# Math381 Assignment 5

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For assignment5, we want do an analysis of a dice game with Markov chains.
Consider with following game:
You start with a score of zero and set the goal score of M. On each turn, you will roll a six-sided, fair dice and the outcome of that roll will be your new score. You add the new score to your total score. You continue rolling and changing the score until you reach a score of M or more.
Now, you first roll a dice and get a number. That number is your current score.
Then, starting with the second roll, the sum of score is determined by the following situations:
1. If the sum is M or greater, the game ends.
2. If not, and the sum of score is a prime number or twice a prime number, you divide your sum by 2 to get your new score.
3. Otherwise, your new score is the sum.

We investigate this game using Markov chains. Suppose our target score is 11. We can model this with a Markov chain with twelve states:0,1,2,3,4,5,6,7,8,9,10,11. The states 0 through 10 indicated our current sum, and state 11 indicates that we have a reached a sum of 11 or more. We start in state 0 with a sum of zero. The state 11 is the one absorbing state. If we get to the 11 state , then the game is over. We never change states ever again. The following matrix shows that the probability of transitioning from 11 to 11 is 1, which means that we are guaranteed to stay in that state.
We use the Sage to generate the transition matrix.
Let A be the transition matrix for this chain:

$$
\mathrm{A} = \begin{pmatrix}
0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 \\
0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 \\
0 & 0 & \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} \\
0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{3} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & 0 & \frac{2}{3} \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 & \frac{5}{6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

The transition matrix A is in canonical form. We can divide the matrix A into four sub-matrices,J,O,Q,R. J is an identity matrix. O is a matrix of all zeros. Q and R are non-negative matrices. See the following picture for details. P means matrix A in this case.

$$
P = \left(\begin{array}{c|c} Q & R \\ \hline O & J \end{array}\right)
$$

We can find non-negative matrix Q.

$$
Q = \begin{pmatrix}
0 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & 0 & 0 & 0 & 0 \\
0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 \\
0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 \\
0 & 0 & \frac{1}{6} & \frac{1}{3} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 \\
0 & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 \\
0 & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
$$

Let $N = (I - Q)^{-1}$.

$$
N = \begin{pmatrix}
1 & \frac{4997}{8304} & \frac{2921}{2076} & \frac{4509}{2768} & \frac{1}{6} & \frac{4145}{8304} & \frac{1}{6} & 0 & \frac{2677}{4152} & \frac{1433}{2768} & 0 \\
0 & \frac{351}{173} & \frac{366}{173} & \frac{381}{173} & 0 & \frac{45}{173} & 0 & 0 & \frac{132}{173} & \frac{93}{173} & 0 \\
0 & \frac{963}{1384} & \frac{963}{346} & \frac{2589}{1384} & 0 & \frac{363}{1384} & 0 & 0 & \frac{567}{692} & \frac{681}{1384} & 0 \\
0 & \frac{501}{1384} & \frac{501}{346} & \frac{3507}{1384} & 0 & \frac{357}{1384} & 0 & 0 & \frac{489}{692} & \frac{807}{1384} & 0 \\
0 & \frac{753}{2768} & \frac{753}{692} & \frac{3887}{2768} & 1 & \frac{1241}{2768} & 0 & 0 & \frac{909}{1384} & \frac{1619}{2768} & 0 \\
0 & \frac{27}{173} & \frac{108}{173} & \frac{189}{173} & 0 & \frac{243}{173} & 0 & 0 & \frac{90}{173} & \frac{87}{173} & 0 \\
0 & \frac{265}{2768} & \frac{265}{692} & \frac{1855}{2768} & 0 & \frac{1001}{2768} & 1 & 0 & \frac{557}{1384} & \frac{1123}{2768} & 0 \\
0 & \frac{49}{1384} & \frac{49}{346} & \frac{343}{1384} & 0 & \frac{441}{1384} & 0 & 1 & \frac{197}{692} & \frac{427}{1384} & 0 \\
0 & \frac{21}{692} & \frac{21}{173} & \frac{147}{692} & 0 & \frac{189}{692} & 0 & 0 & \frac{381}{346} & \frac{183}{692} & 0 \\
0 & \frac{9}{346} & \frac{18}{173} & \frac{63}{346} & 0 & \frac{81}{346} & 0 & 0 & \frac{15}{173} & \frac{375}{346} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

We sum the first row of N to get the expected number of turns until absorbing state 11.
The sum is

