Mathematical Programming 70 (1995) 73-89

On structure and stability in stochastic programs with random technology matrix and complete integer recourse [1]

Rüdiger Schultz *,[2]

Konrad-Zuse-Zentrum für Informationstechnik Berlin, Heilbronner Strasse 10, D-10711 Berlin, Germany

Received February 1993; revised manuscript received 22 June 1994

Abstract

For two-stage stochastic programs with integrality constraints in the second stage, we study continuity properties of the expected recourse as a function both of the first-stage policy and the integrating probability measure.

Sufficient conditions for lower semicontinuity, continuity and Lipschitz continuity with respect to the first-stage policy are presented. Furthermore, joint continuity in the policy and the probability measure is established. This leads to conclusions on the stability of optimal values and optimal solutions to the two-stage stochastic program when subjecting the underlying probability measure to perturbations.

Keywords: Stochastic integer programming; Parametric integer programming; Continuity; Stability; Weak convergence of probability measures

1. Introduction

Consider a probability space $(\Omega, \mathfrak{A}, P)$ and measurable mappings $z : \Omega \to \mathbb{R}^s$, $A : \Omega \to \mathbb{R}^{ms}$ where the images of A are understood as $s \times m$ matrices. A two-stage stochastic integer program with random technology matrix is then given by

$$\min\{f(x) + Q_P(x) : \; x \in C\},$$

* e-mail: schultz@zib-berlin.de.

[1] This research is supported by the Schwerpunktprogramm "Anwendungsbezogene Optimierung und Steuerung" of the Deutsche Forschungsgemeinschaft.

[2] The main part of the paper was written while the author was an assistant at the Department of Mathematics at Humboldt University Berlin.

SSDI 0025-5610(94)00078-6

where

$$Q_P(x) = \int_\Omega \Phi(z(\omega) - A(\omega)x)P(\mathrm{d}\omega)$$

and

$$\Phi(b) = \min\{q^\mathrm{T}y + q'^\mathrm{T}y':\ Wy + W'y' = b, y' \geqslant 0, y \geqslant 0, y' \in \mathbb{R}^{s'}, y \in \mathbb{Z}^{\bar{s}}\}. \quad (1.1)$$

Basically, we assume that $f : \mathbb{R}^m \to \mathbb{R}$ is continuous, $C \subset \mathbb{R}^m$ nonempty, closed, $q \in \mathbb{R}^{\bar{s}}$, $q' \in \mathbb{R}^{s'}$ and that the matrices $W \in L(\mathbb{R}^{\bar{s}}, \mathbb{R}^s)$, $W' \in L(\mathbb{R}^{s'}, \mathbb{R}^s)$ have rational entries. By $\mathbb{Z}^{\bar{s}}$ we denote the subset of integer vectors in $\mathbb{R}^{\bar{s}}$. Further assumptions making the above expressions well-defined are presented below.

The measurable mappings z and A induce a probability measure $\mu := P \circ (z, A)^{-1}$ on $\mathbb{R}^{(m+1)s}$. Our model then reads

$$\mathrm{P}(\mu) \qquad \min\{f(x) + Q(x, \mu):\ x \in C\},$$

where

$$Q(x, \mu) = \int_{\mathbb{R}^{(m+1)s}} \Phi(z - Ax)\mu(\mathrm{d}(z, A)). \quad (1.2)$$

The stochastic program $\mathrm{P}(\mu)$ is an appropriate model for an optimization process where, in a first stage, a decision x must be taken under uncertainty on the problem data z and A, and, in a second stage, a decision (y, y') is made after realization of (z, A). The second stage is formalized via the optimization problem behind Φ (see [11]). The latter, for instance, may model an optimal compensation of the surplus (shortfall) $z - Ax$ or may deliver detailed scheduling decisions (based on $z - Ax$) in a hierarchical system. The integral Q given by (1.2) models the expected additional costs due to the second-stage action (y, y'). The peculiarities of $\mathrm{P}(\mu)$ are two-fold: in the first stage we allow for randomness not only in the vector z but also in the technology matrix A and in the second (or recourse) stage we restrict some decisions y to be integral.

Restricting decisions in the second stage of a stochastic program with recourse to be integers is interesting from the theoretical viewpoint since the implications of the "smoothing effect" of the integral in (1.2) are not obvious. Moreover, in applications a proper modelling often requires integer variables which, obviously, is especially relevant when the second-stage program is a combinatorial optimization problem [18,22,30].

For these reasons we keep the second stage a general linear mixed-integer program. We study the "smoothing effect" of the integral in (1.2), i.e., we derive continuity properties of Q both with respect to x and μ. The joint continuity of Q in x and μ leads to conclusions on the stability of $\mathrm{P}(\mu)$ when μ varies in a certain set of probability measures. Like in other branches of stochastic programming, stability considerations of the model under perturbations of the integrating probability measure are important prerequisites when justifying approximation schemes [5,13] or when replacing incompletely known distributions by suitable estimates [9].

To obtain a first impression on the difficulties that appear when implanting integrality constraints into the second stage of $\mathrm{P}(\mu)$, consider the simple example

$$Q(x,\mu) = \int_{\mathbb{R}} \Phi(z-x)\mu(\mathrm{d}z), \quad \Phi(b) = \min\{y:\ y \geqslant b, y \in \mathbb{Z}\}, \tag{1.3}$$

where μ is the uniform distribution on the interval $[0, \frac{1}{4}]$. Of course,

$$Q(x,\mu) = \int_{\mathbb{R}} \lceil z-x \rceil \mu(\mathrm{d}z),$$

where $\lceil a \rceil$ denotes the smallest integer greater than or equal to a. Simple calculations show that $Q(\cdot,\mu)$ is neither convex nor differentiable. Moreover, for a discrete distribution μ the function $Q(\cdot,\mu)$ becomes discontinuous. So, in contrast to recourse problems without integer variables, neither convexity nor differentiability of $Q(\cdot,\mu)$ can be expected to hold under reasonably comprehensible assumptions.

Compared to the rich literature on structure and stability for stochastic programs without integer requirements (as a collection take, for instance, [11,31] and [9,12,24,26,29]) there are only a few contributions to the problems addressed in the present paper. The first one seems to be due to Stougie [30] who established that $Q(\cdot,\mu)$ is continuous, provided that μ has a uniformly continuous density and that Φ fulfils some boundedness requirement (cf. also [22]).

The more recent papers [15–17,19] focus on simple integer recourse where Φ specifies to

$$\Phi(b) = \min\{q^{+\mathrm{T}}y^{+} + q^{-\mathrm{T}}y^{-}:\\ y^{+} \geqslant b, y^{-} \geqslant -b, y^{+} \geqslant 0, y^{-} \geqslant 0, y^{+} \in \mathbb{Z}^{s}, y^{-} \in \mathbb{Z}^{s}\}.$$

Exploiting the inherent separability, the authors derive explicit formulae, (sharp) convex lower bounds and sufficient conditions for continuity, convexity and differentiability of $Q(\cdot,\mu)$. Whereas [15,16,19] treat the case of random right-hand side, problems containing also a random technology matrix are considered in [17].

As in the present paper, the more general setting of Φ was adopted in [27,28] where continuity of $Q(\cdot,\mu)$ and stability of $\mathrm{P}(\mu)$ were analyzed for the case of a nonstochastic technology matrix A.

