Chapter 2

THE BUFFON'S NEEDLE PROBLEM: FIRST MONTE CARLO SIMULATION

9/13/2013

2.1 INTRODUCTION

George Louis Leclerc, also known as Le Comte de Buffon lived during the period of 1707-1788 in France. Buffon, as he came to be known, was an able mathematician and scientist. He estimated the age of the Earth to be about 75,000 years, when the eighteenth century consensus was that it could not be older than 6,000 years.

He is famous for a test of the feasibility of Archimedes' mirror system used in the defense of Syracuse. Plutarch had described plausible actions of cranes and missile throwers built by Archimedes. By the Middle Ages, his exploits were much embellished, and they had grown into a legend. The Byzantine author John Tzetzes (1120-1183) reported a story that was not included in Plutarch's initial description, that Archimedes had burned the Roman ships laying siege to Syracuse at a distance of a bow's shot by focusing the sun's rays using a system of mirrors.

Buffon undertook the testing of such a scheme by using 168 flat mirrors of side lengths of 6 by 8 inches, placed in a flexible framework. He was able to ignite planks of wood at a distance of 150 feet, and satisfied himself that the Archimedes' scheme was feasible. He did not satisfy his critics though, who suggest that the Syracusans would not have had the luxury of having stationary ships floating on the sea, nor the time to focus 168 light beams at a single spot. Archimedes' and Buffon's light focusing concept is still well alive in modern days' directed energy beams (lasers or charged particles) of "Star Wars" fame.

Buffon reserves more fame with a problem that he posed and solved in 1777, known as the Buffon's needle problem. Laplace ingeniously used it for the estimation of the value of π, in what can be considered as the first documented application of the Monte Carlo method. The Monte Carlo simulation method offers a creative solution to the Buffon's needle problem using modern computers as a tool.

2.2 THE BUFFON'S NEEDLE PROBLEM

The statement of the Buffon's needle problem, shown in Fig. 1, is as follows:

> *"Let a needle of length L be thrown at random onto a horizontal plane ruled with parallel straight lines spaced by a distance d from each other, with d > L. What is the probability p that the needle will intersect one of these lines?"*

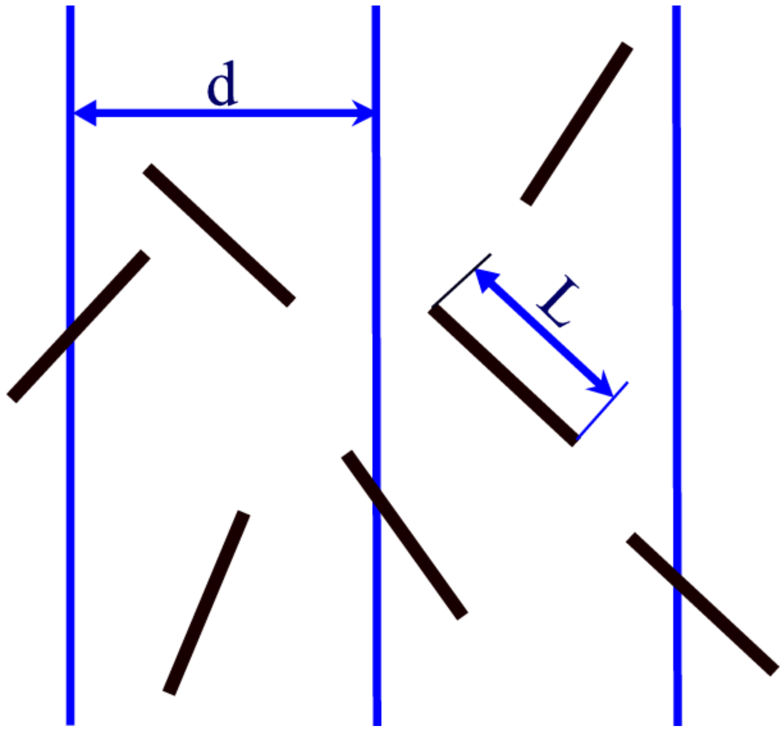

Figure 1. In The Buffon's needle experiment, needles of length L are tossed randomly on a horizontal plane ruled with parallel lines spaced by a distance d, with L<d.

