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A GEOMETRIC BUCHBERGER ALGORITHM FOR INTEGER PROGRAMMING

by

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Abstract

Let $\mathrm{IP}_{\{A,c\}}$ denote the family of integer programs of the form $Min\ cx : Ax = b,\ x \in \mathbf{N}^n$ obtained by varying the right hand side vector b but keeping A and c fixed. A test set for $\mathrm{IP}_{\{A,c\}}$ is a set of vectors in $\mathbf{Z}^n$ such that for each non-optimal solution α to a program in this family, there is at least one element g in this set such that $\alpha - g$ has an improved cost value as compared to α. We describe a unique minimal test set for this family called the reduced Gröbner basis of $\mathrm{IP}_{\{A,c\}}$. An algorithm for its construction is presented which we call a geometric Buchberger algorithm for integer programming and we show how an integer program may be solved using this test set. The reduced Gröbner basis is then compared with some other known test sets from the literature. We also indicate an easy procedure to construct test sets with respect to all cost functions for a matrix $A \in \mathbf{Z}^{(n-2)\times n}$ of full row rank.

KEY WORDS: INTEGER PROGRAMMING, TEST SETS, REDUCED GRÖBNER BASIS, BUCHBERGER ALGORITHM, FIBER, HILBERT BASES.

1 Introduction

In this paper we study integer programs of the form $Min\ cx : Ax = b,\ x \in \mathbf{N}^n$ where A is an $m \times n$ integer matrix of full row rank, b an integral m-vector and c an integral n-vector. Several techniques for solving these problems exist (see [16]) and in this paper we investigate one such solution method.

A natural approach to solving an integer program is to devise a method to search in the neighborhood of a given solution for another solution with an improved cost value. If such a solution is found, we update the current solution with the improved solution and repeat the search. In order to make this search well defined, we define the neighbors of a solution to be all the feasible solutions obtained by adding a finite set of vectors to the current solution. We call such a finite set of vectors a *test set* for the integer program if each non-optimal solution has at least one neighbor (obtained as above) with an improved cost value. If such a test set exists, then the program can be solved to optimality provided an initial solution is known. In this paper we establish the existence of a test set for an entire family of integer programs to which the given program belongs and present an algorithm for its construction. This gives an obvious algorithm for solving integer programs. In a companion paper [10] we compute these test sets and the exact optimal solutions for a family of chance constrained integer programs that arose in a manufacturing setting. In this case, we were able to exploit the structure of these integer programs and certain features of the algorithm used to compute test sets, to considerably speed up the computation.

An important concept in polynomial ideal theory is that of a *Gröbner basis* [2] for a given polynomial ideal with respect to some fixed term order. Gröbner bases are special generators of polynomial ideals which can be computed using *Buchberger's algorithm* [2]. In [4] Conti and Traverso describe an algebraic method to solve integer programs of the above form, using ideas from Gröbner basis theory and commutative algebra. Related work on the application of Gröbner bases to find non-negative integer solutions to systems of linear equations can be found in [11],[12] and [13]. A geometric interpretation of the Conti-Traverso algorithm provides a unique minimal test set for integer programs of the form $Min\ cx : Ax = b,\ x \in \mathbb{N}^n$ as b varies. Denote this family of programs by $\mathrm{IP}_{\{A,c\}}$. We call this test set the *reduced Gröbner basis* of $\mathrm{IP}_{\{A,c\}}$ due to its algebraic origins. A geometric version of the Buchberger algorithm follows from the above interpretation. Since Gröbner bases of polynomial ideals can be computed in practice, these test sets can be generated using a computer algebra package like *Macaulay* [1], making it possible to implement the solution method proposed.

This paper is organized as follows. In Section 2.1 we describe the reduced Gröbner basis of the family of integer programs $\mathrm{IP}_{\{A,c\}}$ and prove that it is the unique minimal test set for this family. We outline an algorithm for solving integer programs in $\mathrm{IP}_{\{A,c\}}$ using this test set. In Section 2.2 we develop the *geometric Buchberger algorithm* for the construction of the reduced Gröbner basis of a larger family of integer programs that contains the reduced Gröbner basis of $\mathrm{IP}_{\{A,c\}}$. Construction of this larger family takes care of initialization issues both in the geometric Buchberger algorithm and in solving a given integer program in $\mathrm{IP}_{\{A,c\}}$ using this test set. In Section 3, we compare the reduced Gröbner basis with some known test sets for integer programs. More specifically, in Section 3.1, the reduced Gröbner basis is compared with the *neighbors of the origin* introduced by Scarf [14], [15] to solve integer programs of the form $Min\ cx : Ax \leq b,\ x \in \mathbb{Z}^n$. In Section 3.2 we introduce a universal test set that is independent of cost and compare it with a similar test set studied by Graver [8]. This section also contains a result that can be used to construct a universal test set for matrices of the form $A \in \mathbb{Z}^{(n-2)\times n}$ of rank $n-2$. Finally in Section 3.3 we compare the above universal test set with another universal test set described in Section 17.3 [16] for the family of integer programs $Min\ cx : Ax \geq b,\ x \in \mathbf{Z}^n$ as b varies. This test set is originally due to Cook, Gerards, Schrijver and Tardos [6]. In each of the above cases, the "Gröbner test set" is contained in the known test set it is compared with.

2 A Test Set for an Integer Program

2.1 The Reduced Gröbner Basis of an Integer Program

Consider an integer program of the form $Min\ cx : Ax = b\ ,\ x \in \mathbf{N}^n$ where $A \in \mathbf{Z}^{m \times n}$ is a matrix of full row rank, $b \in \mathbf{Z}^m$ and $c \in \mathbf{Z}^n$. Let $\mathrm{IP}_{\{A,c\}}$ denote the family of such programs that arise by varying the right hand side vector b, but keeping A and c fixed. Let $\mathrm{IP}_{\{A,c\}}(b)$ denote the specific member of the family with right hand side vector b. We may assume that each $\mathrm{IP}_{\{A,c\}}(b)$ in the family is bounded with respect to the cost function c. Since all data are integral and hence rational, $\mathrm{IP}_{\{A,c\}}(b)$ is unbounded with respect to c if and only if its linear relaxation $Min\ cx : Ax = b\ ,\ x \in \mathbf{Q}^n$ is unbounded with respect to c. As this may be checked using linear programming techniques, we do not lose any generality by making this boundedness assumption.

Let $\pi : \mathbf{N}^n \rightarrow \mathbf{Z}^m$ denote the map $x \mapsto Ax$. Given a right hand side vector b in $\mathbf{Z}^m$, the set of feasible solutions to $\mathrm{IP}_{\{A,c\}}(b)$ constitute $\pi^{-1}(b)$, the pre-image of b under this map. In the rest of this paper, we identify the discrete set of points $\pi^{-1}(b)$ with its convex hull and call it the $b - fiber$ of $\mathrm{IP}_{\{A,c\}}$. Thus $\pi^{-1}(b)$ or the $b - fiber$ of $\mathrm{IP}_{\{A,c\}}$ is the polyhedron that is the convex hull of all solutions to $\mathrm{IP}_{\{A,c\}}(b)$.

Definition 2.1.1 *We call $>$ a term order on $\mathbf{N}^n$ if $>$ has the following properties:*
(1) $>$ is a total order on $\mathbf{N}^n$,
(2) $\alpha > \beta \Rightarrow \alpha + \gamma > \beta + \gamma$ for all $\gamma \in \mathbf{N}^n$ (i.e., $>$ is compatible with sums),
(3) $\alpha > 0$ for all $\alpha \in \mathbf{N}^n \backslash \{0\}$ (i.e., 0 is the minimum element).

Each point in $\mathbf{N}^n$ is a feasible solution in a unique fiber of $\mathrm{IP}_{\{A,c\}}$ - i.e., α in $\mathbf{N}^n$ is a feasible solution in the $A\alpha$-fiber of $\mathrm{IP}_{\{A,c\}}$. Suppose we now group points in $\mathbf{N}^n$ according to increasing cost value $c \cdot x$. Since more than one lattice point can have the same cost value, this ordering is not a total order on $\mathbf{N}^n$. We may however refine this to create a total order by adopting some term order $>$, to break ties among points with the same cost value. Let $>_c$ denote the composite total order on $\mathbf{N}^n$ that first compares two points by cost and breaks ties according to $>$. We may think of $>_c$ as a refinement of the cost function c. It can be checked that $>_c$ satisfies properties (1) and (2) of Definition 2.1.1. However $>_c$ may not satisfy (3) and therefore, $>_c$ is not always a term order on $\mathbf{N}^n$.

Example 2.1.2 *Consider* $\mathbf{N}^2$ *and* $c = (-1, 2)$.

$$\left.\begin{array}{l} c \cdot (0,0) = 0 \\ c \cdot (1,0) = -1 \end{array}\right\} \Rightarrow (0,0) >_c (1,0).$$

We replace the objective function of $\mathrm{IP}_{\{A,c\}}$ by its refinement $>_c$. This change does not affect the optimal value of a member $\mathrm{IP}_{\{A,c\}}(b)$, but since $>_c$ is a total order on $\mathbf{N}^n$, we now have a unique optimum in every fiber of $\mathrm{IP}_{\{A,c\}}$. We denote the family of integer programs under study by $\mathrm{IP}_{\{A,>_c\}}$ to mark this change. Let $\mathcal{S}_{>_c}$ denote the set of all points in $\mathbf{N}^n$ that are non-optimal with respect to $>_c$ in the various fibers of $\mathrm{IP}_{\{A,>_c\}}$. The structure of $\mathcal{S}_{>_c}$ is characterized in Lemma 2.1.4. We first state Lemma 2.1.3 which is a refinement of Lemma 1 in [9]. This is a geometric version of the algebraic Gordan-Dickson Lemma (Theorem 5, Section 2.4 in [7]) and is used to prove Lemma 2.1.4.

Lemma 2.1.3 *If* $P \subseteq \mathbf{N}^n$, $P \neq \phi$, *then there exists a minimal subset* $\{p_1, ..., p_m\} \subseteq P$ *that is finite and unique such that* $p \in P$ *implies* $p_j \leq p$ *(component-wise) for at least one* $j = 1, ..., m$.

Lemma 2.1.4 *There exists a unique, minimal, finite set of vectors* $\alpha(1), \ldots, \alpha(t) \in \mathbf{N}^n$ *such that the set of all non-optimal solutions in all fibers of* $IP_{\{A,>_c\}}$ *is a subset of* $\mathbf{N}^n$ *of the form*

$$\mathcal{S}_{>_c} = \bigcup_{i=1}^t (\alpha(i) + \mathbf{N}^n).$$

Proof: First we show that $\mathcal{S}_{>_c} \subseteq \mathbf{N}^n$ has the property that if $\alpha \in \mathcal{S}_{>_c}$, then $\alpha + \mathbf{N}^n \subset \mathcal{S}_{>_c}$. If $\alpha \in \mathcal{S}_{>_c}$, then α is a non-optimal point with respect to $>_c$ in the fiber $\pi^{-1}(A\alpha)$. Let β be the unique optimum in this fiber with respect to $>_c$. Consider $\alpha + \gamma$, $\beta + \gamma$ for $\gamma \in \mathbf{N}^n$. Then

1. $A(\alpha + \gamma) = A(\beta + \gamma)$ since $A\alpha = A\beta$.
2. $(\alpha + \gamma), (\beta + \gamma) \in \mathbf{N}^n$ since $\alpha, \beta, \gamma \in \mathbf{N}^n$.
3. $\alpha >_c \beta \Rightarrow \alpha + \gamma >_c \beta + \gamma$.

Properties (1) - (3) imply that $\alpha + \gamma$ is not the optimal lattice point in the fiber $\pi^{-1}(A(\alpha + \gamma))$. Therefore $\alpha + \gamma$ belongs to $\mathcal{S}_{>_c}$. This implies that $\alpha + \mathbf{N}^n \subset \mathcal{S}_{>_c}$ since γ was an arbitrary element of $\mathbf{N}^n$. By Lemma 2.1.3 we conclude that there exists a minimal set of elements $\alpha(1), ..., \alpha(t)$ in $\mathcal{S}_{>_c}$ that is finite and unique such that $\mathcal{S}_{>_c} = \bigcup_{i=1}^t (\alpha(i) + \mathbf{N}^n)$.□

We now define a test set for $\mathrm{IP}_{\{A,>_c\}}$ which gives an algorithm for finding the optimal solution of each member $\mathrm{IP}_{\{A,>_c\}}(b)$ of this family of integer programs.

