Assignment 2.

Question 1.

(a) This problem is just like a colouring problem in graph. It can be converted into colouring problem like below.

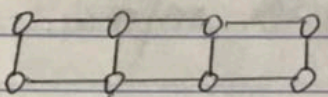
① vertices — Every different houses.

② Edges — Houses which ~~should~~ have different channels should be connected. Here they are the houses next to each other or over the road from one another

③ How to convert:

Colouring	This question.
Assigning a 'colour' to vertex	⟶ set up different channels to houses.
Vertices connected by edge have different colour	⟶ Houses are connected have different channels

④ A graph can be drawed like this:

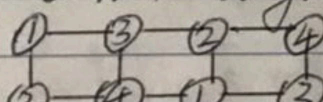


$G(E, V)$

⑤ We need to find out the chromatic number of the graph G, which is denoted by $X(G)$. The number is ~~ex~~ the minimum channels.

Formally: A mapping $c: V \mapsto [1 \dots 6]$ such that for every $e(v,w) \in E$ $c(v) \neq c(w)$, we need to find out the chromatic number $X(G)$

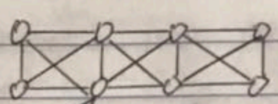
(b) Here I want to give an example of minimum number of wi-fi channels.



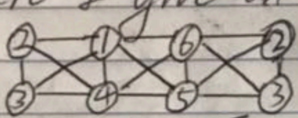
channels: {1, 2, 3, 4}
number of channels: 4

Explaination: For the middle four hourses, they all have three edges connected. So there must have at least four different channels. Becouse the houses and the houses connected to them should have different channels. And then ~~It~~ The four chonnels can be set up to all the houses without the same channel of houses are connected. So the minimum number of channels should be 4.

(c) For this question we can just change the graph like below. (Added some edges for the houses to the left and right of the house over the road)



Here I give an example of ~~the~~ a successful assignment of channels



channel : $\{1, 2, 3, 4, 5, 6\}$
Number of channels : 6.

Explaination: Four middle vetices, each have five edges are connected to them. So for the same reason as last questions. There are at least 6 channels for the set up. Because each houses connected should have ~~the~~ different channel. And also we can set up the six channel to all the houses without interfere. As a result the minimum channel should be 6.

Question 2:

(a) A_v: $P_{v,red} \vee P_{v,green} \vee P_{v,blue}$ $(v \in V)$

(b) B_v: $\bigwedge_{v\in V} \neg(P_{v,red} \wedge P_{v,green}) \wedge \neg(P_{v,red} \wedge P_{v,blue}) \wedge \neg(P_{v,blue} \wedge P_{v,green})$

The answer should be in CNF, so:

$B_v \equiv \bigwedge_{v\in V} (\neg P_{v,red} \vee \neg P_{v,green}) \wedge (\neg P_{v,red} \vee \neg P_{v,blue}) \wedge (\neg P_{v,blue} \vee \neg P_{v,green})$

(c) $C_{u,v}$: $\bigwedge_{\{u,v\}\in E} \neg(P_{u,red} \wedge P_{v,red}) \wedge \neg(P_{u,green} \wedge P_{v,green}) \wedge \neg(P_{u,blue} \wedge P_{v,blue})$

In CNF:

$C_{u,v}$: $\bigwedge_{\{u,v\}\in E} (\neg P_{u,red} \vee \neg P_{v,red}) \wedge (\neg P_{u,green} \vee \neg P_{v,green}) \wedge (\neg P_{u,blue} \vee \neg P_{v,blue})$

(d). ϕG: $\bigwedge_{v\in V} \big((P_{v,red} \vee P_{v,green} \vee P_{v,blue}) \wedge \neg(P_{v,red} \wedge P_{v,green}) \wedge \neg(P_{v,red} \wedge P_{v,blue}) \wedge \neg(P_{v,blue} \wedge P_{v,green}) \big)$

$\wedge \big(\bigwedge_{\{u,v\}\in E} \neg(P_{u,red} \wedge P_{v,red}) \wedge \neg(P_{u,green} \wedge P_{v,green}) \wedge \neg(P_{u,blue} \wedge P_{v,blue}) \big)$

ϕG means that: "for each vertices should have and only have one colour And the colour should be different from the vertice share the same edge"

Question 3:

(a) If a relation $\sqsubseteq$ on a set A satisfies the following conditions (R) reflexive (AS) antisymmetric (T) transive

~~Then~~ $\sqsubseteq$ is a partial order.

