Stochastic integer programming: General models and algorithms

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We survey structural properties of and algorithms for stochastic integer programming models, mainly considering linear two-stage models with mixed-integer recourse (and their multi-stage extensions).

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1. Introduction

We give an overview of recent results for stochastic integer problems. We restrict the presentation to properties and solution methods that are both typical for the problem class under consideration and general, in the sense that they are not restricted to a particular type of practical problems. However, we believe that special purpose algorithms will turn out to be necessary to obtain good computational results for many real-life applications, as is the case for deterministic problems. See the annotated bibliography by Stougie and Van der Vlerk [36] for several publications on applications and special purpose algorithms; an even more extensive bibliography is available on the Internet [42].

The models that we discuss are mostly special cases (or multi-stage extensions) of the following linear two-stage model with *(mixed-)integer recourse*:

$$\inf_x \{c\,x + \mathcal{Q}(x) : x \in X\}, \tag{1}$$

where

$$\mathcal{Q}(x) = E_\omega v(x,\omega), \quad x \in X,$$

$$v(x,\omega) = \inf_y \{q(\omega)y : W(\omega)y = h(\omega) - T(\omega)x,\ y \in \mathbb{Z}_+^{n_2} \times \mathbb{R}_+^{\bar{n}_2}\}. \tag{2}$$

The set X of feasible first-stage decisions is the polyhedron $\{x \in \mathbb{R}_+^{n_1} : Ax \geq b\}$, and may contain integrality requirements on x. We denote the dimension of b and $h(\omega)$ by

m_1 and m_2, respectively; the fixed matrices c and A and the random matrices $q(\omega)$, $W(\omega)$ and $T(\omega)$ have compatible dimensions. E_ω denotes the expectation with respect to the random vector ω, which has support Ω. Without loss of generality, we assume that the random parameters all depend linearly on ω.

The interpretation of this model is as follows. We have to decide on the first-stage variables x here and now, that is, before a realization of the random vector ω becomes known. However, such a decision will usually be infeasible with respect to a second set of constraints $T(\omega)x = h(\omega)$, for which the parameters – which are functions of ω – are only known in distribution. To compensate for this infeasibility, recourse actions (represented by the second-stage variables y) have to be taken; the impact and unit costs of y may also depend on ω, as indicated by the notation $W(\omega)$ and $q(\omega)$, respectively. Hence, given a first-stage decision x, each realization of ω leads to recourse costs given by the value function υ of the second-stage problem (2). To account for these recourse costs, their expected value is taken into the objective function, denoted by the *expected value function* $\mathcal{Q}$.

This model covers the whole range of linear two-stage recourse models. Indeed, if there are no integrality restrictions in the first-stage problem and $n_2 = 0$, we obtain the continuous recourse model. At the other extreme, if all first-stage variables are integer and $\bar{n}_2 = 0$, we have a pure integer recourse model. Solving this problem is at least as hard as solving the continuous recourse problem, since in addition to possible difficulties caused by calculation of the multivariate expectation that defines $\mathcal{Q}$, we have integrality constraints in one or both stages. Since (mixed-)integer problems are NP-hard (see [11]), this causes serious additional problems. If there are integer variables only in the first stage, we obtain an integer problem with a convex objective function (at least if W is fixed). If there are integer variables in the second stage, the structural properties of the model are even less promising in general, as discussed in section 2.

Dyer and Stougie [10] discuss complexity results for various recourse models. It turns out that the complexity caused by integrality is dominated by the complexity of evaluating the multi-dimensional integral, as called for in evaluating the expected value function. This means that in a worst-case setting, our incapability to solve these problems will not be caused by the fact that they contain integer requirements. However, in practice, integrality restrictions provide a serious challenge, as in deterministic optimization.

Given these gloomy prospects, it is even more important to motivate our interest in stochastic integer problems. The main motivation comes from their wide practical applicability, as witnessed by a variety of models treated in the literature. Among the applications having in common the need to explicitly model uncertainty and the use of discrete variables are combinatorial problems such as scheduling [3], routing [22] and location [23] problems, problems related to efficient production of electricity [4,37], production planning [39], environmental control [29], finance [9], and telecommunications [40]. In addition, it is interesting to investigate to what extent smoothing, caused by the integration defining the expected value function $\mathcal{Q}$, can compensate for disturbing properties such as discontinuity and non-convexity which are inherited from the mixed-integer value function υ.

In section 2, we review structural properties of the expected value function $\mathcal{Q}$. Following the general case, we will also present properties for the so-called simple integer recourse model, which is much more tractable due to its special structure. In section 3, we give an overview of general purpose algorithms. Related problems that do not fit in the framework of (1) are reported in section 3.3, and we conclude this paper with some considerations on possible future research in this field.

2. Structural properties

The most general mixed-integer model for which structural properties are known is the fixed recourse model with complete recourse. In this model, only $h(\omega)$ and $T(\omega)$ are random, and the recourse matrix W has the complete recourse property which, analogous to the continuous case, is defined as

$$\text{For all } t \in \mathbb{R}^{m_2} \text{ there exists } y \in \mathbb{Z}_+^{n_2} \times \mathbb{R}_+^{\bar{n}_2} \text{ such that } Wy = t. \tag{3}$$

In section 2.2, we define the *simple integer recourse* model. For this special case, more properties are known.

2.1. *Complete mixed-integer recourse*

All results in this subsection apply to the model with complete mixed-integer recourse and are due to Schultz. For details, we refer to [30–33].