Definition 2.1.5 *A set* $\mathcal{G} \subseteq \{x \in \mathbf{Z}^n : Ax = 0\}$ *is a test set for* $IP_{\{A,>_c\}}$ *if*
(I) for each non-optimal solution α *to each program in* $IP_{\{A,>_c\}}$, *there exists* g *in* $\mathcal{G}$ *such that* $\alpha - g$ *is a feasible solution to the same program with* $\alpha >_c \alpha - g$ *and*
(II) for the optimal solution β *to a program in* $IP_{\{A,>_c\}}$, $\beta - g$ *is infeasible for every* g *in* $\mathcal{G}$.

Given a test set for $\mathrm{IP}_{\{A,>_c\}}$, there is an obvious algorithm to find the unique optimal solution of each integer program in this family. Suppose we wish to solve $\mathrm{IP}_{\{A,>_c\}}(b)$ for which α is a feasible solution. If α is non-optimal then by Definition 2.1.5, there will exist an element g in $\mathcal{G}$ such that $\alpha - g$ is another feasible solution to $\mathrm{IP}_{\{A,>_c\}}(b)$ with the property that $\alpha >_c \alpha - g$. We then update the current solution α by the less expensive solution $\alpha - g$ and repeat. If α is the optimal solution to $\mathrm{IP}_{\{A,>_c\}}(b)$, then for all g in $\mathcal{G}$, $\alpha - g$ will be infeasible for $\mathrm{IP}_{\{A,>_c\}}(b)$ and we stop. Finiteness of the procedure is guaranteed by the assumption that each $\mathrm{IP}_{\{A,>_c\}}(b)$ is bounded with respect to c and that $>$ is a legitimate term order on $\mathbf{N}^n$. The algorithm outlined above assumes that $\mathrm{IP}_{\{A,>_c\}}(b)$ is feasible and that we know an initial feasible solution to this program. In Section 2.2 we describe how infeasibility can be detected. We get around the initialization issue by solving a larger integer program for which an initial solution is obvious, an idea similar to Phase I in the simplex method.

We now define a finite set $\mathcal{G}_{>_c} \subset \{x \in \mathbf{Z}^n : Ax = 0\}$ and prove that it is indeed a test set for $\mathrm{IP}_{\{A,>_c\}}$. Recall the set $\mathcal{S}_{>_c}$ and its unique minimal generators from Lemma 2.1.4.

Definition 2.1.6 *Let $\mathcal{G}_{>_c} = \{g_i = (\alpha(i) - \beta(i)),\ i = 1, ..., t\}$, where $\alpha(1), ..., \alpha(t)$ are the unique minimal elements of $\mathcal{S}_{>_c}$ and $\beta(i)$ is the unique optimum to $IP_{\{A,>_c\}}(A\alpha(i))$.*

In Section 2.2 we give an algorithm that constructs $\mathcal{G}_{>_c}$ without explicitly determining $\alpha(i)$ and $\beta(i)$ beforehand. We call the set $\mathcal{G}_{>_c}$, the *reduced Gröbner basis* of $\mathrm{IP}_{\{A,>_c\}}$ since $\mathcal{G}_{>_c}$ is the geometric equivalent of the reduced Gröbner basis with respect to $>_c$ of a certain polynomial ideal associated with A. In this paper we study $\mathcal{G}_{>_c}$ from a completely geometric point of view. The relevant algebraic connections can be found in [10]. Notice that $(\alpha(i) - \beta(i))$ is a lattice point in the subspace $S = \{x \in \mathbf{Q}^n : Ax = 0\}$. Geometrically we think of $(\alpha(i) - \beta(i))$ as the directed line segment $\vec{g_i} = \overrightarrow{[\alpha(i), \beta(i)]}$ in the $A\alpha(i)$-fiber of $\mathrm{IP}_{\{A,>_c\}}$. The vector is directed from the non-optimal point $\alpha(i)$ to the optimal point $\beta(i)$ due to the minimization criterion in $\mathrm{IP}_{\{A,>_c\}}$ which requires us to move away from expensive points. Note that this orientation of $[\alpha(i), \beta(i)]$ is opposite to the usual orientation of the vector $\alpha(i) - \beta(i)$. In this paper we always think of $\alpha(i) - \beta(i)$ as having the orientation prescribed above. We call $\alpha(i)$ the *tail* of $\vec{g_i}$ and $\beta(i)$ the *head*. We go back and forth between the two interpretations of g_i depending on the context. Subtracting the point $g_i = \alpha(i) - \beta(i)$ from the feasible solution γ gives the new solution $\gamma - \alpha(i) + \beta(i)$ which is equivalent to translating $\vec{g_i}$ by a non-negative integer vector such that its tail meets γ and then moving to the head of the translated vector. Notice that any pair of points $\alpha(i)$ and $\beta(i)$ in the set $\mathcal{G}_{>_c}$ have disjoint supports. Suppose this was not so. Let w be the non-negative, non-zero, integer vector such that $w_j = min\{\alpha(i)_j, \beta(i)_j\}$ for $j = 1, ..., n$. Then $\alpha(i) - w$ and $\beta(i) - w$ are feasible solutions in the $(A(\alpha(i) - w))$-fiber of $\mathrm{IP}_{\{A,>_c\}}$ and $\alpha(i) - w >_c \beta(i) - w$ since $\alpha(i) >_c \beta(i)$. Thus $\alpha(i) - w$

is non-optimal in its fiber and hence belongs to $\mathcal{S}_{>_c}$. Further $\alpha(i) - w \leq \alpha$ which contradicts that $\alpha(i)$ was a minimal generator of $\mathcal{S}_{>_c}$.

Observation 2.1.7 *Adding a vector $v \in \mathbf{N}^n$ to both end points of $\vec{g_i}$ translates $\vec{g_i}$ from the $A\alpha(i)$-fiber of $IP_{\{A,>_c\}}$ to the $A(\alpha(i)+v)$-fiber of $IP_{\{A,>_c\}}$.*

Consider an arbitrary fiber of $\mathrm{IP}_{\{A,>_c\}}$ and a feasible lattice point δ in this fiber. For each vector $\vec{g_i}$ in $\mathcal{G}_{>_c}$, check whether it can be translated by some v in $\mathbf{N}^n$ such that the tail of the translated vector is incident at δ. At δ draw all such possible translations of vectors from $\mathcal{G}_{>_c}$. The head of a translated vector is also incident at a feasible lattice point in the same fiber as δ. We do this construction for all feasible lattice points in all fibers of $\mathrm{IP}_{\{A,>_c\}}$. From Lemma 2.1.4 and the definition of $\mathcal{G}_{>_c}$, it follows that no vector in $\mathcal{G}_{>_c}$ can be translated by a v in $\mathbf{N}^n$ such that its tail meets the optimum on a fiber.

Theorem 2.1.8 *The above construction builds a connected, directed graph in every fiber of $IP_{\{A,>_c\}}$. The nodes of the graph are all the lattice points in the fiber and the edges are the translations of elements in $\mathcal{G}_{>_c}$ by non-negative integral vectors. The graph in a fiber has a unique sink at the unique optimum in the fiber.*

Proof: Pick a fiber of $\mathrm{IP}_{\{A,>_c\}}$ and at each feasible lattice point construct all possible translations of the vectors $\vec{g_i}$ from the set $\mathcal{G}_{>_c}$ as described earlier. This results in a possibly disconnected directed graph in this fiber where nodes are the feasible lattice points and edges the translations of elements in $\mathcal{G}_{>_c}$. Let α be a non-optimal lattice point in this fiber. By Lemma 2.1.4, $\alpha = \alpha(i) + v$ for some $i \in \{1, ..., t\}$ and $v \in \mathbf{N}^n$. Now $\alpha' = \beta(i) + v$ also lies in this fiber and $\alpha >_c \alpha'$ since $\alpha(i) >_c \beta(i)$. Therefore $\vec{g_i}$ translated by $v \in \mathbf{N}^n$ is an edge in this graph and we can move along it from α to the less expensive point α' in the same fiber. This proves that from every non-optimal lattice point in the fiber we can reach an improved point (with respect to $>_c$) in the same fiber by moving along an edge of the graph. By comments made earlier the outdegree of the optimum is zero. Therefore a directed path from a non-optimal point terminates precisely at the unique optimum of the fiber. This in turn implies connectedness of the graph.□

We call the graph in the b-fiber of $\mathrm{IP}_{\{A,>_c\}}$ built from elements in $\mathcal{G}_{>_c}$, the $>_c$ *–skeleton* of that fiber.

Corollary 2.1.9 *In the $>_c$-skeleton of a fiber, there exists a directed path from every non-optimal point α to the unique optimum β. The objective function value (with respect to $>_c$) of successive points in the path decreases monotonically from α to β.*

Corollary 2.1.10 *The reduced Gröbner basis $\mathcal{G}_{>_c}$ is the unique minimal test set for $IP_{\{A,>_c\}}$. It depends only on the matrix A and the refined objective function $>_c$.*

Proof: From Theorem 2.1.8 it follows that the reduced Gröbner basis is a test set for $\mathrm{IP}_{\{A,>_c\}}$. Removing an element $\vec{g_j}$ from $\mathcal{G}_{>_c}$ will result in $\mathcal{G}_{>_c}\backslash\{\vec{g_j}\}$ not being a test set since no segment in $\mathcal{G}_{>_c}\backslash\{\vec{g_j}\}$ can be translated through a non-negative vector in $\mathbf{N}^n$ such that its tail meets $\alpha(j)$. This is because $\alpha(j)$ was an essential generator of $\mathcal{S}_{>_c}$. Therefore $\mathcal{G}_{>_c}$ is a minimal test set for $\mathrm{IP}_{\{A,>_c\}}$. Moreover the generators $\{\alpha(1), ..., \alpha(t)\}$ of $\mathcal{S}_{>_c}$ and the corresponding optimal points $\beta(1), ..., \beta(t)$ were shown to be unique. Therefore $\mathcal{G}_{>_c}$ is the unique minimal test set of $\mathrm{IP}_{\{A,>_c\}}$. Definition 2.1.6 implies that $\mathcal{G}_{>_c}$ depends only on A and $>_c$.$\square$

By Observation 2.1.7, it is clear that no element in $\mathcal{G}_{>_c}$ can be translated by some $v \in \mathbf{N}^n$ such that it connects two lattice points in two distinct fibers of $\mathrm{IP}_{\{A,>_c\}}$. Therefore the $>_c$-skeletons of two distinct fibers of $\mathrm{IP}_{\{A,>_c\}}$ are not connected.

Example 2.1.11 $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ $c = (1,3,14,17)$.
Let $x = (x_1, x_2, x_3, x_4)$ be the vector of variables. The elements in the reduced Gröbner basis $\mathcal{G}_{>_c}$ are $\vec{g_1} = [\overrightarrow{(0,0,2,0),(0,1,0,1)}]$, $\vec{g_2} = [\overrightarrow{(1,0,0,1),(0,1,1,0)}]$ and $\vec{g_3} = [\overrightarrow{(1,0,1,0),(0,2,0,0)}]$. We used Macaulay to calculate $\mathcal{G}_{>_c}$. The tie-breaking term order $>$ is the reverse lexicographic order on the variables x_1, x_2, x_3, x_4.
The $>_c$-skeleton of the fiber corresponding to the right hand side vector $(10,15)^t$ is shown in Figure 1 by projecting it on the x_1, x_2-plane. The unique optimum in this fiber is $(0,7,1,2)$. The shaded points are all the feasible lattice points in this fiber. Recall that since the b-fiber of $IP_{\{A,>_c\}}$ is taken to be the convex hull of solutions to $IP_{\{A,c\}}(b)$, it is a polyhedron. In this case it is the convex hull of all shaded points shown. The vertices of this polyhedron are labeled in Figure 1. Using translations of the elements in $\mathcal{G}_{>_c}$ we draw the $>_c$-skeleton of this fiber. It can be seen that this is a connected directed graph with a unique sink at the optimal point. Also there exists a monotone path from every lattice point in this fiber to the unique optimum. Such a graph exists in each fiber of the family of integer programs with the above coefficient matrix and refined cost function.

We conclude this section with a few remarks about the algorithm that solves programs in $\mathrm{IP}_{\{A,>_c\}}$ using the reduced Gröbner basis $\mathcal{G}_{>_c}$.