Our paper is organized as follows. In Section 2 we collect a few prerequisites from probability theory and basic results on the value function of a mixed-integer linear program with parameters in the right-hand side of the constraints [2,6]. In Section 3 we present sufficient conditions for lower semicontinuity, continuity, Lipschitz continuity of $Q(\cdot,\mu)$ and for the joint continuity of $Q(\cdot,\cdot)$. In Section 4, we derive continuity of the optimal-value function and Berge upper semicontinuity of the solution set mapping when understanding $\mathrm{P}(\mu)$ as a parametric program with respect to μ. Finally, we have a conclusions section.

2. Prerequisites

Recall that the integrand Φ in (1.2) is the value function of a linear mixed-integer program with parameters in the right-hand side of the equality constraints. Properties of such value functions are derived, for instance, in the monograph [2] and in the article [6] from where we quote the propositions below. Basically, we assume that for each $b \in \mathbb{R}^s$ the constraint set of the program defining $\Phi(b)$ is nonempty and that $\Phi(0) = 0$. Then $\Phi(b) \in \mathbb{R}$ for all $b \in \mathbb{R}^s$ (cf., e.g., [20, Proposition I.6.7]) and the following proximity result holds.

Proposition 2.1. (Bank and Mandel [2, Theorem 8.1]; Blair and Jeroslow [6, Theorem 2.1]). *There exist constants $\alpha > 0$, $\beta > 0$ such that for all $b', b'' \in \mathbb{R}^s$ we have*

$$|\Phi(b') - \Phi(b'')| \leqslant \alpha \|b' - b''\| + \beta.$$

Moreover, the value function Φ admits the following representation.

Proposition 2.2 (Blair and Jeroslow [6, Theorem 3.3]). *There exist constants $\gamma > 0$, $\delta > 0$ and vectors $d_1, \ldots, d_\ell \in \mathbb{R}^s$, $\tilde{d}_1, \ldots, \tilde{d}_{\ell'} \in \mathbb{R}^s$ such that for all $b \in \mathbb{R}^s$,*

$$\Phi(b) = \min_y \left\{ q^{\mathrm{T}} y + \max_{j \in \{1,\ldots,\ell\}} d_j^{\mathrm{T}}(b - Wy) \colon \ y \in Y(b) \right\},$$

where

$$Y(b) = \left\{ y \in \mathbb{Z}^{\bar{s}} \colon \ y \geqslant 0, \sum |y_i| \leqslant \gamma \sum |b_r| + \delta, \tilde{d}_k^{\mathrm{T}}(b - Wy) \geqslant 0, k = 1, \ldots, \ell' \right\}.$$

Straightforward duality considerations and a proximity argument for optimal solutions led to the above result. In fact, the vectors $d_1, \ldots, d_\ell$ come up as the vertices of the polyhedron $\{u \in \mathbb{R}^s \colon \ W'^{\mathrm{T}} u \leqslant q'\}$, and the vectors $\tilde{d}_1, \ldots, \tilde{d}_{\ell'}$ stem from an inequality description of the polyhedral cone $W'(\mathbb{R}_+^{s'})$.

Continuity properties of Φ can be derived from Proposition 2.2. Namely, if $Y(\cdot)$ remains constant on an open neighbourhood of some point $\bar{b} \in \mathbb{R}^s$, then, on this neighbourhood, Φ is the pointwise minimum of finitely many continuous (piecewise linear) functions and, hence, continuous at $\bar{b}$. If $\tilde{b} \in \mathbb{R}^s$ is such that $Y(\cdot)$ does not remain constant on any open neighbourhood of $\tilde{b}$, then there must exist $\tilde{y} \in \mathbb{Z}^{\bar{s}}$, $\tilde{y} \geqslant 0$, such that at least one of the inequalities

$$\sum |\tilde{y}_i| \leqslant \gamma \sum |\tilde{b}_r| + \delta$$

and

$$\tilde{d}_k^{\mathrm{T}}(\tilde{b} - W\tilde{y}) \geqslant 0, \quad k = 1, \ldots, \ell',$$

holds as an equation.

In fact, only the second group of inequalities is relevant, as we will explain now. Using only duality arguments, we obtain

$$\Phi(b) = \min_y \{ q^{\mathrm{T}} y + \max_j d_j^{\mathrm{T}} (b - Wy) : \\ y \geqslant 0, y \in \mathbb{Z}^{\bar{s}}, \tilde{d}_k^{\mathrm{T}} (b - Wy) \geqslant 0, k = 1, \ldots, \ell' \}.$$

The merit of [6, Theorem 3.3] (Proposition 2.2) is to restrict the above minimization (over an infinite set) to the finite set $Y(b)$. If $\tilde{b} \in \mathbb{R}^s$ is such that, for some $\tilde{y} \in \mathbb{Z}^{\bar{s}}$, $\tilde{y} \geqslant 0$, the inequality

$$\sum |\tilde{y}_i| \leqslant \gamma \sum |\tilde{b}_r| + \delta$$

holds as an equation, then $Y(b)$ changes on any neighbourhood of $\tilde{b}$. But this has no impact on the result of the minimization, if we assume that the constants γ, δ were selected in such a way (sufficiently large) that the minimum in Proposition 2.2 is attained for a $y \in \mathbb{Z}^{\bar{s}}$ such that

$$\sum |y_i| < \gamma \sum |b_r| + \delta.$$

Hence, the discontinuities of Φ are concentrated in points $b \in \mathbb{R}^s$ where, for some $y \in \mathbb{Z}^{\bar{s}}_+$, at least one of the inequalities

$$\tilde{d}_k^{\mathrm{T}} (b - Wy) \geqslant 0, \quad k = 1, \ldots, \ell',$$

holds as an equation.

The set of discontinuity points of Φ is thus contained in a countable union of hyperplanes in $\mathbb{R}^s$; more specifically, in a union of translates of hyperplanes determined by the facets of the cone $W'(\mathbb{R}^{s'}_+)$.

By the rationality of W', the vectors $\tilde{d}_k$, $k = 1, \ldots, \ell'$, are rational, too. Since also W is rational, this implies that there exists a constant $\varepsilon_0 > 0$ such that (for all $k = 1, \ldots, \ell'$)

$$|\tilde{d}_k^{\mathrm{T}} W y_1 - \tilde{d}_k^{\mathrm{T}} W y_2| > \varepsilon_0, \quad \text{whenever } y_1, y_2 \in \mathbb{Z}^{\bar{s}}, \ \tilde{d}_k^{\mathrm{T}} W y_1 \neq \tilde{d}_k^{\mathrm{T}} W y_2.$$

Hence, for any $b \in \mathbb{R}^s$, there exists a neighbourhood $U(b)$ such that $Y(b') \subseteq Y(b)$ for any $b' \in U(b)$. This implies that $\liminf_{b' \to b} \Phi(b') \geqslant \Phi(b)$, i.e., Φ is a lower semicontinuous function on $\mathbb{R}^s$ (cf. also [6, p. 133]). An example (for the pure-integer case) showing how this lower semicontinuity is lost if the constraint matrix contains irrational entries can be constructed from the example given in [1, p. 58].

One key point in our analysis is that, in (1.2), not only the right-hand side z but also the technology matrix A can be stochastic. Therefore, the stochastic program $\mathrm{P}(\mu)$ contains a joint probability distribution μ of z and A. Moreover, marginal and conditional distributions of μ will be important for our purposes. For convenience, we collect these notions here; further details can be found in textbooks on probability theory [8,10].