Finiteness of the expected value function $\mathcal{Q}$ is obtained under the same conditions as needed in the continuous recourse case. In addition to the complete recourse condition (3) above which ensures $\upsilon < +\infty$, one needs dual feasibility of the LP relaxation of the second-stage problem ("sufficiently expensive recourse") to have $\upsilon > -\infty$. Since only $h(\omega)$ and $T(\omega)$ are random, finiteness of $\mathcal{Q}$ then follows if the random vector ω has finite expectation.

The following property is inherited from the value function υ of the mixed-integer second-stage problem.

Theorem 1. The function $\mathcal{Q}$ is lower semicontinuous on $\mathbb{R}^{n_1}$.

Conditions for continuity of $\mathcal{Q}$ are based on the fact that the discontinuity points of the value function υ are contained in a countable union of hyperplanes. Let $D(x)$, $x \in \mathbb{R}^{n_1}$, denote the set containing all $\omega \in \Omega$ such that $h(\omega) - T(\omega)x$ is a discontinuity point of υ.

Theorem 2. If $\Pr\{\omega \in D(x)\} = 0$, then $\mathcal{Q}$ is continuous at x.

Corollary 3. If the conditional distribution of $h(\omega)$ given $T(\omega) = T$ is continuous for almost all T (with respect to the distribution of $T(\omega)$), then $\mathcal{Q}$ is continuous on $\mathbb{R}^{n_1}$.

If $h(\omega)$ and $T(\omega)$ are independent, then continuity of $\mathcal{Q}$ follows from corollary 3 if $h(\omega)$ is continuously distributed. Similarly, if the joint distribution of $h(\omega)$ and $T(\omega)$ is continuous, then $\mathcal{Q}$ is continuous on $\mathbb{R}^{n_1}$.

Theorem 2 and corollary 3 extend the first continuity result for $\mathcal{Q}$ due to Stougie [35], who used additional assumptions on υ and the distribution of $h(\omega)$ and $T(\omega)$.

The conditions for Lipschitz continuity of the function $\mathcal{Q}$ given in the following theorem may seem rather technical. However, they are necessary, as shown by Schultz in two examples [31]. We present the result for a fixed T matrix; it can be extended to the random T case using conditional distributions.

Theorem 4. Let $h(\omega)$ follow a continuous distribution. Assume that, for any non-singular linear transformation $B : \mathbb{R}^{m_2} \mapsto \mathbb{R}^{m_2}$, each one-dimensional marginal distribution of $Bh(\omega)$ has a bounded probability density function $f(s)$ that is monotone for $|s|$ sufficiently large. Then $\mathcal{Q}$ is Lipschitz continuous on any bounded subset of $\mathbb{R}^{n_1}$.

For example, the (non-degenerate) multivariate normal distribution and the t-distribution satisfy the assumptions of this theorem.

To evaluate the expected value function at least approximately, it may be necessary to approximate the distribution of $h(\omega)$ and $T(\omega)$. This leads to the analysis of stability of optimal solutions and the optimal value under such approximations. In [30], Schultz shows that, under conditions which are not too restrictive in practice, the problem is stable in this sense if the approximating distributions converge weakly to the original distribution. In particular, this result justifies the use of discrete approximations of continuous distributions. Rates of convergence are discussed in [33]. Next to qualitative statements, it is shown that approximation by means of the empirical distribution gives convergence of (local) optimal values and optimal solutions at a rate slightly worse than $\sqrt{n}$. Related results in a more general setting can be found in Artstein and Wets [1].

2.2. *Simple integer recourse*

The simple integer recourse model is obtained from (1) by taking as the second-stage problem

$$\begin{aligned} \upsilon(x,\omega) = \inf\{q^+ y^+ + q^- y^- : \; & y^+ \geq h(\omega) - T(\omega)x, \\ & y^- \geq -(h(\omega) - T(\omega)x), \\ & y^+ \in \mathbb{Z}_+^{m_2},\, y^- \in \mathbb{Z}_+^{m_2}\}. \end{aligned}$$

Compared to (1), this corresponds to taking $y = (y^+, y^-, \bar{y}^+, \bar{y}^-) \in \mathbb{Z}_+^{n_2} \times \mathbb{R}_+^{\bar{n}_2}$, with $n_2 = \bar{n}_2 = 2m_2$, $q = (q^+, q^-, 0, 0)$, $W = (I, -I)$, with I the $n_2 \times n_2$ identity matrix, and replicating the rows of $T(\omega)$ and $h(\omega)$ with a minus sign.

From the structure of the recourse matrix W, we see that this is indeed a simple recourse model. Hence, for this model the well-known condition to ensure $\upsilon > -\infty$ reads $(q^+, q^-) + (0, 0) \geq 0$, i.e., $q^+ \geq 0$ and $q^- \geq 0$.

As suggested by its name, this is the simplest integer recourse model, both conceptually and analytically. This has allowed us to find structural properties beyond those known for the more general case discussed above. Below, we will review results for the model with only $h(\omega)$ random; corresponding results for the model with also $T(\omega)$ random are presented in [19].

