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On the positive sum property and the computation of Graver test sets

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Abstract. In the first part of this paper we discuss the positive sum property, a property inherent to LP, IP, and MIP Graver test sets, and develop, based on that property, a common notational and algorithmic framework for Graver test sets. Moreover, we show how to find an initial feasible solution with the help of universal test sets and deal with termination of the augmentation algorithm that improves this solution to optimality.

1. Introduction

Many important classes of optimization problems arising in practical applications can be modelled as (usually) large mixed-integer linear programming problems. During the last few years, there has been a renewed interest in a different solution approach. These so-called primal methods are based on a simple augmentation procedure that repeatedly improves a given feasible solution as long as this solution is not optimal. In contrast to the simplex algorithm that moves along edges of the feasible region from vertex to vertex in order to find an optimal solution to a given linear program, augmentation steps also through the interior of the set of feasible solutions are allowed.

Test sets are sets of vectors (or directions) that provide improving directions to given non-optimal feasible solutions of our problem, the major ingredient for the augmentation algorithm.

Let $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ denote the integers, rationals, and reals, respectively. Moreover, let $\mathbb{X} = \mathbb{Z}^{d_1} \times \mathbb{R}^{d_2}$, and denote by $\mathbb{X}_+$ the corresponding non-negative orthant. For given $d = d_1 + d_2$, $A \in \mathbb{Z}^{l \times d}$, $c \in \mathbb{R}^d$, and $b \in \mathbb{R}^l$, let

$$(\mathrm{P})_{c,b} : \qquad \min\{c^\intercal z : Az = b,\ z \in \mathbb{X}_+\}$$

be the family of mixed-integer linear optimization problems as $c \in \mathbb{R}^d$ and $b \in \mathbb{R}^l$ vary. Instead of $(\mathrm{P})_{c,b}$ we write $(\mathrm{LP})_{c,b}$ if $\mathbb{X} = \mathbb{R}^d$ and $(\mathrm{IP})_{c,b}$ if $\mathbb{X} = \mathbb{Z}^d$. By abuse of notation we refer to subsets of the problem family $(\mathrm{P})_{c,b}$ as $(\mathrm{P})_{c,b}$ as well, but state which data is kept fixed and which is allowed to vary. Thus, a single instance, that is where b and c are given, is also denoted by $(\mathrm{P})_{c,b}$. In the present work we address solving $(\mathrm{P})_{c,b}$ by test set methods.

Definition 1. *(Test Set)*
A set $\mathcal{T} \subseteq \mathbb{R}^d$ is called a test set for $(\mathrm{P})_{c,b}$ *if*

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1. *$c^\intercal t > 0$ for all $t \in \mathcal{T}$, and*
2. *for every $b \in \mathbb{R}^l$ and for every non-optimal feasible point z_0 of $(\mathrm{P})_{c,b}$ there exist a vector $t \in \mathcal{T}$ and a scalar $\alpha > 0$ such that $z_0 - \alpha t$ is feasible.*

A vector $t \in \mathcal{T}$ satisfying these two conditions is called an improving vector or an improving direction.

A set is called a universal test set for $(\mathrm{P})_{c,b}$ if it contains a test set for $(\mathrm{P})_{c,b}$ for every cost vector $c \in \mathbb{R}^d$.

In contrast to the common definition of test sets we do not impose finiteness on $\mathcal{T}$. This allows a treatment of test sets for mixed-integer programs which need not be finite in general (see for example Cook et al. [2]).

Once a test set $\mathcal{T}$ for and a feasible solution z_0 to $(\mathrm{P})_{c,b}$ are available, the following augmentation algorithm can be employed in order to solve the optimization problem $(\mathrm{P})_{c,b}$.

Algorithm 1. *(Augmentation Algorithm)*

<u>*Input:*</u> *$\mathbb{X}$, A, $\mathcal{T}$, c, a feasible solution z_0 to $(\mathrm{P})_{c,b}$*
<u>*Output:*</u> *an optimum $z_{\min}$ of $(\mathrm{P})_{c,b}$*

<u>*while*</u> *there are $t \in \mathcal{T}$, $\alpha \in \mathbb{R}_{>0}$ such that $c^\intercal t > 0$ and $z_0 - \alpha t$ is feasible* <u>*do*</u>
  $z_0 := z_0 - \alpha t$, where α is maximal such that $z_0 - \alpha t$ is still feasible
<u>*return*</u> *z_0*

In 1975, Graver [5] introduced finite universal test sets for linear programs (LP) and for integer linear programs (IP). However, he provided no algorithm to compute them. He also presented a similar test set for linear mixed-integer programs (MIP) and already pointed out that these sets need not be finite in general [4]. We will discuss the positive sum property, a property inherent to LP, IP, and MIP Graver test sets, and develop, based on this property, a common notational and algorithmic framework for Graver test sets. This simplification of notation and algorithms is an important basis for the development and presentation of fast algorithms to compute Hilbert bases and extremal rays of pointed rational cones [7], and of a novel decomposition approach to two- and multi-stage stochastic programs [6]. In both applications even the LP cases turn out to be important and of interest.

Definition 2. *(Positive Sum Property)*

A set G has the positive sum property with respect to $S \subseteq \mathbb{R}^d$ if $G \subseteq S$ and if any non-zero $v \in S$ can be written as a finite linear combination $v = \sum \alpha_i g_i$ with

- *$g_i \in G$, $\alpha_i > 0$, $\alpha_i g_i \in S$, and*
- *for all i, g_i and v belong to the same orthant, that is, $g_i^{(k)} v^{(k)} \geq 0$ for every component $k = 1, \ldots, d$.*

Graver test sets are inclusion minimal sets having the positive sum property with respect to the set $\ker_{\mathbb{X}}(A) := \{v \in \mathbb{X} : Av = 0\}$, the (mixed-integer) kernel of A. Already Graver [4, 5] showed that the positive sum property implies the universal test set property.

In Lemmas 2 and 5 we present two criteria characterizing those sets that have the positive sum property with respect to $\ker_{\mathbb{R}^d}(A)$ or with respect to an integer lattice

$\Lambda \subseteq \mathbb{Z}^d$, of which $\ker_{\mathbb{Z}^d}(A)$ is just one example. These criteria immediately lead us to algorithms for the computation of LP and IP Graver test sets using the algorithmic pattern of a completion procedure [1]: step by step an initial set of vectors is completed with respect to the positive sum property by adding vectors to the set as long as necessary. The algorithm for the IP case computes minimal points (with respect to a certain relation $\sqsubseteq$ defined below) in integer lattices $\Lambda \subseteq \mathbb{Z}^d$. For the special case $\Lambda = \ker_{\mathbb{Z}^d}(A)$, this algorithm can already be found in [11] in the disguised form of a multi-dimensional Euclidean algorithm. Another algorithm for the computation of LP Graver test sets can be found for example in [13].

Then, in Section 3, we separately treat LP, IP, and MIP Graver test sets in more detail. For the MIP case we present an example similar to the one by Cook et al. [2] which implies that MIP test sets need not be finite in general. In [2], the existence of finite MIP tests is shown if considerations are restricted to a smaller family of optimization problems. In contrast to this approach we introduce a finite set of integer vectors which is no test set by the above definition. However, we can nonetheless reconstruct from this set an improving vector to any non-optimal feasible solution to a given problem. This reconstruction property enables us to use the Augmentation Algorithm 1 as well. Finally, we give an algorithm to compute this finite set of integer vectors.

In Section 4 we include a discussion on termination of the Augmentation Algorithm 1. We demonstrate that particularly in the LP case some caution is appropriate to avoid zig-zagging (even to non-optimal solutions) and we present a strategy which ensures termination.

A test set has to give improving directions for any right-hand side b and hence cannot depend on the specific choice of b. The only and, if integrality constraints on variables are present, usually already hard step specific to b is to find some feasible solution of the given problem. Although there are algorithmic alternatives, we show in Section 5 that universal test sets can be used to solve this feasibility problem as well. Again, some care has to be taken in the LP case to ensure termination of this procedure.

Before starting our analysis, let us mention that other test sets besides Graver test sets were introduced and studied in the literature. We refer to [15] for a recent survey on all currently known test sets in IP.

Some recent work has been done on MIP test sets. Following a geometric approach, Köppe et al. [8, 9] obtained a finiteness result similar to the one we obtain by an algebraic argument in Section 3.

2. Positive sum property

In this section we introduce the positive sum property. Already Graver showed that this property implies the universal test set property. Then we present two criteria which allow us to decide whether a given set G has the positive sum property with respect to an integer lattice $\Lambda \subseteq \mathbb{Z}^d$, for example $\ker_{\mathbb{Z}^d}(A)$, or with respect to $\ker_{\mathbb{R}^d}(A)$.

These two criteria immediately lead us to the concept of a completion procedure: a given generating set G of an integer lattice Λ over $\mathbb{Z}$ or of $\ker_{\mathbb{R}^d}(A)$ over $\mathbb{R}$ is completed with respect to the positive sum property by adding new elements to G as long as necessary. Once the completion procedure terminates, the two criteria below already imply its correctness, that is, the set of vectors returned has indeed the positive sum property with respect to Λ or $\ker_{\mathbb{R}^d}(A)$.

2.1. Positive sum property implies universal test set property

To simplify the subsequent proofs it is convenient to introduce the following relation.

Definition 3. *We define the relation $\sqsubseteq$ on $\mathbb{R}^d$ by $u \sqsubseteq v$ if $u^{(j)}v^{(j)} \geq 0$ and $|u^{(j)}| \leq |v^{(j)}|$ for all components $j = 1, \ldots, d$, that is, u belongs to the same orthant as v and its components are not greater in absolute value than the corresponding components of v.*

Clearly, if $u \sqsubseteq v$ then also $v - u \sqsubseteq v$. The following lemma was already proved by Graver [5].

Lemma 1. *(Positive Sum Property implies Universal Test Set Property)*
If G has the positive sum property with respect to $\ker_{\mathbb{X}}(A)$ *then G is a universal test set for* $(\mathrm{P})_{c,b}$.

2.2. Criteria to check positive sum property

Let us start with the criterion for the IP situation. The following result is not only valid for the (saturated) lattice $\ker_{\mathbb{Z}^d}(A)$ but holds for arbitrary integer lattices Λ, as well. This more general version allows us to generalize Graver test sets and their computation to optimization problems of the form

$$(\mathrm{MOD})_{c,b,\bar{b}}: \qquad \min\{c^{\intercal}z : Az = b, \bar{A}z \equiv \bar{b} \pmod{p}, z \in \mathbb{Z}^d_+\},$$

where $\bar{A}z \equiv \bar{b} \pmod{p}$ abbreviates the relations $\bar{a}_i z \equiv \bar{b}_i \pmod{p_i}$ for $p_i \in \mathbb{Z}_+$, $i = 1, \ldots, \bar{l}$. These problems have a close connection for example to the "group problem in integer programming" ([12], pp. 363-367) and to the integer Fourier-Motzkin elimination [16–18], respectively.