Let $\pi_{\mathbb{R}^s}$ and $\pi_{\mathbb{R}^{ms}}$ denote the projections from $\mathbb{R}^{(m+1)s}$ to $\mathbb{R}^s$ and $\mathbb{R}^{ms}$, respectively. The induced measures $\mu_1 = \mu \circ \pi_{\mathbb{R}^s}^{-1}$, $\mu_2 = \mu \circ \pi_{\mathbb{R}^{ms}}^{-1}$ are then referred to as the marginal distributions of μ with respect to z and A, respectively. By $\mu_1^2(A, \cdot)$ we denote the (regular) conditional distribution of z given A. It has the following properties:

$$\mu_1^2(A, \cdot) \text{ is a probability measure on } \mathbb{R}^s \text{ for any } A \in \mathbb{R}^{ms}; \tag{2.1}$$

the function $\mu_1^2(\cdot, B_1) : \mathbb{R}^{ms} \to [0,1]$ is measurable for any Borel set B_1 in $\mathbb{R}^s$; (2.2)

for any Borel set B in $\mathbb{R}^{(m+1)s}$, $\mu(B) = \int_{\mathbb{R}^{ms}} \int_{\mathbb{R}^s} \mathbf{1}_B(z, A)\mu_1^2(A, \mathrm{d}z)\mu_2(\mathrm{d}A)$ holds, where $\mathbf{1}_B$ denotes the indicator function of B. (2.3)

The above family of probability measures $\mu_1^2(A, \cdot)$ indeed exists, since μ, as a probability measure on a Euclidean space, satisfies the general assumptions for the existence of a (regular) conditional distribution (cf. [8, Theorem 10.2.2], [10, Satz 5.3.21]).

3. Continuity properties of the expected recourse function

We impose the following general assumptions to have (1.1), (1.2) well-defined.

(A1) There exists a $u \in \mathbb{R}^s$ such that $W^{\mathrm{T}}u \leqslant q$, $W'^{\mathrm{T}}u \leqslant q'$.

(A2) For all $t \in \mathbb{R}^s$ there exist $y \in \mathbb{Z}^{\bar{s}}$, $y' \in \mathbb{R}^{s'}$ such that $y \geqslant 0$, $y' \geqslant 0$ and $Wy + W'y' = t$.

(A3) $\int_{\mathbb{R}^{(m+1)s}} (\|z\| + \|A\|)\mu(\mathrm{d}(z, A)) < +\infty$.

In (A3), $\|z\|$ denotes the Euclidean norm of z and $\|A\|$ the induced matrix norm of A. In the context of (stochastic) linear programming, (A1) is called "dual feasibility". Assumption (A2) is the natural extension of the complete-recourse assumption for stochastic programs with (noninteger) recourse and, therefore, it is referred to as "complete (mixed-)integer recourse". Assumption (A3), i.e., the finiteness of the first moment of μ, is basic for (noninteger) stochastic linear programs too (cf. [11,31]).

Proposition 3.1. *Assume* (A1)–(A3); *then* $Q(\cdot, \mu)$ *is a real-valued lower semicontinuous function on* $\mathbb{R}^m$.

Proof. Assumptions (A1), (A2) together with the duality theorem of linear programming and [2, Lemma 7.1] imply that $\Phi(z - Ax) \in \mathbb{R}$ for all $z \in \mathbb{R}^s$, $A \in \mathbb{R}^{ms}$, $x \in \mathbb{R}^m$ (see also [20, Proposition I.6.7]). Furthermore, Φ is measurable as a lower semicontinuous function on $\mathbb{R}^s$ (see Section 2). Assumption (A2) implies that $\Phi(0) = 0$, and we obtain, in light of Proposition 2.1,

$$
\begin{aligned}
|Q(x, \mu)| &\leqslant \int_{\mathbb{R}^{(m+1)s}} |\Phi(z - Ax) - \Phi(0)|\mu(\mathrm{d}(z, A)) \\
&\leqslant \alpha \int_{\mathbb{R}^{(m+1)s}} \|z - Ax\|\mu(\mathrm{d}(z, A)) + \beta \int_{\mathbb{R}^{(m+1)}} \mu(\mathrm{d}(z, A)) \\
&\leqslant \alpha \int_{\mathbb{R}^{(m+1)s}} \|z\|\mu(\mathrm{d}(z, A)) + \alpha\|x\| \int_{\mathbb{R}^{(m+1)s}} \|A\|\mu(\mathrm{d}(z, A)) + \beta.
\end{aligned}
$$

Hence, $Q(\cdot, \mu)$ is a real-valued function on $\mathbb{R}^m$.

To verify the lower semicontinuity, let $x \in \mathbb{R}^m$ and let $\{x_n\}$ be a sequence in $\mathbb{R}^m$ converging to x. Denote $r := \max_{n\in\mathbb{N}} \|x_n\| < +\infty$.

In view of Proposition 2.1 and $\Phi(0) = 0$, we have

$$\begin{aligned}\Phi(z - Ax_n) &\geqslant \Phi(0) - |\Phi(z - Ax_n) - \Phi(0)| \\ &\geqslant -\alpha\|z - Ax_n\| - \beta \\ &\geqslant -\alpha\|z\| - \alpha r\|A\| - \beta.\end{aligned}$$

Therefore, and by (A3), the function $h_0(z, A) := -\alpha\|z\| - \alpha r\|A\| - \beta$ is an integrable minorant of all the functions $h_n(z, A) := \Phi(z - Ax_n)$, $n \in \mathbb{N}$.

Now we have

$$\begin{aligned}Q(x, \mu) &= \int \Phi(z - Ax)\mu(\mathrm{d}(z, A)) \\ &\leqslant \int \liminf_{n\to\infty} \Phi(z - Ax_n)\mu(\mathrm{d}(z, A)) \\ &\leqslant \liminf_{n\to\infty} \int \Phi(z - Ax_n)\mu(\mathrm{d}(z, A)) \\ &= \liminf_{n\to\infty} Q(x_n, \mu).\end{aligned}$$

Here, the first estimate follows from the lower semicontinuity of Φ and the second is a consequence of Fatou's Lemma which works since we have the above minorant. Thus, $Q(\cdot, \mu)$ is lower semicontinuous at x. □

Let us remark that, by the above proposition, $\mathrm{P}(\mu)$ is a "proper" model in the sense that one minimizes a lower semicontinuous function and, if the feasible set C is compact, for instance, the infimum of the objective is finite and actually attained.

To formulate a sufficient condition for the continuity of $Q(\cdot, \mu)$ at some point $x \in \mathbb{R}^m$, we introduce the set $E(x)$ of all those $(z, A) \in \mathbb{R}^{(m+1)s}$ such that Φ is discontinuous at $z - Ax$. $E(x)$ is measurable for all $x \in \mathbb{R}^m$ [4, p. 225].

Proposition 3.2. *Assume* (A1)-(A3) *and let* $x \in \mathbb{R}^m$ *be such that* $\mu(E(x)) = 0$; *then* $Q(\cdot, \mu)$ *is continuous at* x.

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence converging to x. Denote $r := \max_{n\in\mathbb{N}} \|x_n\|$. Proposition 2.1 yields

$$|\Phi(z - Ax_n)| = |\Phi(z - Ax_n) - \Phi(0)| \leqslant \alpha\|z - Ax_n\| + \beta \leqslant \alpha\|z\| + \alpha r\|A\| + \beta.$$

In view of (A3), therefore, the function $h_0(z, A) := \alpha\|z\| + \alpha r\|A\| + \beta$ is an integrable majorant of all the functions $h_n(z, A) := |\Phi(z - Ax_n)|$.