Remark 2.1.12 *Since $>_c$ is a total order on $\mathbf{N}^n$, a directed path from the non-optimal point α to the unique optimum β in a fiber, cannot pass through any lattice point in this fiber more than once. This implies that the algorithm for solving a given $IP_{\{A,>_c\}}(b)$ using $\mathcal{G}_{>_c}$ cannot cycle.*

Remark 2.1.13 *At a given non-optimal point α, there may be several edges in the $>_c$-skeleton with tail incident at α. Moving along any of these edges will improve the current solution. We have so far not specified any rule by which a possible edge may be chosen. Such a rule would be*

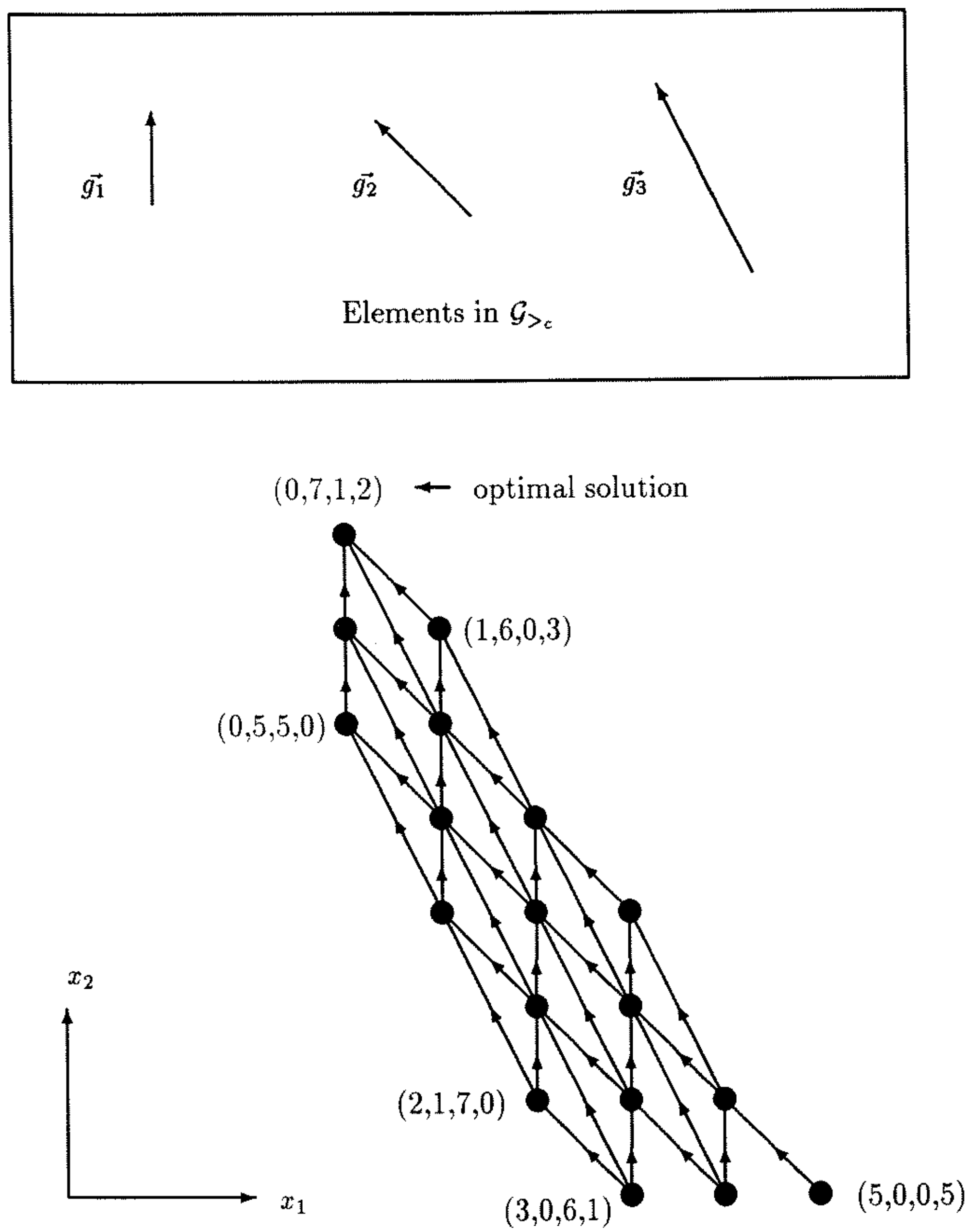

Figure 1: The $>_c$-skeleton of the $(10,15)^t$-fiber of $\mathrm{IP}_{\{A,>_c\}}$ projected on the x_1, x_2 plane.

analogous to a "pivot rule" in the simplex method for linear programming. Two possibilities might be to move along the edge which is the translation of that $\vec{g}_i$ with (1) the most expensive tail or (2) the maximum difference between the cost of its tail and head. The former would be similar to the "largest - coefficient " rule in the simplex method and the latter to the "largest - decrease" rule.

2.2 A Geometric Buchberger Algorithm for Computing $\mathcal{G}_{>_c}$

In the previous section we established the existence of the set $\mathcal{G}_{>_c}$ which was the unique minimal test set for the family of integer programs $\mathrm{IP}_{\{A,>_c\}}$. As mentioned earlier, $\mathcal{G}_{>_c}$ is the geometric equivalent of the reduced Gröbner basis with respect to $>_c$, of a polynomial ideal associated with A. The reduced Gröbner basis with respect to a term order of any polynomial ideal can be computed using Buchberger's algorithm [2]. In the context of integer programming, Buchberger's algorithm has a special geometric interpretation which we describe in this section. No familiarity with the theory of Gröbner bases is assumed.

Recall the generic integer program $\mathrm{IP}_{\{A,c\}}(b)$ which was $Min\ cx : Ax = b, x \in \mathbf{N}^n$, where $A \in \mathbf{Z}^{m \times n}$ is a matrix of full row rank, $b \in \mathbf{Z}^m$ and $c \in \mathbf{Z}^n$. We may assume without loss of generality that all entries in A, c and b are non-negative integers. Consider the following extended integer program $\mathrm{EIP}_{\{A,(M,c)\}}(b)$, associated with $\mathrm{IP}_{\{A,c\}}(b)$:

$$\begin{gathered}\text{Min } My + cx \\ \text{s.t. } Iy + Ax = b \\ (y, x) \in \mathbf{N}^{m+n}\end{gathered}$$

where I is the $m \times m$ identity matrix and $M \in \mathbf{N}^m$ is a vector whose components have large magnitude. Let $\mathrm{EIP}_{\{A,(M,c)\}}$ denote the family of extended integer programs corresponding to the family $\mathrm{IP}_{\{A,c\}}$. The extended program $\mathrm{EIP}_{\{A,(M,c)\}}(b)$ has the obvious solution, $y = b, x = 0$ because of the non-negativity assumption above and hence is always feasible. As before, we call the convex hull of feasible solutions to $\mathrm{EIP}_{\{A,(M,c)\}}(b)$ the b-fiber of $\mathrm{EIP}_{\{A,(M,c)\}}$. The b-fiber of $\mathrm{IP}_{\{A,c\}}$ is the face of the b-fiber of $\mathrm{EIP}_{\{A,(M,c)\}}$ in the $y = 0$ subspace of $\mathbf{R}^{m+n}$. As for $\mathrm{IP}_{\{A,c\}}$, we replace the objective function of $\mathrm{EIP}_{\{A,(M,c)\}}$ by its refinement $>_{(M,c)}$, which first sorts points according to the cost function (M, c) and breaks ties according to a term order $>$ on $\mathbf{N}^{m+n}$. We choose $>$ such that it is an extension of the old term order on $\mathbf{N}^n$. We now consider the family of integer programs $\mathrm{IP}_{\{A,>_c\}}$ and the corresponding family of extended programs $\mathrm{EIP}_{\{A,>_{(M,c)}\}}$.

Since M is assumed to be large, $\mathrm{IP}_{\{A,>_c\}}(b)$ is feasible and has optimal solution x^* if and only if $y = 0, x = x^*$ is optimal for $\mathrm{EIP}_{\{A,>_{(M,c)}\}}(b)$. This implies that $\mathrm{EIP}_{\{A,>_{(M,c)}\}}(b)$ is bounded with respect to its cost function (M, c) since $\mathrm{IP}_{\{A,>_c\}}(b)$ was bounded with respect to c. If $\mathrm{IP}_{\{A,>_c\}}(b)$ is infeasible, then $\mathrm{EIP}_{\{A,>_{(M,c)}\}}(b)$ has an optimal solution for which $y \neq 0$. Therefore by solving

$\text{EIP}_{\{A,>_{(M,c)}\}}(b)$ to optimality, we can detect whether $\text{IP}_{\{A,>_c\}}(b)$ is feasible and if so, its optimal solution can be read off from the optimal solution of $\text{EIP}_{\{A,>_{(M,c)}\}}(b)$.

From the previous section, it follows that $\text{EIP}_{\{A,>_{(M,c)}\}}$ has a unique reduced Gröbner basis $\mathcal{G}_{>_{(M,c)}}$ with respect to its refined objective function $>_{(M,c)}$. Since an initial solution to each $\text{EIP}_{\{A,>_{(M,c)}\}}(b)$ is readily available, $\text{EIP}_{\{A,>_{(M,c)}\}}(b)$ can be solved to optimality using the algorithm described in the previous section, provided $\mathcal{G}_{>_{(M,c)}}$ is known. As in Lemma 2.1.4, the non-optimal points with respect to $>_{(M,c)}$, in the various fibers of $\text{EIP}_{\{A,>_{(M,c)}\}}$ is a set $\mathcal{S}_{>_{(M,c)}}$ of the form $\mathcal{S}_{>_{(M,c)}} = \bigcup_{i=1}^{T}(\alpha(i)' + \mathbf{N}^{m+n})$ where $\alpha(1)', ..., \alpha(T)'$ are unique vectors in $\mathbf{N}^{m+n}$. A solution α in the $A\alpha$-fiber of $\text{IP}_{\{A,>_c\}}$ may be identified with the solution $y = 0, x = \alpha$ in the $[I, A](0, \alpha)^t$-fiber of $\text{EIP}_{\{A,>_{(M,c)}\}}$. Under this identification, $\mathcal{S}_{>_c}$ is the intersection of $\mathcal{S}_{>_{(M,c)}}$ with the $y = 0$ subspace of $\mathbf{R}^{m+n}$ since an element α is in $\mathcal{S}_{>_c}$ if and only if $y = 0, x = \alpha$ is non-optimal in the $[I, A](0, \alpha)^t$-fiber of $\text{EIP}_{\{A,>_{(M,c)}\}}$. Therefore the reduced Gröbner basis $\mathcal{G}_{>_c}$ of $\text{IP}_{\{A,>_c\}}$ consists of those elements in the reduced Gröbner basis $\mathcal{G}_{>_{(M,c)}}$ of $\text{EIP}_{\{A,>_{(M,c)}\}}$ which lie in the $y = 0$ subspace of $\mathbf{R}^{m+n}$. We describe an algorithm to compute $\mathcal{G}_{>_{(M,c)}}$ which contains $\mathcal{G}_{>_c}$ by the above arguments. Using this test set we can solve $\text{EIP}_{\{A,>_{(M,c)}\}}(b)$ and hence $\text{IP}_{\{A,>_c\}}(b)$ to optimality.

Consider the line segments $d_j = [((a^j)^t, \mathbf{0}), (\mathbf{0}, e_j)]$ in $\mathbf{R}^{m+n}$ for $j = 1, ..., n$, where a^j is the jth column of A and e_j (taken to be a row vector) is the jth unit vector in $\mathbf{R}^n$. The $\mathbf{0}$ in $((a^j)^t, \mathbf{0})$ is a row vector of size n with all components 0 while the $\mathbf{0}$ in $(\mathbf{0}, e_j)$ is a row vector of size m with all components 0. The segment d_j lies in the a^j-fiber of $\text{EIP}_{\{A,>_{(M,c)}\}}$ and connects two feasible integer points in this fiber. We call d_j the *jth fundamental segment* of $\text{EIP}_{\{A,>_{(M,c)}\}}$. As for $\text{IP}_{\{A,>_c\}}$ and $\mathbf{N}^n$, each point in $\mathbf{N}^{m+n}$ is a feasible solution to a unique $\text{EIP}_{\{A,>_{(M,c)}\}}(b)$. Given two points α, β in $\mathbf{N}^{m+n}$, we join them by an edge if the segment $[\alpha, \beta]$ is the translation of a fundamental segment by some v in $\mathbf{N}^{m+n}$. This construction allows one to think of each fiber of $\text{EIP}_{\{A,>_{(M,c)}\}}$ as an undirected forest in which the nodes are all the lattice points in the fiber and the edges are all possible translations of fundamental segments connecting two nodes. We call the graph in the b-fiber the skeleton of the fiber. As before, the skeletons of two distinct fibers are not connected. However, the skeleton in any given fiber is a connected graph.