Employing similar arguments as in Graver's proof of Lemma 1, we can conclude that every set of vectors that has the positive sum property with respect to the integer lattice $\Lambda := \{z : Az = 0, \bar{A}z \equiv 0 \pmod{p}, z \in \mathbb{Z}^d\}$ is also a universal test sets for the family of optimization problems $(\mathrm{MOD})_{c,b,\bar{b}}$ as $c \in \mathbb{R}^d$, $b \in \mathbb{R}^l$, and $\bar{b} \in \mathbb{R}^{\bar{l}}$ vary.

Universal test sets (universal Gröbner bases) for $(\mathrm{MOD})_{c,b,\bar{b}}$ were already considered in [14]. Here, we only want to emphasize that all our algorithmic results for $(\mathrm{IP})_{c,b}$ readily extend to $(\mathrm{MOD})_{c,b,\bar{b}}$, since only the lattice structure of $\ker_{\mathbb{Z}^d}(A)$ and the positive sum property with respect to this lattice are exploited in the proofs. However, for ease of exposition, we will later restrict our attention to the special case $\Lambda = \ker_{\mathbb{Z}^d}(A)$.

Lemma 2. *(Criterion for Positive Sum Property with respect to Integer Lattice)*
Let Λ be an integer sublattice of $\mathbb{Z}^d$. A symmetric set $G \subseteq \Lambda$ has the positive sum property with respect to Λ if and only if the following two conditions hold:

- *G finitely generates Λ over $\mathbb{Z}$, and*

– *for every pair $v, w \in G$, the vector $v+w$ can be written as a finite linear combination $v + w = \sum \alpha_i g_i$, where for all i we have $g_i \in G$, $\alpha_i \in \mathbb{Z}_{>0}$, and $g_i \sqsubseteq v + w$.*

Herein, a set G is called symmetric if $v \in G$ implies $-v \in G$.

Proof. We have to show that any non-zero $z \in \Lambda$ can be written as a finite positive linear integer combination of elements from G where each vector in this combination belongs to the same orthant as z. Since G generates Λ over $\mathbb{Z}$ and since G is symmetric, the vector z can be written as a linear combination $z = \sum \alpha_i v_i$ for finitely many $\alpha_i \in \mathbb{Z}_{>0}$ and $v_i \in G$. Note that $\sum \alpha_i \|v_i\|_1 \geq \|z\|_1$ with equality if and only if $v_i \sqsubseteq z$ for all i.

From the set of all such linear integer combinations $\sum \alpha_i v_i$ choose one such that $\sum \alpha_i \|v_i\|_1$ is minimal and assume that $\sum \alpha_i \|v_i\|_1 > \|z\|_1$. Otherwise $v_i \sqsubseteq z$ for all i and we are done. Therefore, there have to exist vectors v_{i_1}, v_{i_2} in this representation which have some component $k = k_0$ of different signs.

By the assumptions on the set G, the vector $v_{i_1} + v_{i_2}$ can be written as $v_{i_1} + v_{i_2} = \sum \beta_j v'_j$ for finitely many $\beta_j \in \mathbb{Z}_{>0}$, $v'_j \in G$, and $\beta_j v'_j \sqsubseteq v_{i_1} + v_{i_2}$ for all j. The latter implies that we have for each component $k = 1, \ldots, d$,

$$\sum_j \beta_j |{v'_j}^{(k)}| = |\sum_j \beta_j {v'_j}^{(k)}| = |(v_{i_1} + v_{i_2})^{(k)}| \leq |v_{i_1}^{(k)}| + |v_{i_2}^{(k)}|,$$

where the last inequality is strict for $k = k_0$ by construction. Summing up over $k = 1, \ldots, d$, yields $\sum \beta_j \|v'_j\|_1 = \|v_{i_1} + v_{i_2}\|_1 < \|v_{i_1}\|_1 + \|v_{i_2}\|_1$. But now z can be represented as

$$\begin{aligned} z &= \alpha_{i_1} v_{i_1} + \alpha_{i_2} v_{i_2} + \sum_{i \neq i_1, i_2} \alpha_i v_i \\ &= \sum \beta_j v'_j + (\alpha_{i_1} - 1) v_{i_1} + (\alpha_{i_2} - 1) v_{i_2} + \sum_{i \neq i_1, i_2} \alpha_i v_i \end{aligned}$$

and it holds

$$\sum \beta_j \|v'_j\|_1 + (\alpha_{i_1} - 1)\|v_{i_1}\|_1 + (\alpha_{i_2} - 1)\|v_{i_2}\|_1 + \sum_{i \neq i_1, i_2} \alpha_i \|v_i\|_1 < \sum \alpha_i \|v_i\|_1$$

in contradiction to the minimality required on $\sum \alpha_i \|v_i\|_1$. Thus, we conclude that $\|z\|_1 = \sum \alpha_i \|v_i\|_1$, and therefore, G has the positive sum property with respect to Λ. □

Now let us turn our attention to the LP situation. In order to prove the criterion of Lemma 5 below we need to introduce the set of circuits associated with a matrix A. This set will turn out to be a minimal set having the positive sum property with respect to $\ker_{\mathbb{R}^d}(A)$.

Definition 4. *For $v \in \mathbb{R}^d$ denote by* $\operatorname{supp}(v) := \{j : v^{(j)} \neq 0\}$ *the support of v.*

Definition 5. *(Circuits of a Matrix)*

A circuit of a matrix $A \in \mathbb{Z}^{l \times d}$ is a vector $q \in \ker_{\mathbb{Z}^d}(A)$ of inclusion minimal support in $\ker_{\mathbb{R}^d}(A) \setminus \{0\}$ whose components have greatest common divisor 1.

Note that if A has only integer (or rational) entries then there is always a rational (and thus also an integer) vector for every minimal support in $\ker_{\mathbb{R}^d}(A)$. Proofs for the following three statements can be found in [5].

Lemma 3. *If v is a circuit of A and $w \in \ker_{\mathbb{R}^d}(A)$ satisfies $\operatorname{supp}(w) \subseteq \operatorname{supp}(v)$ then $w = \alpha v$ for some $\alpha \in \mathbb{R}$. In particular, we have $w = \pm v$ if v and w are circuits with the same support.*

Corollary 1. *Every matrix $A \in \mathbb{Z}^{l \times d}$ has only finitely many circuits.*

Lemma 4. *The set of all circuits of A has the positive sum property with respect to $\ker_{\mathbb{R}^d}(A)$.*

Now we are in the position to state and to prove the main result of this subsection.

Lemma 5. *(Criterion for Positive Sum Property with respect to $\ker_{\mathbb{R}^d}(A)$)*
A set $G \subseteq \ker_{\mathbb{R}^d}(A)$ has the positive sum property with respect to $\ker_{\mathbb{R}^d}(A)$ if and only if the following two conditions hold:

- *G finitely generates $\ker_{\mathbb{R}^d}(A)$ over $\mathbb{R}$, and*
- *for every pair $v, w \in G$, and all $\alpha \in \mathbb{R}$, the vector $v + \alpha w$ can be written as a finite linear combination $v + \alpha w = \sum \alpha_i g_i$, where for all $i = 1, \ldots, d$, we have $g_i \in G$ and $\operatorname{supp}(g_i) \subseteq \operatorname{supp}(v + \alpha w)$. Herein, only those values for $\alpha \in \mathbb{R}$ need to be considered, for which $v + \alpha w$ contains a zero entry at some component k at which neither v nor w have a zero entry, that is, $(v + \alpha w)^{(k)} = 0$ but $v^{(k)} w^{(k)} \neq 0$.*

Proof. It suffices to prove that G contains some non-zero scalar multiple of every circuit q of A. Lemma 5 then follows by application of Lemma 4.

Let q be an arbitrary circuit of A. Without loss of generality we may assume that the zero components of q are the first m components. Otherwise we may rearrange the variables to ensure this property. Since the vectors in G generate $\ker_{\mathbb{R}^d}(A)$ over $\mathbb{R}$ we have $q = \sum \alpha_i g_i$ for finitely many non-zero $\alpha_i \in \mathbb{R}$ and $g_i \in G$.

To prove our claim, it suffices to rewrite this linear combination into a linear combination where all appearing vectors from G have a zero first component. Repeating this process with the second, third, $\ldots$, m^{th} components, we arrive at a linear representation of q by elements from G which have zeros in their first m components. Choose any $g_j \in G$ that occurs in this last linear combination. Since $\operatorname{supp}(g_j) \subseteq \operatorname{supp}(q)$ and since q is a circuit, we conclude by Lemma 3 that $g_j \in G$ is a non-zero scalar multiple of q and our claim follows. Thus, it remains to show that this rewriting step is always possible.

If all g_i in $q = \sum \alpha_i g_i$ have already a zero first component we can go on with the second, third, $\ldots$, m^{th} components. If not, $0 = q^{(1)} = \sum \alpha_i g_i^{(1)}$ implies that there exist at least two vectors g_j and g_k with non-zero first components. Take g_j and g_k and rewrite $\alpha_j g_j + \alpha_k g_k$ as follows:

$$\begin{aligned}
\alpha_j g_j + \alpha_k g_k &= \alpha_j g_j + \alpha_k \left(g_k - \frac{g_k^{(1)}}{g_j^{(1)}} g_j \right) + \alpha_k \frac{g_k^{(1)}}{g_j^{(1)}} g_j \\
&= \left(\alpha_j + \alpha_k \frac{g_k^{(1)}}{g_j^{(1)}} \right) g_j + \alpha_k \left(g_k - \frac{g_k^{(1)}}{g_j^{(1)}} g_j \right).
\end{aligned}$$

Since the first component of $(g_k - g_k^{(1)} g_j / g_j^{(1)})$ vanishes, the assumptions on G yield a representation of $(g_k - g_k^{(1)} g_j / g_j^{(1)})$ as a linear combination of elements in G whose support is contained in the support of $(g_k - g_k^{(1)} g_j / g_j^{(1)})$. Thus, each vector in this linear combination has a zero first component. Substituting this representation back into the above linear combination for q we arrive at a new representation $q = \sum \alpha_i' g_i'$, where the number of vectors in this linear combination with non-zero first component is at least one less than the corresponding number for $q = \sum \alpha_i g_i$. Hence this process terminates with a finite linear representation of q using only elements from G with a zero first component. Now we can go on with the second, third, ..., m^{th} components. Clearly, zero entries on previous components (that are already considered) will not be destroyed by our rewriting steps. This finally concludes the proof. □

2.3. Completion algorithm

Lemmas 2 and 5 reduce the decision on whether a finite symmetric set F has the positive sum property with respect to an integer lattice Λ or with respect to $\ker_{\mathbb{R}^d}(A)$ to a finite number of representability tests which may be treated algorithmically. In order to complete a set of vectors with respect to the positive sum property, these lemmas suggest the following procedure which adds further elements to the set as long as it does not have the desired positive sum property. Such procedures are known as completion procedures [1]. In our case, the set F is completed with respect to the positive sum property.

