Chap 5: Initial Problems for ODE

Due on May, 2022

Yuchen Ge

1. Introduction

The chapter mianly deals with the problem of initial problems for ODE.

First we claim that all function and variable names are self-clear. And a/b is interpreted to be section a question b, e.g. 5.4/7.

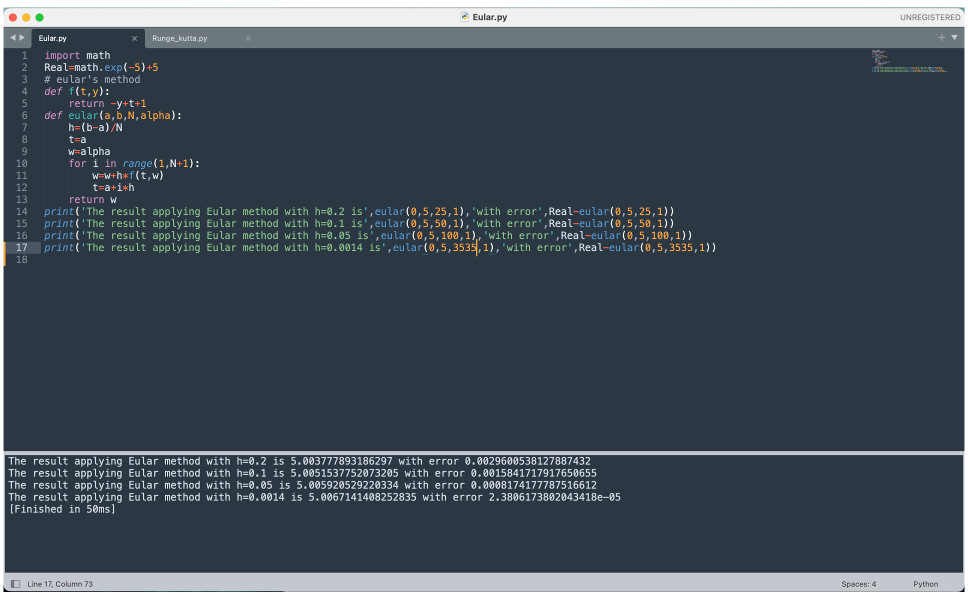
2. Application of Eular's Method (5.2/7)

2.1 Program Running and Data Analysis

Question 5.2/7 a. requires the best approximation of the initial problem. First we calculate the real value

$$y(5) = e^{-5} + 5 = 5.006737946999086.$$

Then we apply Eular's Method with parameters $h = 0.2, 0.1, 0.05$. We run the program as follows. (Meanwhile we display the running time.)



We write down the results in the table below, where we **reserve seven significant digits** for the result.

Table 1: Results of 5.2/7

h	**Approximation of y(5)**	**Error**
0.2	5.003778	0.002960
0.1	5.005154	0.001584
0.05	5.005921	0.000817

For question 5.2/7 b., we apply formula (5.14)

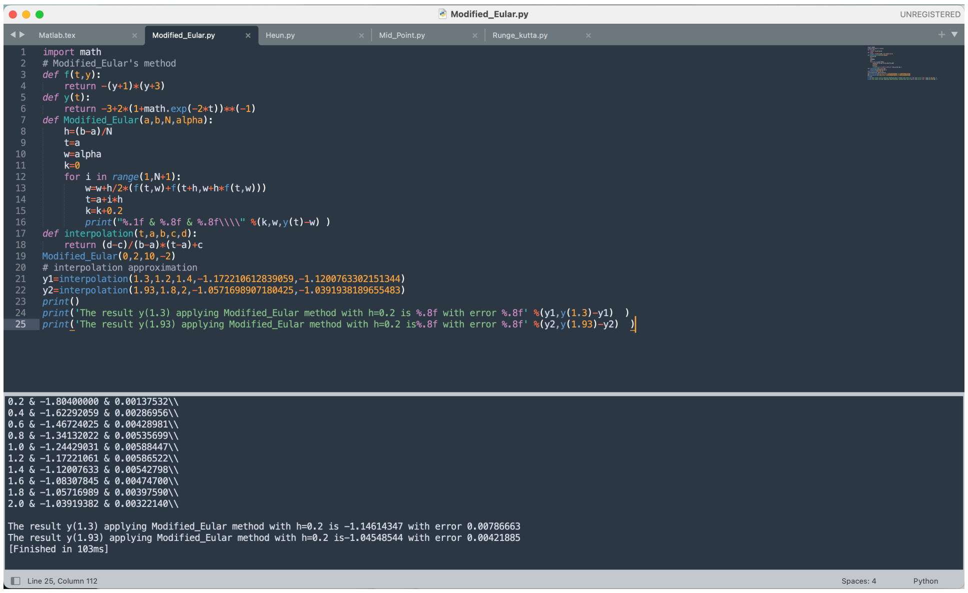
$$h = \sqrt{\frac{2\delta}{M}} = (\frac{2 \cdot 10^{-6}}{1})^{1/2} = 0.001414213562373095 \approx 0.0014142.$$