Proof:

① (R) $\because$ For every $x \in A$ that $x \vee x = x$, so $x \sqsubseteq x$

$\therefore$ It satisfy (R)

② (AS) Assume For $\forall x, y \in A$: $x \sqsubseteq y$ if $x \vee y = y$

$y \sqsubseteq x$ if $y \vee x = x$

According to commutative law that:

$$x \vee y = y \vee x$$

As a result if $x \sqsubseteq y$, $y \sqsubseteq x \in A$ then:

$$x \vee y = y = y \vee x = x$$

$$x = y$$

Proof hold $\therefore$ $\sqsubseteq$ satisfy (AS)

③ (T) Assume For $\forall x, y, z \in A$: $x \sqsubseteq y$ if $x \vee y = y$

$y \sqsubseteq z$ if $y \vee z = z$

$\because$ $x \vee y = y$

$\therefore$ $y \vee z = (x \vee y) \vee z = z$

According to associative law $(x \vee y) \vee z = x \vee (y \vee z)$

$= x \vee z = z$

As a result: $x \vee z = y \vee z = z$

$\therefore$ It satisfy (T)

④ In conclusion, this relation $\sqsubseteq$ satisfy (R), (AS), (T), so $\sqsubseteq$ is a partial order.

(b) Because it satisfy reflexivity, antisymmetry and transitivity (According to the proof in last question)

So $\sqsubseteq$ is a non-strict partial order.

Proof:

① (R) Assume for $\forall$ set $x \in Pow(X)$ $x \vee x = x$ so $x \sqsubseteq x$

② (AS) Assume $\forall x, y \in Pow(X)$ $x \sqsubseteq y$, so $x \vee y = y$ $\because x \vee y = y \vee x = y$, $\therefore y \vee x = y$

$\therefore$ $y \sqsubseteq x$. It satisfy (AS)

③ (T) Assume $\forall x, y, z \in Pow(X)$, $x \vee y = y$ and $y \vee z = z$ $\therefore (x \vee y) \vee z = z$, $\therefore x \vee (y \vee z) = z$ $\therefore x \vee z = z$ proof hold

(c) Because $\equiv$ is not interest in $\leftrightarrow$

So $(x \vee y) \leftrightarrow y$

$\equiv ((x \vee y) \rightarrow y) \wedge (y \rightarrow (x \vee y))$

$\equiv (\neg (x \vee y) \vee y) \wedge (\neg y \vee (x \vee y))$

$\equiv (\neg x \wedge \neg y \vee y) \wedge (\neg y \vee y \vee x)$

$\equiv (\neg x \wedge T) \wedge (T \vee x)$

$\equiv \neg x \wedge T \wedge T$

$\equiv \neg x$

Question 4 (a) Proof:

Base case (B): $P(0)$: add(0,0) = add(0,0)

It is obviously that it is true.

(Inductive step): Assume $P(n)$: add(n,0) = add(0,n) hold.

then $P(n+1)$: add(n+1,0) = add(0,n+1) is also hold

Proof:

left side: add(n+1,0) = n+1

Right side: add(0,n+1) = add(0,n)+1

Because add(0,n) = add(n,0) = n.

$\therefore$ add(0,n+1) = add(0,n)+1 = n+1

As a result add(n+1,0) = add(0,n+1) hold.

Conclusion: $P(n)$ holds for all $n \in N$

(b) $Q(n)$: If $a+b=n$ and $a,b>0$ then add(a,b) = add(b,a)

Assume: $R(n) \equiv P(n) \wedge Q(n)$

So: $R(n)$: If $a+b=n$ and $a,b \geq 0$ then add(a,b) = add(b,a)

We want to proof $Q(n)$ is holds for all $n \in N$

Because $P(n)$ holds for all $n \in N$ (Already proofed in last question)

As long as $R(n)$ holds for all $n \in N$, $Q(n)$ must holds.

Now we just proof $R(n)$ holds for all $n \in N$.

Base case (B): ~~a~~ $R(0)$: $a+b=0$ so $a=b=0$.

then add(0,0) = add(0,0) is obviously holds.

Inductive case [ ]:

Assume $R(n)$: $a+b=n$ $a,b\geq 0$ $add(a,b)=add(b,a)$ holds

To prove $P(n+1)$: $1+a+b=n+1$ $a,b\geq 0$

There are two cases ① $add(a+1,b)=add(b,a+1)$

② $add(a,b+1)=add(b+1,a)$

① left side $add(a+1,b)=add(a+1,b-1)+1$

$=add(a+1,b-2)+2$

$\vdots$

$=add(a+1,b-b)+b$

$=add(a+1,0)+b$

$=a+1+b.$

Right side $add(b,a+1)=add(b,a)+1$

$=add(b,a-1)+2$

$\vdots$

$=add(b,a-a)+a+1=add(b,0)+a+1$

$=b+a+1$

As a result $add(a+1,b)=add(b,a+1)=a+b+1$ proof hold.