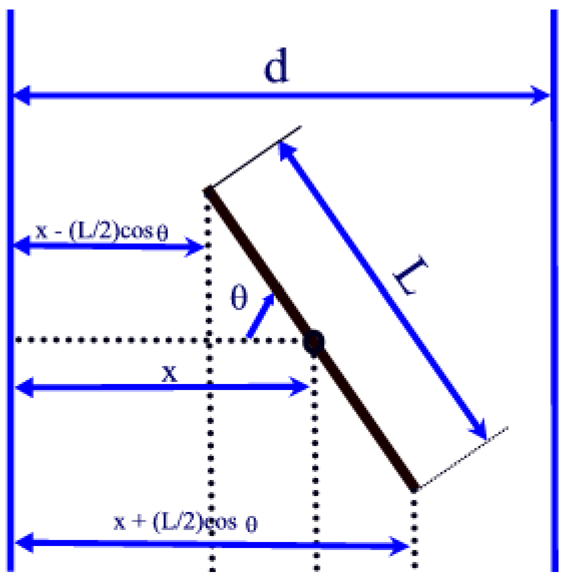

Figure 2. Geometry of needle position relative to the parallel lines in the Buffon needle problem.

The implied randomness here means that any position of the center of the needle from the nearest line be x, and any orientation angle, θ of the needle are equally probable, and that these two random variables x and θ are independent of each other.

As shown in Fig. 2, the angle θ occurrence is uniformly distributed over the interval:

$$\theta \in [-\frac{\pi}{2}, +\frac{\pi}{2}].$$

The position x of the center of the needle is uniformly distributed over the interval:

$$x \in [0, d].$$

Two conditions must be simultaneously satisfied for the needle *not* to intersect the parallel lines:

$$(x - \frac{L}{2}\cos\theta \geq 0)\, AND\, (x + \frac{L}{2}\cos\theta \leq d)$$

These can be rewritten as:

$$(x \geq \frac{L}{2}\cos\theta)\, AND\, (x \leq d - \frac{L}{2}\cos\theta)$$

The curves:

$$x = \frac{L}{2}\cos\theta, \qquad (1)$$

and:

$$x = d - \frac{L}{2}\cos\theta \qquad (2)$$

define the boundary for the intersection by the needle with the parallel lines, as shown in Fig. 3.

We can express the probability of non-intersection of the needle in the Buffon's needle problem in terms of the shaded area A bounded by the two curves of Eqs. 1 and 2, and the rectangular enclosing area πd in Fig. 3 as follows:

$$p_{\text{non-intersection}} = \frac{A}{\pi d} \qquad (3)$$

where:

$$A = \pi d - 2\int_{-\pi/2}^{+\pi/2} \frac{L}{2}\cos\theta \, d\theta$$

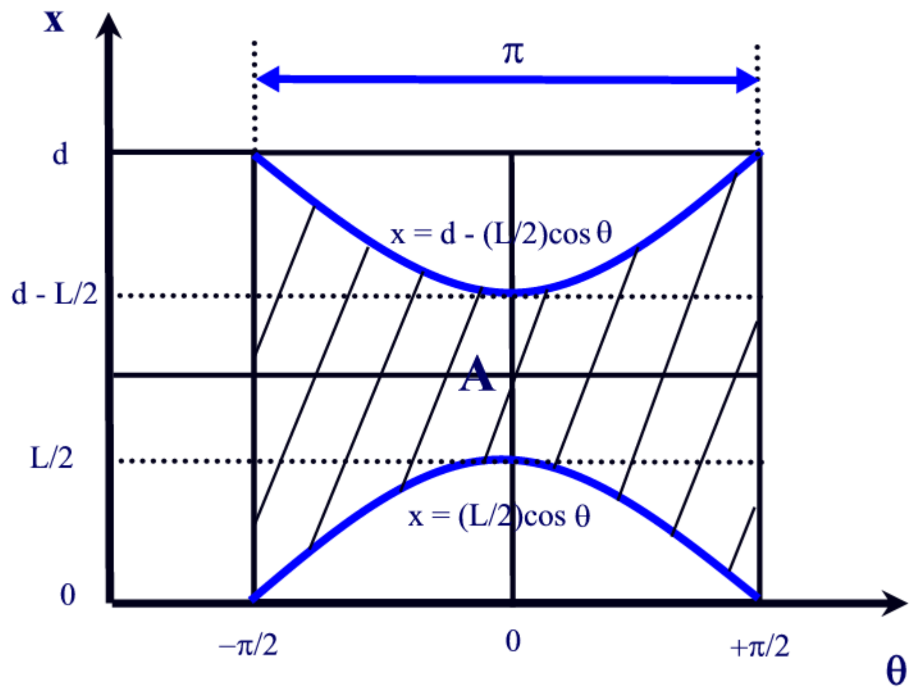

Figure 3. Boundaries for the intersection of the needles with the parallel lines as a function of the lines spacing d and needle orientation angle θ.

Carrying out the integration yields:

$$A = \pi d - 2L \tag{4}$$

Substitution of Eqn. 4 into Eqn. 3 yields:

$$p_{\text{non-intersection}} = \frac{\pi d - 2L}{\pi d} = 1 - \frac{2L}{\pi d} \tag{5}$$

The probability p of intersection of the dropped needles with the parallel lines becomes:

$$p = 1 - p_{\text{non-intersection}} = \frac{2L}{\pi d} \tag{6}$$

This is the result obtained by Buffon. He attempted to calculate the probability p by throwing a needle a number of times onto ruled paper and observing the fraction of intersections out of the total number of trials.

2.3 LAPLACE'S ESTIMATION OF THE VALUE OF Pi (π)

Buffon's solution remained dormant for 35 years, until another great French mathematician: Pierre Simon Laplace (1736-1813) gave it a novel twist.

Laplace and his two contemporaries Joseph Louis Lagrange (1749-1827) and Adrien Marie Legendre (1752-1833) survived the French Revolution as members of the committee of Weights and Measures, which introduced the metric system of units as used today. Lazare Carnot was at this time a member of the Committee for Public Safety, in charge of military affairs, became "organisateur de la victoire," a hero of the French Revolution. Carnot's chair of geometry at the Institut National was voted to Napolean Bonaparte, and he went into exile as a royalist, observing that: "A revolution replaces a tyranny by another." The chemist Lavoisier, discoverer of the oxidation process, was less lucky and his neck tragically met with the guillotine as a royalist.

Napoleon appointed Laplace as Minister of Interior, and then promptly dismissed him after six weeks. Laplace authored two books: "Mécanique Céleste" in 5 volumes from 1799 to 1825, considered as a great successor in celestial mechanics to Newton's "Principia," and "Théorie Analytique des Probabilitées" in 1812. In the former book, Laplace introduced the theory of potential. The latter book is the foundation of modern probability theory. It introduced the integral transform known as the Laplace Transform, which is the basis of modern control theory, electrical circuits design, and systems analysis. In the same book he included a discussion of the Buffon's needle problem, which he saw in a new light. From Eqn. 6, Laplace suggested that a method of estimating the value of π could be used as:

$$\pi = \frac{2L}{pd} \tag{7}$$

If the length L and the spacing d are known, one needs to estimate p to obtain the value of π. Some people would choose L = d, but the choice is arbitrary.