Then we run the program as above and we have the approximation of $y(5) \approx 5.00671414$.

3. Application of Runge-Kutta's Method (5.4/4c,5c,6,7,8,9,11,12)

3.1 Program Running and Data Analysis of 5.4/4c,5c

Applying Modified Eular's method for $y' = -(y+1)(y+3)$ with initial value $y(0) = -2$ (5.4/4c), we run the program as follows: (Meanwhile we also display the running time as a byproduct.)



And therefore we give a table containing results and errors: (reserve nine significant digits)

Table 2: Results of 5.4/4c (Modified Eular's method)

t	**Approximation of y(t)**	**Error**
0.2	-1.80400000	0.00137532
0.4	-1.62292059	0.00286956
0.6	-1.46724025	0.00428981
0.8	-1.34132022	0.00535699
1.0	-1.24429031	0.00588447
1.2	-1.17221061	0.00586522
1.4	-1.12007633	0.00542798
1.6	-1.08307845	0.00474700
1.8	-1.05716989	0.00397590
2.0	-1.03919382	0.00322140

And we find that the error term grows until $t = 1.2$, which is contractdict to the earlier claim for Eular's method that the error monotonically increases with t increases. The reason behind this is that Eular's method requires only the term $f(t_i, w_i)$, while the other methods like the Modified Eular Method requires terms below which is much more comlicated,

$$\frac{h}{2}[f(t_i, w_i) + f(t_{i+1}, w_i + hf(t_i, w_i))].$$

Then for 5.4/5c we use the approximation value of $y(1.2)$ and $y(1.4)$ to linear interpolate $y(1.3)$, which leads to

$$\tilde{y}(1.3) = \frac{\tilde{y}(1.4) - \tilde{y}(1.2)}{1.4 - 1.2}(1.3 - 1.2) + \tilde{y}(1.2) = -1.1461434715270966$$

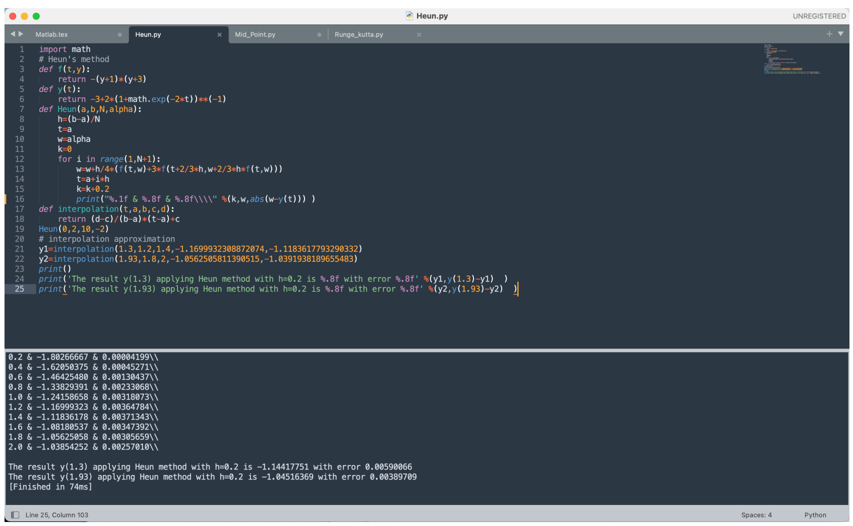
where we denote the approximation value of $y(1.3)$ by $\tilde{y}(1.3)$. Similarly, we have $\tilde{y}(1.93) = -1.0454854440789212$. And thus gather the information for the table (reserve nine significant digits):

Table 3: Results of 5.4/5c (Modified Eular's method)

t	**Approximation of y(t)**	**Error**
1.3	-1.14614347	0.00786663
1.93	-1.04548544	0.00421885

3.2 Program Running and Data Analysis of 5.4/6,7

Similarly we apply Heun method and run the program as follows:



And therefore we can give a table containing results and errors, where we reserve nine significant digits for the results.