Lemma 2.2.1 *Given any two feasible integer points α, β in a fiber of $EIP_{\{A,>_{(M,c)}\}}$, there exists an undirected path joining them in the skeleton of this fiber.*

Proof: Let $\alpha = (v_1, ..., v_m, w_1, ..., w_n)$ and $\beta = (v_1', ..., v_m', w_1', ..., w_n')$ be two integer points in the b-fiber of $\text{EIP}_{\{A,>_{(M,c)}\}}$. Then $[I, A]\alpha^t = [I, A]\beta^t = b$ which implies

$$v_i + a_{i1}w_1 + \cdots + a_{in}w_n = v_i' + a_{i1}w_1' + \cdots + a_{in}w_n' \quad \text{for } i = 1, ..., m \qquad (*)$$

We prove the lemma by explicitly constructing a path from α to β in the skeleton of $\text{EIP}_{\{A,>_{(M,c)}\}}(b)$. Denote the translation of a segment $[a, b]$ by a vector v as $v * [a, b]$. The resulting segment is

$[a+v, b+v]$. Beginning at α, we first move along edges of the skeleton which are translations of d_1, to points for which the component w_1 decreases successively by one, until we arrive at a solution to $\mathrm{EIP}_{\{A,>_{(M,c)}\}}(b)$ for which $w_1 = 0$. More explicitly, the edges in this path are

$$\begin{aligned}
\text{1st edge} &:= [(v_1, ..., v_m, w_1, ..., w_n), (v_1 + a_{11}, ..., v_m + a_{m1}, w_1 - 1, w_2, ..., w_n)] \\
&= (v_1, ..., v_m, w_1 - 1, w_2, ..., w_n) * [(\mathbf{0}, e_1), ((a^1)^t, \mathbf{0})] \\
\text{2nd edge} &:= [(v_1 + a_{11}, ..., v_m + a_{m1}, w_1 - 1, w_2, ..., w_n), \\
&\qquad (v_1 + 2a_{11}, ..., v_m + 2a_{m1}, w_1 - 2, w_2, ..., w_n)] \\
&= (v_1 + a_{11}, ..., v_m + a_{m1}, w_1 - 2, w_2, ..., w_n) * [(\mathbf{0}, e_1), ((a^1)^t, \mathbf{0})] \\
&\vdots \\
\text{last edge} &:= [(v_1 + (w_1 - 1)a_{11}, ..., v_m + (w_1 - 1)a_{m1}, 1, w_2, ..., w_n), \\
&\qquad (v_1 + w_1 a_{11}, ..., v_m + w_1 a_{m1}, 0, w_2, ..., w_n)] \\
&= (v_1 + (w_1 - 1)a_{11}, ..., v_m + (w_1 - 1)a_{m1}, 0, w_2, ..., w_n) * [(\mathbf{0}, e_1), ((a^1)^t, \mathbf{0})]
\end{aligned}$$

Thus we have moved from the point α to the point $(v_1 + w_1 a_{11}, ..., v_m + w_1 a_{m1}, 0, w_2, ..., w_n)$ for which $w_1 = 0$. Clearly all intermediate points reached are solutions to $\mathrm{EIP}_{\{A,>_{(M,c)}\}}(b)$. We continue this procedure for $w_2, ..., w_n$, in this order, driving w_j to zero by moving along edges of the skeleton that are translations of the jth fundamental segment d_j. The last point reached by this process is $((v_1 + \sum_{j=}^{n} w_j a_{1j}), ..., (v_m + \sum_{j=1}^{n} w_j a_{mj}), 0, 0, ..., 0)$ for which the last n components are all zero. In a similar manner, we can construct a path from β to the point $((v_1' + \sum_{j=1}^{n} w_j' a_{1j}), ..., (v_m' + \sum_{j=1}^{n} w_j' a_{mj}), 0, 0, ..., 0)$. However, $(*)$ implies that the two points reached are in fact the same and we have a path from α to β in the skeleton of the b-fiber of $\mathrm{EIP}_{\{A,>_{(M,c)}\}}$.$\square$

In particular, there exists an undirected path (in the skeleton of the fiber) from the initial solution $(y = b, x = 0)$ of $\mathrm{EIP}_{\{A,>_{(M,c)}\}}(b)$ to the unique optimal solution with respect to $>_{(M,c)}$. However we lack a systematic procedure to find this path. Suppose we now direct the edges in the skeletons of $\mathrm{EIP}_{\{A,>_{(M,c)}\}}$ according to the objective function $>_{(M,c)}$ so that the more expensive end of an edge with respect to $>_{(M,c)}$ is the tail and the less expensive end, the head. By Definition 2.1.5, if the set of directed fundamental segments were to be a test set for $\mathrm{EIP}_{\{A,>_{(M,c)}\}}$, then, for each non-optimal point in each fiber, there must be some edge in the corresponding directed skeleton with tail incident at this point. This is however not true for the directed skeletons constructed above.

Example 2.2.2 *Consider the integer program*

$$Min\ 100y + x_1 + 5x_2 : y + 2x_1 + 3x_2 = b,\ y, x_1, x_2 \geq 0, integer$$

which is the extended integer program of $Min\ x_1 + 5x_2 : 2x_1 + 3x_2 = b,\ x_1, x_2 \in \mathbb{N}$. *The fundamental segments for* $EIP_{\{A,>_{(M,c)}\}}$ *are* $d_1 = [(2,0,0),(0,1,0)]$ *and* $d_2 = [(3,0,0),(0,0,1)]$. *The cost function* *directs these segments : d_1 has tail $(2,0,0)$ and head $(0,1,0)$; d_2 has tail $(3,0,0)$ and head $(0,0,1)$. Consider the directed skeleton in the fiber corresponding to $b=6$, shown in Figure 2. The optimal solution of $EIP_{\{A,>_{(M,c)}\}}(6)$ is $(0,3,0)$ i.e., $y=0$, $x_1=3$ and $x_2=0$. The shaded points are all the solutions to $EIP_{\{A,>_{(M,c)}\}}(6)$. The point $(1,1,1)$ is a lattice point in this fiber and hence a node in the directed skeleton of the fiber made up of translations of $\vec{d_1}$ and $\vec{d_2}$. This point is non-optimal with respect to the cost function. However, its outdegree is zero, making it impossible to move from this point to an improved solution along the edges of the directed skeleton.*

In fact, the set of fundamental segments directed according to $>_{(M,c)}$ almost always fail to be a test set for $\text{EIP}_{\{A,>_{(M,c)}\}}$. The above example suggests that more edges may be needed in the directed skeleton of a fiber to create monotone paths from non-optimal points in the fiber to the unique optimum. The geometric Buchberger algorithm constructs the edges necessary to achieve this property. We first describe two sub-routines used in the algorithm. All vectors used as input in the procedures described below have two integer points in the same fiber of $\text{EIP}_{\{A,>_{(M,c)}\}}$ as end points. Vectors are directed according to $>_{(M,c)}$.

Construction of the S-vector $\overrightarrow{S(f,g)}$ of two vectors $\vec{f}=\overrightarrow{[a,b]}$ and $\vec{g}=\overrightarrow{[c,d]}$

Step 1: Translate $\vec{f}$ and $\vec{g}$ through the smallest possible non-negative integral vectors to a fiber of $\text{EIP}_{\{A,>_{(M,c)}\}}$ on which their tails meet.

Step 2: Take as $\overrightarrow{S(f,g)}$ the line segment which is the difference of their heads, oriented from the more expensive end to the less expensive end.

Reducing a vector $\vec{f}=\overrightarrow{[a,b]}$ by a set of vectors $\mathcal{T}=\{\vec{f_i}=\overrightarrow{[a_i,b_i]},\ \ i=1,\ldots,t\}$.

INITIALIZE		$\vec{r}=\vec{f}$ where $\vec{r}$ is called the *gap*.
		flag = UP.
REPEAT		
	IF	there exists $\vec{f_i}$ in $\mathcal{T}$ such that $\vec{f_i}$ translated by some v in $\mathbf{N}^{m+n}$ has tail incident at <u>tail</u> of $\vec{r}$ THEN update $\vec{r}$ to be the line segment joining the head of the old $\vec{r}$ and the head of the translated $\vec{f_i}$ directed from the expensive to cheap end.
	ELSE IF	there exists $\vec{f_i}$ in $\mathcal{T}$ such that $\vec{f_i}$ translated by some v in $\mathbf{N}^{m+n}$ has tail incident at <u>head</u> of $\vec{r}$ THEN update $\vec{r}$ to be the line segment joining the tail of the old $\vec{r}$ and the head of the translated $\vec{f_i}$ directed from expensive to cheap end.
	ELSE	flag = DOWN. Output $\vec{r}$ which is the current gap.
UNTIL	flag = DOWN.	

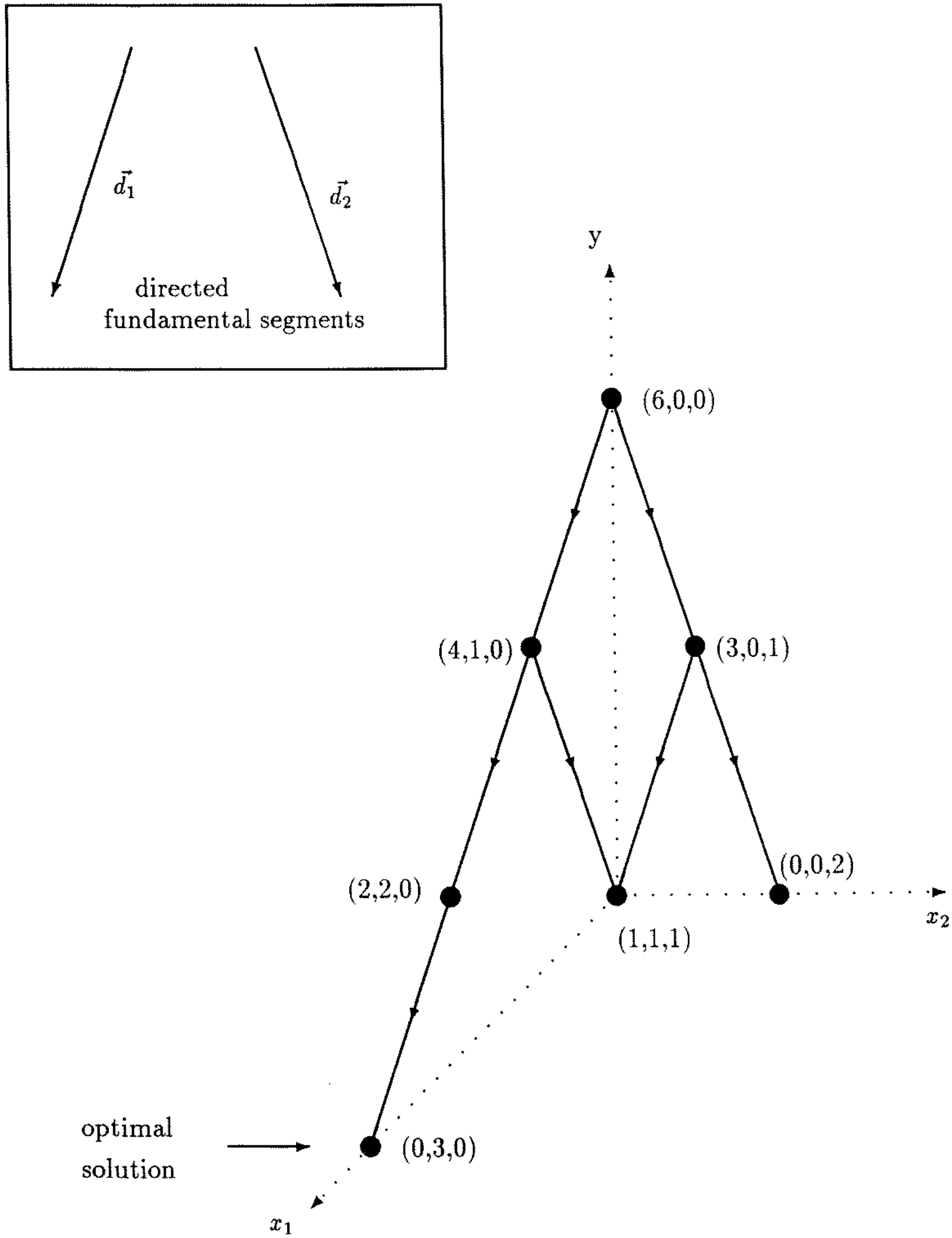

Figure 2: The directed skeleton in the 6-fiber of $\mathrm{EIP}_{\{A,>_{(M,c)}\}}$.