In the special case where the length of the needle is equal to half the spacing between the parallel lines:

$$L = \frac{d}{2},$$

The intersection probability becomes simply:

$$p = \frac{1}{\pi} \tag{8}$$

According to the last equation, for a sufficiently large number of droppings of the needle on the parallel lines, the ratio of hits over the number of trials would be equal to 1/π, and hence offers an ingenious way of calculating the value of π:

$$\pi = \frac{1}{p} = \frac{1}{Lim_{N\to\infty}(\frac{n}{N})} \tag{9}$$

where N is the total number of trials, and n is the number of intersections of the needle with the parallel lines. It is not difficult to estimate p hence the value of π correct to a few decimal places with N experiments.

Equation 9 can be written in the general case as:

$$\pi = \frac{2L}{pd} = \frac{2L/d}{Lim_{N\to\infty}(\frac{n}{N})} \tag{10}$$

Laplace generalized the method for a sheet of paper with two sets of mutually perpendicular lines. It has been used as a game to calculate the first decimal places of the value of π using thousands of throws.

Recovering from wounds incurred in the American Civil War, an officer known as Captain Fox killed his boredom applying the Laplace's method for estimating π. His experimentation was reported by A. Hall in 1873 in: "On an experimental determination of the value of Pi."

2.4 COMPUTER ESTIMATION OF THE VALUE OF Pi (π)

Laplace had discovered a powerful method of computation that was well ahead of its time. It came to its own with the advent of modern computers. What Laplace proposed was a method of numerically estimating a mean value, or a mathematical expectation, by generating a random event that is sampled a large number of times N, and observing its outcome experimentally. This is what became to be known as the Monte Carlo method. Modern computers allow the numerical realization of the experiment with a speed and a number of trials N unrealizable as a hand experiment. Hand experiments would in fact be very inefficient; since the probability of obtaining a correct answer to five decimal places in 3,400 throws would be less than 1.5 percent.

A computer procedure to estimate π written in the Fortran programming language is shown in Fig. 4. Table 1 shows the results for a number of estimates ranging from 100 throws to 1 billion throws, which was easily done on a laptop machine.

The procedure uses a random number r_1 uniformly distributed over the unit interval:

$$r_1 \in [0,1],$$

to sample a value of the angle θ uniformly distributed over the interval from zero to ninety degrees:

$$\theta = \frac{\pi}{2} r_1 ,$$

resulting in an angle:

$$\theta \in \left[0, \frac{\pi}{2} \right]$$

The cosine of the sampled angle θ is calculated and the projection of half the length of the needle horizontally is calculated as:

$$S = \frac{L}{2} \cos\theta , \tag{11}$$

as given by Eqn. 1 and Fig. 2.

Another random number r_2 is generated to sample the position of the center of the needle by multiplying it by half the lines spacing d/2:

$$T = \frac{d}{2} r_2 \tag{12}$$

The sampled position T of the center of the needle is next compared to the sampled projection of half the length of the needle. If the position of the center is less or equal than the sampled half length projection, or:

$$T \leq S \Rightarrow (hits = hits + 1) ,$$

the needle is considered as touching the parallel lines, and the hits counter is incremented by one hit. Otherwise no hit is scored, and a new throw of the needle is sampled. After N trials, the number of hits n is used to calculate the probability p and the value of π according to Eqns. 9 or 10.

```
!	buffon_needle f90
!	Buffon Needle Problem
!	Monte Carlo Simulation of needle dropping on parallel lines
!	Simulation is used to determine the value of pi
!	M. Ragheb

	program buffon_needle

!	Using double precision variables
	real(8) pi,spacing,needle_length,ninety_degrees,position,theta,hits,trials
!	Initialization of variables
!	Value of Pi, 17 decimal places can be stored for double precision constants
	pi=3.14159265358979324
!	Alternative value generated by the compiler as: pi= 4 arctan(1)
!	pi=4.0*datan(1.0D0)
!	Number of hits
```

```
        hits=0.0
!       Number of experiments or trials
        trials = 1.0e+08
!       Spacing between lines
        spacing = 2.0
!       needle_length (e. g. spacing /2.0)
        needle_length = 1.0
!       Ninety degrees as pi/2.0
        ninety_degrees = pi/2.0
!       Simulate needle dropping
        ntrials=trials
        do i = 1 , ntrials
                call random (rr)
                position =  (spacing / 2.0) * rr
                call random (rr)
                theta = ninety_degrees * rr
                costheta = cos (theta)
                projection = (needle_length / 2.0 ) * costheta
!       Test for position of needle point
                        if( position .LE. projection ) then
!       Score a hit
                                hits = hits + 1.0
                        endif
        end do
!       Evaluation of pi
        prob=hits/trials
        pi = (2.0*needle_length)/(spacing*prob)
!       Write results
        write(*,*) 'Variables:'
        write(*,*) 'needle_length = ', needle_length
        write(*,*) 'spacing = ', spacing
        write(*,*) 'trials  = ', trials
        write(*,*) 'hits    = ', hits
        write(*,*) 'prob = ', prob
        write(*,*) 'pi      = ', pi
        end
        stop
```