Due to the simple recourse structure, the expected value function $\mathcal{Q}$ is separable:

$$\mathcal{Q}(x) = \sum_{i=1}^{m_2} E_{h_i(\omega)} \inf_{y_i^+, y_i^-} \{q_i^+ y_i^+ + q_i^- y_i^- : \; \begin{aligned}[t] & y_i^+ \geq h_i(\omega) - T_i x, \\ & y_i^- \geq -(h_i(\omega) - T_i x), \\ & y_i^+ \in \mathbb{Z}_+, \, y_i^- \in \mathbb{Z}_+\}. \end{aligned}$$

Each of the m_2 minimization problems on the right is trivially solved by $y_i^+ = \lceil h_i(\omega) - T_i x \rceil^+$, $y_i^- = \lfloor h_i(\omega) - T_i x \rfloor^-$, where $\lceil s \rceil^+ = \max\{0, \lceil s \rceil\}$, $\lfloor s \rfloor^- = \max\{0, -\lfloor s \rfloor\}$, for $s \in \mathbb{R}$. Hence, $\mathcal{Q}$ is completely characterized by the one-dimensional generic function Q, given by

$$Q(z) = q^+ E_\xi \lceil \xi - z \rceil^+ + q^- E_\xi \lfloor \xi - z \rfloor^-, \quad z \in \mathbb{R}, \tag{4}$$

where ξ is a random variable taking the place of $h_i(\omega)$, z represents a tender variable for $T_i x$, and q^+, q^- with $q^+ + q^- > 0$ are nonnegative constants. Below we present results for the one-dimensional function Q; the extension to the n_1-dimensional case is straightforward.

Obviously, conditions for (Lipschitz) continuity of Q follow from the general results cited above. However, as shown in [25], using the closed form expression for the function Q given below, it is possible to obtain sharper results for this special case. Some of these results have been elaborated in [41]; they are summarized in the following theorem.

Theorem 5. Let Q be the one-dimensional expected value function of an integer simple recourse program, defined by (4), and let ξ be a random variable with finite expected value. Then

(a) For all $z \in \mathbb{R}$,

$$Q(z) = q^+ \sum_{k=0}^{\infty} \Pr\{\xi > z + k\} + q^- \sum_{k=0}^{\infty} \Pr\{\xi < z - k\}. \tag{5}$$

(b) The function Q is lower semicontinuous. It is continuous on $\mathbb{R}$ if and only if ξ is continuously distributed. If ξ is discretely distributed with support Ξ, then the set of discontinuity points is given by $\Xi + \mathbb{Z}$, and in between two successive points in $\Xi + \mathbb{Z}$, the function Q is constant.

If ξ has a probability density function f with bounded total variation $|\Delta| f$, then

(c) Q is Lipschitz continuous with a constant that is at most $\max\{q^+, q^-\}(1 + |\Delta| f/2)$.

(d) Q is differentiable at $z \in \mathbb{R}$ if f is continuous at $z + k$ for all $k \in \mathbb{Z}$. Hence, Q is differentiable if f is continuous.

Consider now convexity properties of the function Q. It follows from (5) that

$$Q(z+1) - Q(z) = -q^+ \Pr\{\xi > z\} + q^- \Pr\{\xi < z+1\}, \quad z \in \mathbb{R},$$

so that Q is convex on any grid $\alpha + \mathbb{Z}$, $\alpha \in [0, 1)$. However, in general, the function Q is not convex on $\mathbb{R}$. In [18], it is shown that under some mild technical assumptions on the probability density function f, denoted by $f \in \mathcal{F}$, the following result holds.

Theorem 6. Assume ξ has a probability density function $f \in \mathcal{F}$. Then the function Q, defined by (4), is convex on $\mathbb{R}$ if and only if $f(s) = G(s+1) - G(s)$, $s \in \mathbb{R}$, where G is an arbitrary cumulative distribution function with finite mean value.

For example, if G corresponds to the degenerate distribution in (1), then f is a probability density function of the uniform distribution on [0, 1]. Shifted versions of this distribution play a central role in the convexity of the function Q, as can be seen from the formulation of theorem 6 in terms of random variables: Q is convex if and only if there exists a random variable η with finite mean value, such that for all $s \in \mathbb{R}$, the conditional distribution of ξ given $\eta = s$ is uniform on $[s-1, s]$.

3. Solution methods

We now give an overview of solution methods for stochastic integer problems. The first subsection on algorithms for general mixed-integer recourse models mainly reviews methods that are based on decomposition (by time stages or by scenarios) or (partial) enumeration. Next, we survey algorithms for the simple integer recourse that are all based on convex approximations. In the last subsection, we consider algorithms for three related problems that do not fit in the framework of our integer recourse model (1).

3.1. Algorithms for general mixed-integer recourse models

Assume that the random variables in (1) are all discretely distributed, and that their joint distribution has a finite number of possible realizations (q^r, W^r, T^r, h^r), each with probability p^r, $r = 1, \dots, N$. Then our mixed-integer recourse model can be represented as

$$\inf_{x, y^r} \left\{ cx + \sum_{r=1}^{N} p^r q^r y^r : W^r y^r + T^r x = h^r \ \forall r, \quad x \in X,\ y^r \in \mathbb{Z}_+^{n_2} \times \mathbb{R}_+^{\bar{n}_2} \ \forall r \right\}, \tag{6}$$

which is a large-scale, dual block-angular mixed-integer problem with at least $N \cdot m_2$ constraints and $n_1 + N(n_2 + \overline{n}_2)$ variables, of which at least $N \cdot n_2$ are required to be integer. For realistic sizes of N and n_2, it is impossible to solve this problem by methods that do not make use of the special structure in the problem.

Another approach that is clearly limited to small problem instances is dynamic programming, as proposed by Lageweg et al. [20]. Again, a finite discrete distribution is assumed, and the second-stage problem has to be a combinatorial optimization problem that is solvable by dynamic programming. The stochastic integer programming problem is then solved by a single "giant" recursion that combines the separate dynamic programming computations for all the individual realizations. In [20], two-stage scheduling, bin packing, and multi-knapsack problems are solved in this way.