Due to $\mu(E(x)) = 0$, it holds

$$h_n(z, A) \xrightarrow{n\to\infty} h(z, A) := \Phi(z - Ax), \quad \mu\text{-almost surely,}$$

and Lebesgue's dominated convergence theorem works:

$$\begin{aligned}\lim_{n\to\infty} Q(x_n,\mu) &= \lim_{n\to\infty}\int_{\mathbb{R}^{(m+1)s}} \Phi(z-Ax_n)\mu(\mathrm{d}(z,A))\\ &= \int_{\mathbb{R}^{(m+1)s}} \lim_{n\to\infty}\Phi(z-Ax_n)\mu(\mathrm{d}(z,A))\\ &= \int_{\mathbb{R}^{(m+1)s}} \Phi(z-Ax)\mu(\mathrm{d}(z,A)) = Q(x,\mu). \qquad \Box\end{aligned}$$

Corollary 3.3. *Assume* (A1)–(A3) *and let the conditional distribution* $\mu_1^2(A,\cdot)$ *of* z *given* A *be absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}^s$ *for* μ_2*-almost all* $A\in\mathbb{R}^{ms}$ $(\mu_2 := \mu\circ\pi_2^{-1})$; *then* $Q(\cdot,\mu)$ *is a continuous function on* $\mathbb{R}^m$.

Proof. As a consequence of Proposition 2.2, we obtained in Section 2 that the set of discontinuity points of Φ is contained in a countable union $\mathcal{H}$ of hyperplanes in $\mathbb{R}^s$. Therefore, for any $x\in\mathbb{R}^m$, $E(x)\subset E_1(x)$ where $E_1(x) := \{(z,A)\in\mathbb{R}^{(m+1)s}\colon z-Ax\in\mathcal{H}\}$. By (2.3),

$$\begin{aligned}\mu(E_1(x)) &= \int_{\mathbb{R}^{ms}}\int_{\mathbb{R}^s} \mathbf{1}_{E_1(x)}(z,A)\mu_1^2(A,\mathrm{d}z)\mu_2(\mathrm{d}A)\\ &= \int_{\mathbb{R}^{ms}}\int_{Ax+\mathcal{H}} \mu_1^2(A,\mathrm{d}z)\mu_2(\mathrm{d}A).\end{aligned}$$

Since $\mu_1^2(A,\cdot)$ is absolutely continuous μ_2-almost surely, we now have

$$\int_{Ax+\mathcal{H}} \mu_1^2(A,\mathrm{d}z) = 0, \qquad \text{for } \mu_2\text{-almost all } A\in\mathbb{R}^{ms}.$$

Hence, $\mu(E_1(x)) = 0$. This implies $\mu(E(x)) = 0$ for arbitrary $x\in\mathbb{R}^m$, and Proposition 3.2 yields the assertion. $\Box$

Remark 3.4. If z and A are independent random variables, then $\mu_1^2(A,\cdot)$ is absolutely continuous (for μ_2-almost all $A\in\mathbb{R}^{ms}$) if already the marginal distribution μ_1 has this property. Indeed, $\mu_1^2(A,\cdot)$ then coincides μ_2-almost surely with μ_1. Another instance where Corollary 3.3 works is given when there is a joint density of z and the random components of A (i.e., those which are not constant μ-almost surely).

If z and A are not independent, then it is not sufficient to claim that μ_1 is absolutely continuous when wishing to satisfy the assumptions of Corollary 3.3. Indeed, let $\mu\in\mathcal{P}(\mathbb{R}^2)$ be the uniform distribution concentrated on the line segment $\mathrm{conv}\{(0,0),(\frac{1}{2}\sqrt{2},\frac{1}{2}\sqrt{2})\}$. Then both μ_1 and μ_2 coincide with the uniform distribution on $\mathrm{conv}\{0,\frac{1}{2}\sqrt{2}\}$ and, for $0\leqslant A\leqslant\frac{1}{2}\sqrt{2}$, $\mu_1^2(A,\cdot)$ coincides with the measure concentrated at A.

However, if μ_1 is absolutely continuous and μ_2 is discrete (with countably many mass points), then $\mu_1^2(A,\cdot)$ is absolutely continuous μ_2-almost surely. To see this, let $B_1\subset\mathbb{R}^s$ have Lebesgue measure zero. Then,

$$0=\mu_1(B_1)=\mu(B_1\times\mathbb{R}^{ms})=\int_{\mathbb{R}^{ms}}\int_{\mathbb{R}^s}\mathbf{1}_{B_1\times\mathbb{R}^{ms}}\mu_1^2(A,\mathrm{d}z)\mu_2(\mathrm{d}A)$$
$$=\sum_{j=1}^{\infty}\int_{\mathbb{R}^s}\mathbf{1}_{B_1}(z)\mu_1^2(A_j,\mathrm{d}z)\cdot p_j=\sum_{j=1}^{\infty}p_j\cdot\mu_1^2(A_j,B_1),$$

with suitable mass points A_j and probabilities $p_j > 0$, $j = 1,2,\ldots$. This implies $\mu_1^2(A_j, B_1) = 0$ for all j, and $\mu_1^2(A,\cdot)$ is absolutely continuous μ_2-almost surely.

Remark 3.5. Proposition 3.2 and Corollary 3.3 extend [30, Theorem 5.1] where additional assumptions on Φ and μ are made and [28, Theorem 3.2] where only random right-hand sides in $\mathrm{P}(\mu)$ are allowed. In the recent paper [17] a similar analysis is carried out for simple integer recourse. The authors also show how to verify the crucial assumption in Proposition 3.2 when having information about conditional distributions given certain components of the random technology matrix A.

Proposition 3.6. *Adopt the setting of Corollary* 3.3 *and assume further that for any nonsingular linear transformation $B \in L(\mathbb{R}^s,\mathbb{R}^s)$ and for μ_2-almost all $A \in \mathbb{R}^{ms}$ the one-dimensional marginal distributions of $\mu_1^2(A,\cdot)\circ B$ have densities which are uniformly bounded with respect to A and which, outside some bounded interval not depending on A, are monotonically decreasing with growing absolute value of the argument. Then $Q(\cdot,\mu)$ is Lipschitz continuous on any bounded subset of $\mathbb{R}^m$.*

Proof. Let $D\subset\mathbb{R}^m$ be bounded and $x', x''\in D$. Then we have

$$\begin{aligned}&|Q(x',\mu)-Q(x'',\mu)|\\&=\left|\int_{\mathbb{R}^{(m+1)s}}(\Phi(z-Ax')-\Phi(z-Ax''))\mu(\mathrm{d}(z,A))\right|\\&=\left|\int_{\mathbb{R}^{ms}}\int_{\mathbb{R}^s}(\Phi(z-Ax')-\Phi(z-Ax''))\mu_1^2(A,\mathrm{d}z)\mu_2(\mathrm{d}A)\right|\\&\leqslant\int_{\mathbb{R}^{ms}}\left|\int_{\mathbb{R}^s}\Phi(z-Ax')\mu_1^2(A,\mathrm{d}z)-\int_{\mathbb{R}^s}\Phi(z-Ax'')\mu_1^2(A,\mathrm{d}z)\right|\mu_2(\mathrm{d}A).\end{aligned}$$