Table 4: Results of 5.4/6 (Heun method)

t	**Approximation of y(t)**	**Error**
0.2	-1.80266667	0.00004199
0.4	-1.62050375	0.00045271
0.6	-1.46425480	0.00130437
0.8	-1.33829391	0.00233068
1.0	-1.24158658	0.00318073
1.2	-1.16999323	0.00364784
1.4	-1.11836178	0.00371343
1.6	-1.08180537	0.00347392
1.8	-1.05625058	0.00305659
2.0	-1.03854252	0.00257010

Also we find that the error term grows until $t = 1.6$, which is contractdict to the earlier claim for Eular's method that the error monotonically increases with t increases. The reason behind this is the same.

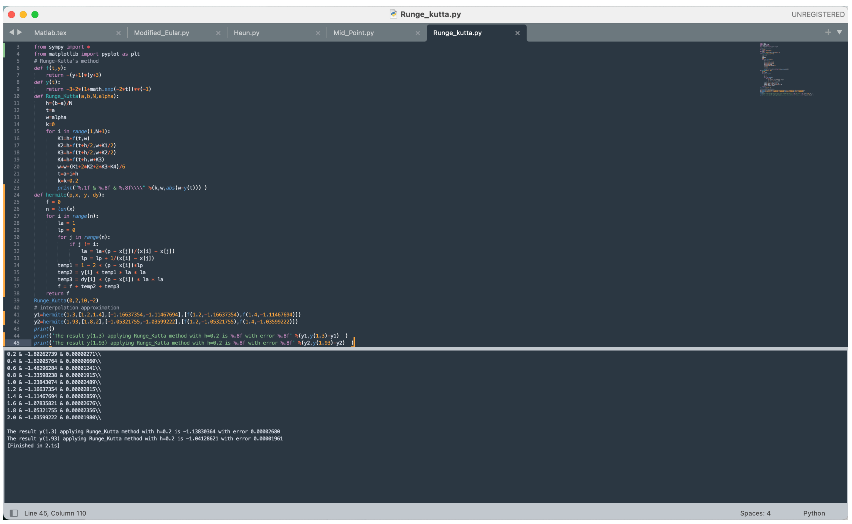
Then for 5.4/7, similarly, we can interpolate in the same way as above and thus gives the table (reserve nine significant digits):

Table 5: Results of 5.4/7 (Heun method)

t	**Approximation of y(t)**	**Error**
1.3	-1.14417751	0.00590066
1.93	-1.04516369	0.00389709

3.3 Program Running and Data Analysis of 5.4/8,9

Similarly we apply Midpoint method and run the program as follows:



And therefore we give a table containing results and errors.

Table 6: Results of 5.4/8 (Midpoint method)

t	**Approximation of y(t)**	**Error**
0.2	-1.80200000	0.00062468
0.4	-1.61929656	0.00075448
0.6	-1.46276690	0.00018353
0.8	-1.33678999	0.00082676
1.0	-1.24024699	0.00184114
1.2	-1.16889759	0.00255219
1.4	-1.11751649	0.00286814
1.6	-1.08117882	0.00284737
1.8	-1.05579872	0.00260473
2.0	-1.03822270	0.00225028

Also we find that the error term grows until $t = 1.6$, which is contractdict to the earlier claim for Eular's method that the error monotonically increases with t increases. The reason behind this is the same.

