Assignment 3

1. For all $n \geq 1$ and $1 \leq k < n$

Right side $= \binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1} =$ left side.

① $RHS = \dfrac{n \times (n-1) \times \cdots \times (n-k+1)}{k!} + \dfrac{n \times (n-1) \times \cdots \times (n-k-1+1)}{(k+1)!}$

$= \dfrac{(k+1) \times n \times (n-1) \times \cdots \times (n-k+1)}{k! \times (k+1)} + \dfrac{n \times \cdots \times (n-k+1) \times (n-k)}{(k+1)!}$

$= \dfrac{(n+1) \times \cdots \times (n-k+1)}{(k+1)!}$

② $LHS = \binom{n+1}{k+1} = \dfrac{(n+1) \times n \times \cdots \times (n+1-k-1+1)}{(k+1)!} = \dfrac{(n+1) \times \cdots \times (n-k+1)}{(k+1)!}$

③ $RHS = LHS = \dfrac{(n+1) \times \cdots \times (n-k+1)}{(k+1)!}$

Proof done.

2. (a) Formulas be constructed from $\vee, \wedge, (,), P, \neg P, q$

Permutations of $P, \neg P, q$: $3! = 6$

$\vee, \wedge$: $2! = 2$

$(,)$: 2

Number of formulas is: $6 \times 2 \times 2 = 24$.

(b). 5 equivalent classes can be found by all those famulas.

3.(a) Because the choice is uniformly at random

As a result, each of one choice has 0.25 probability

$E = \frac{1}{4} \times 5 + (E+3)\frac{1}{4} + \frac{1}{4} \times 8 + \frac{1}{4}(E+2)$

$E = 9$

The expected time is 9 minutes

(b) Assume Expected time E:

$E = \frac{1}{4} \times 5 + \frac{1}{4} \times 8 + \frac{1}{4} \times (E \text{ of option } B) + \frac{1}{4} \times (E \text{ of option } D)$

Expected time of choose option B:

$$\begin{cases} 3 + E_2 \\ E_2 = \frac{1}{3} \times 5 + \frac{1}{3} \times 8 + \frac{1}{3} \times (2 + E_3) \\ E_3 = \frac{1}{2} \times 5 + \frac{1}{2} \times 8 \end{cases}$$

As a result Expected time is: $\frac{61}{6}$ mins

Expected time of choose option D:

$$\begin{cases} 2 + E_2 \\ E_2 = \frac{1}{3} \times 5 + \frac{1}{3} \times 8 + \frac{1}{3} \times (3 + E_3) \\ E_3 = \frac{1}{2} \times 5 + \frac{1}{2} \times 8 \end{cases}$$

Expected time is: $\frac{57}{6}$ mins

So the expected time is $\frac{1}{4} \times 5 + \frac{1}{4} \times 8 + \frac{1}{4} \times \frac{61}{6} + \frac{1}{4} \times \frac{57}{6}$

$\cancel{= \frac{57}{6} \approx 9.8 \text{ mins}}$

$= \frac{61+57+30+48}{4 \times 6} \approx 8.17$ mins.

4.(a) $P_1(n+1) = \frac{P_2(n)}{2}$ $P_2(n+1) = P_1(n) + \frac{P_3(n)}{2}$

$P_3(n+1) = \frac{P_2(n)+P_4(n)}{2}$ $P_4(n+1) = P_5(n) + \frac{P_3(n)}{2}$

$P_5(n+1) = \frac{P_4(n)}{2}$

(b)

	P_1	P_2	P_3	P_4	P_5
0	1	0	0	0	0
1	0	1	0	0	0
2	$\frac{1}{2}$	0	$\frac{1}{2}$	0	0
3	0	$\frac{3}{4}$	0	$\frac{1}{4}$	0
4	$\frac{3}{8}$	0	$\frac{1}{2}$	0	$\frac{1}{8}$
5	0	$\frac{5}{8}$	0	$\frac{3}{8}$	0
6	$\frac{5}{16}$	0	$\frac{1}{2}$	0	$\frac{3}{16}$
7	0	$\frac{9}{16}$	0	$\frac{7}{16}$	0
8	$\frac{9}{32}$	0	$\frac{1}{2}$	0	$\frac{7}{32}$