Our assumptions and [28, Theorem 3.4] (cf. also its proof) imply that there exists a constant $\tilde{L}>0$ (independent of A) such that

$$\left|\int_{\mathbb{R}^s}\Phi(z-Ax')\mu_1^2(A,\mathrm{d}z)-\int_{\mathbb{R}^s}\Phi(z-Ax'')\mu_1^2(A,\mathrm{d}z)\right|\leqslant\tilde{L}\cdot\|A\|\cdot\|x'-x''\|.$$

Hence we obtain

$$|Q(x',\mu)-Q(x'',\mu)|\leqslant\tilde{L}\int_{\mathbb{R}^{ms}}\|A\|\mu_2(\mathrm{d}A)\cdot\|x'-x''\|.$$

Note that

$$\int_{\mathbb{R}^{ms}}\|A\|\mu_2(\mathrm{d}A)=\int_{\mathbb{R}^{ms}}\|A\|\int_{\mathbb{R}^s}\mu_1^2(A,\mathrm{d}z)\mu_2(\mathrm{d}A)=\int_{\mathbb{R}^{(m+1)s}}\|A\|\mu(\mathrm{d}(z,A)).$$

Therefore, assumption (A3) yields the assertion. □

Since in the independent case conditional and marginal distributions coincide, we have the following straightforward consequence of Proposition 3.6.

Corollary 3.7. *Adopt the setting of Corollary* 3.3 *and let the random variables* z *and* A *be independent. Assume further that for any nonsingular linear transformation* $B \in L(\mathbb{R}^s, \mathbb{R}^s)$ *the one-dimensional marginal distributions of* $\mu_1 \circ B$ *have bounded densities which, outside some bounded interval, are monotonically decreasing with growing absolute value of the argument. Then* $Q(\cdot, \mu)$ *is Lipschitz continuous on any bounded subset of* $\mathbb{R}^m$.

If the support of μ_1 is the whole of $\mathbb{R}^s$ and if μ_1 is r-convex for some $r \in (-\infty, 0]$ (for the definition see [28] and the references therein), then the assumptions of Corollary 3.7 are satisfied [28, Proposition 3.8]. Distributions sharing these properties are, for instance, the (nondegenerate) multivariate normal and t-distributions [7, p. 113].

Clearly, if the marginal distribution μ_1 is one-dimensional, then the superposition with the linear transformation B can be skipped in Proposition 3.6 and Corollary 3.7, and the assumptions are in terms of $\mu_1^2(A, \cdot)$ and μ_1, only. In the multidimensional case it is sufficient to fulfill the assumptions of Proposition 3.6 only for specific transformations $B \in L(\mathbb{R}^s, \mathbb{R}^s)$ that depend on the problem (1.1), cf. [28]. Indeed, only such nonsingular linear transformations have to be considered that send the facets of $W'(\mathbb{R}_+^{s'})$ to the (linear) hyperplane orthogonal to the first coordinate axis (cf. Section 2 and the proof of [28, Theorem 3.4]).

The one-dimensional case including several refinements is treated in detail in [15-17,19].

The indispensability of both the boundedness and the monotonicity assumptions in the above statements is proved by [28, Examples 3.6 and 3.7] (for one-dimensional random right-hand side).

In the case of dependent random variables z and A, the verification of the assumptions in Proposition 3.6 is not so obvious. However, at least for the situation where we have a joint density for z and the random components of A, we can calculate a density of $\mu_1^2(A, \cdot)$ as a quotient of the joint density and the marginal density for A.

Let us now study the continuity of Q as a function jointly of $x \in \mathbb{R}^m$ and $\mu \in \mathcal{P}(\mathbb{R}^{(m+1)s})$ — the set of all Borel probability measures on $\mathbb{R}^{(m+1)s}$. While at $\mathbb{R}^m$ we have the usual convergence, a suitable notion on $\mathcal{P}(\mathbb{R}^{(m+1)s})$ is that of weak convergence of probability measures which covers a number of specific convergence modes for probability measures (e.g., pointwise convergent densities, discretizations via conditional expectations, convergence of empirical measures). A sequence $\{\mu_n\}$ of probability measures in $\mathcal{P}(\mathbb{R}^{(m+1)s})$ is said to *converge weakly* to $\mu \in \mathcal{P}(\mathbb{R}^{(m+1)s})$, i.e., $\mu_n \xrightarrow{w} \mu$, if for any bounded continuous function $g : \mathbb{R}^{(m+1)s} \to \mathbb{R}$ we have

$$\int_{\mathbb{R}^{(m+1)s}} g(z')\mu_n(\mathrm{d}z') \longrightarrow \int_{\mathbb{R}^{(m+1)s}} g(z')\mu(\mathrm{d}z'), \quad \text{as } n \to \infty.$$

A detailed description of the topology of weak convergence of probability measures can be found in the monograph [4].

For notational convenience we introduce the following subset of probability measures:

$$\Delta_{p,K}(\mathbb{R}^{(m+1)s}) = \left\{\nu \in \mathcal{P}(\mathbb{R}^{(m+1)s}) \colon \int_{\mathbb{R}^{(m+1)s}} \|(z,A)\|^p \nu(\mathrm{d}(z,A)) \leqslant K\right\},$$

where $p > 1$ and $K > 0$ are fixed real numbers.

Proposition 3.8. *Assume* (A1), (A2) *and let* $\mu \in \Delta_{p,K}(\mathbb{R}^{(m+1)s})$ *for some* $p > 1$, $K > 0$. *If the conditional distribution* $\mu_1^2(A,\cdot)$ *of* z *given* A *is absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}^s$ *for* μ_2*-almost all* $A \in \mathbb{R}^{ms}$, *then* Q, *as a function from* $\mathbb{R}^m \times \Delta_{p,K}(\mathbb{R}^{(m+1)s})$ *to* $\mathbb{R}$, *is continuous on* $\mathbb{R}^m \times \{\mu\}$.

Proof. Take an arbitrary $x \in \mathbb{R}^m$ and consider sequences $\{x_n\}$, $\{\mu_n\}$ in $\mathbb{R}^m$ and $\Delta_{p,K}(\mathbb{R}^{(m+1)s})$, respectively, such that $x_n \to x$ and $\mu_n \xrightarrow{w} \mu$ as $n \to \infty$. We introduce functions $h_n : \mathbb{R}^{(m+1)s} \to \mathbb{R}$ and $h : \mathbb{R}^{(m+1)s} \to \mathbb{R}$ defined by

$$h_n(z,A) = \Phi(z - Ax_n) \quad \text{and} \quad h(z,A) = \Phi(z - Ax),$$

which are measurable due to the lower semicontinuity of Φ. Consider the set $E_0(x)$ of all those $(z,A) \in \mathbb{R}^{(m+1)s}$ such that there exists a sequence $\{(z_n,A_n)\}_{n=1}^{\infty}$ in $\mathbb{R}^{(m+1)s}$ with

$$(z_n,A_n) \to (z,A) \quad \text{and} \quad h_n(z_n,A_n) \not\to h(z,A).$$

In our situation, $E_0(x) = E(x)$ with $E(x)$ as in Proposition 3.2. Indeed, the inclusion $E_0(x) \subseteq E(x)$ is easy to see. For the reverse inclusion, let $(z,A) \in E(x)$ and $\{b_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^s$ such that $b_n \to z - Ax$ and $\Phi(b_n) \not\to \Phi(z - Ax)$ as $n \to \infty$. Now consider the sequence $\{z_n,A_n)\}_{n=1}^{\infty}$ given by $A_n = A$ and $z_n = b_n + Ax_n$. Then,

$$(z_n,A_n) \to (z,A)$$

and

$$h_n(z_n,A_n) = \Phi(z_n - A_n x_n) = \Phi(b_n) \not\to \Phi(z - Ax) = h(z,A).$$

Hence, $(z,A) \in E_0(x)$.