Algorithm 2. *(Completion Procedure)*

Input: a finite symmetric set $F \subseteq \ker_{\mathbb{Z}^d}(A)$ *generating* Λ *over* $\mathbb{Z}$ *or* $\ker_{\mathbb{R}^d}(A)$ *over* $\mathbb{R}$

Output: a set $G \supseteq F$ *that has the positive sum property with respect to* Λ *or* $\ker_{\mathbb{R}^d}(A)$

$G := F$
$C := \bigcup_{f,g \in G} \text{S-vectors}(f, g)$
while $C \neq \emptyset$ *do*
  $s :=$ *an element in* C
  $C := C \setminus \{s\}$
  $f := \text{normalForm}(s, G)$
  if $f \neq 0$ *then*
    $C := C \cup \bigcup_{g \in G} \text{S-vectors}(f, g)$
    $G := G \cup \{f\}$
return G.

In this algorithm, the set S-vectors(f, g) corresponds to the "critical" vectors that are described in Lemmas 2 and 5, respectively. That is, we have $\text{S-vectors}(f, g) = \{f + g\}$ for the integer case and $\text{S-vectors}(f, g) = \bigcup_\alpha \{f + \alpha g\}$ for the continuous case. As specified in Lemma 5, only those values for $\alpha \in \mathbb{R}$ need to be considered, for which $f + \alpha g$ contains a zero entry at some component k at which neither f nor g have a zero entry, that is, $(f + \alpha g)^{(k)} = 0$ but $f^{(k)} g^{(k)} \neq 0$. Note that in the continuous case, $s \in$ S-vectors(v, w) if and only if $\mathrm{supp}(s) \subseteq (\mathrm{supp}(v) \cup \mathrm{supp}(w)) \setminus \{e\}$ for some $e \in \mathrm{supp}(v) \cap \mathrm{supp}(w)$. This construction is also used in matroid theory to characterize circuits (see for example Oxley [10]).

Behind the function normalForm(s, G) there is the following algorithm which returns 0 if a representation $s = \sum \alpha_i g_i$ with finitely many coefficients $\alpha_i \in \mathbb{Z}_{>0}$ (or $\alpha_i \in \mathbb{R}_{>0}$), $g_i \in G$, and $\alpha_i g_i \sqsubseteq s$ is found, or it returns a vector t such that a representation of this kind is possible for $G \cup \{t\}$.

The normalForm algorithm aims at finding a representation $s = \sum \alpha_i g_i$ with finitely many $\alpha_i \in \mathbb{Z}_{>0}$ (or $\alpha_i \in \mathbb{R}$), $g_i \in G$, and $\alpha_i g_i \sqsubseteq s$ (or $\mathrm{supp}(g_i) \subseteq \mathrm{supp}(s)$) by reducing s by elements of G in such a way that, if at some point of this reduction the zero vector is reached, a desired representation $s = \sum \alpha_i g_i$ has been found. If the reduction process terminates with a vector $t \neq 0$ then a desired representation $s = \sum \alpha_i g_i$ with vectors from $G \cup \{t\}$ has been constructed. The vector t is called a normal form of s with respect to the set G.

Algorithm 3. *(Normal form algorithm)*
Input: a vector s, a set G of vectors
Output: a normal form of s with respect to G

while there is some $g \in G$ such that s is reducible by g do
  $s :=$ reduce s by g
return s

The reduction involved in the normalForm algorithm has to be defined separately for the integer and the continuous cases. For the special situation that $\Lambda = \ker_{\mathbb{Z}^d}(A)$, our definition for the integer case leads to an algorithm already presented by Pottier [11] in the disguised form of a multi-dimensional Euclidean algorithm.

In the integer situation we say that $s \in \mathbb{Z}^d$ can be reduced by $g \in \mathbb{Z}^d$ to $s - g$ if $g \sqsubseteq s$. Thus, in case of reducibility, we have $s = g + (s - g)$ with $g \sqsubseteq s$ and $s - g \sqsubseteq s$. Since $\|s - g\|_1 < \|s\|_1$, we conclude that normalForm(s, G) always terminates.

To prove termination of the completion algorithm in the integer case we need the Gordan-Dickson Lemma (see for example Section 4.2 in [3]). The following is an equivalent geometric formulation.

Lemma 6. *(Gordan-Dickson Lemma)*
Let $\{p_1, p_2, \ldots\}$ be a sequence of points in $\mathbb{Z}_+^d$ such that $p_i \not\leq p_j$ whenever $i < j$. Then this sequence is finite.

Lemma 7. *With the above definitions of* S-vectors *and of* normalForm *for the integer case, the Completion Procedure 2 terminates and satisfies its specifications.*

Proof. Termination of the above algorithm follows immediately by application of the Gordan-Dickson Lemma to the sequence $\{(v^+, v^-) : v \in G \setminus F\} \subseteq \mathbb{Z}_+^{2d}$, where for $v \in \mathbb{R}^d$ we component-wise define $v^+ := \max(0, v)$ and $v^- := \max(0, -v)$. To this end, note that $f = \mathrm{normalForm}(s, G)$ implies that there is no $g \in G$ with $g \sqsubseteq f$, or in other words, there is no $g \in G$ with $(g^+, g^-) \leq (f^+, f^-)$. Thus, the algorithm produces a sequence $\{(v^+, v^-) : v \in G \setminus F\} = \{f_1, f_2, \ldots\}$, where $(f_i^+, f_i^-) \not\leq (f_j^+, f_j^-)$ for $i < j$. This sequence is finite by the Gordan-Dickson Lemma. Correctness of the algorithm follows immediately from Lemma 2, since upon termination normalForm$(v + w, G) = 0$

for all $v, w \in G$, giving a representation $v + w = \sum \alpha_i g_i$ with $\alpha_i \in \mathbb{Z}_{>0}$, $g_i \in G$, and $g_i \sqsubseteq v + w$. □

In the LP situation we say that vector $s \in \mathbb{R}^d$ can be reduced by $g \in \mathbb{R}^d$ if $\operatorname{supp}(g) \subseteq \operatorname{supp}(s)$. In case of reducibility, s is reduced to $s - \alpha g$ where $\alpha \in \mathbb{R}$ is chosen in such a way that $\operatorname{supp}(s - \alpha g) \subsetneq \operatorname{supp}(s)$, which implies that normalForm(s, G) always terminates. Moreover, $s = \alpha g + (s - \alpha g)$ where $\operatorname{supp}(\alpha g) \subseteq \operatorname{supp}(s)$ and $\operatorname{supp}(s - \alpha g) \subseteq \operatorname{supp}(s)$.

Lemma 8. *With the above definitions of* S-vectors *and of* normalForm *for the continuous case, the Completion Procedure 2 terminates and satisfies its specifications.*

Proof. Each vector in $G \setminus F$ must have a different support. Hence the algorithm terminates. Correctness of the above algorithm follows immediately from Lemma 5, as all the required linear representations were again constructed in the calls of the normal form algorithm. □

3. Graver test sets

In the sequel we discuss, separately for each situation, Graver test sets in LP, IP, and MIP, which are minimal sets of vectors having the positive sum property with respect to $\ker_{\mathbb{X}}(A)$, where $\mathbb{X} = \mathbb{R}^d$, $\mathbb{X} = \mathbb{Z}^d$, and $\mathbb{X} = \mathbb{Z}^{d_1} \times \mathbb{R}^{d_2}$, $d = d_1 + d_2$, respectively.

LP and IP Graver test sets are always finite as was already shown by Graver in 1975 [5]. We show how the two completion procedures presented in Section 2.3 for the integer and the continuous cases can be used to compute both sets. MIP Graver test sets, however, need not be finite in general. To demonstrate this, we present a small example similar to the one by Cook et al. [2]. Thereafter, we give a solution to this finiteness problem by presenting a finite set of integer vectors which is inherent to the MIP Graver test set and from which an improving vector to a non-optimal solution can be reconstructed by solving a finite number of pure LP's. Moreover, we will again employ a completion procedure to compute these finitely many integer vectors.

3.1. IP Graver test sets

The notion of a Graver test set in IP is strongly related to the notion of a Hilbert basis, which we will recall now. For further details we refer to Schrijver [12] and Weismantel [15].

Definition 6. *(Hilbert Basis)*

Let C be a rational cone. A finite set $H = \{h_1, \ldots, h_t\} \subseteq C \cap \mathbb{Z}^d$ is called a Hilbert basis of C if every $z \in C \cap \mathbb{Z}^d$ has a representation of the form

$$z = \sum_{i=1}^{t} \lambda_i h_i$$

with non-negative integral multipliers $\lambda_1, \ldots, \lambda_t$.

Note that every pointed rational cone possesses a unique Hilbert basis that is minimal with respect to inclusion.

Definition 7. *Let $\mathbb{O}_j$ be the j^{th} orthant of $\mathbb{R}^d$ and H_j the unique minimal Hilbert basis of* $\ker_{\mathbb{R}^d}(A) \cap \mathbb{O}_j$. *Then we define $\mathcal{G}_{\text{IP}}(A) := \bigcup H_j \setminus \{0\}$ to be the IP Graver test set (or IP Graver basis) of A.*

As $\mathcal{G}_{\text{IP}}(A)$ is a finite union of finite sets, $\mathcal{G}_{\text{IP}}(A)$ has finite cardinality. By construction, the elements of $\mathcal{G}_{\text{IP}}(A)$ are exactly all elements of $\ker_{\mathbb{Z}^d}(A) \setminus \{0\}$ that are minimal with respect to the partial ordering $\sqsubseteq$ on $\mathbb{Z}^d$. Moreover, if $v \in \mathcal{G}_{\text{IP}}(A)$ then also $-v \in \mathcal{G}_{\text{IP}}(A)$, that is, G is symmetric.