~~For P_1:~~ ~~$P_1(n+2) = P(n)$~~

~~$P_1(n+2) = P(n)$~~

For P_1 just take n is even into consideration, then there is a sequence: $1, \frac{1}{2}, \frac{3}{8}, \frac{5}{16}, \frac{9}{32}, \ldots\ldots$

When ~~$n \geq 2$~~ $n \geq 2$: $P(n+1) = \frac{P(n) + \frac{1}{4}}{2} = \frac{P(n)}{2} + \frac{1}{8}$

Assume $P(1) = \frac{1}{2}$ then $P(n)$ will converges to:

$$P(n) = \frac{1}{2} \times P(n-1) + \frac{1}{8}$$

$$= \frac{1}{2^2} \times P(n-2) + \frac{1}{8} + \frac{1}{16}$$

$$= \frac{1}{2^3} \times P(n-3) + \frac{1}{8} + \frac{1}{2\times 8} + \frac{1}{2\times 2\times 8}$$

$$\vdots$$

$$= \frac{1}{2^{n-1}} \times P(1) + \frac{1}{2^3} + \cdots + \frac{1}{2^{n+1}} = \frac{1}{2^3} + \cdots + \frac{1}{2^{n+1}} + \frac{1}{2^n}$$

$$= \lim_{n\to\infty} \frac{1-(\frac{1}{2})^{n+1}}{4} + \frac{1}{2^n}$$

$$= \frac{1}{4}$$

The same as P_1, for P_2. $\frac{3}{4}, \frac{5}{8}, \frac{9}{16}$ ...

$$P_n = \frac{P(n-1)}{2} + \frac{1}{4}$$

$$= \frac{1}{2^{n-1}} \times P(1) + \frac{1}{2^2} + \cdots + \frac{1}{2^n}$$

$$= \lim_{n\to\infty} \left(\frac{1}{4} \times \frac{1-(\frac{1}{2})^{n-1}}{1-\frac{1}{2}}\right) + \frac{3}{4} \times \frac{1}{2^{n-1}} = \frac{1}{2}$$

For P_3 it converge to $\frac{1}{2}$

For P_4 : $\frac{1}{4}, \frac{3}{8}, \frac{7}{16}$

$$P_n = \frac{P(n-1)}{2} + \frac{1}{4} = \frac{1}{2^{n-1}} \times P(1) + \frac{1}{2^2} + \cdots + \frac{1}{2^n}$$

$$= \lim_{n\to\infty} \left(\frac{1}{4} \times \frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}}\right) + \frac{1}{2^{n-1}} \times \frac{1}{4} = \frac{1}{2}$$

For P_5: $\frac{1}{8}, \frac{3}{16}, \frac{7}{32}$

$$P(n) = \frac{1}{2} \times P(n-1) + \frac{1}{8}$$

$$= \frac{1}{2^{n-1}} \times P(1) + \frac{1}{2^3} + \cdots + \frac{1}{2^{n+1}} = \lim_{n\to\infty} \frac{1-(\frac{1}{2})^{n+1}}{4} + \frac{1}{2^{n+2}}$$

$$= \frac{1}{4}$$

In conclusion the steady state probabilities for V_1, V_2, V_3, V_4, V_5 are $\frac{1}{4}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}$ respectively.

(C). ~~Expected distance is equal to~~ $\cancel{\frac{1}{4} \times 0 + \frac{1}{2} \times 1 + \frac{1}{2} \times 2 + \frac{1}{2} \times 3 + \frac{1}{4} \times 4}$

Because when n is even P_2, P_4 are always 0

$\cancel{= 0 + \frac{1}{2} + 1 + \frac{3}{2} + 1}$

when n is odd

$\cancel{= 4}$.

P_1, P_3, P_5 is always 0.

So the Expected distance is when n is even:

$0 + 0 + 1 + 0 + 1 = 2$

when n is odd:

$0 + \frac{1}{2} + 0 + \frac{3}{2} + 0 = 2$.

5. (a) $|\Omega| = 3^4 = 81$

The number of possible 3-colouring the graph by 3 colours is: $\binom{3}{2} = 6$

The probability is $\frac{6}{81} = \frac{2}{27}$

(b) $|\Omega| = 6^5 = 7776$

We assign six "colour pairs" to diagonal line. For each assignment only have one possible valids 3-colouring. As a result the answer is:

$$\frac{6}{7776} = \frac{1}{6^4} = \frac{1}{1296}$$