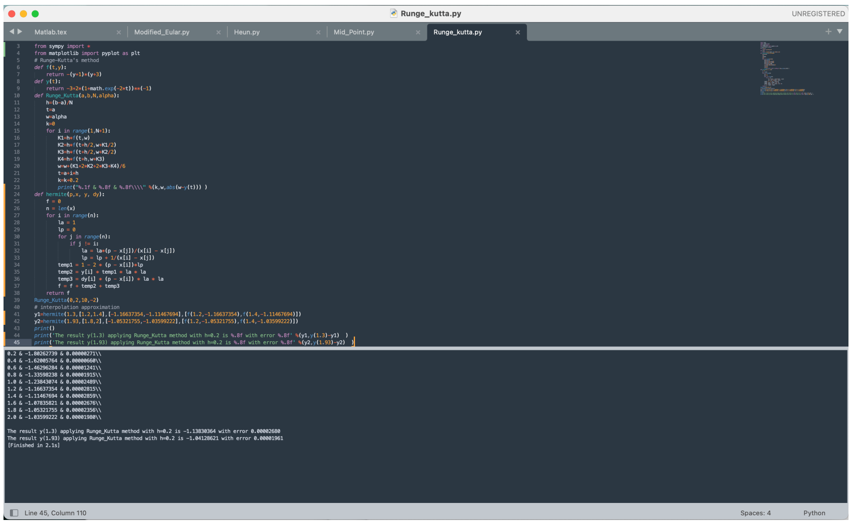
Then for 5.4/9 , similarly, we can interpolate in the same way as above and thus gives the table (reserve nine significant digits):

Table 7: Results of 5.4/9 (Midpoint method)

t	**Approximation of y(t)**	**Error**
1.3	-1.14320704	0.00493020
1.93	-1.04437431	0.00310771

3.4 Program Running and Data Analysis of 5.4/11,12

And similarly we apply Runge-Kutta's method method and run the program as follows:



And therefore we give a table containing results and errors (reserve nine significant digits):

Table 8: Results of 5.4/11 (Runge-Kutta's method)

t	**Approximation of y(t)**	**Error**
0.2	-1.80262739	0.00000271
0.4	-1.62005764	0.00000660
0.6	-1.46296284	0.00001241
0.8	-1.33598238	0.00001915
1.0	-1.23843074	0.00002489
1.2	-1.16637354	0.00002815
1.4	-1.11467694	0.00002859
1.6	-1.07835821	0.00002676
1.8	-1.05321755	0.00002356
2.0	-1.03599222	0.00001980

Also we find that the error term grows until $t = 1.4$, which is contractdict to the earlier claim for Eular's method that the error monotonically increases with t increases. The reason behind this is the same.

Then for 5.4/12 we use the approximation value of $y(1.2)$ $y(1.4)$ $y'(1.2)$ and $y'(1.4)$ to hermite interpolate $y(1.3)$ as above, which leads to

$$\tilde{y}(1.3) = H_3(1.3) = -1.13830364$$

where we denote the approximation value of $y(1.3)$ by $\tilde{y}(1.3)$ and hermite polynomial is being calculated as in the program below

```python
def hermite(p,x, y, dy):
f = 0
n = len(x)
for i in range(n):
    la = 1
    lp = 0
    for j in range(n):
        if j != i:
            la = la*(p - x[j])/(x[i] - x[j])
            lp = lp + 1/(x[i] - x[j])
    temp1 = 1 - 2 * (p - x[i])*lp
    temp2 = y[i] * temp1 * la * la
    temp3 = dy[i] * (p - x[i]) * la * la
    f = f + temp2 + temp3
return f
```

Similarly, we have $\tilde{y}(1.93) = -1.04128621$. And thus gather the information for the table (reserve nine significant digits)

Table 9: Results of 5.4/12 (Runge-Kutta's method)

t	**Approximation of y(t)**	**Error**
1.3	-1.13830364	0.00002680
1.93	-1.04128621	0.00001961

3.5 Conclusion of Above Methods

Generally speaking, by comparing the results we see that:

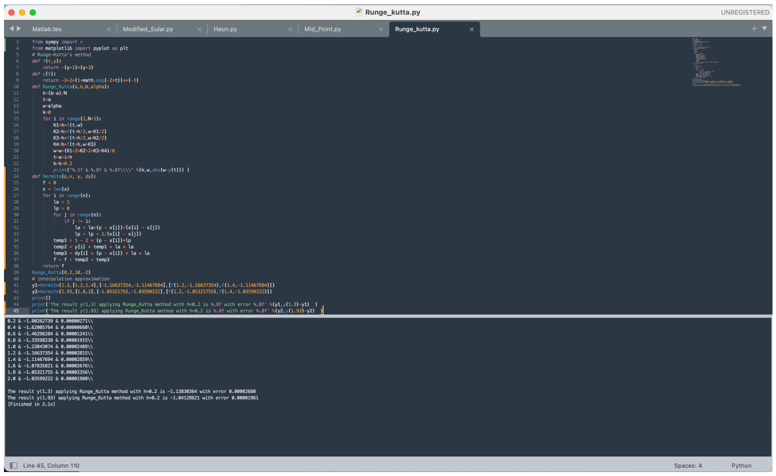
1. the Runge-Kutta method gives the best aprroximation but it has the most calculations.
2. Heun method gives good approximations near the initial values but gives no approximations further.
3. Midpoint/Modified-Eular method gives good approximations everywhere and has mild calculations.
4. Euler's Method has the smallest calculations but sacrifies its accuracy.