Lemma 9. *$\mathcal{G}_{\text{IP}}(A)$ is the inclusion minimal subset of* $\ker_{\mathbb{Z}^d}(A)$ *that has the positive sum property with respect to* $\ker_{\mathbb{Z}^d}(A)$.

Proof. Take any non-zero element $z \in \ker_{\mathbb{Z}^d}(A)$. Then z belongs to some orthant $\mathbb{O}_j$ and thus can be written as a positive integer linear combination of elements of the Hilbert basis $H_j \subseteq \mathcal{G}_{\text{IP}}(A)$ of $\ker_{\mathbb{R}^d}(A) \cap \mathbb{O}_j$. As, by construction, each element $z \in \mathcal{G}_{\text{IP}}(A)$ cannot be written as a non-trivial sum of two vectors from $\ker_{\mathbb{Z}^d}(A)$ that lie in the same orthant as z, z must be contained in every set that has the minimal sum property with respect to $\ker_{\mathbb{Z}^d}(A)$. This proves minimality of $\mathcal{G}_{\text{IP}}(A)$. □

Corollary 2. *$\mathcal{G}_{\text{IP}}(A)$ is a universal test set for* $(\text{IP})_{c,b}$.

In Section 2.3 we have already seen how a (symmetric) generating set of $\ker_{Z^d}(A)$ over $\mathbb{Z}$ can be completed with respect to the positive sum property. The resulting set G has to contain the IP Graver test set which consists of all $\sqsubseteq$-minimal elements in G.

3.2. LP Graver test sets

For every orthant $\mathbb{O}_j$ of $\mathbb{R}^d$ consider the pointed rational cone $\ker_{\mathbb{R}^d}(A) \cap \mathbb{O}_j$. Up to scalar factors this cone has a unique inclusion minimal generating set over $\mathbb{R}$. Moreover, as the cone is rational, we may scale each generator to have only integer components with greatest common divisor 1.

Definition 8. *For every orthant $\mathbb{O}_j$ of $\mathbb{R}^d$ let H_j be the unique minimal generating set over $\mathbb{R}$ of the pointed rational cone* $\ker_{\mathbb{R}^d}(A) \cap \mathbb{O}_j$, *where the components of each generator are scaled to integers with greatest common divisor* 1. *Then we define $\mathcal{G}_{\text{LP}}(A) := \bigcup H_j$ to be the LP Graver test set (or LP Graver basis) of A.*

As $\mathcal{G}_{\text{LP}}(A)$ is a finite union of finite sets, $\mathcal{G}_{\text{LP}}(A)$ has finite cardinality.

Lemma 10. *$\mathcal{G}_{\text{LP}}(A)$ has the positive sum property with respect to* $\ker_{\mathbb{R}^d}(A)$. *Thus, $\mathcal{G}_{\text{LP}}(A)$ is a universal test set for* $(\text{LP})_{c,b}$.

Proof. Take any non-zero element $z \in \ker_{\mathbb{R}^d}(A)$. Thus, z belongs to some orthant $\mathbb{O}_j$ and can be written as a positive linear combination of elements of the generating set $H_j \subseteq \mathcal{G}_{\text{LP}}(A)$ of $\ker_{\mathbb{R}^d}(A) \cap \mathbb{O}_j$. The second part follows from Lemma 1. □

Lemma 11. *$\mathcal{G}_{\mathrm{LP}}(A)$ coincides with the set of all circuits of A.*

Proof. We show that $\mathcal{G}_{\mathrm{LP}}(A)$ contains every circuit. The claim then follows from the positive sum property of the set of circuits, Lemma 4.

Let q be a circuit of A belonging to some orthant $\mathbb{O}_j$ of $\mathbb{R}^d$. Then q can be written as $q = \sum \alpha_i g_i$ where $\alpha_i > 0$ and $g_i \in H_j$. This implies $\alpha_i g_i \sqsubseteq q$ for all i. We conclude that $\mathrm{supp}(g_i) \subseteq \mathrm{supp}(q)$ and thus $q = g_i$ by Lemma 3. □

In Section 2.3 we have already seen how a generating set of $\ker_{\mathbb{R}^d}(A)$ over $\mathbb{R}$ can be completed with respect to the positive sum property. The resulting set G has to contain a scalar multiple of each circuit of A. Thus, the LP Graver test set consists of all support minimal elements in G, normalized to integer vectors whose components have a greatest common divisor 1. In that way we may compute the LP Graver test set.

3.3. MIP Graver test sets

Let $A = (A_1|A_2)$, where the columns of A_1 and A_2 correspond to the integer and continuous variables, respectively. Throughout this section, A_1 and A_2 are assumed to be integer matrices of sizes $l \times d_1$ and $l \times d_2$. Analogously, we subdivide $c = (c_1, c_2)$, where $c_1 \in \mathbb{R}^{d_1}$ and $c_2 \in \mathbb{R}^{d_2}$. Let $z \in \mathbb{Z}^{d_1}$ and $q \in \mathbb{R}^{d_2}$ denote the integer and continuous variables, respectively, and let $d = d_1 + d_2$. Our aim is to construct a universal test set for the family of optimization problems $(\mathrm{P})_{c,b}$ as $c \in \mathbb{R}^d$ and $b \in \mathbb{R}^l$ vary.

Again, a Graver test set in the mixed-integer situation is defined to be an inclusion minimal subset of $\ker_{\mathbb{X}}(A)$ which has the positive sum property with respect to $\ker_{\mathbb{X}}(A)$. This leads us to the following set of vectors.

Definition 9. *The MIP Graver test set $\mathcal{G}_{\mathrm{MIP}}(A)$ contains all vectors*

- $(0, q)$, $q \in \mathcal{G}_{\mathrm{LP}}(A_2)$, *and*
- $(z, q) \in \ker_{\mathbb{X}}(A)$, $z \neq 0$, *and such that there is no* $(z', q') \in \ker_{\mathbb{X}}(A)$ *satisfying* $(z', q') \sqsubseteq (z, q)$.

The proof of the following lemma can be found in [4]. (To be self-contained we reproduce its proof in the appendix, Section 6.)

Lemma 12. *$\mathcal{G}_{\mathrm{MIP}}(A)$ is an inclusion minimal set which has the positive sum property with respect to* $\ker_{\mathbb{X}}(A)$. *Consequently, $\mathcal{G}_{\mathrm{MIP}}(A)$ is a universal test set for* $(\mathrm{P})_{c,b}$.

The set $\mathcal{G}_{\mathrm{MIP}}(A)$, however, need not be finite in general. To demonstrate this, we present an example similar to the one by Cook et al. [2].

Example. Consider the family of optimization problems

$$\min\{c^{(0)}z + c^{(1)}q_1 + c^{(2)}q_2 : z + q_1 + q_2 = b, z \in \mathbb{Z}_+, q_1, q_2 \in \mathbb{R}_+\}$$

as $c = (c^{(0)}, c^{(1)}, c^{(2)}) \in \mathbb{R}^3$ and $b \in \mathbb{R}$ vary. Thus, we have $A = (A_1|A_2)$ where $A_1 = (1)$ and $A_2 = (1\ 1)$. Every element in $\ker_{\mathbb{Z}\times\mathbb{R}^2}(A)$ can be written as linear combination $\alpha_1(1, -1, 0) + \alpha_2(0, 1, -1)$ with $\alpha_1 \in \mathbb{Z}$ and $\alpha_2 \in \mathbb{R}$. We will prove that for every $\beta \in \mathbb{R}$, $0 < \beta < 1$, the vector

$$u := (1, -1, 0) + \beta(0, 1, -1) = (1, -1 + \beta, -\beta) \in \ker_{\mathbb{Z}\times\mathbb{R}^2}(A) \setminus \{0\}$$

cannot be written as a sum $v + w$ with $v, w \in \ker_{\mathbb{Z}\times\mathbb{R}^2}(A) \setminus \{0\}$ and $v, w \sqsubseteq u$.

Suppose on the contrary that such vectors v, w exist. Since $v, w \sqsubseteq u$, one of the two vectors, say v, must have a zero integer component. Hence $v = \gamma(0, 1, -1)$ for some $\gamma \in \mathbb{R}$. But, clearly, for no choice of $\gamma \in \mathbb{R}$ and of $\beta \in \mathbb{R}$, $0 < \beta < 1$, the vectors $(1, -1+\beta, -\beta)$ and $\gamma(0, 1, -1)$ belong to the same orthant of $\mathbb{R}^3$, which is a contradiction to $v \sqsubseteq u$. We conclude that for all $\beta \in \mathbb{R}$, $0 < \beta < 1$, the vector $(1, -1 + \beta, -\beta)$ is $\sqsubseteq$-minimal and thus, it belongs to $\mathcal{G}_{\mathrm{MIP}}(A)$. □

Having only an infinite test set available, we cannot yet make algorithmic use of the Augmentation Algorithm 1 to improve a feasible initial solution to optimality. However, we will construct a finite set $\mathcal{G}_{\mathbb{Z}}(A) \subseteq \mathbb{Z}^{d_1}$ from which an improving vector to a non-optimal solution of the given problem can be reconstructed in finitely many steps. Thus, the Augmentation Algorithm 1 can be employed again to find an optimal solution.

Consider the projection $\phi : \mathbb{Z}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{Z}^{d_1}$ which maps each mixed-integer vector onto its d_1 integer components. Define $\mathcal{G}_{\mathbb{Z}}(A) = \mathcal{G}_{\mathbb{Z}}(A_1|A_2) := \phi(\mathcal{G}_{\mathrm{MIP}}(A))$ to be the set of images of the elements in $\mathcal{G}_{\mathrm{MIP}}(A)$.

Lemma 13. *$\mathcal{G}_{\mathbb{Z}}(A_1|A_2)$ is finite for every matrix $A \in \mathbb{Z}^{l\times d}$ and for any subdivision $A = (A_1|A_2)$.*

Proof. Each element $(z, q) \in \mathcal{G}_{\mathrm{MIP}}(A)$ satisfies $\|z\|_1 \leq \Delta(A_1, A_2)$ for some scalar $\Delta(A_1, A_2)$ which depends only on A_1 and A_2 ([4], again, to be self-contained, this is reproduced in the appendix, Section 6). This inequality can be true only for finitely many vectors $z \in \mathbb{Z}^{d_1}$. □

In [9], Köppe obtained the above finiteness result by a geometric description of the rational parts of mixed-integer test set elements when the integer parts are kept fixed. The analysis in [9] then also led to the conclusions we obtain in the following lemma.