Given $\vec{f}=\overrightarrow{[a,b]}$, the above reduction algorithm constructs a path from tail to head of $\vec{f}=\overrightarrow{[a,b]}$, made up of translations of the vectors $\{\vec{f_i},\ \ i=1,...,t\}$ in $\mathcal{T}$ and the gap vector $\vec{r}$. The tail and head of $\vec{f}$ are moved to successively less expensive points, along appropriate translations of vectors in $\mathcal{T}$. By Observation 2.1.7, the path lies entirely within the fiber in which $\vec{f}$ lies. The gap at any stage is the line segment joining the two points to which the tail and head of $\vec{f}$ have been advanced thus far, oriented from the more expensive to the less expensive end.

We now describe the geometric Buchberger algorithm which constructs the reduced Gröbner basis $\mathcal{G}_{>_{(M,c)}}$ of $\mathrm{EIP}_{\{A,>_{(M,c)}\}}$. We call a test set (not necessarily unique, minimal) for $\mathrm{EIP}_{\{A,>_{(M,c)}\}}$ a *Gröbner basis* of $\mathrm{EIP}_{\{A,>_{(M,c)}\}}$.

A Geometric Buchberger Algorithm (for Integer Programming)

(On the lines of presentation in [7], for a Gröbner basis.)

Step 1 : *Finding a Gröbner basis for $EIP_{\{A,>_{(M,c)}\}}$.*
Input: $F=\{\vec{d_1},...,\vec{d_n}\}$, the fundamental segments of $\mathrm{EIP}_{\{A,>_{(M,c)}\}}$ directed according to $>_{(M,c)}$.
Output: A set of vectors $\mathcal{H}_{>_{(M,c)}}=\{\vec{h_1},...,\vec{h_t}\}$ that is a Gröbner basis for $\mathrm{EIP}_{\{A,>_{(M,c)}\}}$.
$\mathcal{H}_{>_{(M,c)}}=F$
REPEAT
  $F_1=\mathcal{H}_{>_{(M,c)}}$
  FOR each pair $\vec{p}\neq\vec{q}$ in F_1 DO

    $\overrightarrow{S(p,q)}=S$-vector of $\vec{p}$ and $\vec{q}$.

    $\vec{r}=$ gap obtained by reducing $\overrightarrow{S(p,q)}$ by members of F_1.
    If $\vec{r}\neq 0$, THEN $\mathcal{H}_{>_{(M,c)}}=\mathcal{H}_{>_{(M,c)}}\cup\{\vec{r}\}$
UNTIL $\mathcal{H}_{>_{(M,c)}}=F_1$.

Step 2: *Finding a minimal Gröbner basis of $EIP_{\{A,>_{(M,c)}\}}$.*
REPEAT for $\vec{h_i}\in\mathcal{H}_{>_{(M,c)}}$
  IF there exists some $\vec{h_j}$ in $\mathcal{H}_{>_{(M,c)}}$, $j\neq i$ such that the translation of $\vec{h_j}$ by some $v\in\mathbf{N}^{m+n}$ has its tail incident at the tail of $\vec{h_i}$ THEN remove $\vec{h_i}$ from $\mathcal{H}_{>_{(M,c)}}$.
UNTIL all vectors in $\mathcal{H}_{>_{(M,c)}}$ have been checked.

Step 3: *Finding the reduced Gröbner basis of $EIP_{\{A,>_{(M,c)}\}}$.*
Reduce each $\vec{h_i}\in\mathcal{H}_{>_{(M,c)}}$ by elements in $\mathcal{H}_{>_{(M,c)}}\backslash\vec{h_i}$. Replace $\vec{h_i}\in\mathcal{H}_{>_{(M,c)}}$ with the gap obtained.□

The set output at the end of Step 1 of the above algorithm contains all the fundamental segments of $\mathrm{EIP}_{\{A,>_{(M,c)}\}}$ although some or all of them may be removed in Steps 2 and 3. Consider

$\mathcal{H}_{>_{(M,c)}} = \{\vec{h_i}, i = 1, ..., t\}$, the final output of the geometric Buchberger algorithm when $>_{(M,c)}$ is used as the term order. We show below that $\mathcal{H}_{>_{(M,c)}}$ is indeed the reduced Gröbner basis $\mathcal{G}_{>_{(M,c)}}$ of $\mathrm{EIP}_{\{A,>_{(M,c)}\}}$. Recall the set $\mathcal{S}_{>_{(M,c)}} = \bigcup_{i=1}^{T}(\alpha(i)' + \mathbf{N}^{m+n})$ of all non-optimal solutions with respect to $>_{(M,c)}$ in the various fibers of $\mathrm{EIP}_{\{A,>_{(M,c)}\}}$. By definition, $\mathcal{G}_{>_{(M,c)}} = \{[\overrightarrow{\alpha(i)', \beta(i)'}], i = 1, ..., T\}$ where $\alpha(i)'$ are the unique minimal generators of $\mathcal{S}_{>_{(M,c)}}$ and $\beta(i)'$ is the unique optimum in the $[I, A]\alpha(i)'$-fiber of $\mathrm{EIP}_{\{A,>_{(M,c)}\}}$. Consider the S-vector, $\overrightarrow{S(h_i, h_j)}$ of the two elements $\vec{h_i}$ and $\vec{h_j}$ in $\mathcal{H}_{>_{(M,c)}}$. Step 1 of the geometric Buchberger algorithm insures that $\overrightarrow{S(h_i, h_j)}$ reduces to zero when reduced by elements in $\mathcal{H}_{>_{(M,c)}}$ which implies that there exists a path from tail to head of $\overrightarrow{S(h_i, h_j)}$ made up of translations of elements in $\mathcal{H}_{>_{(M,c)}}$, lying in the same fiber as this S-vector. Call this the S-path of $\vec{h_i}$ and $\vec{h_j}$. The reduction algorithm guarantees that the S-path of $\vec{h_i}$ and $\vec{h_j}$ does not pass through any integer point that is strictly more expensive than the tail of $\overrightarrow{S(h_i, h_j)}$.

We update the skeletons in the fibers of $\mathrm{EIP}_{\{A,>_{(M,c)}\}}$ which were directed according to $>_{(M,c)}$, by drawing in all possible translations of elements in $\mathcal{H}_{>_{(M,c)}}$ and removing those edges which are not such translations. Call the updated skeleton in a fiber the $\mathcal{H}_{>_{(M,c)}}$-skeleton of the fiber. Let the underlying undirected graph be called the $\mathcal{H}$-skeleton. Since some of the fundamental segments may be removed in Steps 2 and 3 of the geometric Buchberger algorithm, it is no longer clear whether the $\mathcal{H}$-skeleton of a fiber is a connected graph. However, if a directed fundamental segment was removed, it was because there exists a path joining its head and tail made up of translations of elements in $\mathcal{H}_{>_{(M,c)}}$, lying in the same fiber as this fundamental segment. This implies that the $\mathcal{H}$-skeleton is a connected graph.

Lemma 2.2.3 *Consider the $\mathcal{H}_{>_{(M,c)}}$-skeleton of a fiber of $EIP_{\{A,>_{(M,c)}\}}$. Each non-optimal integer point α' in this fiber has outdegree at least one and the unique optimum has outdegree zero.*

Proof: Let β' be the unique optimum with respect to $>_{(M,c)}$ in the fiber of $\mathrm{EIP}_{\{A,>_{(M,c)}\}}$ in which α' lies. Since the $\mathcal{H}$-skeleton is connected, there exists a path $P(\alpha', \beta')$ (not necessarily directed), joining α' and β' in the $\mathcal{H}_{>_{(M,c)}}$-skeleton of this fiber. If the edge of this path incident at α' has its tail at α', we have the required property. Suppose this is not so. Let p, the most expensive point in this path, be called the peak of $P(\alpha', \beta')$. By the above assumption, $p \neq \alpha'$. Let the two edges of the path whose tails are incident at p be translations of $\vec{h_i}$ and $\vec{h_j}$. The line segment joining the two neighbors of p in the path $P(\alpha', \beta')$ is a translation of the S-vector of $\vec{h_i}$ and $\vec{h_j}$. Therefore we may replace the two edges incident at p by the translation of the S-path of $\vec{h_i}$ and $\vec{h_j}$ to get a new path $P(\alpha', \beta')$ (again not necessarily directed), in the $\mathcal{H}_{>_{(M,c)}}$-skeleton of this fiber. The peak of this new path has strictly smaller cost value with respect to $>_{(M,c)}$ than the former peak. Since

$\text{EIP}_{\{A,>_{(M,c)}\}}$ is bounded with respect to $>_{(M,c)}$, after finitely many repetitions of this procedure we obtain a path $P(\alpha', \beta')$ in which α' is the peak as required. The uniqueness of the optimum β' guarantees that the outdegree of this point in the $\mathcal{H}_{>_{(M,c)}}$-skeleton is zero.□

Lemma 2.2.3 proves that $\mathcal{H}_{>_{(M,c)}}$ is a test set for $\text{EIP}_{\{A,>_{(M,c)}\}}$ with respect to $>_{(M,c)}$. Since $\alpha(1)', ..., \alpha(T)'$ are the unique minimal generators of $\mathcal{S}_{>_{(M,c)}}$ there exists an element $\vec{h_i}$ in $\mathcal{H}_{>_{(M,c)}}$ with tail $\alpha(i)'$, for $i = 1, ..., T$. Further, Steps 2 and 3 of the geometric Buchberger algorithm insure that such a $\vec{h_i}$ can only have the unique optimum in the fiber of $\alpha(i)'$ as head. Therefore $\mathcal{G}_{>_{(M,c)}} \subseteq \mathcal{H}_{>_{(M,c)}}$. To establish equality here, suppose $\vec{h} = \overrightarrow{[a, b]}$ in $\mathcal{H}_{>_{(M,c)}}$ does not belong to $\mathcal{G}_{>_{(M,c)}}$. Say the tail a, is not an essential generator of $\mathcal{S}_{>_{(M,c)}}$. This implies $a = \alpha(k)' + v$ for some $k \in \{1, ..., T\}$ and $v \in \mathbb{N}^{m+n}$. Since $\mathcal{G}_{>_{(M,c)}} \subseteq \mathcal{H}_{>_{(M,c)}}$, Step 2 of the geometric Buchberger algorithm would have then removed $\vec{h}$ from $\mathcal{H}_{>_{(M,c)}}$. Therefore a is an essential generator of $\mathcal{S}_{>_{(M,c)}}$ which then forces b to be the unique optimum in this fiber as before. We therefore have the following theorem.