In the large-scale formulation (6), we can consider the decision vector x to be the complicating factor, since if x were fixed, the remaining problem would be separable and therefore much easier to solve. For general deterministic mixed-integer problems, this approach of fixing complicating variables (which are chosen to be the integer variables, so that the remaining problem is an LP) has led to Benders' decomposition. Similarly, for continuous two-stage recourse models we may consider the first-stage variables x as "complicating" the separability. Based on this observation, Van Slyke and Wets [43] proposed the L-shaped algorithm. In both cases, the part of the objective function corresponding to the remaining variables (i.e., the expected value function in the two-stage model) is represented by a single variable θ, whose optimal value is determined iteratively by using LP duality to construct a piecewise linear outer approximation of this convex part of the objective function as well as its effective domain. The cuts that are generated to these ends are known as optimality cuts and feasibility cuts, respectively.

Given these very similar solution methods for these two problem classes, it is only natural to combine them to obtain an algorithm for mixed-integer recourse problems. This was first done by Wollmer [44] for models with 0–1 first-stage variables and continuous second-stage problems. Laporte and Louveaux [21] extend this to models with binary first-stage variables and arbitrary (but easily computable) second-stage problems. Of course, in general, the expected value function of such a problem is non-convex, so that it cannot be described using linear cuts as in the continuous case. However, due to the fact that the first-stage variables are binary, it is possible to construct a valid set of linear optimality cuts for this problem. For $x^k \in X \subset \{0, 1\}^{n_1}$, define the index set $S^k = \{i : x_i^k = 1\}$. Then an optimality cut at x^k is given by

$$\theta \geq (\mathcal{Q}(x^k) - L)\left(\sum_{i \in S^k} x_i - \sum_{i \notin S^k} x_i\right) + (\mathcal{Q}(x^k) - L)(|S^k| - 1) + L,$$

where L is a global lower bound for $\mathcal{Q}$. Since there can be a large number of these optimality cuts (2^{n_1} at most), they are generated iteratively in a branch and cut scheme. In addition, it is observed that cuts obtained from the continuous relaxation of the second-stage problem can be used to speed up the computations, since the expected value function of the latter problem is a lower bound for $\mathcal{Q}$. This so-called integer L-shaped method has been applied successfully to several problems, see [22,23].

Carøe and Tind [6] consider L-shaped decomposition for integer recourse problems in a general setting: they allow arbitrary first-stage variables in combination with a pure integer second-stage problem; $T(\omega)$ and $h(\omega)$ are discretely distributed, q is fixed. Optimality cuts for the non-convex function $\mathcal{Q}$ (as well as feasibility cuts) are derived from duality theory for integer programming (see e.g. [26]). For a current solution $\bar{x}$ and realizations (T^r, h^r), $r = 1,\dots, N$, the second-stage problems are

$$\inf_y \{q\,y : W\,y \geq h^r - T^r\bar{x},\ y \in \mathbb{Z}_+^{n_2}\},$$

whose respective so-called $(\mathcal{F})$-dual problems are

$$\sup_F \{F(h^r - T^r\bar{x}) : F(W\,y) \leq q\,y \quad y \in \mathbb{Z}_+^{n_2},\ F \in \mathcal{F}\}, \tag{7}$$

where $\mathcal{F}$ is a suitable subset of the set of dual price functions (nondecreasing functions $F : \mathbb{R}^{m_2} \mapsto \mathbb{R} \cup \{-\infty, \infty\}$, with $F(0) = 0$). The choice of the set $\mathcal{F}$ is governed by the algorithm that is used to solve the second-stage problems. It should be large enough so that the duality gap is closed.

Theorem 7. Suppose that F^r, $r = 1,\dots,N$, are optimal dual price functions for (7). Then an optimality cut for $\mathcal{Q}$ at $\bar{x}$ is given by the inequality $\theta \geq \sum_{r=1}^N p^r F^r(h^r - T^r\bar{x})$.

With these optimality cuts (and similarly defined feasibility cuts) the usual L-shaped scheme of repeatedly solving the current problem and adding cuts can be applied. Clearly, the performance of the method depends heavily on the choice of the set $\mathcal{F}$. Carøe and Tind show finite convergence of the method in the case a branch and bound or a cutting plane technique is used to solve the second-stage problems. However, it has to be assumed that the nonlinear master problems can be solved by a finite method. This assumption seriously limits the practicability of the method. The authors give some suggestions to arrive at more tractable master problems, at the expense of optimality.

We conclude the overview of L-shaped type algorithms with a paper by Averbakh [2], who considers the related mixed-integer recourse problem

$$\begin{aligned} \max\{G(x) + E_\omega C(\omega, y(\omega)) :\ & E_\omega A_i(\omega, y(\omega)) \leq B_i(x),\quad i = 1, \ldots, m_2, \\ & y(\omega) \in Y(\omega) \subset \mathbb{R}^{n_2} \text{ a.s.}, \\ & x \in X\}, \end{aligned}$$

where ω is continuously distributed. No convexity conditions are imposed on the functions C, A_i, or the sets $Y(\omega)$. In particular, there can be integrality restrictions on the variables y. In comparison with our usual model (1), there are only statistical constraints on the second-stage variables here. It is shown that under some technical conditions this second-stage problem is convex, and a dual problem is formulated. Assuming that the latter problem can be solved, this provides the information needed in the proposed L-shaped type algorithm.

The following two algorithms are in a sense dual to the L-shaped algorithms discussed above. Whereas the latter algorithms decompose the problem by time stages and operate by searching for increasingly better solutions x, the algorithms below consider subproblems corresponding to scenarios (i.e., realizations in the two-stage setting) and are governed by finding good dual multipliers. Therefore, the first group can be named primal decomposition algorithms, and the second group dual decomposition algorithms.

The dual decomposition algorithm proposed by Carøe and Schultz [5] applies to two-stage problems with mixed-integer variables in both stages, and a finite discrete distribution for $(q(\omega), T(\omega), h(\omega))$. As indicated by the authors, the method can be applied to multi-stage problems of the same type; elaborations of algorithmic and implementational details is a field of ongoing research.