Lemma 14. *Given vectors b, c, and a non-optimal feasible solution (z_0, q_0) to $A_1z + A_2q = b$, there exists $z_1 \in \mathcal{G}_{\mathbb{Z}}(A)$ from which an improving vector (z_1, q_1) to (z_0, q_0) can be constructed algorithmically.*

Proof. Since (z_0, q_0) is not optimal, there must exist an improving vector $(z', q') \in \mathcal{G}_{\mathrm{MIP}}(A)$. Therefore, an improving vector can be reconstructed from $z_1 := z' \in \mathcal{G}_{\mathbb{Z}}(A)$. It remains to show that we can find z_1 and a suitable q_1 algorithmically.

For every $z_1 \in \mathcal{G}_{\mathbb{Z}}(A)$ with $z_0 - z_1 \geq 0$ we try to find a vector q_1 such that $(z_0 - z_1, q_0 - q_1)$ is feasible and such that the objective value $c^\intercal(z_0 - z_1, q_0 - q_1)$ is as small as possible. If $c^\intercal(z_0 - z_1, q_0 - q_1) < c^\intercal(z_0, q_0)$ then (z_1, q_1) is an improving vector. This problem is equivalent to the optimization problem

$$q_1 \in \arg\max_q \{c_1^\intercal z_1 + c_2^\intercal q : A_1(z_0 - z_1) + A_2(q_0 - q) = b, q_0 - q \geq 0, q \in \mathbb{R}^{d_2}\}. \tag{1}$$

Clearly, there is an improving vector for (z_0, q_0) in $\mathcal{G}_{\mathrm{MIP}}(A)$ starting with z_1 if and only if (z_1, q_1) is an improving vector, that is, if and only if the above maximization problem (1) has a feasible solution with strictly positive cost function value. □

Corollary 3. *If for all $z_1 \in \mathcal{G}_{\mathbb{Z}}(A)$ the maximal value of (1) is non-positive then (z_0, q_0) is optimal.*

Proof. If the solution (z_0, q_0) is not optimal, then there has to exist an improving vector $(z_1, \bar{q}) \in \mathcal{G}_{\mathrm{MIP}}(A_1|A_2)$. But for this z_1, the maximal value of (1) is at least $c_1^\intercal z_1 + c_2^\intercal \bar{q} > 0$, contradicting the assumption that no such z_1 exists. □

We need to solve many LP subproblems in order to reconstruct an improving vector. However, compared to the original mixed-integer problem, these LP computations can be considered to be cheap. The advantages of our approach are clear. $\mathcal{G}_{\mathbb{Z}}(A)$ is always a finite set of vectors from which for every given $b \in \mathbb{R}^l$ improving vectors to non-optimal solutions can be reconstructed. Moreover, this approach allows us to use powerful LP solvers to take care of the continuous part of our mixed-integer problem once the set $\mathcal{G}_{\mathbb{Z}}(A)$ has been computed. Finally, note that this project-and-lift approach is not restricted to MIP Graver test sets. It can be used for any other type of mixed-integer test sets, as well.

In the following, we show how to compute a superset of $\mathcal{G}_{\mathbb{Z}}(A)$. To this end, we assume that we have available an algorithm to solve the following problem.

Problem 1. For $D \in \mathbb{Z}^{l\times\bar{d}}$ and $b \in \mathbb{R}^l$ define $P_{D,b} := \{z : Dz = b, z \in \mathbb{R}_+^{\bar{d}}\}$. For given $b_1, b_2 \in \mathbb{R}^l$ decide, whether $P_{D,b_1+b_2} = P_{D,b_1} + P_{D,b_2}$, where $P_{D,b_1} + P_{D,b_2}$ denotes the Minkowski sum of P_{D,b_1} and P_{D,b_2}.

In order to prove termination of the algorithm below, the following sufficient condition to check $P_{D,b_1+b_2} = P_{D,b_1} + P_{D,b_2}$ will turn out very useful.

Lemma 15. *Let $D \in \mathbb{Z}^{l\times\bar{d}}$ be of full row rank, and let $b_1, b_2 \in \mathbb{R}^l$, such that $P_{D,b_1} \neq \emptyset$ and $P_{D,b_2} \neq \emptyset$. Moreover, suppose that for every invertible $l \times l$ submatrix B of D the vectors $B^{-1}b_1$ and $B^{-1}b_2$ lie in the same orthant of $\mathbb{R}^l$. Then $P_{D,b_1+b_2} = P_{D,b_1} + P_{D,b_2}$.*

A proof of this lemma can be found for example in [9]. To compute $\mathcal{G}_{\mathbb{Z}}(A)$, we will again employ the Completion Algorithm 2 by defining the necessary notions.

Definition 10. *Let $\bar{F}$ be a symmetric integer generating set for the integer lattice*

$$\phi(\ker_{\mathbb{Z}^{d_1}\times\mathbb{R}^{d_2}}(A)) = \{z \in \mathbb{Z}^{d_1} : \exists q \in \mathbb{R}^{d_2} \text{ with } (z, q) \in \ker_{\mathbb{Z}^{d_1}\times\mathbb{R}^{d_2}}(A)\}.$$

Let $F := \{(v, P_{(A_2|-A_2),-A_1 v}) : v \in \bar{F}\}$ be the input set to the completion algorithm. For better readability, we will write P_v instead of $P_{(A_2|-A_2),-A_1 v}$.

We apply the completion procedure to more complicated structures than vectors. Therefore, the test $f \neq 0$ in the Completion Algorithm 2 has to be replaced by $f \neq (0, P_0)$.

Definition 11. *We define* S-vectors$((u, P_u), (u', P_{u'})) := \{(u + u', P_{u+u'})\}$. *We say that $(u', P_{u'})$ reduces (u, P_u), or $(u', P_{u'}) \sqsubseteq (u, P_u)$ for short, if $u' \sqsubseteq u$ and $P_u = P_{u'} + P_{u-u'}$. In case of reducibility, (u, P_u) is reduced to $(u - u', P_{u-u'})$.*

We will now collect some results that will turn out useful when proving termination of the completion algorithm. A consequence of Lemma 15 is the following.

Corollary 4. *Let $A = (A_1|A_2) \in \mathbb{Z}^{l\times d}$ and let $A_2 \in \mathbb{Z}^{l\times d_2}$ be of full row rank. Denote by $B_1, \ldots, B_M$ all invertible $l \times l$ submatrices of A_2 and define for each $z \in \mathbb{Z}^{d_1}$ the vector*

$$f(A,z) := (z, \det(B_1)B_1^{-1}(-A_1z), \ldots, \det(B_M)B_M^{-1}(-A_1z)) \in \mathbb{Z}^{d_1+Ml}.$$

Then the relation $(z, P_z) \not\sqsubseteq (z', P_{z'})$ implies $f(A,z) \not\sqsubseteq f(A,z')$.

Proof. Suppose that $f(A,z) \sqsubseteq f(A,z')$. Therefore, the vectors $f(A,z)$, $f(A,z')$, and $f(A,z'-z) = f(A,z') - f(A,z)$ all lie in the same orthant of $\mathbb{R}^{d_1+Ml}$. Thus, $z \sqsubseteq z'$, and the vectors $\det(B_i)B_i^{-1}(Az)$ and $\det(B_i)B_i^{-1}(A(z'-z))$ lie in the same orthant of $\mathbb{R}^l$ for all $i = 1, \ldots, M$. Hence, $P_{z'} = P_{z'-z} + P_z$, by Lemma 15, and therefore $(z, P_z) \sqsubseteq (z', P_{z'})$. Thus, $(z, P_z) \not\sqsubseteq (z', P_{z'})$ implies $f(A,z) \not\sqsubseteq f(A,z')$, as claimed. □

Corollary 5. *Let $A = (A_1|A_2) \in \mathbb{Z}^{l\times d}$ and let $\{u_1, u_2, \ldots\}$ be a sequence in $\mathbb{Z}^{d_1}$. Then every sequence $\{(u_1, P_{u_1}), (u_2, P_{u_2}), \ldots\}$ with $(u_i, P_{u_i}) \not\sqsubseteq (u_j, P_{u_j})$ whenever $i < j$, is finite.*

Proof. Consider the sequence $\{f(A,u_1), f(A,u_2), \ldots\}$ of vectors in $\mathbb{Z}^{d_1+Ml}$, which satisfies $f(A,u_i) \not\sqsubseteq f(A,u_j)$ whenever $i < j$, by Corollary 4. Applying the Gordan-Dickson Lemma to the sequence

$$\{(f(A,u_1)^+, f(A,u_1)^-), (f(A,u_2)^+, f(A,u_2)^-), \ldots\} \subseteq \mathbb{Z}_+^{2(d_1+Ml)},$$

which satisfies $(f(A,u_i)^+, f(A,u_i)^-) \not\leq (f(A,u_j)^+, f(A,u_j)^-)$ whenever $i < j$, we conclude that this sequence and, thus, also $\{f(A,u_1), f(A,u_2), \ldots\}$ and $\{(u_1, P_{u_1}), (u_2, P_{u_2}), \ldots\}$ are finite. □

Proposition 1. *If the input set, the procedure* normalForm*, and the set* S-vectors *are defined as above, then the Completion Algorithm 2 terminates and computes a set G such that $\mathcal{G}_{\mathbb{Z}}(A) \subseteq \{u : (u, P_u) \in G\} \cup \{0\}$.*

Proof. In the course of the algorithm, a sequence of pairs in $G \setminus F$ is generated that satisfies the conditions of Corollary 5. Therefore, the algorithm terminates.

Let G denote the set that is returned by the algorithm. To show correctness, we have to prove that

$$\bar{G} := \{(\bar{u}, \bar{v}) \in \mathbb{Z}^{d_1} \times \mathbb{R}^{d_2} : (\bar{u}, P_{\bar{u}}) \in G, (\bar{v}^+, \bar{v}^-) \in P_{\bar{u}}\}$$

has the positive sum property with respect to $\ker_{\mathbb{Z}^{d_1}\times\mathbb{R}^{d_2}}(A)$. To this end, we construct for arbitrarily chosen vector $(z,q) \in \ker_{\mathbb{Z}^{d_1}\times\mathbb{R}^{d_2}}(A)$ from some pairs $(u, P_u) \in G$ a representation

$$(z,q) = \sum_u \alpha_u(u, q_u),$$

where $\alpha_u \in \mathbb{Z}_{>0}$, $(q_u^+, q_u^-) \in P_u$, and $\alpha_u(u, q_u) \sqsubseteq (z,q)$ for all u. Since G contains F, at least a positive integer linear combination

$$(z,q) = \sum_{u\in G} \alpha_u(u, q_u)$$

with $\alpha_u \in \mathbb{Z}_{>0}$, $(u, P_u) \in G$, and $(q_u^+, q_u^-) \in P_u$ for all u, is possible. For each such integer linear combination consider the value $\sum \alpha_u \|(u, q_u)\|_1$, which is always non-negative. Thus, this sum is bounded from below by some non-negative infimum. Let us assume for the moment that there is a linear integer combination $\sum_u \alpha_u (u, q_u)$ that attains this infimum.