Our assumption now implies $\mu(E_0(x)) = 0$ (cf. the proof of Corollary 3.3) and we can apply Rubin's Theorem [4, Theorem 5.5]. This yields

$$\mu_n \circ h_n^{-1} \xrightarrow{w} \mu \circ h^{-1}, \quad \text{as } n \to \infty. \tag{3.1}$$

To end up with

$$\int_{\mathbb{R}^{(m+1)s}} h_n(z,A)\mu_n(\mathrm{d}(z,A)) \xrightarrow{n\to\infty} \int_{\mathbb{R}^{(m+1)s}} h(z,A)\mu(\mathrm{d}(z,A)), \tag{3.2}$$

which, of course, is just the assertion, we will show that

$$\lim_{a\to\infty} \sup_n \int_{\{(z,A)\colon |h_n(z,A)|\geqslant a\}} |h_n(z,A)|\mu_n(\mathrm{d}(z,A)) = 0. \tag{3.3}$$

Since $p > 1$,

$$\begin{aligned}&\int_{\mathbb{R}^{(m+1)s}} |h_n(z,A)|^p \mu_n(\mathrm{d}(z,A))\\&\geqslant \int_{\{|h_n(z,A)|\geqslant a\}} |h_n(z,A)| \cdot |h_n(z,A)|^{p-1} \mu_n(\mathrm{d}(z,A))\\&\geqslant a^{p-1} \int_{\{|h_n(z,A)|\geqslant a\}} |h_n(z,A)| \mu_n(\mathrm{d}(z,A)).\end{aligned}$$

Therefore,

$$\int_{\{|h_n(z,A)|\geqslant a\}} |h_n(z,A)| \mu_n(\mathrm{d}(z,A)) \leqslant a^{1-p} \int_{\mathbb{R}^{(m+1)s}} |h_n(z,A)|^p \mu_n(\mathrm{d}(z,A)). \tag{3.4}$$

Proposition 2.1 and $h_n(0) = 0$ imply

$$|h_n(z,A)|^p \leqslant (\alpha\|z\| + \alpha\|x_n\| \cdot \|A\| + \beta)^p.$$

Since $\{x_n\}$ is bounded and all μ_n belong to $\Delta_{p,K}(\mathbb{R}^{(m+1)s})$, we now have a positive constant c such that

$$\int_{\mathbb{R}^{(m+1)s}} |h_n(z,A)|^p \mu_n(\mathrm{d}(z,A)) \leqslant c, \quad \text{for all } n \in \mathbb{N}.$$

Using (3.4), we thus obtain (3.3). Finally, (3.1) and [4, Theorem 5.4] yield (3.2), and the proof is complete. □

Remark 3.9. It is straightforward to replace the above assumption on $\mu_1^2(A,\cdot)$ by $\mu(E(x)) = 0$ (cf. Proposition 3.2) and to end up with continuity of $Q : \mathbb{R}^m \times \Delta_{p,K}(\mathbb{R}^{(m+1)s}) \to \mathbb{R}$ at (x,μ).

Proposition 3.8 extends corresponding results for noninteger stochastic programs in [12,24]. From an example in [24] it is also clear that the joint continuity of Q is lost if there is no assumption finally leading to the uniform integrability in (3.3). We have achieved this by claiming that $\mu \in \Delta_{p,K}(\mathbb{R}^{(m+1)s})$, $p > 1$, $K > 0$.

We close this section with an example illustrating the difficulties that occur when aiming at quantitative continuity results for $Q(x,\cdot)$ as a function on a suitable (metric) space of probability measures. For noninteger stochastic programs such results can be obtained when equipping a suitable subset of $\mathcal{P}(\mathbb{R}^{(m+1)s})$ with the Wasserstein metric [26, Proposition 2.1]. We will give an example that, for stochastic integer programs, there is no Hölder continuity estimate for $Q(x,\cdot)$ with respect to the Wasserstein distance. This also means that there cannot be a Hölder estimate with respect to the Prokhorov and the Dudley (or β-)metric, respectively [21,25].

For the comfort of the reader, we briefly introduce the Wasserstein distance $W_1(\mu,\nu)$ of two probability measures μ and ν belonging to

$$\mathfrak{M}_1(\mathbb{R}^{(m+1)s}) := \left\{\mu' \in \mathcal{P}(\mathbb{R}^{(m+1)s}): \int_{\mathbb{R}^{(m+1)s}} \|z'\|\mu'(\mathrm{d}z') < +\infty\right\}.$$

It is given by

$$W_1(\mu,\nu) = \inf\left\{\int_{\mathbb{R}^{(m+1)s}\times\mathbb{R}^{(m+1)s}} \|z'-z''\|\eta(\mathrm{d}z',\mathrm{d}z''): \eta \in D(\mu,\nu)\right\},$$

where

$$D(\mu,\nu) := \{\eta \in \mathcal{P}(\mathbb{R}^{(m+1)s}\times\mathbb{R}^{(m+1)s}): \eta\circ\pi_1^{-1} = \mu, \eta\circ\pi_2^{-1} = \nu\}$$

and π_1, π_2 denote the first and second projections, respectively.

For details we refer to [21] where it is also shown that for $\mu,\nu \in \mathfrak{M}_1(\mathbb{R})$,

$$W_1(\mu,\nu) = \int_{-\infty}^{\infty} |F_\mu(t) - F_\nu(t)|\,\mathrm{d}t, \tag{3.5}$$

where F_μ, F_ν denote the distribution functions of μ and ν, respectively.

Example 3.10. Let in (1.2) $\Phi(z-x) = \min\{y\colon\ y \geqslant z-x, y \in \mathbb{Z}\}$, i.e., $A = 1$ nonrandom, and let μ be a distribution of z with support contained in the closed interval $[-\frac{1}{2},\frac{1}{2}]$. It can be computed (cf. also the explicit formulae in [15,19]) that for $x \in [-\frac{1}{4},\frac{1}{4}]$ we have $Q(x,\mu) = 1 - F_\mu(x)$, where F_μ again denotes the distribution function of μ. Consider $\mu_n \in \mathcal{P}(\mathbb{R})$, $n \geqslant 1$, with support in $[-\frac{1}{2},\frac{1}{2}]$ and continuous distribution function F_{μ_n} fulfilling

$$F_{\mu_n}(t) = \begin{cases} t^{1/n} + \frac{1}{2}, & \text{for } 0 \leqslant t \leqslant r_n, \\ -|t|^{1/n} + \frac{1}{2}, & \text{for } -r_n \leqslant t \leqslant 0, \end{cases}$$

with a suitably fixed real number $r_n > 0$, $r_n < (\frac{1}{2})^n$. Let $\varepsilon_n > 0$ such that $\varepsilon_n < r_n$. We construct perturbations μ_{n,ε_n} of μ_n whose distribution functions coincide with those of μ_n for $t \notin [-\varepsilon_n,\varepsilon_n]$ and which are defined on $[-\varepsilon_n,\varepsilon_n]$ as follows:

$$F_{\mu_{n,\varepsilon_n}}(t) = \begin{cases} \frac{1}{2} - \varepsilon_n^{1/n}, & \text{for } -\varepsilon_n \leqslant t \leqslant 0, \\ \frac{1}{2} - \varepsilon_n^{1/n} + 2\varepsilon_n^{(1-n)/n}\cdot t, & \text{for } 0 \leqslant t \leqslant \varepsilon_n. \end{cases}$$

Using (3.5), we compute $W_1(\mu_n,\mu_{n,\varepsilon_n}) = \varepsilon_n^{(n+1)/n}$. On the other hand, $|Q(0,\mu_n) - Q(0,\mu_{n,\varepsilon_n})| = |F_{\mu_n}(0) - F_{\mu_{n,\varepsilon_n}}(0)| = \varepsilon_n^{1/n}$. Hence,

$$|Q(0,\mu_n) - Q(0,\mu_{n,\varepsilon_n})| = W_1(\mu_n,\mu_{n,\varepsilon_n})^{1/n+1}.$$

Since the construction was possible for any $n \in \mathbb{N}$, $n \geqslant 1$, there is no W_1-based Hölder estimate for $Q(x,\cdot)$.