② left side $add(a,b+1)=add(a,0)+b+1$

$=a+b+1$

Right side $add(b+1,a)=add(b+1,0)+a$

$=a+b+1$

As a result $add(a,b+1)=add(b+1,a)$ proof hold.

Proof done.

In conclusion: Because $R(n)$ holds for all $n\in N$

$Q(n)$: If $a+b=n$ and $a,b\geq 0$ then $add(a,b)=add(b,a)$

is also hold for all $n\in N$

Proof done.

Question 5:

(a) For rec_a(n), and iter_a give asymptotic upper bounds for the running time is actually to calculate "~~Θ~~ big-O"

For rec_a(n):

$T(n)$ denote the total cost of ~~running time~~

$T(n) = T(n-1) + T(n-2) + C$

$T(0) = T(1) = 1$

Because we want to know the upper bound, and $T(n-1) > T(n-2)$

So we just need to calculate upper bound for $T(n) = 2T(n-1) + C$

$$
\begin{aligned}
T(n) &= 2T(n-1) + C \\
&= 4T(n-2) + 3C \\
&= 8T(n-3) + 7C \\
&\vdots \\
&= 2^k T(n-k) + (2^k - 1)C
\end{aligned}
$$

$n-k=0 \Rightarrow k=n$

$$
\begin{aligned}
&= 2^n + (2^n - 1)C = 2^n + C \times 2^n - C = (C+1)2^n - C \\
&= O(2^n)
\end{aligned}
$$

For iter_a(n):

$T(n)$ is also denote the total cost of running time.

$T(n) = T(n-1) + C$

$T(0) = T(1) = 1$

So

$$
\begin{aligned}
T(n) &= T(n-2) + 2C \\
&= T(n-3) + 3C \\
&= C \cdot n \\
&= O(n)
\end{aligned}
$$

So the asymptotic upper bounds for rec_a is $O(2^n)$

for iter_a is $O(n)$

(b) Next page.

First of all, I want to give the example answer of several a_n a_n+2^n

n	a_n	a_n+2^n
0	0	1
1	1	3
2	5	9
3	19	27
4	65	81
5	211	243
⋮	⋮	⋮

From the example, we can easily find out that $a_n+2^n=3^n$

So I guess $a_n=3^n-2^n$

Proof: (Base case): when $n=0$, $n=1$

$$a_0=3^0-2^0=0 \quad a_1=3^1-2^1=1$$

$$a_2=3^2-2^2=5\times 1-6\times 0=5.$$

(Inductive case): Assume that $a_n=3^n-2^n$ $n\in N$ hold

To prove $a_{n+1}=3^{n+1}-2^{n+1}$ also hold.

$$\begin{aligned}
a_{n+1} &= 5a_n-6a_{n-1} \\
&= 5\times(3^n-2^n)-6\times(3^{n-1}-2^{n-1}) \\
&= 5\times 3^n-5\times 2^n-2\times 3\times 3^{n-1}+3\times 2\times 2^{n-1} \\
&= 5\times 3^n-5\times 2^n-2\times 3^n+3\times 2^n \\
&= 3\times 3^n-2\times 2^n \\
&= 3^{n+1}-2^{n+1}
\end{aligned}$$

So $a_{n+1}=3^{n+1}-2^{n+1}$ hold

In conclusion: $a_n=3^n-2^n$ for all $n\in N$

and my guess is right.

(C) Yes, I can find a more efficient method for computing a_n.

a_n can be calculated by matrix

In this form:

$$\begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix}=\begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a_{n-1} \\ a_{n-2} \end{pmatrix}=\cdots=\begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix}^{n-1}\begin{pmatrix} a_1 \\ a_0 \end{pmatrix}$$

To calculate a_n, we only need to calculate

$$\begin{pmatrix} 5 & -6 \\ 1 & 0 \end{pmatrix}^{n-1} \begin{pmatrix} a_1 \\ a_0 \end{pmatrix}$$

Algorithm can be writen as below:

~~if n<2~~

```
matrix_a(n):
    if n<2: return n
    else:
        matrix element = [5 -6; 1 0]
        matrix answer.
        whiln(n):
            if (n%2):
                answer = answer * element;
            element = element * element.
            n = n/2.
    return answer.
```

To calculate the running time:

$count(n=0) = count(n=1) = 1$

$count(n \geq 2) = 5 + Count(\frac{n}{2}) \rightarrow$ if n odd

$count(n \geq 2) = 4 + Count(\frac{n}{2}) \rightarrow$ if n even

Assume running time is $T(n)$:

$T(n) = 5 + T(\frac{n}{2})$ $T(1) = 1$

$T(n) = \boxed{O(\log n)}$

As a result my answer is more efficient.