If $\sum_u \alpha_u \|(u, q_u)\|_1 = \|(z, q)\|_1$ then all vectors (u, q_u) lie in the same orthant as (z, q), and thus, $\alpha_u (u, q_u) \sqsubseteq (z, q)$ as desired. But since (z, q) was chosen arbitrarily, this implies that $\bar{G}$ has the positive sum property with respect to $\ker_{\mathbb{Z}^{d_1} \times \mathbb{R}^{d_2}}(A)$ and thus, $\bar{G}$ contains $\mathcal{G}_{\mathrm{MIP}}(A)$. The claim then follows.

Assume on the contrary that we have $\sum_u \alpha_u \|(u, q_u)\|_1 > \|(z, q)\|_1$ for such a minimal combination. Then there exist u_1, u_2 in this representation, and a component m, such that $(u_1, q_{u_1})^{(m)} (u_2, q_{u_2})^{(m)} < 0$. Thus,

$$\|(u_1 + u_2, q_{u_1} + q_{u_2})\|_1 < \|(u_1, q_{u_1})\|_1 + \|(u_2, q_{u_2})\|_1.$$

During the run of the algorithm, $(u_1 + u_2, P_{u_1+u_2})$ was reduced to $(0, P_0)$ by elements $(v_1, P_{v_1}), \ldots, (v_s, P_{v_s}) \in G$, giving a representation $u_1 + u_2 = \sum_{j=1}^{s} v_j$, $v_j \sqsubseteq u_1 + u_2$, and $P_{u_1+u_2} = P_0 + P_{v_1} + \ldots + P_{v_s}$. The latter implies that there are vectors q_{v_j} with $(q_{v_j}^+, q_{v_j}^-) \in P_{v_j}$, $j = 1, \ldots, s$, and $(q_{v_0}^+, q_{v_0}^-) \in P_0$ such that

$$(q_{u_1} + q_{u_2})^+ = \sum_{j=0}^{s} q_{v_j}^+$$

and

$$(q_{u_1} + q_{u_2})^- = \sum_{j=0}^{s} q_{v_j}^-.$$

Thus $q_{v_j} \sqsubseteq q_{u_1} + q_{u_2}$, $j = 0, \ldots, s$, and therefore,

$$\|q_{u_1} + q_{u_2}\|_1 = \sum_{j=0}^{s} \|q_{v_j}\|_1.$$

Altogether, we obtain

$$\begin{aligned}\sum_{j=0}^{s} \|(v_j, q_{v_j})\|_1 = \|\sum_{j=0}^{s}(v_j, q_{v_j})\|_1 &= \|(u_1 + u_2, q_{u_1} + q_{u_2})\|_1 \\ &< \|(u_1, q_{u_1})\|_1 + \|(u_2, q_{u_2})\|_1,\end{aligned}$$

where $v_0 = 0$. Now rewrite $(z, q) = \sum_u \alpha_u (u, q_u)$ as

$$(z, q) = \sum_{u, u \neq u_1, u_2} \alpha_u (u, q_u) + (\alpha_{u_1} - 1)(u_1, q_{u_1}) + (\alpha_{u_2} - 1)(u_2, q_{u_2}) + \sum_{j=0}^{s} (v_j, q_{v_j})$$

where

$$\sum_{u \neq u_1, u_2} \alpha_u \|(u, q_u)\|_1 + (\alpha_{u_1} - 1)\|(u_1, q_{u_1})\|_1 + (\alpha_{u_2} - 1)\|(u_2, q_{u_2})\|_1 + \sum_{j=0}^{s} \|(v_j, q_{v_j})\|_1$$

is strictly less than $\sum_u \alpha_u \|(u, q_u)\|_1$, in contradiction to the minimality assumption on the integer linear combination $\sum_u \alpha_u (u, q_u)$.

It remains to show that the infimum of $\sum_u \alpha_u \|(u, q_u)\|_1$ is indeed attained by some integer linear combination. This can be seen as follows. Since z, q, and all u with $(u, P_u) \in G$ are given, we have to find $\alpha_u \in \mathbb{Z}_+$ and vectors $q_u \in \mathbb{R}^{d_2}$ with $(q_u^+, q_u^-) \in P_u$ such that $\sum_u \alpha_u \|(u, q_u)\|_1$ attains the infimum value.

First, since there is at least one such linear combination, we have an upper bound K for the infimum. Thus, there are at most finitely many choices for the non-negative integers α_u such that $\sum_u \alpha_u \|u\|_1 \leq K$. It remains to show that for fixed integers α_u the infimum of $\sum_u \alpha_u \|(u, q_u)\|_1$ is indeed attained for some choice of the vectors q_u. Then also the global infimum, as the least value of the finitely many minimal values of $\sum_u \alpha_u \|(u, q_u)\|_1$, wherein the α_u are fixed, is attained by some combination.

Therefore, suppose in the following that the integers α_u are fixed and that there exists at least one linear integer representation of (z, q) for this fixed choice of the numbers α_u. Since α_u and all u are fixed, we have to minimize $\sum_u \alpha_u \|q_u\|_1$ where $\sum_u \alpha_u q_u = q$ and $A_2 q_u = -A_1 u$, for all u with $(u, P_u) \in G$. Writing $q_u = q_u^+ - q_u^-$ we obtain a linear minimization problem with non-empty feasible region, whose objective is bounded from below by 0. Thus, the minimum is attained for some choice of the vectors q_u. This concludes the proof. $\square$

4. Termination of Augmentation Algorithm

Let us now have a look at termination of the Augmentation Algorithm 1 in the LP, IP, and MIP situations. Assume that we are given a feasible solution z_0 to our problem. Moreover, assume that $(\text{LP})_{c,b}$ is bounded with respect to c, which can be checked by standard techniques from LP. Therefore, if we assume integer (or rational) entries in A and b, also $(\text{IP})_{c,b}$ and $(\text{P})_{c,b}$ are bounded with respect to c, see Schrijver [12].

In the LP case, circuits provide improving directions to non-optimal solutions. However, a zig-zagging effect, even to a non-optimal point, is possible and we have to take some care on how to choose the next circuit for improvement.

Example. Consider the problem

$$\min\{z_1 + z_2 - z_3 : 2z_1 + z_2 \leq 2, z_1 + 2z_2 \leq 2, z_3 \leq 1, (z_1, z_2, z_3) \in \mathbb{R}^3_{\geq 0}\}$$

with optimal solution $(0, 0, 1)$. Introducing slack variables z_4, z_5, z_6 we obtain the problem $\min\{c^\intercal z : Az = (2, 2, 1)^\intercal, z \in \mathbb{R}^6_{\geq 0}\}$ with $c^\intercal = (1, 1, -1, 0, 0, 0)$ and

$$A = \begin{pmatrix} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}.$$

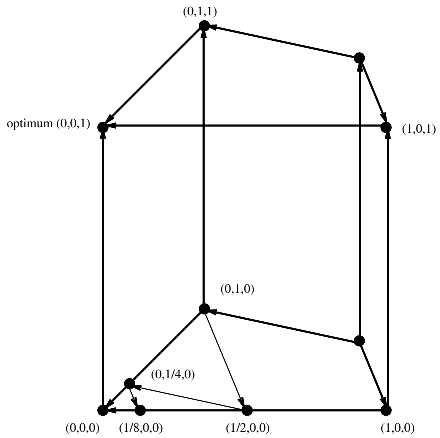

Fig 1. Zig-zagging to non-optimal solution

$\ker_{\mathbb{R}^6}(A)$ is generated over $\mathbb{R}$ by the vectors $(1, 0, 0, -2, -1, 0)$, $(0, 1, 0, -1, -2, 0)$, and $(0, 0, 1, 0, 0, -1)$. We obtain as the circuits of A the vectors $(1, 0, 0, -2, -1, 0)$, $(0, 1, 0, -1, -2, 0)$, $(1, -2, 0, 0, 3, 0)$, $(2, -1, 0, -3, 0, 0)$, $(0, 0, 1, 0, 0, -1)$ together with their negatives. The improving directions are given by all circuits v for which $c^\intercal v > 0$. These are the following: $(1, 0, 0, -2, -1, 0)$, $(0, 1, 0, -1, -2, 0)$, $(-1, 2, 0, 0, -3, 0)$, $(2, -1, 0, -3, 0, 0)$, $(0, 0, -1, 0, 0, 1)$.

Now start with the feasible solution $z_0 = (0, 1, 0, 1, 0, 1)$. Going along the directions $(0, 1, 0, -1, -2, 0)$ and $(0, 0, -1, 0, 0, 1)$ as far as possible, we immediately arrive at $(0, 0, 1, 2, 2, 0)$ which corresponds to the desired optimal solution $(0, 0, 1)$ of our problem. However, alternatively choosing only the vectors $(-1, 2, 0, 0, -3, 0)$ and $(2, -1, 0, -3, 0, 0)$ as improving directions, the augmentation process does not terminate. In our original space $\mathbb{R}^3$ this reads as

$$(0, 1, 0) \to \left(\frac{1}{2}, 0, 0\right) \to \left(0, \frac{1}{4}, 0\right) \to \left(\frac{1}{8}, 0, 0\right) \to \left(0, \frac{1}{16}, 0\right) \to \dots$$

clearly showing the zig-zagging effect to the non-optimal point $(0, 0, 0)$. Thus, we have to impose certain constraints on the circuit which is chosen next in the augmentation algorithm. □

To ensure termination of the Augmentation Algorithm 1 we split the augmentation process into two phases.

Algorithm 4. *(Augmentation Strategy to reach Optimum in finitely many Steps)*

1. *In phase* 1 *only those circuits are allowed for improvement which lead to a better feasible solution with an additional zero entry. Repeat this step until there is no circuit with this property and go to phase* 2.
2. *In phase* 2 *any appropriate circuit is chosen for improvement. Stop, if there is no such circuit, otherwise improve and go back to phase* 1.