Theorem 2.2.4 *The geometric Buchberger algorithm constructs the reduced Gröbner basis $\mathcal{G}_{>_{(M,c)}}$ of $EIP_{\{A,>_{(M,c)}\}}$.*

Example 2.2.5 *We continue Example 2.2.2 to show how the geometric Buchberger algorithm calculates the edges necessary to build connected directed graphs in all fibers, each with a unique sink at the optimal point in the fiber. Recall that the integer program was $Min\ 100y + x_1 + 5x_2$: $y + 2x_1 + 3x_2 = b,\ y, x_1, x_2 \geq 0, integer$. The fundamental segments directed according to the cost function $(100, 1, 5)$ were $\vec{d_1} = \overrightarrow{[(2,0,0),(0,1,0)]}$ and $\vec{d_2} = \overrightarrow{[(3,0,0),(0,0,1)]}$. These directed segments are also the input to the geometric Buchberger algorithm. In Step 1 of the algorithm the S-vector of $\vec{d_1}$ and $\vec{d_2}$ is calculated (see Figure 3). The common fiber on which the tails of both vectors meet is the fiber corresponding to $b = 3$. This is achieved by translating $\vec{d_1}$ by $(1, 0, 0)$. The S-vector produced is $\overrightarrow{[(1,1,0),(0,0,1)]}$. It cannot be reduced by elements in the current partial Gröbner basis and hence it is added to the current basis. The partial Gröbner basis now consists of the three vectors $\overrightarrow{[(2,0,0),(0,1,0)]}$, $\overrightarrow{[(3,0,0),(0,0,1)]}$ and $\overrightarrow{[(1,1,0),(0,0,1)]}$. The geometric Buchberger algorithm continues to calculate S-vectors of pairs of elements in the current partial Gröbner basis until no new gap vectors are produced. The final output of the algorithm is the reduced Gröbner basis $\mathcal{G}_{>_{(M,c)}} = \{\ \overrightarrow{[(2,0,0),(0,1,0)]}, \overrightarrow{[(1,1,0),(0,0,1)]}, \overrightarrow{[(1,0,1),(0,2,0)]}, \overrightarrow{[(0,0,2),(0,3,0)]}\ \}$. The skeleton in each fiber is updated using elements of $\mathcal{G}_{>_{(M,c)}}$. The updated skeleton in the 6-fiber is shown in Figure 4.*

It may be checked that now there exists a monotone path from every feasible lattice point in this fiber to the unique optimum. This property holds for all fibers corresponding to this coefficient

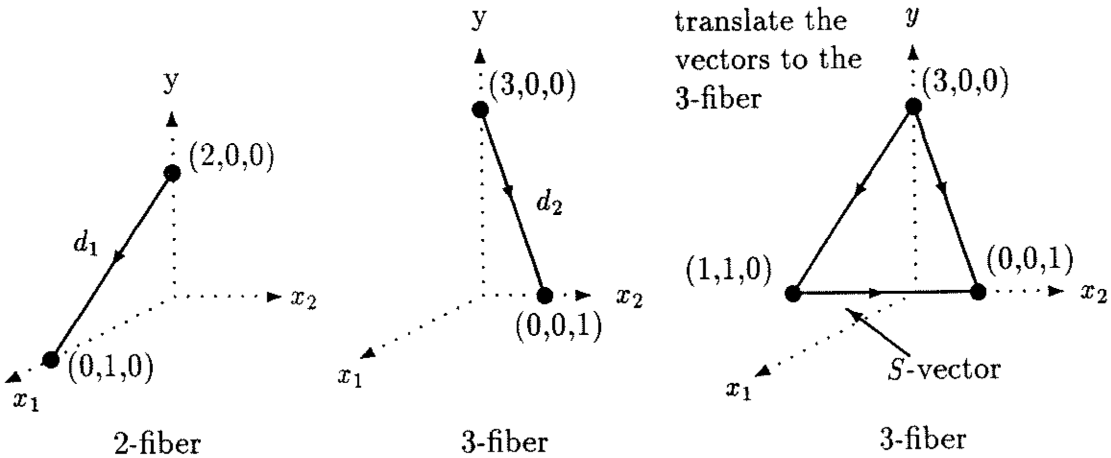

Figure 3: The S-vector of $\vec{d_1}$ and $\vec{d_2}$.

matrix and cost function. Note that the directed fundamental segment $\vec{d_2}$ has been removed; that move can be achieved by first moving along $\overrightarrow{[(2,0,0),(0,1,0)]}$ and then along $\overrightarrow{[(1,1,0),(0,0,1)]}$.

We now outline an argument for why the geometric Buchberger algorithm should terminate in finitely many steps. In order to do this we first show that the reduction of a vector $\vec{f}$ with both end points on a fiber, by elements in a Gröbner basis $\mathcal{G}$ or a partial Gröbner basis terminates in finitely many steps. Recall that reducing $\vec{f}$ by elements in $\mathcal{G}$ was equivalent to advancing the head and tail of $\vec{f}$ to less expensive points by moving along translations of elements in $\mathcal{G}$. The reduction stops either when the current head and tail of the gap vector are the same point or they are both component-wise smaller than the tail of any element in $\mathcal{G}$. Therefore in order for the reduction to not terminate, there must exist some element $\vec{g}$ in $\mathcal{G}$ whose tail is component-wise smaller than an infinite number of points in the fiber. But this contradicts Lemma 2.1.3 and hence the result. Since the reduction algorithm as used in the geometric Buchberger algorithm is finite, we need only prove finite termination of Step 1 of the algorithm. At an intermediate stage of Step 1, we have a finite, partial Gröbner basis which requires the reduction of finitely many S-vectors which can be done in finitely many steps. Suppose $\mathcal{S}$ is the set of all non-optimal points in all fibers of the family of integer programs under study. Let $\mathcal{S}' = \bigcup(\alpha(p)' + \mathbb{N}^{m+n})$ where $\{\alpha(p)'\}$ are the tails of the vectors in the current partial Gröbner basis. Clearly $\mathcal{S}' \subseteq \mathcal{S}$. The reduction of an S-vector can contribute at most one new element to the current partial Gröbner basis. Each time a new vector is added, we enlarge $\mathcal{S}'$ while still maintaining $\mathcal{S}' \subseteq \mathcal{S}$. By Lemma 2.1.3, we cannot get an infinite ascending chain of sets of the form $\mathcal{S}'$ all contained in $\mathcal{S}$. Therefore this process terminates in finitely many steps and hence the geometric Buchberger algorithm terminates in finitely many steps.

We wish to remark that the Buchberger algorithm in the case of integer programming can be

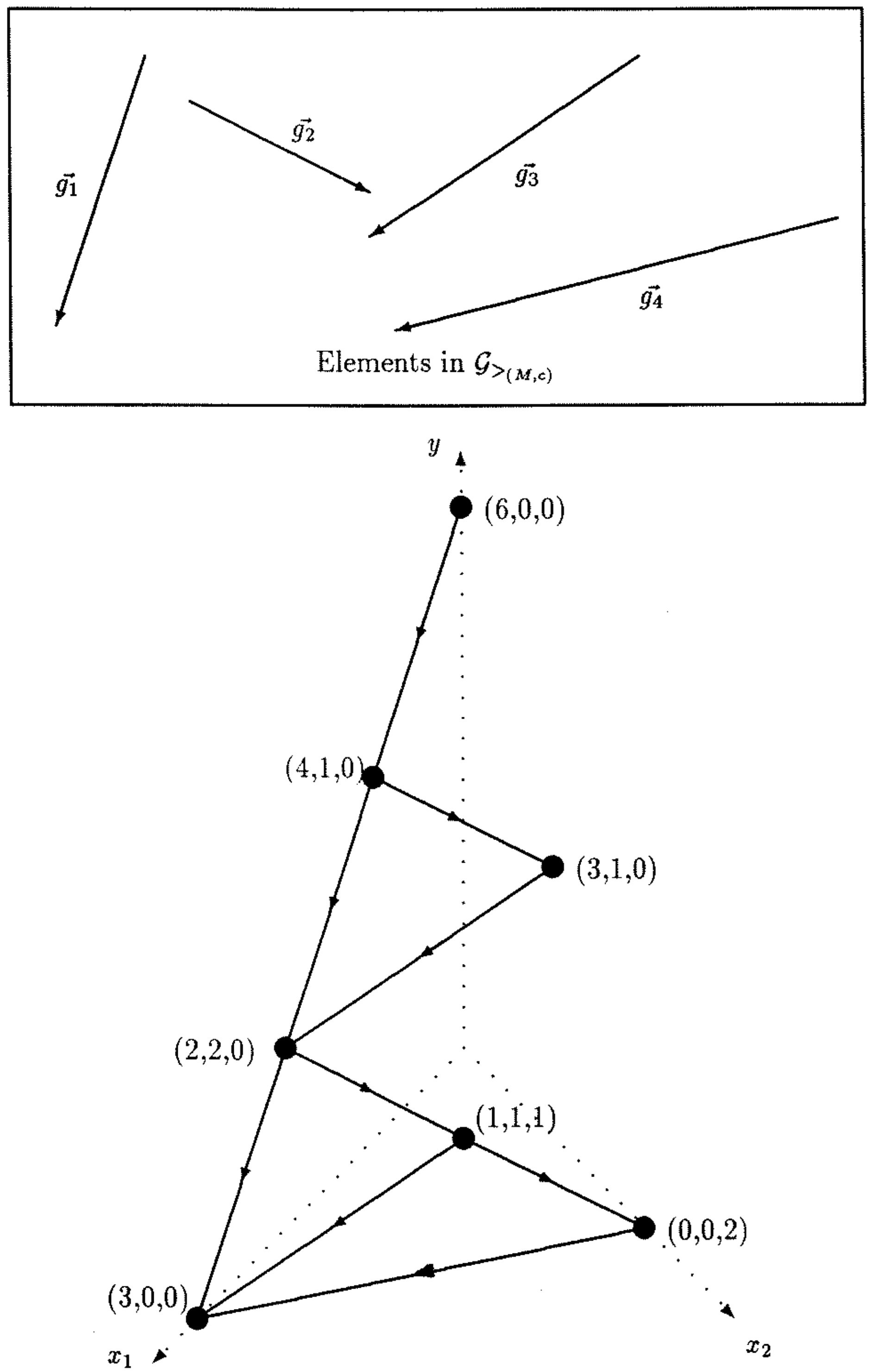

Figure 4: The updated skeleton in the 6-fiber.

sped up considerably in practice by exploiting certain features of the underlying algebraic theory. Such improvements to the algorithm are crucial in order to make this method feasible in practice. An exposition of some such speed up techniques as well as the algebraic methods to calculate $\mathcal{G}_{>_c}$ can be found in [4] and [5]. These methods can be implemented using a computer algebra package, to solve integer programs. These papers also describe efficient ways to compute the reduced Gröbner basis when the coefficient matrix has negative entries.

3 Comparison with Other Known Test Sets for Integer Programs

Several examples of test sets associated with integer programs can be found in the literature. However the setup and definitions vary considerably. In this section we examine the relationship between the reduced Gröbner bases described in the previous section and some of these test sets.

3.1 Neighbors of the Origin

Consider the family $\mathcal{F}$, of integer programs of the form $Min\ cx : Ax \leq b,\ x \in \mathbf{Z}^n$ obtained by varying the right hand side vector b where $A \in \mathbf{Z}^{m \times n}$, $b \in \mathbf{Z}^m$ and $c \in \mathbf{Z}^n$. Scarf [14], [15] describes a test set for this family of integer programs called the *neighbors of the origin*. In fact the neighbors of the origin are described for integer programs with arbitrary real data but we shall restrict attention to the case in which all data is integral. The neighbors of the origin are computed under the following assumptions on the family $\mathcal{F}$.

Assumption 1 : The set $\{x : cx \leq c_0, Ax \leq b\}$ is a polytope for all choices of the right hand side vector $(c_0, b)^t$.

Assumption 2 : The cost function c has been refined by some term order.

Under an additional general positioning condition on the rows of the matrix A (see [14], [15]), the neighbors of the origin is a unique minimal test set for $\mathcal{F}$. Since no such condition holds for the reduced Gröbner basis, we work with the above assumptions while comparing the two test sets. Consider the following definitions from [17].

Definition 3.1.1 *A point x dominates a point y if x lies in the interior of the smallest convex body of the form $\{x : cx \leq c_0, Ax \leq b\}$ that contains both y and 0.*

Definition 3.1.2 *An integer point x is a neighbor of the origin if and only if no integer point dominates x.*

In order to compare the reduced Gröbner basis with the neighbors of the origin we first compute the reduced Gröbner basis of the equivalent family $\mathcal{F}'$ comprised of programs of the form:

$$\text{Min } cw - cu$$
$$Aw - Au + Is = b$$
$$w \in \mathbf{N}^n,\ u \in \mathbf{N}^n,\ s \in \mathbf{N}^m$$

A solution (w, u, s) of an integer program in $\mathcal{F}'$ is interpreted as the solution $w - u$ to the corresponding integer program in $\mathcal{F}$. An element in the reduced Gröbner basis of $\mathcal{F}'$ has the form $\vec{g} = [\overrightarrow{(\alpha, \beta, \gamma), (\alpha', \beta', \gamma')}]$ where $\alpha, \beta \in \mathbf{N}^n$ and $\gamma \in \mathbf{N}^m$. By the above correspondence, we may think of $\vec{g}$ as pointing from $\alpha - \beta$ to $\alpha' - \beta'$ which is the same as the vector pointing from $(\alpha - \beta) - (\alpha' - \beta')$ to the origin. We now show that $(\alpha - \beta) - (\alpha' - \beta')$ is a neighbor of the origin.