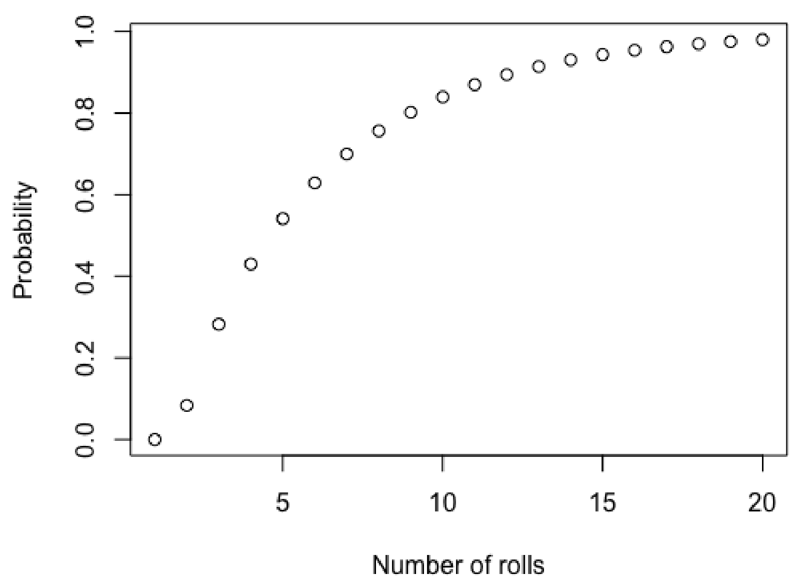
$$\tfrac{27539}{4152} \approx 6.63270712909$$

So, on average, it will take about 6.633 rolls until the sum is 11 or greater.
By using Markov chain transition matrix, we calculate the probability that the game ends on the first turn, on the second turn, on the third turn, etc. I made a table to show the results. Column A gives the probability that we are in state 11 after i throws. Column B gives the probability that we reach state 11 on throw i.

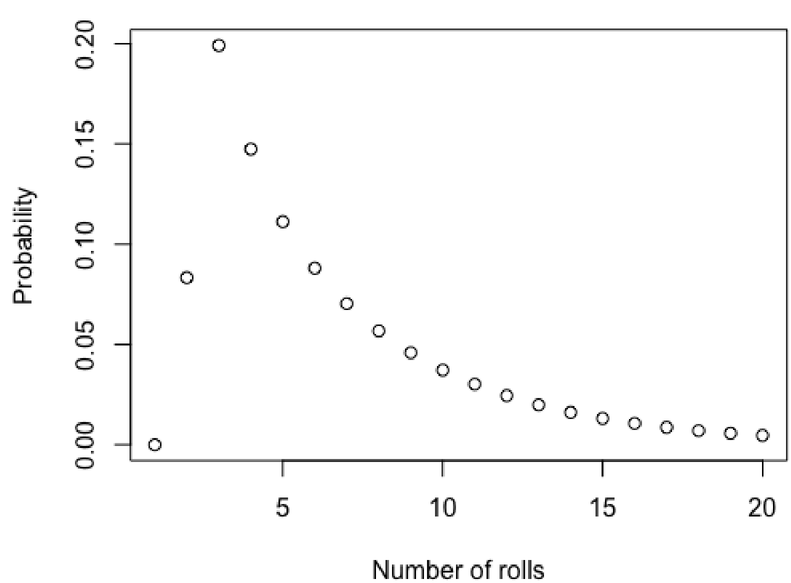
| i | A | B |
|---|---|---|
| 1 | 0 | 0 |
| 2 | 0.08333 | 0.08333 |
| 3 | 0.28241 | 0.19908 |
| 4 | 0.42978 | 0.14737 |
| 5 | 0.54102 | 0.11124 |
| 6 | 0.62905 | 0.08803 |
| 7 | 0.69943 | 0.07038 |
| 8 | 0.75616 | 0.05673 |
| 9 | 0.80205 | 0.04590 |
| 10 | 0.83926 | 0.03722 |
| 11 | 0.86946 | 0.03020 |
| 12 | 0.89398 | 0.02452 |
| 13 | 0.91389 | 0.01991 |
| 14 | 0.93006 | 0.01617 |
| 15 | 0.94318 | 0.01313 |
| 16 | 0.95385 | 0.01067 |
| 17 | 0.96252 | 0.00867 |
| 18 | 0.96955 | 0.00703 |
| 19 | 0.97527 | 0.00572 |
| 20 | 0.97991 | 0.00464 |

We find that the game is most likely to end on the 3rd roll because the probability to reach state 11 is highest on 3rd roll. After the 3rd roll, the probability to reach state 11 decreases as the number of rolls increases. For example, it is rarely for a player to roll a dice 20 times to end the game. The median game length is 5 because the the probability that we are in state 11 after 5 throws is 54.1%. That means that the probability that we are in state 11 is about 50% if the number of throws less than 5 or greater than 5. We'll use two plots to show the results visually and gain a better understanding of the data.

The probability that we are in state 11 after i throws.




The probability that we reach state 11 on throw i.

