Frugal Hawkes Processes

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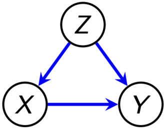
Idea of Frugal Parameterization



Theorem (theorem 3.1 in [ED23].)

Consider an outcome Y, and causally prior variables Z, X (see the figure above). Then we can smoothly parameterize the joint distribution $P(z, x, y)$ with models for

$$P(z, x) \qquad P^*(y \mid x) \qquad \phi^*_{ZY|X}(z, y \mid x)$$

where P^* is the intervened distribution and $\phi^*_{ZY|X}(z, y \mid x)$ is some dependence measure.

Note that Z, X, Y can be vector valued.
This gives us the best of both worlds: a coherent joint distribution and a marginal specification of our choice. For example, causal inference researchers often consider the marginal model $P(y \mid do(x))$.

Proof Sketch of Frugal Parameterization

Proof.

Here is a sketch of the algorithm we use.

- Construct $P^*(z \mid x) = P(z)$ from $P(z, x)$.
- Then combine with $P^*(y \mid x)$ and $\phi^*_{ZY|X}$ to obtain $P^*(y, z \mid x)$. (In general, if $\phi^*_{ZY|X}$ is a copula we use inverse CDFs.)
- Then obtain $P(x, z)/P^*(z \mid x)$, and multiply by $P^*(y, z \mid x)$. This gives $P(z, x, y)$.

□

The third step follows from the following fact: given that $\phi^*_{ZY|X}$ is a copula density, the likelihood is

$$P(z, x, y) = P(x, z) \cdot P^*(y \mid x) \cdot \phi^*_{ZY|X}(z, y \mid x).$$

This follows from

$$P(z) \cdot P(y \mid z, x) = P(z) \cdot P^*(y \mid x) \cdot \phi^*_{YZ|X}(y, z \mid x).$$

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Preliminary on Point Processes

Let $N = \left(N^k\right)_{k\in[d]}$ denote a collection of point processes where

$$N^k(A) = \sum_i \delta_{\tau_i^k}(A)$$

is a simple and non-exploding point process for $k \in [d]$.

Definitions

The intensity $\lambda_t^k = \mathbb{E}\left(N^k(\mathrm{d}t) \mid \mathcal{F}_{t-}^V\right)$ describes the conditional rate of new events at time t.

We consider a special intensity process

$$\lambda_t^k = \beta_0^k + \sum_{j\in V} \int_{-\infty}^{t-} g^{jk}(t-s) N^j(\mathrm{d}s) \tag{1}$$

where $g^{jk} : [0,\infty) \to [0,\infty)$ is the kernel, $j, k \in [d]$.

Definitions of Multivariate Hawkes Processes

Definitions

A d-dimensional point process $N = \left(N^k\right)_{k\in[d]}$ with intensity processes $\lambda^k, k \in [d]$ as defined by eq. (1) is called a multivariate linear Hawkes process.

Nonlinear Hawkes processes is defined via substituting λ_t^k with $\eta(\lambda_t^k)$ for some link function $\eta(\cdot)$ in eq. (1). Define

$$\mathbf{g}^{jk} = \int_0^\infty g^{jk}(t)\mathrm{d}t$$

and the matrix $G = \left(\mathbf{g}^{jk}\right)_{j,k}$.

Definitions

For a multivariate linear Hawkes process, the local independence graph is a graph with vertices $[d]$ and an edge $j \to k$ iff $\mathbf{g}^{jk} > 0$.

Note that $\exists$ a natural extension of the local independence graph to nonlinear Hawkes process, even general point processes, see [MH20].

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Simulation of Multivariate Hawkes Processes

After the introduction, a natural question arises.

Problem

Given the local independence graph, kernels g^{jk}, and baseline intensities β_0^k of the the Multivariate Hawkes Process $N = \left(N^k\right)_{k\in[d]}$, how can we simulate N and its marginal model?

Many model classes, for example Hawkes processes, are not closed under marginalization. This is a challenge for simulation.
However, we shall attempt this via a counterpart of the frugal parameterization in multivariate Hawkes processes.

Solution

Generalize ideas of the frugal parameterization from [ED23] to multivariate (linear) Hawkes processes for simulation.

If successful, the algorithm will be implemented with R.

References

[ED23] Robin J Evans and Vanessa Didelez. "Parameterizing and Simulating from Causal Models". In: *Journal of the Royal Statistical Society Series B: Statistical Methodology* (May 2023), qkad058. ISSN: 1369-7412. DOI: 10.1093/jrsssb/qkad058. URL: https://doi.org/10.1093/jrsssb/qkad058.

[MH20] Søren Wengel Mogensen and Niels Richard Hansen. "Markov equivalence of marginalized local independence graphs". In: *The Annals of Statistics* 48.1 (2020), pp. 539–559. DOI: 10.1214/19-AOS1821. URL: https://doi.org/10.1214/19-AOS1821.