Consider the large-scale mixed-integer problem (6). Introducing copies of the first-stage variables x^r corresponding to realizations (q^r, T^r, h^r), $r = 1, \ldots, N$, and writing the non-anticipativity constraints as $\sum_r H^r x^r = 0$, with $H = (H^1, \ldots, H^N)$ a suitable $(l \times n_1 N)$ matrix, the Lagrangian relaxation with respect to these constraints is separable according to scenarios:

$$D(\lambda) = \sum_{r=1}^{N} \min_{x^r,\, y^r} \{ p^r (c x^r + q^r y^r) + \lambda (H^r x^r) : (x^r, y^r) \in S^r \}, \quad \lambda \in \mathbb{R}^l,$$

where $S^r = \{(x, y) : x \in X,\ W y + T^r x \le h^r,\ y \in Y\}$ and Y determines the type of the variables y. By a well-known weak duality result (see e.g. [26]), the optimal value of the Lagrangian dual $\max_\lambda D(\lambda)$, which is a convex problem, provides a lower bound for the optimal value of (6). Moreover, if for some λ the solutions (x^r, y^r) are feasible, then they are optimal and so is λ.

In general the scenario solutions (x^r, y^r) of the Lagrangian relaxation will not coincide in their x-component. To enforce this, a branch and bound scheme is proposed that uses the Lagrangian relaxation as bounding procedure. Possible first-stage solutions are constructed from the scenario solutions by taking the weighted average, combined with a rounding heuristic in order to satisfy integrality restrictions. Several numerical examples are given. Moreover, the method is applied successfully to a realistic unit commitment problem as reported in [4].

Next, we briefly consider application of the well-known progressive hedging algorithm by Rockafellar and Wets [28] to multi-stage mixed-integer 0–1 problems. Løkketangen and Woodruff [24] propose to use progressive hedging in combination with tabu search to solve the quadratic mixed-integer 0–1 scenario problems. Although a formal justification is lacking, they observe convergence to an implementable solution for several test problems. To speed up convergence to a (in general non-optimal) solution, they use the concept of integer convergence: as soon as all integer variables have converged they are fixed, and the resulting continuous problem is solved by any standard method to obtain values for the remaining variables. Similarly, Takriti et al. [37] obtained reasonably good solutions when applying progressive hedging to a multi-stage unit commitment problem.

Next, we consider two enumerative algorithms. Ruszczyński et al. [29] propose a stochastic version of the deterministic branch and bound algorithm, which uses exact computation of the objective function at many possible solutions. The basic idea of the method is to use statistical estimates of function values instead, since exact computations are in general far too expensive. The results are presented in a much more general setting than our two-stage recourse problems; within the latter context, mixed-integer variables in both stages and arbitrary distributions for all parameters can be handled. Consider the problem

$$\min\{F(x) : x \in X\},$$

where $F(x) = E_\omega f(x,\omega)$, with f an arbitrary function, and $X = Y \cap Z$, with Y some (possibly infinite) subset of $\mathbb{R}^{n_1}$ and Z a finite set, e.g. a subset of $\mathbb{Z}^{n_1}$. The algorithm constructs increasingly finer partitions of Y, denoted by $\mathcal{P}^k = \{Y^1, \dots, Y^{n(k)}\}$ at iteration k. Let $F^*(Y^p)$ denote the optimal value of F over the subset Y^p, and $F^*(Y) = \min_p F^*(Y^p)$ the optimal value of the original problem. Assume that there exist functions L and U such that, for each $Y^p \subset Y$, $L(Y^p) \le F^*(Y^p) \le U(Y^p)$ and $L(Y^p) = F(\bar{x}) = U(Y^p)$ for some $\bar{x} \in Y^p$, and if Y^p is a singleton, then $L(Y^p) = F^*(Y^p) = U(Y^p)$. Instead of using these bounds L and U, the algorithm works with estimates $\xi^l(Y^p, \theta)$ and $\eta^m(Y^p, \theta)$, respectively, assuming that almost surely $\lim_{l\to\infty} \xi^l(Y^p) = L(Y^p)$ and $\lim_{m\to\infty} \eta^m(Y^p) = U(Y^p)$. For example, taking $L(Y^p) = E \min\{f(x,\omega) : x \in Y^p\}$ and $U(Y^p) = E f(\bar{x})$ for some $\bar{x} \in Y^p$, $\xi^l(Y^p) = l^{-1}\sum_{i=1}^{l} \min\{f(x,\omega^i) : x \in Y^p\}$ and $\eta^m(Y^p) = m^{-1}\sum_{i=1}^{m} f(\bar{x}, \omega^i)$, where the ω^i are sampled from the distribution of ω, satisfies the assumptions. Several other possibilities are discussed in the paper.

At iteration k the stochastic branch and bound algorithm proceeds as follows:

(i) Select the so-called record set $\overline{Y} \in \arg\min_p \{\xi_k(Y^p)\}$ and an approximate solution $x^k \in Y^k \in \arg\min_p \{\eta_k(Y^p)\}$, where ξ_k and η_k denote the current estimates of the bounds.

(ii) If the record set is not a singleton, partition the record set and update the partition $\mathcal{P}^k$ accordingly.

(iii) Update the estimates $\xi_k(Y^p)$ and $\eta_k(Y^p)$ for all $Y^p \in \mathcal{P}^k$ (see below).

(iv) Remove subsets Y^p that contain no feasible solutions. Bounding out can only be applied if the estimates of the bounds are exact.

Proceed with the next iteration unless some stopping criterion is met.