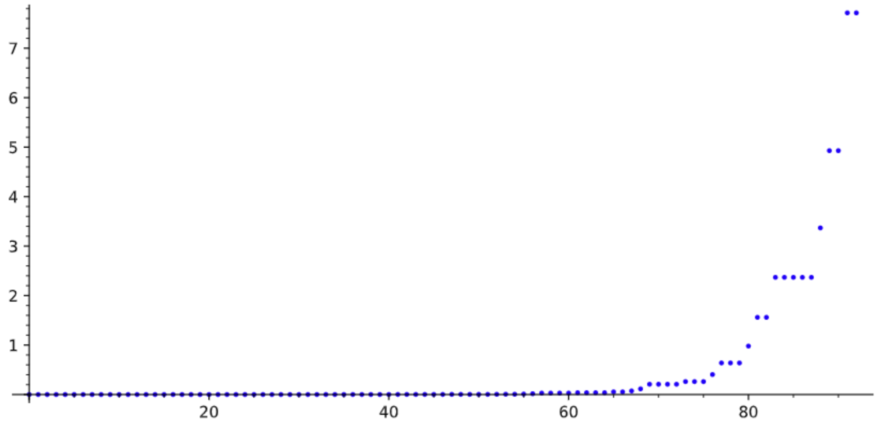



Now, we want to investigate how the expected number of turns in the game changes with the value of

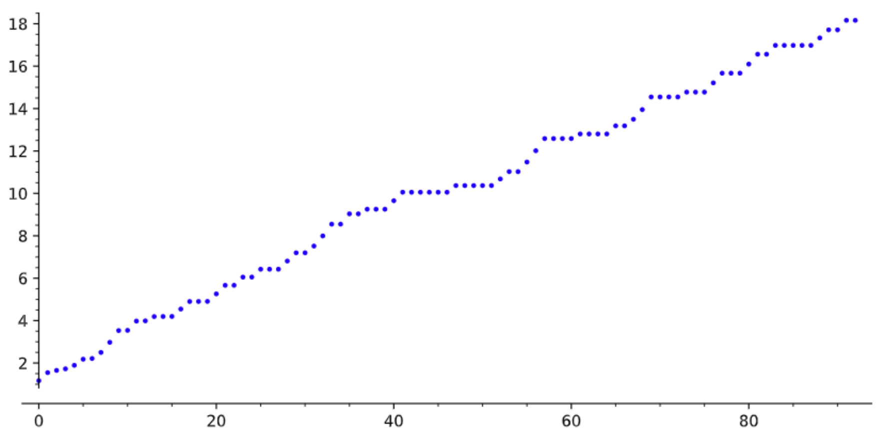
M. By changing M, I got different expected number of turns. I created a table and a plot to present the data.

| M | Expected number of turns |
|---|---|
| 7 | 3.21666666666667 |
| 8 | 4.69444444444444 |
| 9 | 5.21604938271605 |
| 10 | 5.61471193415638 |
| 11 | 6.63270712909441 |
| 12 | 8.83373786407767 |
| 13 | 9.14252966558792 |
| 14 | 12.1767869282315 |
| 15 | 19.6035714285714 |
| 16 | 34.1369047619048 |
| 17 | 34.4702380952381 |
| 18 | 53.3490259740260 |
| 19 | 53.9853896103896 |
| 20 | 66.0488544474394 |
| 21 | 66.2941374663073 |
| 22 | 66.5803009883199 |
| 23 | 94.1469561257021 |
| 24 | 134.376348769248 |
| 25 | 134.863563560751 |
| 26 | 135.146772320258 |
| 27 | 192.721007162810 |
| 28 | 288.296701919756 |
| 29 | 288.593649179113 |
| 30 | 424.882070068740 |
| 31 | 425.411576597165 |
| 32 | 616.050360276331 |
| 33 | 616.295643295199 |
| 34 | 616.581806817211 |
| 35 | 906.404029183090 |
| 36 | 1336.36970822159 |
| 37 | 1336.86500233924 |
| 38 | 1836.65439867993 |
| 39 | 2961.80914097008 |
| 40 | 5161.98531030702 |
| 41 | 5162.31864364035 |
| 42 | 8397.66625611477 |
| 43 | 8398.30261975113 |
| 44 | 10441.1752609488 |
| 45 | 10441.4205439676 |
| 46 | 10441.7067074896 |
| 47 | 15614.5815342510 |
| 48 | 23288.7985743155 |
| 49 | 23289.2938684332 |
| 50 | 23289.5752409822 |
| 51 | 23289.9035089561 |
| 52 | 23290.1947235530 |
| 53 | 23290.4274150926 |
| ... | ... |
| ... | ... |
| 98 | 7.71404157329257e7 |
| 99 | 7.71404161106427e7 |