Figure 4. Procedure for estimating the value of Pi, π using the Buffon's needle problem.

Table 1. Estimates of the value of π, for different numbers of trials N.

Number of trials, N	Estimate of π
10^1	4.999999
10^2	3.703704
10^3	3.194888
10^4	3.193868
10^5	3.147326
10^6	3.138978
10^7	3.140663
10^8	3.141302
10^9	3.141682

Notice that we cannot hope to obtain a better value of:

$$\pi = 3.14159265358979324,$$

than the one that was used in the sampling of the angle θ, even in double precision which keeps 17 significant digits, because of the introduction of round-off error, particularly in the estimation of the cosine function.

If better approximations are needed, the Lambert's and Euler's methods offer ways of getting better approximations to the value of π.

2.5 LAMBERT'S METHOD FOR ESTIMATING π

The Swiss mathematician Johann Heinrich Lambert (1728-1777) gave a continued fraction for π, which gives good approximations as:

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \ldots}}}}}}}} \tag{13}$$

The first ten convergents of this continued fraction are:

3:1
22:7
333:106
355:113
103993:33102
104348:33215
208341:66317
312689:99532
833719:265381
1146408:364913
… : …

They offer valuable approximations. For instance, the value 355/113 is correct to six decimal places, and was suggested by Adriaan Anthoniszoon, a Dutch mathematician and fortification engineer.

2.6 EULER'S METHOD FOR ESTIMATING π

Leonhard Euler(1707-1783) was born in Basel, Switzerland, and published 886 books and mathematical memoirs, even after he lost his eyesight in 1771. He provided several formulae for the estimation of π in his book: “Introductio in Analysin infinitorium” in 1748. This book standardized the mathematical notation used today, and with other Euler work, introduced the symbols of modern mathematics: π, f(x), Σ, ∫, e, and i.

In particular, he considered the summation of the series of inverse squares:

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+... \qquad (14)$$

The summation of this series had baffled other earlier mathematicians. Euler in 1736 considered the trigonometric function series:

$$\sin x = x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+... \qquad (15)$$

This can be considered as an equation of degree infinity as:

$$\sin x = x(1-\frac{x^2}{3!}+\frac{x^4}{5!}-\frac{x^6}{7!}+...)=0 \qquad (16)$$

The roots of Eqn. 16, excluding zero, are:

$$\pm\pi,\pm 2\pi,\pm 3\pi,...$$

Substituting $x^2 = y$ into Eqn. 16 yields:

$$\sin y^{\frac{1}{2}} = y^{\frac{1}{2}}(1-\frac{y}{3!}+\frac{y^2}{5!}-\frac{y^3}{7!}+...)=0 \qquad (17)$$