When updating the estimates in step (iii), most attention is paid to (subsets of) the record set. Loosely speaking, this means that several observations are made for the record set, whereas other subsets are observed once or even only "every now and then". Precise conditions are given under which approximate solutions x^k converge to an optimal solution. A solution obtained after a finite number of iterations is in general an approximation; probabilistic bounds on the accuracy of the solution are given. Several examples and numerical results are presented.

The second enumerative algorithm is due to Schultz et al. [34], and applies to the two-stage model (1) with continuous first stage, pure integer second stage, and a finite discrete distribution of $h(\omega)$. Using structural properties of the expected value function $\mathcal{Q}$, a countable subset of the feasible set X containing an optimal solution is constructed. Assuming that the recourse matrix W is integral (or rational, so that integrality can be obtained by scaling), it follows that, for every $\bar{x} \in \mathbb{R}^{n_1}$, the function $\mathcal{Q}$ is constant on

$$\bigcap_{r=1}^{N} \{x \in \mathbb{R}^{n_1} : \lceil T x - h^r \rceil = \lceil T\bar{x} - h^r \rceil\}. \tag{8}$$

Using that $\mathcal{Q}$ is lower semicontinuous, it is shown that the so-called set of candidates V, which consists of all vertices of the sets (8) intersected with X, contains an optimal solution. Conditions are given under which level sets of the continuous relaxation of the problem can be used to further restrict V to a finite set. Based on the structure of the set V, a complete enumeration scheme is proposed.

Since the cardinality of V is in general very high, it is important to identify subsets of candidate points that cannot be optimal solutions. To this end, two improvements of the algorithm are proposed. The first one consists of intersecting V with increasingly smaller level sets, as mentioned above. Secondly, based on the identification of common directions of increase of the first-stage objective cx and the expected value function $\mathcal{Q}(x)$, certain groups of candidate points need not be evaluated.

For each remaining candidate point, the function $\mathcal{Q}$ needs to be evaluated, which implies that N integer second-stage problems have to be solved. Since all these integer problems only differ in their right-hand side coefficients, it is suggested to solve them by means of Gröbner basis methods, a technique from computational algebra. This appears to be appropriate in this setting, since after computing the Gröbner basis which is independent of the right-hand side of the problem, solving each instance is very cheap. However, the algorithm does not depend on how the function evaluations are performed. Numerical results for a small test problem are given.

Inspired by its success in the deterministic setting, Carøe and Tind [7] investigate the use of cutting planes for two-stage mixed-integer 0–1 recourse models with finite discrete distributions. Starting from the deterministic equivalent (6), the feasible set S can be written as the intersection of sets S^r, $r = 1,\dots, N$, each constraining only x and y^r. Instead of trying to construct the convex hull of S, which would allow the solution of (6) as a continuous problem, here the aim is to construct the so-called hull-relaxation

$\bigcap_r \operatorname{conv} S^r$. The optimal value over the latter set is equal to the optimal value of the Lagrangian dual with respect to the non-anticipativity constraints as formulated above. The motivating idea is that a lot of computation time can be saved if a valid inequality obtained for some S^k can be transformed – without too much effort – into valid inequalities for all sets S^r. Carøe and Tind show that this is indeed possible in the case W is fixed, but also that the resulting cuts are not likely to be very strong. The results are implemented in a conceptual L-shaped algorithm using so-called Lift-and-Project cuts, which are preferable to other classes of cuts since they do not disturb the structure of the problem. The paper concludes with a theoretical result on lifting valid inequalities for second-stage problems (with x fixed) to valid inequalities for conv S^r for the case that (some of) the first stage variables are binary. No numerical results are reported.

3.2. Algorithms for simple integer recourse models

We now review algorithms for the simple integer recourse model defined in section 2.2. Clearly, the above algorithms developed for more general models can be used, but the following special purpose algorithms benefit from the additional structure in the problem. Analogously to section 2.2, the results will be presented in a one-dimensional setting. They all apply to models with randomness only on the right-hand side parameter ξ.

Since the function Q is convex on any grid $\alpha + \mathbb{Z}$, $\alpha \in [0, 1)$, a convex approximating model is obtained if Q is replaced by the piecewise linear function generated by the function values on such a grid. Louveaux and Van der Vlerk [25] propose an L-shaped algorithm to solve this approximating problem, which gives exact solutions if either the tender variable is restricted to be integer or if ξ is distributed on a subset of $\alpha + \mathbb{Z}$ for some $\alpha \in [0, 1)$.

Klein Haneveld et al. [15] show that for any reasonable convex approximation $\hat{Q}$ of Q, it holds that

$$\hat{Q}(z) = q^+ E_\psi (\psi - z)^+ + q^- E_\psi (\psi - z)^- + \hat{c}, \quad z \in \mathbb{R}, \tag{9}$$

where the random variable ψ has cdf $W(s) = (\hat{Q}'_+(s) + q^+)/(q^+ + q^-)$, $s \in \mathbb{R}$, $\hat{Q}'_+$ denotes the right derivative of $\hat{Q}$, and $\hat{c}$ is a constant. That is, Q equals the one-dimensional expected value function of some continuous simple recourse program (plus a constant). Consequently, we can solve simple integer recourse problems (at least approximately) by algorithms developed for continuous simple recourse problems if the distribution of ψ is known. The following algorithms for two particular cases give this distribution as output.

If the right-hand side parameter ξ follows a finite discrete distribution, it follows from the structural properties of Q that its convex hull is determined completely by the function values in discontinuity points of Q. Based on this observation, Klein Haneveld et al. [16] propose a strongly polynomial algorithm that finds the corresponding constant $\hat{c}$ and distribution of ψ, which turns out to be discrete in all cases. However, as shown in [14], this one-dimensional approach leads to the convex hull of the n_1-dimensional objective function $cx + \mathcal{Q}(x)$ only if the matrix T has full row rank. Even in that case, the function obtained does not reflect the restriction of the objective to the feasible set X. In any event, the result is a convex lower bound for the objective.