4. Stability

In this section we study consequences of the above continuity results for the stability of

$$\mathrm{P}(\mu) \qquad \min\{f(x)+Q(x,\mu):\ x\in C\},$$

when the underlying measure μ is subjected to perturbations. Of course, $\mathrm{P}(\mu)$ is a non-convex program, and, hence, also local minimizers should be included into the analysis. Therefore, beside Berge's classical stability theory for abstract parametric programs [3], Robinson's results on local stability ([23], cf. also [14]) will be the main tools for our investigations. We will see that, having the continuity properties of Section 3 at one's disposal and using the techniques of [3,23], it is only a small step to arrive at the desired stability of $\mathrm{P}(\mu)$.

Let $V\subset\mathbb{R}^m$ be an arbitrary subset and let $\operatorname{cl}V$ denote the closure of V. Then we introduce the following localized versions for the optimal-value function and the solution set mapping:

$$\varphi_V(\mu) := \inf\{f(x)+Q(x,\mu):\ x\in C\cap\operatorname{cl}V\},$$

$$\psi_V(\mu) := \{x\in C\cap\operatorname{cl}V:\ f(x)+Q(x,\mu)=\varphi_V(\mu)\}.$$

A central observation in [14,23] is that local minimizers of parametric programs may behave unstable when directly transferring assumptions from global stability analysis. For local stability analysis it turns out crucial that considerations include all local minimizers that are, in some sense, nearby the minimizers one is interested in. This leads to the concept of a complete local minimizing set (CLM set), which can be formulated in our terminology as follows.

Given $\mu\in\mathcal{P}(\mathbb{R}^{(m+1)s})$, a nonempty set $M\subset\mathbb{R}^m$ is called a *CLM set* for $\mathrm{P}(\mu)$ with respect to an open set $V\subset\mathbb{R}^m$ if $M\subset V$ and $M=\psi_V(\mu)$. Of course, the set of global minimizers is always a CLM set; further examples are strict local minimizers. For more details consult [14,23].

Considering $\mathrm{P}(\mu)$ as a parametric program whose parameter space is $\mathcal{P}(\mathbb{R}^{(m+1)s})$ endowed with the topology of weak convergence of probability measures (cf. Section 3 and [4]) we have the following general stability theorem.

Proposition 4.1. *Assume* (A1), (A2), *let* $\mu\in\Delta_{p,K}(\mathbb{R}^{(m+1)s})$ *for some* $p>1$, $K>0$ *and let the conditional distribution* $\mu_1^2(A,\cdot)$ *of* z *given* A *be absolutely continuous with respect to the Lebesgue measure on* $\mathbb{R}^s$ *for* μ_2*-almost all* $A\in\mathbb{R}^{ms}$. *Suppose further that* $M\subset\mathbb{R}^m$ *is a CLM set for* $\mathrm{P}(\mu)$ *with respect to some bounded open set* $V\subset\mathbb{R}^m$, *i.e.,* $M=\psi_V(\mu)$. *Then,*

(i) *the function* φ_V *(from* $\Delta_{p,K}(\mathbb{R}^{(m+1)s})$ *to* $\mathbb{R}$*) is continuous at* μ;

(ii) *the multifunction* ψ_V *(from* $\Delta_{p,K}(\mathbb{R}^{(m+1)s})$ *to* $\mathbb{R}^m$*) is Berge upper semicontinuous at* μ, *i.e., for any open set* G *in* $\mathbb{R}^m$ *with* $G\supset\psi_V(\mu)$ *there exists a neighbourhood* U *of* μ *in* $\Delta_{p,K}(\mathbb{R}^{(m+1)s})$ *such that* $\psi_V(\mu')\subset G$ *whenever* $\mu'\in U$;

(iii) *there exists a neighbourhood* U' *of* μ *in* $\Delta_{p,K}(\mathbb{R}^{(m+1)s})$ *such that for all* $\mu'\in U'$ *we have that* $\psi_V(\mu')$ *is a CLM set for* $\mathrm{P}(\mu')$ *with respect to* V.

Proof. Using Proposition 3.8, the proof of (i) and (ii) follows the lines of Berge's theory (cf. also [1, proof of Theorem 4.2.2]) and is therefore not repeated here.

When verifying (iii), the nonemptiness of $\psi_V(\mu')$ is gained via the lower semicontinuity of $Q(\cdot,\mu')$ (Proposition 3.1); the CLM property then follows from (ii). □

Let us add a few comments on the above proposition.

If the location of the (bounded) CLM set $\psi_V(\mu)$ is known, it could be helpful to know that the assumption on $\mu_1^2(A,\cdot)$ can be relaxed to claiming that $\mu(E(x)) = 0$ for any $x \in C \cap \mathrm{cl}\, V$ (cf. Remark 3.9). Indeed, for the mentioned analysis along the lines of Berge, the continuity of Q is only needed on $(C \cap \mathrm{cl}\, V) \times \{\mu\}$.

If one relaxes the CLM property of M to assuming that M is a bounded set of local minimizers to $\mathrm{P}(\mu)$, then it is not difficult to construct counterexamples where the perturbed programs have no local minimizers at all near M how "small" the perturbation is ever taken. Such an example can be found in [27] for two-stage stochastic integer programs with random right-hand side and nonstochastic technology matrix.

When analyzing (iii), it is clear that in view of the lower semicontinuity of $Q(\cdot,\mu')$ (Proposition 3.1) and the compactness of $\mathrm{cl}\, V$, the sets $\psi_V(\mu')$ are always nonempty. In this context, the essence of (iii) is that nonemptiness of $\psi_V(\mu')$ is not enforced by restricting the objective to a compact, but that, for μ' sufficiently close to μ, the sets $\psi_V(\mu')$ again consist of local minimizers to $\mathrm{P}(\mu')$.

Proposition 4.1 extends the corresponding stability result [27, Theorem 3.1] from two-stage stochastic integer programs with random right-hand side to those where also the technology matrix may be random. The analogous stability results for two-stage stochastic programs without integer requirements were derived in [12,24].

Proposition 4.1 may also be read as a general justification for numerical procedures that rely on approximating the distribution μ by simpler ones. For instance, discretizing μ via conditional expectations [5,13] yields a weakly convergent sequence of probability measures, provided that support partitions become arbitrarily small. Proposition 4.1 then ensures convergence of local optimal values and optimal solutions. Of course, up to now there are no comprehensive algorithms to solve stochastic integer programs with discrete probability distributions. However, in the recent paper [16] some substantial progress was made for stochastic programs with simple integer recourse.