For y not equal to zero, Eqn. 17 becomes:

$$(1-\frac{y}{3!}+\frac{y^2}{5!}-\frac{y^3}{7!}+...)=0 \qquad (18)$$

In similarity to Eqn. 16, the roots of Eqn. 18 are the squared values:

$$\pm\pi^2,\pm(2\pi)^2,\pm(3\pi)^2,...$$

According to the theory of equations, a branch of higher algebra, the negative coefficient of the linear term y in Eqn. 18 is the sum of the reciprocals of the roots of the equation, from which:

$$\frac{1}{3!} = \frac{1}{\pi^2} + \frac{1}{(2\pi)^2} + \frac{1}{(3\pi)^2} + \dots \tag{19}$$

Equation 19 can be rearranged as:

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \dots \tag{20}$$

This provides a solution to the problem of the sum of the inverse squares, and as an added bonus, a series for calculating the value of π to any desired number of significant digits.

There may not be a compelling scientific or practical purpose in using more than the 17 decimal places that double precision provides. The calculation of π to 100, 500 or 10,265 decimal places is only of theoretical importance, except maybe for checking that a new computer design is performing its calculations accurately or as a cipher key.

Incidentally, the March 13 date is usually designated as "Pi Day."

A value of π that is correct to 100 decimal digits is:

π =3. 14159 26535 89793 23846 26433
83279 50288 41971 69399 37510
58209 74944 59230 78164 06286
20899 86280 34825 34211 70679.

A value of π that is correct to 10,000 decimal digits using the Wolfram Alpha computational engine is listed in the Appendix.

2.7 DISCUSSION

Laplace showed us how to use a simulation procedure to estimate the value of π. He used an analog computer consisting of a needle and a sheet of paper. This was the first known application of what became known as the Monte Carlo method. It is particularly useful for estimating the mean values of complicated functions of random variables.

As an integration method, it is capable of estimating multidimensional integrals, that other methods would have difficulty handling. If the estimation of the integral is too difficult in terms of programming, a direct simulation of the basic physical processes involved, is always possible by the Monte Carlo method. The Monte Carlo method excels where the problems at hand are multidimensional, and where complexity and nonlinearity can only be handled in other methods by linearization and significant approximations to the problems at hand. It finds nowadays wide uses in a variety of fields such as particle transport, fluid flow, heat transfer, operations research, biological systems and game theory, to name a few.

EXERCISES

1. Derive the equations that generalize the Buffon's needle problem to two sets of parallel lines that are perpendicular to each other, as earlier derived by Laplace. Write a procedure for the estimation of π, and compare the results to those obtained from a single set of parallel lines.
2. A generalization that is even more challenging is to a three-dimensional set of perpendicular lines. The ultimate generalization would be to an n-dimensional hyper cube.
3. Estimate the percentage absolute error relative to the double precision value of π, and plot it as a function of the number of trials N.
4. Instead of loading the value of π in double precision to 17 decimal places as:

$$\pi = .3.14159265358979324,$$

let the computer generate its value itself as 4.0 $\tan^{-1}(1)$, in double precision as:

$$\pi = .4.0*\text{datan}(1.0\ D\ 0).$$

Do you notice any difference in the results?
4. Consider the Euler's method for the cosine function instead of the sine function. Prove that it yields another expression for the calculation of π as:

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ...$$

Use it to calculate the value of π to a number of significant digits.
5. Study the effect of the choice of needle length L and spacing d in the Buffon's needle problem. By varying the ratio L/d, are the results affected?

APPENDIX

π TO 10,000 DIGITS

π=3.14159265358979323846264338327950288419716939937510582097494459230781
640628620899862803482534211706798214808651328230664709384460955058223172
535940812848111745028410270193852110555964462294895493038196442881097566
593344612847564823378678316527120190914564856692346034861045432664821339
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3891249721775283479131515574857242454150695950829533116861727855889075 09
8381754637464939319255060400927701671139009848824012858361603563707660 10
4710181942955596198946767837449448255379774726847104047534646208046684 25
9069491293313677028989152104752162056966024058038150193511253382430035 58
7640247496473263914199272604269922796782354781636009341721641219924586 31
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