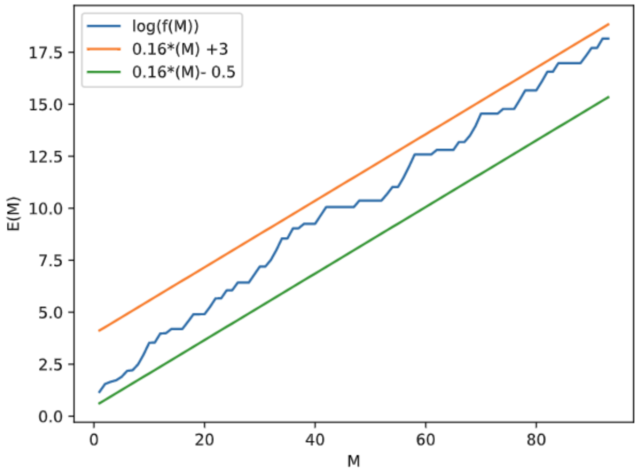
Expected number of turns from M=7 to M =99.




From the plot above, we find that the expected number of turns in the game increases as the value of M increases. Let f(M) be the expected number of turns with a goal score of M. The f(M)increases rapidly but not smoothly and it tends to infinity as M tends to infinity. I can get the expected value when M=299, which is f(M) = 6.6061772873821e19. However, it takes about five minutes to get this number. The long running time makes it difficult or even impossible to calculate expected number of turns for larger value of M. So I will stop this program at M=299.
From the plot above, we can see that the function f(M) has a irregular behavior, which makes it hard to predict its behavior. Thus, to better investigate the behavior of f(M), we try different functions to see what that looks like. I find that log(f(M)) is close to being linear. So log(f(M)) looks like a good match.
The plot of log(f(M)) looks like this:



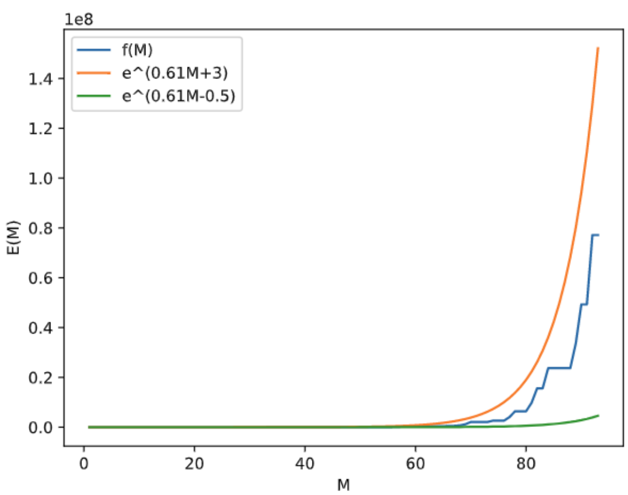

Since log(f(M)) is approximately linear, we can draw two different lines on the plot such that $g(M) < f(log(M)) < h(M)$:




From the graph, we see the expected number of rolls to end the game for each M all lie between the orange line and green line. Then, we can conclude that:

$$\text{for } 7 \le M \le 100$$
$$0.16M + 3 < log(f(M)) < 0.16M - 0.5$$
$$e^{0.16M+3} < f(M) < e^{0.16M-0.5}$$

The plot for $e^{0.16M+3} < f(M) < e^{0.16M-0.5}$ looks like this:




We can conclude that as the value of M get larger, the expected number of turns will get larger. We get the upper bound and lower bound on f(M). The value of the bounds will increase as the value of M increase. Hence, if value of M tends to infinity, the expected number of turns tends to infinity.

```
Appendix:
Here is the Sage code for this assginement:

import matplotlib.pyplot as plt
R=[]
H= []
J = []
## To create transition matrix with different goal score.
for n in range(7,100):
    A = zero_matrix(QQ,n+1,n+1)
```

```
    for i in range(6):
        A[0,i+1]=1/6
    A[n,n]=1;
    for i in range(1,n):
        for j in range(6):
            roll = j+1
            sum  = i + roll
            if sum >= n:
                put = n
            else:
                if(is_prime(sum) or is_prime(floor(sum/2))):
                    put = floor(sum/2)
                else:
                    put = sum
            A[i,put]+=1/6
    Q=A[:n,:n]
    F=(matrix.identity(n)-Q).inverse()
    sum = 0
    for i in range(n):
        sum = sum +F[0][i]
    log_sum = log(sum*1.)
    R.append(log_sum)
    H.append(0.16*(n) +3)
    J.append(0.16*(n)- 0.5)
y1 = R
y2 = H
y3 = J
x= range(1,94)
fig, ax = plt.subplots()
ax.plot(x, y1, label='log(f(M))')
ax.plot(x, y2, label='0.16*(M) +3')
ax.plot(x, y3, label = '0.16*(M)- 0.5')
ax.legend()
ax.set_xlabel('M')
ax.set_ylabel('E(M)')
plt.show()
```