Proposition 2. *Under the assumption that the problem $(LP)_{c,b}$ is bounded with respect to c, the LP Augmentation Algorithm 1 which uses the Augmentation Strategy 4 terminates with an optimal solution.*

Proof. At the end of phase 1 the current feasible solution z' has to be optimal with respect to all feasible solutions whose support is contained in $\text{supp}(z')$, that is, there is no feasible solution z'' with $\text{supp}(z'') \subseteq \text{supp}(z')$ and $c^\intercal z'' < c^\intercal z'$. Assume on the contrary that there is such a solution z''. Since $z' - z'' \in \ker_{\mathbb{R}^d}(A)$, and by the positive sum property of the set of circuits with respect to $\ker_{\mathbb{R}^d}(A)$, there exists some circuit q with $\text{supp}(q) \subseteq \text{supp}(z' - z'') \subseteq \text{supp}(z')$ and $c^\intercal q > 0$. This circuit, however, improves z': Choose the largest positive scalar α such that $z' - \alpha q$ is still feasible. This scalar is finite since $LP_{c,b}$ is bounded with respect to the given cost function. We conclude that $z' - \alpha q$ contains an additional zero entry and $c^\intercal(z' - \alpha q) < c^\intercal z'$, contradicting the fact that the first phase finished with z'.

Now, if z' is optimal then the algorithm terminates in phase 2. Otherwise, the zero pattern of z' will never reappear after z' was improved in phase 2. Thus, no zero pattern reappears at the end of the first phase and the augmentation process has to terminate.

The feasible solution upon termination has to be optimal since in the second phase no augmenting circuit was found. The claim now follows by the universal test set property of circuits. □

Finally, let us look at the IP and the MIP situations. Here, the Augmentation Algorithm 1 always terminates with an optimal solution.

Proposition 3. *If the relaxed problem $(LP)_{c,b}$ is bounded with respect to c, the IP Augmentation Algorithm 1 terminates with an optimal solution.*

Proof. Let $\epsilon := \min\{|c^\intercal g| : g \in \mathcal{G}_{\text{IP}}(A), c^\intercal g \neq 0\}$. Then each time we improve a given feasible solution z_0 by an element $g \in \mathcal{G}_{\text{IP}}(A)$ to $z_0 - \alpha g$ the cost function value drops by at least $\alpha(c^\intercal g) \geq \epsilon > 0$ since α is a positive integer. Since $(\text{LP})_{c,b}$ is bounded with respect to c, this can happen only finitely often and the IP augmentation algorithm has to terminate. The returned solution has to be optimal, since it cannot be improved along some test set direction. □

Proposition 4. *If the relaxed problem $(LP)_{c,b}$ is bounded with respect to c, the MIP Augmentation Algorithm 1 terminates with an optimal solution.*

Proof. First transform the problem $(\text{P})_{c,b}$ by Gaussian elimination into an equivalent optimization problem $(\text{P})'_{c,b'}$ with problem matrix

$$A' := \begin{pmatrix} A'_1 & A'_2 \\ A''_1 & 0 \end{pmatrix},$$

wherein A'_2 has full row rank. Note that the right-hand side b will change to some b' as well, but the cost function c remains fixed. Moreover, since the kernel of the problem matrix does not change, the finite set $\mathcal{G}_{\mathbb{Z}}(A_1|A_2)$ remains unchanged after this transformation.

Now suppose on the contrary that the augmentation process does not terminate, that is, it generates an infinite sequence of feasible solutions to $(P)'_{c,b'}$ with strictly decreasing objective value. Divide this sequence into consecutive subsequences of length $K+1$, where K denotes the number of invertible submatrices B of A'_2 of full rank. As we will see, within each of these subsequences the cost function value drops by at least some constant $\epsilon > 0$ depending only on A'_1, A''_1, A'_2, and c. Since $(LP)_{c,b'}$ is bounded with respect to c, this can happen only finitely often, and the MIP Augmentation Algorithm 1 has to terminate after finitely many steps.

Since A' is an integer matrix, the set $\{q \in \mathbb{R}^{d_2}_+ : A'(z,q) = b'\}$ is a polyhedron for every fixed $z \in \mathbb{Z}^{d_1}$. After each augmentation step the continuous part q_0 of the current feasible solution (z_0, q_0) is a vertex of $\{q \in \mathbb{R}^{d_2}_+ : A'(z_0,q) = b'\}$ (or a point with the same cost function value). Thus, after possible rearrangement of continuous variables, we can write A'_2 as $(B|\bar{A}_2)$ where B is an invertible $l \times l$-matrix and the optimal point is given by $(z_0, B^{-1}(b' - A'_1 z_0), 0)$. Let $c = (c_1, c_{2,1}, c_{2,2})$ be divided analogously. Then the cost function value of $(z_0, B^{-1}(b' - A'_1 z_0), 0)$ is given by

$$c_1^\intercal z_0 + c_{2,1}^\intercal B^{-1}(b' - A'_1 z_0) = (c_1^\intercal - c_{2,1}^\intercal B^{-1} A'_1) z_0 + c_{2,1}^\intercal B^{-1} b' =: \tilde{c}_B^\intercal z_0 + \tilde{c}_{0,B}$$

where $\tilde{c}_B$ and $\tilde{c}_{0,B}$ depend only on the given problem data and on the specific choice of B.

Consider a sequence of $K+1$ consecutive feasible solutions as generated in the MIP Augmentation Algorithm 1. By the pigeon-hole principle there are two solutions $(z_1, B^{-1}(b' - A'_1 z_1), 0)$ and $(z_2, B^{-1}(b' - A'_1 z_2), 0)$ whose continuous parts are determined by the same submatrix B of A'_2 as already indicated by the notation. Moreover, let the second solution have a better cost function value than the first one.

Thus $0 < \tilde{c}_B^\intercal z_1 + \tilde{c}_{0,B} - (\tilde{c}_B^\intercal z_2 + \tilde{c}_{0,B}) = \tilde{c}_B^\intercal (z_1 - z_2)$. In the MIP augmentation algorithm the vector $z_1 - z_2$ was represented as a sum $\sum_{i=1}^{\leq K} g_i$ of at most K elements from $\mathcal{G}_\mathbb{Z}(A') = \mathcal{G}_\mathbb{Z}(A)$, since z_2 was obtained from z_1 by at most K augmentation steps. Since $\mathcal{G}_\mathbb{Z}(A')$ is finite, there are only finitely many possible positive values of $\tilde{c}_B^\intercal(\sum_{i=1}^{\leq K} g_i) > 0$. Thus, there exists a constant $\epsilon_B > 0$ depending only on the choices of B and the given data A'_1, A''_1, A'_2, and c, such that

$$\tilde{c}_B^\intercal (z_1 - z_2) = \tilde{c}_B^\intercal \left(\sum_{i=1}^{\leq K} g_i \right) \geq \epsilon_B > 0$$

for all choices of the $g_i \in \mathcal{G}_\mathbb{Z}(A')$. Therefore, the cost function value drops by at least $\epsilon_B > 0$ between the two mixed-integer solutions $(z_1, B^{-1}(b' - A'_1 z_1), 0)$ and $(z_2, B^{-1}(b' - A'_1 z_2), 0)$ in the augmentation algorithm.

Since there are only finitely many invertible submatrices B of A'_2 of full rank, the value $\epsilon := \min_B\{\epsilon_B\} > 0$ is well defined and depends only on A'_1, A''_1, A'_2, and c, the given problem data. Moreover, we proved that within each subsequence of length $K+1$ the cost function value drops by at least $\epsilon_B \geq \epsilon > 0$, which implies that the MIP Augmentation Algorithm 1 has to terminate provided that the relaxed problem $(LP)_{c,b'}$ is bounded with respect to c. The returned solution has to be optimal, since it cannot be improved along some test set direction. □

5. Feasible initial solutions

So far we have addressed the problem of improving a given feasible solution for $(\mathrm{P})_{c,b}$ using directions given by test set elements. In doing so, the optimization problem can be solved in finitely many augmentation steps. But, particularly in the IP and MIP cases, it is often already a very hard problem to find an initial feasible solution at all.

However, one can at least compute a (mixed-) integer solution to $Az = b$ in polynomial time, ignoring the lower bounds 0 on the variables ([12], Section 5.3). Computing a solution in the LP case can be done for example by Gaussian elimination or by even faster methods.

In the following we will demonstrate how universal test sets can be used to transform any given solution to $Az = b, z \in \mathbb{X}$, into a feasible solution of $(\mathrm{P})_{c,b}$, that is, a solution to $Az = b, z \in \mathbb{X}$, satisfying also the lower bounds $z \geq 0$. The proposed algorithm always terminates and returns either a feasible solution or the answer that no such solution exists.

The algorithm below is similar to phase I in the simplex method. However, we want to stress the fact, that such a step can be done with the help of universal test sets, as well, which may turn out useful when test sets are applied to large stochastic (integer) programs [6].

Algorithm 5. *(Algorithm to find a Feasible Solution)*
Input: a solution $z_1 \in \mathbb{X}$ to $Az = b$, a universal test set $\mathcal{T}$ for $(\mathrm{P})_{c,b}$
Output: a feasible solution or "FAIL" if no such solution exists

while there exist $t \in \mathcal{T}$, $\alpha \in \mathbb{R}_{>0}$ with $z_1 - \alpha t \in \mathbb{X}$, $\|(z_1 - \alpha t)^-\|_1 < \|z_1^-\|_1$,
and $(z_1 - \alpha t)^{(k)} \geq 0$ whenever $z_1^{(k)} \geq 0$ do
$z_1 := z_1 - \alpha t$, where α is maximal such that all conditions are satisfied
if $\|z_1^-\|_1 > 0$ then return "FAIL" else return z_1

In the MIP case we have to decide for every $g_z \in \mathcal{G}_{\mathbb{Z}}(A)$ whether we can find g_q and a scalar $\alpha > 0$ such that $t := (g_z, g_q)$ has the desired properties. For this we distinguish two cases. Either we have $g_z = 0$ or $g_z \neq 0$. In both cases we can fix $\alpha = 1$ and are left with a pure LP problem: Find $(g_z, g_q) \in \ker_{\mathbb{X}}(A)$ minimizing $\|(z_1 - \alpha g)^-\|_1 = \|z_1 - (g_z, g_q)^-\|_1$, where z_1 and g_z are given.

