the stock price. Our earlier work had shown that good models for the prices of vanilla options across all strikes and maturities at a given point of time are provided by the class of Lévy processes associated with infinitively divisible self decomposable limit laws for the variable of unit period returns. Here we investigate the process for quadratic variation implied by this Lévy process as well as the associated Sato process for the log returns.

A number of process properties are described, including infinite activity, variation, complete monotonicity, and various levels of self decomposability. In each case we describe how these properties pass from the Lévy process to the implied process for quadratic variation. We identify a particularly useful parametric class of models derived from the $CGMY$ price process that delivers self decomposability at the second level so that return spreads are limit laws. For the quadratic variation of $CGMY$ we provide closed form formulae for Laplace transforms of quadratic variation and show how these may be used directly to price options on realized variance and realized volatility. We also investigate $CGMY$ as a time changed Brownian motion and show that the time change falls substantially short of the quadratic variation.

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