5. Conclusions

For the expected-recourse function Q of a stochastic program with complete (mixed-) integer recourse we have derived sufficient conditions for various types of continuity and studied their impact on the stability of the model. Q is considered as a function both in the first-stage policy x and in the integrating probability measure μ. Basic assumptions guaranteeing that Q is well-defined simultaneously ensure the lower semicontinuity of $Q(\cdot,\mu)$ (Proposition 3.1). The function $Q(\cdot,\mu)$ is continuous on $\mathbb{R}^m$ if, almost surely, the conditional distribution of the right-hand side vector z given the value of the technology matrix A has a density (Corollary 3.3). A more general result (Proposition 3.2) states that $Q(\cdot,\mu)$ is continuous at $x \in \mathbb{R}^m$ if certain hyperplanes related to x have

μ-measure zero. Properties of certain linear transforms of the above-mentioned conditional distributions guarantee the Lipschitz continuity of $Q(\cdot, \mu)$, namely, densities of one-dimensional marginal distributions must be bounded and have a certain tail property (Proposition 3.6). To study the joint continuity of Q in x and μ, the space of underlying probability measures is equipped with weak convergence. Adding a moment condition (resulting in uniform integrability) to the assumptions for the continuity of $Q(\cdot, \mu)$ (Proposition 3.2, Corollary 3.3) then yields the desired joint continuity (Proposition 3.8, Remark 3.9).

In a stochastic program with complete (mixed-)integer recourse, the function Q is the most essential part. The stability of such models is studied in terms of continuity properties of (multi-)functions assigning to μ local optimal values and local optimal solutions, respectively. The inherent nonconvexity necessitates the exclusion of certain "pathological" types of local minimizers. Assuming that the unperturbed problem has a bounded set of local solutions and imposing the sufficient conditions for the joint continuity of Q (Proposition 3.8) then leads to continuity of local optimal values and upper semicontinuity of local optimal solutions (Proposition 4.1).

Acknowledgements

I wish to thank Willem K. Klein Haneveld (University of Groningen), Werner Römisch (Humboldt University Berlin), Leen Stougie (University of Amsterdam) and Maarten H. van der Vlerk (University of Groningen) for beneficial discussions. Moreover, I am indebted to two anonymous referees for their helpful comments.

References

[1] B. Bank, J. Guddat, D. Klatte, B. Kummer and K. Tammer, *Nonlinear Parametric Optimization* (Akademie-Verlag, Berlin, 1982).
[2] B. Bank and R. Mandel, *Parametric Integer Optimization* (Akademie-Verlag, Berlin, 1988).
[3] C. Berge, *Espaces Topologiques, Fonctions Multivoques* (Dunod, Paris, 1959).
[4] P. Billingsley, *Convergence of Probability Measures* (Wiley, New York, 1968).
[5] J. Birge and R. Wets, "Designing approximation schemes for stochastic optimization problems, in particular for stochastic programs with recourse," *Mathematical Programming Study* 27 (1986) 54–102.
[6] C.E. Blair and R.G. Jeroslow, "The value function of a mixed integer program: I," *Discrete Mathematics* 19 (1977) 121–138.
[7] C. Borell, "Convex set functions in d-space," *Periodica Mathematica Hungarica* 6 (1975) 111–136.
[8] R.M. Dudley, *Real Analysis and Probability* (Wadsworth & Brooks/Cole, Pacific Grove, 1989).
[9] J. Dupačová, "Stochastic programming with incomplete information: A survey of results on postoptimization and sensitivity analysis," *Optimization* 18 (1987) 507–532.
[10] P. Gänssler and W. Stute, *Wahrscheinlichkeitstheorie* (Springer, Berlin, 1977).
[11] P. Kall, *Stochastic Linear Programming* (Springer, Berlin, 1976).
[12] P. Kall, "On approximations and stability in stochastic programming," in: J. Guddat, H.Th. Jongen, B. Kummer and F. Nožička, eds., *Parametric Optimization and Related Topics* (Akademie Verlag, Berlin, 1987) pp. 387–407.
[13] P. Kall, A. Ruszczyński and K. Frauendorfer, "Approximation techniques in stochastic programming," in: Y. Ermoliev and R. Wets, eds., *Numerical Techniques for Stochastic Optimization* (Springer, Berlin, 1988) pp. 33–64.

[14] D. Klatte, "A note on quantitative stability results in nonlinear optimization," in: K. Lommatzsch, ed., *Proceedings of the 19. Jahrestagung Mathematische Optimierung*, Seminarbericht Nr. 90, Sektion Mathematik, Humboldt-Universität Berlin (1987) pp. 77–86.

[15] W.K. Klein Haneveld, L. Stougie and M.H. van der Vlerk, "Stochastic integer programming with simple recourse," Research Memorandum 455, Institute of Economic Research, University of Groningen (1991).

[16] W.K. Klein Haneveld, L. Stougie and M.H. van der Vlerk, "On the convex hull of the simple integer recourse objective function," Research Memorandum 516, Institute of Economic Research, University of Groningen (1993).

[17] W.K. Klein Haneveld and M.H. van der Vlerk, "On the expected value function of a simple integer recourse problem with random technology matrix," *Journal of Computational and Applied Mathematics* 56 (1994) 45–53.

[18] G. Laporte and F.V. Louveaux, "The integer L-shaped method for stochastic integer programs with complete recourse," *Operations Research Letters* 13 (1993) 133–142.

[19] F.V. Louveaux and M.H. van der Vlerk, "Stochastic programs with simple integer recourse," *Mathematical Programming* 61 (1993) 301–326.

[20] G.L. Nemhauser and L.A. Wolsey, *Integer and Combinatorial Optimization* (Wiley, New York, 1988).

[21] S.T. Rachev, *Probability Metrics and the Stability of Stochastic Models* (Wiley, New York, 1991).

[22] A. Rinnooy Kan and L. Stougie, "Stochastic integer programming," in: Y. Ermoliev and R. Wets, eds., *Numerical Techniques for Stochastic Optimization* (Springer, Berlin, 1988) pp. 201–213.

[23] S.M. Robinson, "Local epi-continuity and local optimization," *Mathematical Programming* 37 (1987) 208–222.

[24] S.M. Robinson and R. Wets, "Stability in two-stage stochastic programming," *SIAM Journal on Control and Optimization* 25 (1987) 1409–1416.

[25] W. Römisch and R. Schultz, "Distribution sensitivity in stochastic programming," *Mathematical Programming* 50 (1991) 197–226.

[26] W. Römisch and R. Schultz, "Stability analysis for stochastic programs," *Annals of Operations Research* 30 (1991) 241–266.

[27] R. Schultz, "Continuity and stability in two-stage stochastic integer programming," in: K. Marti, ed., *Stochastic Optimization, Numerical Methods and Technical Applications*, Lecture Notes in Economics and Mathematical Systems 379 (Springer, New York, 1992) pp. 81–92.

[28] R. Schultz, "Continuity properties of expectation functions in stochastic integer programming," *Mathematics of Operations Research* 18 (1993) 578–589.

[29] A. Shapiro, "Quantitative stability in stochastic programming," *Mathematical Programming* 67 (1994) 99–108.

[30] L. Stougie, "Design and analysis of algorithms for stochastic integer programming," Ph.D. Thesis, Center for Mathematics and Computer Science, Amsterdam (1985).

[31] R. Wets, "Stochastic programs with fixed recourse: the equivalent deterministic program," *SIAM Review* 16 (1974) 309–339.