Lemma 16. *Algorithm 5 satisfies its specifications.*

Proof. Suppose that the algorithm terminates and that at the end z_1 is still infeasible although there is some feasible solution z_0 to our problem. Consider the optimization problem

$$\min\{\sum_{i:z_1^{(i)}<0} (-z)^{(i)} : Az = b + Az_1^-, z \in \mathbb{X}_+\}. \tag{2}$$

Then z_1^+ is feasible for problem (2) and it admits an objective function value of 0. Moreover, $z_0 + z_1^-$ is feasible, too, with strictly negative objective function value, since $(z_0 + z_1^-)^{(k)} > 0$ whenever $z_1^{(k)} < 0$ (and $z_0^{(k)} \geq 0$). By the universal test set property of $\mathcal{T}$, there have to exist a vector $t \in \mathcal{T}$ and a scalar $\alpha > 0$ such that $z_1^+ - \alpha t$ is feasible for problem (2) with strictly negative objective function value. In other words, $z^+ - \alpha t \geq 0$ and $\sum_{k:z_1^{(k)}<0} (-t)^{(k)} > 0$. Therefore, $(-t)^{(k)} > 0$ for at least some $k = k_0$ with $z_1^{(k_0)} < 0$. Moreover $(-t)^{(k)} \geq 0$ whenever $z_1^{(k)} < 0$, since $(z^+ - \alpha t)^{(k)} \geq 0$, $\alpha > 0$, and $(z_1^+)^{(k)} = 0$. Thus, $\|(z_1 - \alpha t)^-\|_1 < \|z_1^-\|_1$ and if $z_1^{(k)} \geq 0$ then also $(z_1 - \alpha t)^{(k)} = (z_1^+ - \alpha t)^{(k)} \geq 0$. Thus, the vector t and the scalar α satisfy the conditions of the while-loop which is a contradiction to z_1 being the output of Algorithm 5. □

Lemma 17. *Algorithm 5 terminates in the IP and MIP cases. The algorithm terminates in the LP case if the following strategy similar to the Augmentation Strategy 4 is followed.*

1. *In phase* 1 *only those circuits are allowed for improvement which lead to a better solution with an additional zero entry. Repeat this step until there is no circuit with this property and go to phase* 2.
2. *In phase* 2 *any appropriate circuit is chosen for improvement. Stop, if there is no such circuit, otherwise improve and go back to phase* 1.

Proof. The algorithm terminates in the IP case since in each step $\|z_1^-\|_1$ drops by at least 1. The algorithm terminates in the LP case since, analogously to the termination proof of the LP Augmentation Strategy 4, every support of z_1 appears at most once at the end of phase 1.

Suppose that the algorithm does not terminate in the MIP case and that it generates an infinite sequence $z_1, z_2, \ldots$ of points in $\mathbb{X}$. Note that all non-negative components of a point z_j remain non-negative for all subsequent points during the run of the above algorithm. Thus, since the algorithm does not terminate, there is a number N such that $\emptyset \neq \mathrm{supp}(z_N^-) = \mathrm{supp}(z_j^-)$ for all $j \geq N$. Now consider the problem

$$\min\{ \sum_{i:z_N^{(i)}<0} (-z)^{(i)} : Az = b + Az_N^-, z \in \mathbb{X}_+\}. \tag{3}$$

As can be seen from the proof of Lemma 16, the Algorithm 5 produces a sequence $z_1, z_2, \ldots$ of points in $\mathbb{X}$ such that $z_1^- \geq z_2^- \geq \ldots$, that is, $z_i^- \geq z_j^-$ whenever $i < j$. Thus we have $z_j + z_N^- = z_j^+ + (z_N^- - z_j^-) \geq z_j^+ \geq 0$ and we can conclude that $z_j + z_N^-$, $j \geq N$, are all feasible solutions of (3). Since by our assumption on N, we have $z_j^{(k)} < 0$ whenever $z_N^{(k)} < 0$, all these solutions have a strictly positive objective value. However, since the algorithm does not terminate, we can use a similar argument as in the termination proof of the MIP augmentation algorithm, Proposition 4, to show that the cost function value drops below any given value within a finite number of steps. Thus, in particular, the cost function value becomes eventually negative for some z_j with $j \geq N$. From this contradiction we can conclude that the algorithm terminates in the MIP case, too. □

6. Appendix

Lemma 18. *(Foroudi and Graver [4], General Decomposition Theorem)*
$\mathcal{G}_{\mathrm{MIP}}(A)$ *has the positive sum property with respect to* $\ker_{\mathbb{X}}(A)$.

Proof. Suppose that $(z, q) \in \ker_{\mathbb{X}}(A)$ cannot be written as a positive linear combination of elements in $\mathcal{G}_{\text{MIP}}(A)$. Thus, $(z, q) \notin \mathcal{G}_{\text{MIP}}(A)$. We know that $\|z\|_1 > 0$ since $\mathcal{G}_{\text{LP}}(A_2)$ has the positive sum property with respect to $\ker_{\mathbb{R}^{d_2}}(A_2)$. From all such vectors $(z, q) \in \ker_{\mathbb{X}}(A)$ choose one such that $\|z\|_1 + |\operatorname{supp}(q)|$ is minimal.

Suppose that there is some circuit $q_1 \in \mathcal{G}_{\text{LP}}(A_2)$ such that $\operatorname{supp}(q_1) \subseteq \operatorname{supp}(q)$ and such that $(0, q_1)$ lies in the same orthant as (z, q). Choose the largest scalar $\alpha_1 \in \mathbb{R}_{>0}$ such that $(z, q - \alpha_1 q_1)$ still belongs to the same orthant as (z, q). Therefore, $\operatorname{supp}(q - \alpha_1 q_1) \subsetneq \operatorname{supp}(q)$ and consequently

$$\|z\|_1 + |\operatorname{supp}(q - \alpha_1 q_1)| < \|z\|_1 + |\operatorname{supp}(q)|.$$

By the minimality required on $\|z\|_1 + |\operatorname{supp}(q)|$ we conclude that there is a linear representation $(z, q - \alpha_1 q_1) = \sum \beta_j (z_j, q_j)$ where for all j we have $\beta_j (z_j, q_j) \in \mathbb{X}$, $\beta_j (z_j, q_j) \sqsubseteq (z, q - \alpha_1 q_1)$, and $\beta_j \in \mathbb{R}_{>0}$. Hence $(z, q) = \alpha_1 q_1 + \sum \beta_j (z_j, q_j)$ is a valid representation of (z, q) in contrast to our initial assumption. Thus, we may assume that there is no vector $(0, q_1) \in \ker_{\mathbb{X}}(A)$ with $(0, q_1) \sqsubseteq (z, q)$.

From $(z, q) \notin \mathcal{G}_{\text{MIP}}(A)$ we conclude that there is some $(z', q') \in \ker_{\mathbb{X}}(A) \setminus \{0\}$ with $(z', q') \sqsubseteq (z, q)$, $(z', q') \neq (z, q)$. Hence $(z, q) = (z', q') + (z - z', q - q')$, where also $(z - z', q - q') \sqsubseteq (z, q)$, $(z - z', q - q') \neq (z, q)$. This implies in particular, that $\operatorname{supp}(q') \subseteq \operatorname{supp}(q)$ and $\operatorname{supp}(q - q') \subseteq \operatorname{supp}(q)$.

But neither $z' = 0$ nor $z - z' = 0$, by construction. Therefore, we have $\|z'\|_1 < \|z\|_1$ and $\|z - z'\|_1 < \|z\|_1$. Thus, by the minimality assumption on $\|z\|_1 + |\operatorname{supp}(q)|$, both (z', q') and $(z - z', q - q')$ can be written as valid positive linear combinations of elements from $\mathcal{G}_{\text{MIP}}(A)$, all of which belong to the same orthant as (z', q') and as $(z - z', q - q')$, respectively.

Substituting these representations into $(z, q) = (z', q') + (z - z', q - q')$ gives a valid positive linear combination representing (z, q), contradicting our initial assumption that no such representation exists. □

Lemma 19. *(Foroudi and Graver [4], Lemma* 13*)*

Given a matrix $A = (A_1|A_2) \in \mathbb{Z}^{l \times (d_1 + d_2)}$*, the following inequality holds for any vector* $(z_0, q_0) \in \mathcal{G}_{\text{MIP}}(A_1|A_2)$*:*

$$\|z_0\|_1 \leq \sum_{(z,q) \in \mathcal{G}_{\text{LP}}(A)} \|z\|_1.$$

Proof. Let $(z_0, q_0) \in \mathcal{G}_{\text{MIP}}(A_1|A_2) \subseteq \ker_{\mathbb{R}^d}(A)$. The positive sum property of $\mathcal{G}_{\text{LP}}(A)$ with respect to the set $\ker_{\mathbb{R}^d}(A)$ yields a finite linear representation $(z_0, q_0) = \sum \alpha_i (z_i, q_i)$ where $(z_i, q_i) \in \mathcal{G}_{\text{LP}}(A)$, $\alpha_i (z_i, q_i) \sqsubseteq (z_0, q_0)$, $\alpha_i \in \mathbb{R}_{>0}$.

Suppose that there exists a summand $\alpha_j (z_j, q_j)$ in this representation with $z_j \neq 0$ and $\alpha_j > 1$. From $(z_j, q_j) \in \mathcal{G}_{\text{LP}}(A) \subseteq \mathbb{Z}^d$ we conclude that

$$(z_j, q_j) \in \mathbb{X} \qquad \text{and} \qquad (\alpha_j - 1)(z_j, q_j) + \sum_{i \neq j} \alpha_i (z_i, q_i) = (z_0, q_0) - (z_j, q_j) \in \mathbb{X}.$$

By construction, $z_j \neq 0$ and $z_0 - z_j \neq 0$, $(z_j, q_j), (z_0 - z_j, q_0 - q_j) \sqsubseteq (z_0, q_0)$, and $(z_0, q_0) = (z_j, q_j) + (z_0 - z_j, q_0 - q_j)$. Therefore, $(z_0, q_0) \notin \mathcal{G}_{\text{MIP}}(A_1|A_2)$. Thus, if $(z_0, q_0) \in \mathcal{G}_{\text{MIP}}(A_1|A_2)$ then any linear representation $(z_0, q_0) = \sum \alpha_i (z_i, q_i)$ with

$(z_i, q_i) \in \mathcal{G}_{\mathrm{LP}}(A)$, $\alpha_i(z_i, q_i) \sqsubseteq (z_0, q_0)$, and $\alpha_i \in \mathbb{R}_{>0}$, must satisfy $\alpha_j \leq 1$ whenever $z_j \neq 0$. Consequently,

$$\|z_0\|_1 = \sum_i \|\alpha_i z_i\|_1 = \sum_{i:z_i\neq 0} \|\alpha_i z_i\|_1 \leq \sum_{i:z_i\neq 0} \|z_i\|_1 \leq \sum_{(z,q)\in\mathcal{G}_{\mathrm{LP}}(A)} \|z\|_1$$

and the claim is proved. □

With the additional information that z_0 belongs to some given orthant $\mathbb{O}_j$ the above estimate can be strengthened to

$$\|z_0\|_1 \leq \sum_{(z,q)\in\mathcal{G}_{\mathrm{LP}}(A), z\in\mathbb{O}_j} \|z\|_1.$$

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