Theorem 6 provides the means to obtain convex approximations for Q in the case ξ is continuously distributed. In [17], Klein Haneveld et al. analyze a class of approximations defined by their probability density functions $f_\alpha(s) = F(\lfloor s \rfloor_\alpha + 1) - F(\lfloor s \rfloor_\alpha)$, $s \in \mathbb{R}$. Here, F is the cumulative distribution function of ξ, $\alpha \in [0, 1)$ is a shift parameter, and $\lfloor \cdot \rfloor_\alpha$ denotes round down with respect to the set $\{\alpha + \mathbb{Z}\}$ (the case $\alpha = 0$ corresponds to the usual integer round down). For each $\alpha \in [0, 1)$, the function $Q_\alpha(z) = q^+ E\lceil \xi_\alpha - z \rceil^+ + q^- E\lfloor \xi_\alpha - z \rfloor^-$, $z \in \mathbb{R}$, with the random variable ξ_α distributed according to f_α, is a convex approximation of Q. Since Q_α coincides with Q at all points in $\{\alpha + \mathbb{Z}\}$ and is piecewise linear in between, it is a more tractable representation of the same approximation as considered in Louveaux and Van der Vlerk [25].

The cdf of the random variable ψ_α in the continuous simple recourse representation (9) of $\mathcal{Q}_\alpha$ is given by

$$W_\alpha(s) = \frac{q^+}{q^+ + q^-} F(\lfloor s \rfloor_\alpha) + \frac{q^-}{q^+ + q^-} F(\lfloor s \rfloor_\alpha + 1), \quad s \in \mathbb{R}.$$

That is, ψ_α is a discrete random variable with support in $\{\alpha + \mathbb{Z}\}$. Independent of the choice of α, the constant $\hat{c}$ in (9) is equal to $(q^+ q^-)/(q^+ + q^-)$.

A uniform bound on the error of the approximation (and hence on the optimal value obtained) is given by

$$\| Q_\alpha - Q \|_\infty \leq (q^+ + q^-) \frac{|\Delta| f}{4},$$

where $|\Delta| f$ is the total variation of f. This uniform error bound can be improved by a factor of two at most, which is achieved by taking, for an arbitrary $\alpha \in [0, 1/2)$ and $\beta = \alpha + 1/2$, $f_{\alpha\beta} = (f_\alpha + f_\beta)/2$ as the approximating distribution. For many distributions, the total variation of f decreases as the variance of the distribution increases. In these cases, the approximation becomes better accordingly.

The continuous simple recourse representations of the approximations presented above have discretely distributed right-hand side parameters, and can therefore be solved efficiently. Algorithms to compute these distributions (and standard solution methods) are implemented in the model management system SLP-IOR developed by Kall and Mayer [13].

3.3. *Algorithms for related problems*

In this subsection, we deal with three special models, in which integer variables arise in a natural way in stochastic programming models. First we discuss two recent papers dealing with chance constraints. We conclude with a paper that describes the influence of decisions on the probability distribution of the random parameters by 0–1 variables. In all cases, the ideas behind the models and algorithms are sketched.

In many applications, it is appropriate not only to deal with expected recourse costs as qualitative measure of the risk of violating constraints, but also to take the probability of violation of constraints into account explicitly. This approach leads to chance constraints in stochastic optimization problems. For instance, a chance constraint variant of model (1) is

$$\inf_x \{cx : \Pr\{T(\omega)x \geq h(\omega)\} \geq \alpha,\ x \in X\}, \tag{10}$$

where α is the minimum required probability level that the constraints $T(\omega)x \geq h(\omega)$ are satisfied. Unfortunately, in general chance constrained models are nonconvex, even if X does not include integrality constraints. Only if $(T(\omega), h(\omega))$ are continuously distributed, and if their distribution satisfies additional requirements (that are rather restrictive, in particular if $T(\omega)$ is not deterministic, see [27]), one gets a convex optimization problem.

If ω is discrete (or if its support Ω is approximated by a finite set of sample values, as is often the case), convexity of the feasible set is lost. In this case, however, it is possible to formulate problem (10) as a mixed-integer linear program. Indeed, if the distribution is given by $\Pr\{(T(\omega), h(\omega)) = (T^r, h^r)\} = p^r$, $r = 1,\ldots, N$, then (10) is equivalent to

$$\begin{aligned} \inf_x \{c x : T_i^r x + M_i \delta^r \geq h_i^r, \quad & r = 1, \ldots, N,\ i = 1, \ldots, m_2, \\ \sum_{r=1}^{N} p^r \delta^r \leq 1 - \alpha, \quad & \\ x \in X,\ \delta^r \in \{0,1\}, \quad & r = 1, \ldots, N\}. \end{aligned} \tag{11}$$

Here, M_i is a sufficiently large number, and the indicator δ^r is equal to zero if and only if $T^r x \geq h^r$. (The "if" part holds under the assumption that any solution with $T^r x \geq h^r$ and $\delta^r = 1$ is not optimal.)