Theorem 3.1.3 *An element in the reduced Gröbner basis of $\mathcal{F}'$ is a neighbor of the origin for the family $\mathcal{F}$.*

Proof: By Definition 3.1.2, it suffices to show that the smallest convex body of the form $\{x : cx \leq c_0, Ax \leq b\}$ containing the origin and $(\alpha - \beta) - (\alpha' - \beta')$ does not contain an interior lattice point. This is equivalent to showing that the smallest convex body of the form $\{x : cx \leq c_0, Ax \leq b\}$ containing $p = (\alpha - \beta)$ and $q = (\alpha' - \beta')$, denoted $SCB(p, q)$, does not contain any interior lattice points. By Assumption 2, $cp > cq$. The body $SCB(p, q)$ can be described as follows:

$$SCB(p,q) = \{x \in \mathbf{R}^n : cx \leq max(cp, cq) = cp, \quad a_i x \leq max(a_i p, a_i q) \quad \text{for } i = 1, ..., m\ \}$$

We first show that there exists some $j \in \{1, ..., m\}$ such that the constraint $a_j x \leq max(a_j p, a_j q)$ is binding at q but not at p. Suppose for all $i \in \{1, ..., m\}$, the constraint $a_i x \leq max(a_i p, a_i q)$ was either binding at both p and q or binding at p alone. Let $\mu = p + \lambda(q - p)$ where λ is a non-negative integer. Now $c\mu = cp + \lambda(cq - cp) \leq cp$. For $i \in \{1, ..., m\}$, $a_i \mu = a_i p + \lambda(a_i q - a_i p) \leq a_i p = max(a_i p, a_i q)$ by assumption. Therefore μ lies in $SCB(p, q)$. This implies that $SCB(p, q)$ is not a polytope as λ can be made arbitrarily large. But this contradicts Assumption 1 and therefore there exists some j such that $max(a_j p, a_j q) = a_j q > a_j p$.

Suppose there exists an interior lattice point k in $SCB(p, q)$. Then $ck < cp$ and $a_i k < max(a_i p, a_i q)$ for all $i \in \{1, ..., m\}$. In particular, $a_j k < max(a_j p, a_j q) = a_j q$. Since the jth constraint was binding at q and not at p, $max(a_j p, a_j k) < a_j q$.

We now consider the smallest convex body containing k and p denoted as $SCB(k, p)$. By the above arguments, $q \notin SCB(k, p)$. Since $ck < cp$, p is a non-optimal point in $SCB(k, p)$. This is equivalent to saying that for some $\delta \in \mathbf{N}^m$, (α, β, δ) is a non-optimal solution to the integer program:

$$\text{Min } cw - cu$$
$$a_i w - a_i u + s_i = max(a_i p, a_i k) \quad \text{for } i = 1, ..., m$$
$$w, u \in \mathbf{N}^n,\ s_i \in \mathbf{N}$$

Note that the new slack δ associated with p is component-wise less than or equal to the old slack γ since $SCB(k,p)$ is strictly contained in $SCB(p,q)$. Since (α, β, δ) is a non-optimal solution for the above integer program, there exists some element $\overline{g}$ in the reduced Gröbner basis of $\mathcal{F}'$ such that its tail can be translated through a non-negative integer vector to meet this non-optimal point. If the translation is by a non-zero vector, we have a contradiction to the fact that (α, β, γ) was an essential generator of the set of non-optimal points in the various fibers of $\mathcal{F}'$. If the tail of $\overline{g}$ is exactly the tail of g, then the head of this translated vector lies in $SCB(k,p)$ which prevents $(\alpha', \beta', \gamma')$ from being the head of g. This contradicts the uniqueness of the head of an element in the reduced Gröbner basis. Therefore there exists no interior lattice points in $SCB(p,q)$. $\square$

Theorem 3.1.3 holds under the assumptions made in the beginning of this section and when the tie-breaker used to compute both test sets is the same. For a fixed matrix A, there may be several reduced Gröbner bases associated with a given cost function depending on the choice of the tie-breaking term order. These may not all be of the same cardinality. We now give an example of an integer program for which the reduced Gröbner basis with the reverse lexicographic order as tie-breaker, is strictly contained in the set of neighbors of the origin associated with this program.

Example 3.1.4 *Consider the integer program:*

$$\begin{array}{ll} & \mathit{Max}\ x_1 + x_2 + x_3 \\ \textit{subject to} & \\ & 8x_1 - x_2 - x_3 \geq b_1 \\ & -5x_1 + 4x_2 - 5x_3 \geq b_2 \\ & -4x_1 - 4x_2 + 5x_3 \geq b_3 \\ & x_1,\ x_2,\ x_3 \in \mathbf{Z}. \end{array}$$

The neighbors of the origin for this matrix A and cost function c are $\pm(0,5,4)$, $\pm(0,1,1)$, $\pm(0,4,3)$, $\pm(0,3,2)$, $\pm(0,2,1)$, $\pm(1,5,4)$, $\pm(1,0,0)$, $\pm(1,4,3)$, $\pm(1,1,1)$, $\pm(0,1,0)$, $\pm(1,1,0)$, $\pm(1,5,3)$, $\pm(1,4,4)$, $\pm(1,3,3)$, $\pm(1,2,1)$, $\pm(0,0,1)$, $\pm(1,3,2)$, $\pm(1,2,2)$. (The above example and neighbors were obtained through private communication from Scarf. These neighbors of the origin were calculated using a certain tie-breaking rule. This implies that they are a sufficient test set contained (sometimes strictly) in the set of neighbors given by Definition 3.2.1). In order to calculate the reduced Gröbner basis for this matrix A and cost function c, we first convert the problem to a minimization problem and then force equality on the constraints by adding surplus variables. By introducing additional variables, we obtain non-negativity restrictions on all variables. The reduced Gröbner basis was then computed using Macaulay [1]. It consists of those neighbors from the above list with a negative sign except -(1,5,4).

Consider $SCB((0,0,0), (1,5,4)) = \{ (x_1, x_2, x_3) \in \mathbf{Q}^3 : x_1 + x_2 + x_3 \geq 0,\ 8x_1 - x_2 - x_3 \geq -1,\ -5x_1 + 4x_2 - 5x_3 \geq -5,\ -4x_1 - 4x_2 + 5x_3 \geq -4 \}$. In order to move from (0,0,0) to (1,5,4), we *first move to (1,0,0) by subtracting the Gröbner basis element (-1,0,0) from (0,0,0). From there we move to (1,5,4), using the element (0,-5,-4). These moves keep us inside SCB((0,0,0),(1,5,4)). This implies that (1,5,4) is not an element in the reduced Gröbner basis. However, SCB((0,0,0), (1,5,4)) does not contain any interior lattice points and so (1,5,4) is a neighbor of the origin.*□

3.2 A Universal Test Set

Suppose we assume in the rest of this section that all cost functions have been refined to be linear orders on $\mathbf{N}^n$. Then from Section 2, $\mathcal{G}_c$ is the unique minimal test set for $\mathrm{IP}_{\{A,c\}}$. However $\mathcal{G}_c$ may not be a test set for $\mathrm{IP}_{\{A,c'\}}$ where c' is a cost function different from c. Let $\mathcal{U}$ be a subset of $\{u \in \mathbf{Z}^n : Au = 0\}$. We may think of an element u in $\mathcal{U}$ as the undirected line segment $[u^+, u^-]$ where $u = u^+ - u^-$, $u^+, u^- \in \mathbf{N}^n$ being unique vectors with disjoint supports. We may assume that a cost function directs this line segment from the more expensive end (with respect to this cost) to the less expensive end. Let $\mathcal{U}_c$ denote the set of elements in $\mathcal{U}$ directed according to c. Let $\mathrm{IP}_{\{A\}}$ denote the family of integer programs of the form considered thus far with A as coefficient matrix.

Definition 3.2.1 *We call* $\mathcal{U}$ *a* universal Gröbner basis *for* $IP_{\{A\}}$ *if* $\mathcal{U}$ *has the property that* $\mathcal{U}_c$ *contains a test set for* $IP_{\{A,c\}}$ *whenever* c *is a cost function such that the programs in* $IP_{\{A,c\}}$ *have bounded optima.*

If $\mathcal{U}$ is a universal Gröbner basis for $\mathrm{IP}_{\{A\}}$ then the test set $\mathcal{U}_c$ may not be a unique, minimal test set (i.e., reduced Gröbner basis) for $\mathrm{IP}_{\{A,c\}}$. We describe below a universal Gröbner basis for $\mathrm{IP}_{\{A\}}$ studied by Graver [8]. The proof of the existence of an equivalent test set was also given by Blair and Jeroslow (see Lemma 6.1 in [3]). Define a set $\mathcal{H} \subset \mathbf{Z}^n$ as follows. Let $D(\mathcal{A}) = \{\lambda \in \mathbf{R^n} : A\lambda = 0\}$ and $D_{\mathbf{Z}}(\mathcal{A}) = D(\mathcal{A}) \cap \mathbf{Z}^n$. We call $\lambda \in D_{\mathbf{Z}}(\mathcal{A})$ *elementary* if :

(1) λ is non-zero.

(2) λ is a primitive lattice point. i.e., g.c.d.$(\lambda_1, ..., \lambda_n) = 1$, and

(3) its support, supp(λ), is minimal with respect to inclusion.

Let O_j denote the jth orthant in $\mathbf{R}^n$ where $j \in \{+,-\}^n$. Consider the cones $C_j = D(\mathcal{A}) \cap O_j$ for $j \in \{+,-\}^n$. It can be shown that every non-zero C_j is a pointed cone in $\mathbf{R}^n$ with the elementary vectors in O_j as extreme rays. Let H_j be the unique, minimal Hilbert basis of a non-zero cone C_j and $\mathcal{H} = \cup H_j \backslash \{0\}$ be the union of all these Hilbert bases minus the origin.

In [8], Graver proves that $\mathcal{H}$ is a universal test set for $\mathrm{IP}_{\{A\}}$. We outline a proof of this assertion in the following arguments. Let c be a generic refined cost function and let α be a non-optimal integer point in some fiber of $\mathrm{IP}_{\{A,c\}}$ in which β is the unique optimum. Then $\alpha - \beta$ lies in one of the cones C_j described above. Let H_j be the Hilbert basis of this cone. Then $\alpha - \beta = \sum z_{j_i} h_{j_i}$ where z_{j_i} are non-negative integers and the elements h_{j_i} belong to $H_j \subset \mathcal{H}$. Since $c\alpha > c\beta$, there exists some h_{j_i} from the sum above such that $ch_{j_i} > 0$ which allows movement from α. Therefore $\mathcal{H}$ contains a test set for $\mathrm{IP}_{\{A\}}$ with respect to every cost function c and so is a universal Gröbner basis for $\mathrm{IP}_{\{A\}}$. We call $\mathcal{H}$ the *universal Graver test set* of $\mathrm{IP}_{\{A\}}$.

We now relate the reduced Gröbner bases of $\mathrm{IP}_{\{A\}}$ to the set $\mathcal{H}$. Let $\mathcal{G}_c$ be the reduced Gröbner basis of $\mathrm{IP}_{\{A,c\}}$ and let $\vec{g}$ be an element of $\mathcal{G}_c$. We are as before, assuming that c is already a linear order on $\mathbf{N}^n$. Assume that $\vec{g}$ has tail α and head β. Then $\alpha - \beta$ lies in one of the cones C_j defined above.