Turn to the next page for section 4.

4. Application of Multistep Method (5.6/3c)

4.1 Program Running and Data Analysis of 5.6/3c

We will approximate the solutions to 5.6/3c by applying **Adams-Bashforth Method**. We run the program as below:



Gathering the information, we give the table of results. (reserve nine significant digits)

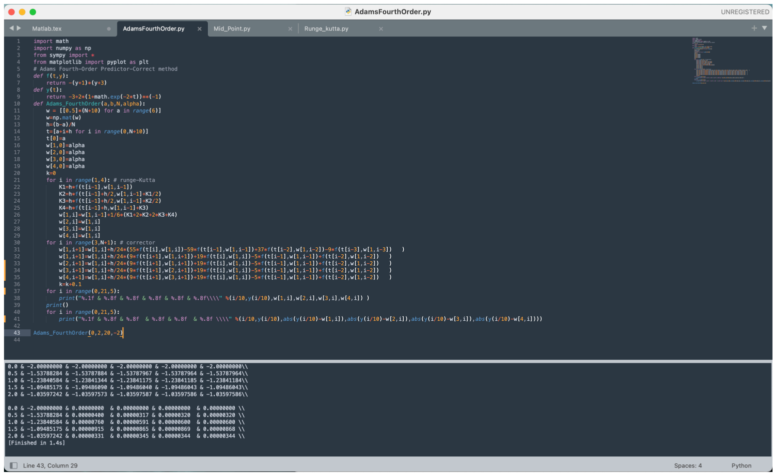
Table 10: Results of 5.6/3c (Adams-Bashforth method)

t	**Real value**	**2-step**	**3-step**	**4-step**	**5-step**
0.1	-1.90033201	-1.90033209	-1.90033209	-1.90033209	-1.90033209
0.2	-1.80262468	-1.80182214	-1.80262486	-1.80262486	-1.80262486
0.3	-1.70868739	-1.70721663	-1.70876712	-1.70868768	-1.70868768
0.4	-1.62005104	-1.61811122	-1.62024327	-1.62008995	-1.62005146
0.5	-1.53788284	-1.53570097	-1.53819884	-1.53793718	-1.53786761
0.6	-1.46295043	-1.46074506	-1.46337907	-1.46300534	-1.46292261
0.7	-1.39563222	-1.39358576	-1.39614605	-1.39567144	-1.39559281
0.8	-1.33596323	-1.33420670	-1.33652621	-1.33597925	-1.33592042
0.9	-1.28370213	-1.28231190	-1.28427720	-1.28369238	-1.28365952
1.0	-1.23840584	-1.23740929	-1.23896052	-1.23837336	-1.23836930
1.1	-1.19950098	-1.19888717	-1.20001057	-1.19945122	-1.19947117
1.2	-1.16634539	-1.16607721	-1.16679398	-1.16628513	-1.16632443
1.3	-1.13827684	-1.13830220	-1.13865663	-1.13821228	-1.13826257
1.4	-1.11464835	-1.11490931	-1.11495817	-1.11458465	-1.11464097
1.5	-1.09485175	-1.09529098	-1.09509518	-1.09479250	-1.09484815
1.6	-1.07833145	-1.07889647	-1.07851510	-1.07827891	-1.07833179
1.7	-1.06459093	-1.06523630	-1.06472296	-1.06454612	-1.06459229
1.8	-1.05319399	-1.05388219	-1.05328300	-1.05315708	-1.05319726
1.9	-1.04376254	-1.04446387	-1.04381686	-1.04373311	-1.04376513

Comparing the results with the real values we find that with the steps increase, the more accurate the Adams-Bashforth method become. (Accuracy is measured by the difference between real value and approximation value.)