As can be expected from the structure of (11), it is possible to deal with the probability of constraint violation in the framework of recourse models, if one allows for mixed-integer recourse. To illustrate this, consider model (6) with all random variables discretely distributed. If one takes $n_2 = 1$, $\bar{n}_2 = m_2$, $q^r = (q_0, 0)$, with $q_0 \in \mathbb{R}_+$, $W^r = (M, -I)$, where $M \in \mathbb{R}^{m_2}$, has sufficiently large elements M_i, and I is the $m_2 \times m_2$ identity matrix, and if one replaces the integer recourse variables by 0–1 variables (i.e., $y^r = (\delta^r, \bar{y}^r)$, with $\delta^r \in \{0, 1\}$ and $\bar{y}^r \in \mathbb{R}_+^{m_2}$), then (6) is equivalent to

$$\inf_x \{c x + q_0 \cdot \Pr\{T(\omega)x \ngeq h(\omega)\} : x \in X\}.$$

A nice example of the inclusion of chance constraints in multi-stage stochastic linear programming by means of 0–1 variables (in fact, the only one that we are aware of) is given by Dert [9] (see also [45]). He formulates an Asset Liability Management (ALM) model for Dutch pension funds. It aims at minimizing the cost of funding while safeguarding the pension fund's ability to meet its obligations. Obviously, it deals with long-term planning, and the returns of assets and the liabilities are uncertain, so that a multi-stage stochastic model is appropriate. The uncertainties are modeled by a scenario tree. At each time, ideally the solvency requirement should hold: the pension fund's asset value should be sufficient to meet all its future obligations. In uncertain circumstances, however, underfunding can not be excluded completely. Therefore, Dert [9] introduces recourse actions, representing, for instance, remedial payments of the working members of the pension fund. In conjunction, also chance constraints are introduced to assure that the probability of becoming underfunded does not exceed a given level. This is implemented by introducing 0–1 variables, in the way indicated above. Actually, such chance constraints are imposed at any point of the scenario tree, using the conditional distribution of the next branching. As a result, the ALM model is a large-scale mixed-integer stochastic multi-stage linear optimization problem. Obviously, it is much too large to solve to optimality by standard methods. Therefore, a heuristic is proposed. First, the scenario tree is reduced so that the number of states increases only linearly in the number of stages. Then, in a backward procedure, decisions are determined solving successively a number of two-stage mixed-integer problems, arising by stage aggregation. In order to find good decisions for stage t, the stages up to and including t are combined to the "first" stage, whereas the future stages are combined in the "second" stage. After this sequence of two-stage mixed-integer programs, a forward procedure is executed which improves the current solution by taking into account information that was not available when the backward procedure was executed. As a result, a feasible approximation of the optimal solution is found. Moreover, importance sampling is applied to reduce the number of realizations without losing too much accuracy.

Another recent paper dealing with chance constrained integer programming is Tayur et al. [39]. Here, the problem of scheduling n job types on m parallel machines is studied, when setup times are needed and the demands of the job types are random. This leads to a model of type (10), where all scheduling decisions x are integer variables. The objective is to minimize total costs (setup and production costs), and the chance constraint requires that the probability of production times and setup times not exceeding the capacities on all machines together is sufficiently large. The following solution technique is proposed. First, the model without the chance constraint is solved. Its solution x_0 does not satisfy the chance constraint. By exploiting the theory of Gröbner bases, a set of directions is generated that are used to trace paths from every nonoptimal solution (as x_0) to the optimal solution. The chance constraint is not modeled by additional integer variables; as a subroutine, only an "oracle" is needed, telling whether or not a candidate solution satisfies it or not. The oracle is based on a discrete distribution, for instance, a large enough sample of the demand distribution.

We conclude by mentioning a special way in which 0–1 variables may enter a multi-stage stochastic recourse model. Jonsbräten et al. [12] assume that the process of information gathering is part of the decision making. To that purpose, they include a binary decision vector $d \in \{0, 1\}^q$ in the first stage. The choice of d determines the probability distribution of the random parameters, in a multi-stage setting the scenario tree, actually, and the constraints of the original decision variables depend on d as well. For instance, the probability distribution of production costs can be specified more accurately if at the first stage the corresponding production is sufficiently large. In such models, one has to find the best of 2^q stochastic programming problems. Jonsbräten et al. [12] provide lower (upper) bounds for the optimal value by assuming that the information to be gained comes earlier (later) than a current partial d-vector specifies. Each lower and upper bound asks for the solution of a stochastic program. They are used in a branch and bound algorithm that is memory intensive, but has the advantage that promising branching decisions can be made, and that the space searched for good upper bounds is quickly reduced. Computational experience indicates that considerable savings are possible but, as expected, that for real-life sized problems, one has to be satisfied with approximations of the optimal solution. Problem-dependent heuristics may improve the behavior of the algorithm.

4. Concluding remarks

As can be seen from the references in this paper, stochastic integer programming has received increasingly more research attention during the last decade. We believe that the following two factors have contributed to this development. The mere fact that it appeared to be possible to obtain initial results for this problem class, which combines the difficulties of stochastic programming and integer programming, may have encouraged researchers to work in this field. However, we believe that the main motivation comes from the practical relevance of these models. Hence, it is important to continue research in this field.

In the introduction of this paper, we indicated our believe that progress can not only be expected from further study of structural properties or general purpose algorithms. As in deterministic integer programming, it is likely that special purpose algorithms will play a major role. Indeed, as suggested by their success in the deterministic setting, development of heuristics seems a promising direction for future research.

On the other hand, there remain several interesting theoretical questions. For example, is it possible to construct non-trivial convex approximations for (mixed-) integer models with non-simple recourse structure? If so, can such approximations always be represented as continuous recourse problems?

Finally, we note that first results have been obtained for multi-stage stochastic integer problems. In addition to some of the papers mentioned above, models of this type are presented and solved (by special purpose methods) in e.g. Dentcheva and

Römisch [8] and Takriti et al. [38]. Development of solution methods for multi-stage stochastic integer programming problems provides a major challenge for future research.

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