Lemma 3.2.2 *An element $\alpha - \beta$ in the reduced Gröbner basis of $IP_{\{A,c\}}$ lies in $\mathcal{H}$.*

Proof: Suppose $\alpha - \beta$ lies in the pointed cone C_j with Hilbert basis H_j. If $\alpha - \beta$ is not an element of H_j, then $\alpha - \beta = v + w$ for $v, w \in C_j \cap \mathbf{Z}^n$. At least one of $cv > 0$ or $cw > 0$ since $c(\alpha - \beta) > 0$. Therefore we can move from α to a less expensive point using one of v or w. This contradicts that α was an essential generator of the set of non-optimal points in the various fibers of $\mathrm{IP}_{\{A,c\}}$.□

Since c was an arbitrary cost function, it follows that every element in any reduced Gröbner basis of $\mathrm{IP}_{\{A\}}$ lies in $\mathcal{H}$. We now consider the set $\mathcal{G}$ defined as follows. Let $\mathcal{G} = \cup \mathcal{G}_c$ be the union of all reduced Gröbner bases of $\mathrm{IP}_{\{A\}}$ as c varies over all refined cost functions for which the fibers in $\mathrm{IP}_{\{A,c\}}$ have bounded optima. Clearly $\mathcal{G}$ is a universal Gröbner basis for $\mathrm{IP}_{\{A\}}$. Using Lemma 3.2.2 we obtain the following relationship between the two sets $\mathcal{G}$ and $\mathcal{H}$.

Theorem 3.2.3 *The universal Gröbner basis $\mathcal{G}$ is contained in the universal Graver test set $\mathcal{H}$.*

Theorem 3.2.3 gives an alternate proof for why the set $\mathcal{H}$ is a universal Gröbner basis for $\mathrm{IP}_{\{A\}}$. It also implies that $\mathcal{G}$ is finite since $\mathcal{H}$ is finite. Further, we have the following corollary.

Corollary 3.2.4 *There exists only finitely many distinct reduced Gröbner bases for $IP_{\{A\}}$ as cost varies.*

We now give an example of an integer program for which $\mathcal{G}$ is properly contained in $\mathcal{H}$.

Example 3.2.5 $A = [1, 1, 2]$
$\mathcal{G} = \{\pm(1, -1, 0), \pm(2, 0, -1), \pm(0, 2, -1)\}$. *The vector* $(1, 1, -1)$ *is in $\mathcal{H}$ but not in $\mathcal{G}$ and hence $\mathcal{G}$ is strictly contained in $\mathcal{H}$.*

A universal test set of an arbitrary matrix is in general difficult to find. However for a matrix $A \in \mathbf{Z}^{d \times n}$ of full row rank with $d = n-2$, we outline an easy procedure to construct the elementary vectors and a universal test set. Let $A \in \mathbf{Z}^{d \times n}$ be a matrix of rank d where $d = n - 2$. Let $S = \{x \in \mathbf{R}^n : Ax = 0\}$ be the kernel of A of dimension two and let $B = [b_1, ..., b_n] \in \mathbf{Z}^{2 \times n}$ be a matrix whose rows form a basis of S. Then $S = \{\sum_{j=1}^{n} \phi(b_j)e_j : \phi \in (\mathbf{R}^2)^*\}$, where e_j is the jth unit vector in $\mathbf{R}^n$. Pick a linear form ϕ such that $\phi(b_j) = 0$. Such a ϕ is uniquely determined by b_j (up to scaling) and is of the form $\phi = e_1^*(b_j)e_2^* - e_2^*(b_j)e_1^*$ where $\{e_1^*, e_2^*\}$ is the dual basis of $\mathbf{R}^2$. The element in S corresponding to this linear form ϕ is $(\phi(b_1), ..., \phi(b_n))$ where $\phi(b_k) = e_1^*(b_j)e_2^*(b_k) - e_2^*(b_j)e_1^*(b_k) =$ determinant of the submatrix $[b_j, b_k]$ of B. We denote the determinant of $[b_j, b_k]$ by $[jk]$. Therefore the element in the kernel given by ϕ is $([j1], [j2], ..., [jj], ..., [jn])$ which has a zero in the jth position. Let $\epsilon_j = (1/k)([j1], [j2], ..., [jj], ..., [jn])$ where k is the g.c.d. of the components $[j1], [j2], ..., [jj], ..., [jn]$. Note that ϵ_j is a non-zero primitive lattice point.

Lemma 3.2.6 *The vectors $\pm\epsilon_j$, $j = 1, ..., n$ are the elementary vectors of the subspace S.*

Proof: Suppose the non-zero, primitive lattice point ϵ_j is not an elementary vector of S. Then there is a vector δ_j in the kernel with a strictly smaller support than ϵ_j. Let $l \in \{1, ..., n\}$ be an index such that ϵ_j has a non-zero entry in the lth position where as δ_j has a zero entry here. Since support of δ_j is strictly smaller than support of ϵ_j, the jth entry in δ_j is also zero. This implies that $\phi(b_l) = 0$. Hence b_l and b_j are collinear and the lth entry of ϵ_j is zero. This contradicts the assumption that δ_j had strictly smaller support than ϵ_j. Therefore ϵ_j is an elementary vector. The same argument implies that all elementary vectors (up to sign) can be constructed in this manner and so there exists at most 2n elementary vectors for A.

Lemma 3.2.7 *Let $C = pos(a, b)$, $a, b \in \mathbf{Z}^n$ be a two dimensional pointed cone. Consider the two dimensional polyhedron $P = conv(C \cap \mathbf{Z}^n \backslash \{0\})$. Let $f_1, ..., f_l$ be all the bounded edges of P. Then the set of all lattice points on the edges $f_1, ..., f_l$ form a Hilbert basis for C.*

Proof: See Proposition 1.19 pp. 24 [19].

Let $\mathcal{E} = \{e_1, ..., e_p\}$ be the set of all elementary vectors of A. We relabel the elements in $\mathcal{E}$ in such a way that $pos(e_{j+1})$ is encountered after $pos(e_j)$ when you walk in a clockwise manner along the circumference of a circle in the kernel of A, centered at the origin and with infinitesimal radius. Let $C_k = pos(e_k, e_{k+1})$ and let $P_k = conv(C_k \cap \mathbf{Z}^n \backslash \{0\})$. Let $f_{k_1}, ..., f_{k_{l_k}}$ be the bounded edges of the two dimensional polyhedron P_k.

Theorem 3.2.8 *The set $\mathcal{T} = \bigcup_{k=1}^{p} \{h : h$ is a lattice point on an edge f_{k_j} of P_k, $j = 1, ..., l_k\}$ is a universal Gröbner basis for $IP_{\{A\}}$.*

Proof: By the description of the universal Graver test set $\mathcal{H}$ and Lemma 3.2.7 we see that $\mathcal{T} = \mathcal{H}$ for these matrices. Now using Theorem 3.2.3 we have that $\mathcal{T}$ is a universal Gröbner basis for $\mathrm{IP}_{\{A\}}$.□

Example 3.2.5 shows that the union of the reduced Gröbner bases of $\mathrm{IP}_{\{A\}}$ may be strictly smaller than $\mathcal{T}$ even for matrices in this family.

3.3 The Schrijver Test Set

A universal test set for the family of integer programs $Min\ cx : Ax \geq b,\ x \in \mathbf{Z}^n$ as c and b varies can be derived from Section 17.3 of [16]. We call this the *Schrijver test set* for simplicity although the result is originally due to Cook, Gerards, Schrijver and Tardos [6]. This test set is constructed as follows. Consider a partition of the matrix A into two submatrices A_1 and A_2 such that $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$. Let $C(A_1, A_2)$ be the cone $\{u \in \mathbf{R}^n : A_1 u \leq 0, A_2 u \geq 0\}$ and let $H(A_1, A_2)$ be a Hilbert basis of $C(A_1, A_2)$. Let $\mathcal{R}$ be the union of all Hilbert bases corresponding to all possible partitions of A into submatrices A_1 and A_2. Then $\mathcal{R}$ is a universal test set for the above family of integer programs (see Theorem 17.3 [16]) which we call the Schrijver test set of this family..

We show that every element in the universal Graver test set of $\mathrm{IP}_{\{A\}}$ is in the corresponding Schrijver test set. Recall that $\mathrm{IP}_{\{A\}}$ is the family of integer programs $Min\ cx : Ax = b,\ x \in \mathbf{N}^n$ as c and b varies. In order to calculate the Schrijver test set of $\mathrm{IP}_{\{A\}}$, we look at the family of integer programs $Min\ cx : Ax \geq b,\ -Ax \geq -b,\ x \geq 0,\ x$ *integer*, as b and c varies. The matrix $B = \begin{bmatrix} A \\ -A \\ I \end{bmatrix}$ plays the role of the matrix A above and I is the $n \times n$ identity matrix. Let h be an element in the universal Graver test set. Then from the previous section we know that h is a member of the Hilbert basis of a cone $C_j = D(\mathcal{A}) \cap O_j$ where O_j is say the orthant $\{u \in \mathbf{R}^n : I_1 u \leq 0, I_2 u \geq 0\}$ and I has been partitioned as $I = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$. This implies that h is a member of the Hilbert basis of the cone $C(B_1, B_2)$ where $B_1 = \begin{bmatrix} A \\ -A \\ I_1 \end{bmatrix}$ and $B_2 = I_2$. Hence h is in the corresponding Schrijver test set. Therefore using Theorem 3.2.3 we have the following result.

Theorem 3.3.1 *The Schrijver test set of $IP_{\{A\}}$ contains the universal Gröbner basis $\mathcal{G}$.*

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References

[1] Bayer D., M. Stillman and M. Stillman. *Macaulay User Manual*, Cornell University 1989.

[2] Buchberger, B. *Gröbner bases - an algorithmic method in polynomial ideal theory*, Chapter 6 in N.K.Bose (ed.): "Multidimensional Systems Theory" D. Riedel Publications 1985.

[3] Blair, C.E. and R.G. Jeroslow. *The Value Function of an Integer Program*, Mathematical Programming 23, 1982, pp. 237-273.

[4] Conti, P. and C. Traverso. *Buchberger Algorithm and Integer Programming*, Proceedings AAECC-9 (New Orleans), Springer Verlag LNCS 539, 1991, pp. 130-139.

[5] Conti, P. and C. Traverso. *Gröbner Bases and Integer Programming*, Algorithms of Computer Algebra, Dagstuhl Seminar Report, December 1991.

[6] Cook, W., A.M.H. Gerards, A. Scrijver and É. Tardos. *Sensitivity Theorems in Integer Linear Programming.* Mathematical Programming 34, 1986, pp. 251-264.

[7] Cox, D., J. Little and D. O'Shea. *Ideals, Varieties and Algorithms: An Introduction to Computational Algebraic Geometry and Commutative Algebra*, Springer Verlag 1992.

[8] Graver, J.E. *On the Foundations of Linear and Integer Programming I*, Mathematical Programming 8, 1975, pp. 207-226.

[9] Jeroslow, R.G. *Some Basis Theorems for Integral Monoids*, Mathematics of Operations Research, Vol 3, No.2, May 1978, pp. 145-153.

[10] Natraj, N.R., S.R. Tayur and R.R. Thomas. *An Algebraic Geometry Algorithm for Scheduling in Presence of Setups and Correlated Demands*, GSIA Working Paper, Carnegie-Mellon University, December 1992, under review.

[11] Ollivier, F. *Canonical Bases: Relations with Standard Bases, Finiteness Conditions and Application to Tame Automorphisms*, Proceedings MEGA-90, Birkhauser, Progress in Mathematics, 1991, pp. 379-400.

[12] Pottier, L. *Minimal Solutions of Linear Diophantine Systems : Bounds and Algorithms*, Proceedings RTA (Como), Springer Verlag LNCS 488, 1991.

[13] Pottier, L. *Sub-groups of Z^n, Standard Bases and Linear Diophantine Systems*, Research Report 1510, I.N.R.I.A. Sophia Antipolis, September 1991.

[14] Scarf, H.E. *Production Sets with Indivisibilities. Part I. Generalities*, Econometrica 49, 1981, pp. 1-32.

[15] SCARF, H.E. *Neighborhood Systems for Production Sets with Indivisibilities*, Econometrica 54, 1986, pp. 507-532.

[16] SCHRIJVER, A. *Theory of Linear and Integer Programming*, Wiley-Interscience Series in Discrete Mathematics and Optimization 1986.

[17] SHALLCROSS, D. *Neighbors of the Origin for Four by Three Matrices*, Mathematics of Operations Research, Vol.17, No.3, August 1992.

[18] STURMFELS, B. *Gröbner Bases of Toric Varieties*, Tôhuku Math. J 43, 1991 pp. 249-261.

[19] ODA, T. *Convex Bodies and Algebraic Geometry : An Introduction to the Theory of Toric Varieties*, Springer - Verlag 1985.