4.2 Program Running and Data Analysis of 5.6/5,6

We will approximate the solutions to 5.6/3c by applying **Adams Fourth-Order Predictor-Corrector Method**. We run the program as below:



In which we change the algorithm (iterate the formula in the algorithm) so that it can make the corrector be iterated for p iterations. (p=2,3,4)

And we shall gather the results in a table as below

Table 11: Results of 5.6/5,6 (Adams Fourth-Order Predictor-Corrector Method)

t	**Real value**	**1-iteration**	**2-iteration**	**3-iteration**	**4-iteration**
0.0	-2.00000000	-2.00000000	-2.00000000	-2.00000000	-2.00000000
0.5	-1.53788284	-1.53787884	-1.53787967	-1.53787964	-1.53787964
1.0	-1.23840584	-1.23841344	-1.23841175	-1.23841185	-1.23841184
1.5	-1.09485175	-1.09486090	-1.09486040	-1.09486043	-1.09486043
2.0	-1.03597242	-1.03597573	-1.03597587	-1.03597586	-1.03597586

and with the errors also calculated in the table below.

Table 12: Results of 5.6/5,6 (Adams Fourth-Order Predictor-Corrector Method)

t	**Errors of 1-iteration**	**Errors of 2-iteration**	**Errors of 3-iteration**	**Errors of 4-iteration**
0.0	0.00000000	0.00000000	0.00000000	0.00000000
0.5	0.00000400	0.00000317	0.00000320	0.00000320
1.0	0.00000760	0.00000591	0.00000600	0.00000600
1.5	0.00000915	0.00000865	0.00000869	0.00000868
2.0	0.00000331	0.00000345	0.00000344	0.00000344

From the above table 12 we find that for almost all values of t, the best approximation is given by 2-iteration. So the best p is 2.

5 Application Runge-Kutta for Systems Algorithms (5.9/2c)

5.1 Program Running and Data Analysis of 5.9/2c Runge-Kutta for Systems Algorithms

We will approximate the solutions to 5.9/2c by applying **Runge-Kutta Method for Systems of Differential Equations**. First observe in chapter 5.9 we can only deal with the numerical approximation of the m-th order system in the form of (5.44). But in 5.9/2c, we will approximate the solutions of

$$y''' + 2y'' - y' - 2y = e^t$$

s.t. the initial conditions: $y(0) = 1, y'(0) = 2, y''(0) = 0$.

As a result of the above argument we can't directly apply the algorithm. But if we first magically suppose

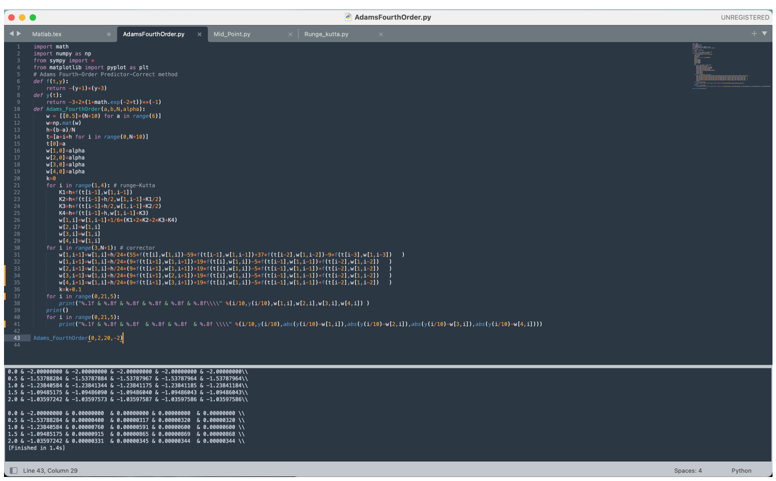
$$y_1 := y \quad y_2 := y' \quad y_3 := y''$$

Then the form of 5.9/2c will be transformed to

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = 2y_1 + y_2 - 2y_3 + e^t \end{cases} \tag{1}$$

s.t. the initial conditions: $y_1(0) = 1, y_2(0) = 2, y_3(0) = 0$.

First and foremost, we run the program as below (Meanwhile we also display the running time as a byproduct.)



We see that approximated values of $y(t)$, $y'(t)$ and $y''(t)$ are all showed in the display table separated by & and \\ finished in 2.0s. And the errors represent the difference between real values of $y(t)$ and approximation values of $y(t)$ in a step of $h = 0.2$ as requested by the question text.

And therefore we shall gather the results into the table below which can represent the results more clearly, where we **reserve nine significant digits**.

Observe that the error accumulates with t grows as usual. But as a windfall we can approximate the solutions of $y'(t)$ and $y''(t)$. Still the details are gathered in the table below.

Table 13: Results of 5.9/2c (Adams Fourth-Order Predictor-Corrector Method)

t	Real value	Approximation of $y(t)$	Error	Approximation of $y'(t)$	Approximation of $y''(t)$
0.0	1.00000000	1.00000000	0.00000000	2.00000000	0.00000000
0.2	1.40637383	1.40633678	0.00003705	2.09439538	0.91961556
0.4	1.84923495	1.84918146	0.00005350	2.36188926	1.74722932
0.6	2.36197037	2.36190903	0.00006134	2.79290064	2.56757686
0.8	2.97762424	2.97755643	0.00006781	3.39310710	3.45012070
1.0	3.73170445	3.73162695	0.00007749	4.18124911	4.45721868
1.2	4.66469806	4.66460440	0.00009366	5.18835555	5.65033178
1.4	5.82454694	5.82442783	0.00011912	6.45808521	7.09509448
1.6	7.26928830	7.26913151	0.00015679	8.04801932	8.86584062
1.8	9.07004289	9.06983275	0.00021014	10.03184298	11.05003348
2.0	11.31452924	11.31424573	0.00028351	12.50243368	13.75296416
2.2	14.11129305	14.11091055	0.00038250	15.57594327	17.10304019
2.4	17.59486416	17.59434982	0.00051435	19.39702265	21.25797699
2.6	21.93209017	21.93140177	0.00068840	24.14540203	26.41222100
2.8	27.32994449	27.32902779	0.00091670	30.04410928	32.80597197
3.0	34.04517155	34.04395688	0.00121468	37.36968748	40.73623289

6. Code Appendix

6.1 Eular's Method

```
    import math
    Real=math.exp(-5)+5
    # eular's method
    def f(t,y):
        return -y+t+1
    def eular(a,b,N,alpha):
        h=(b-a)/N
        t=a
        w=alpha
        for i in range(1,N+1):
            w=w+h*f(t,w)
            t=a+i*h
        return w
    print('The result applying Eular method with h=0.2 is',eular(0,5,25,1),'with ...
                                                 error',Real-eular(0,5,25,1))
    print('The result applying Eular method with h=0.1 is',eular(0,5,50,1),'with ...
                                                 error',Real-eular(0,5,50,1))
    print('The result applying Eular method with h=0.05 is',eular(0,5,100,1),'with ...
                                                 error',Real-eular(0,5,100,1))
    print('The result applying Eular method with h=0.0014 is',eular(0,5,3535,1),'with ...
                                                 error',Real-eular(0,5,3535,1))
```

6.2 Modified Eular's Method

```
import math
# Modified_Eular's method
```

```
def f(t,y):
    return -(y+1)*(y+3)
def y(t):
    return -3+2*(1+math.exp(-2*t))**(-1)
def Modified_Eular(a,b,N,alpha):
    h=(b-a)/N
    t=a
    w=alpha
    k=0
    for i in range(1,N+1):
        w=w+h/2*(f(t,w)+f(t+h,w+h*f(t,w)))
        t=a+i*h
        k=k+0.2
        print("%.1f & %.8f & %.8f\\\\" %(k,w,y(t)-w) )
def interpolation(t,a,b,c,d):
    return (d-c)/(b-a)*(t-a)+c
Modified_Eular(0,2,10,-2)
# interpolation approximation
y1=interpolation(1.3,1.2,1.4,-1.172210612839059,-1.1200763302151344)
y2=interpolation(1.93,1.8,2,-1.0571698907180425,-1.0391938189655483)
print()
print('The result y(1.3) applying Modified_Eular method with h=0.2 is %.8f with error %.8f' ...
                                                %(y1,y(1.3)-y1)  )
print('The result y(1.93) applying Modified_Eular method with h=0.2 is%.8f with error %.8f' ...
                                                %(y2,y(1.93)-y2)  )
```

6.3 Modified Eular's Method

```
import math
# Heun's method
def f(t,y):
    return -(y+1)*(y+3)
def y(t):
    return -3+2*(1+math.exp(-2*t))**(-1)
def Heun(a,b,N,alpha):
    h=(b-a)/N
    t=a
    w=alpha
    k=0
    for i in range(1,N+1):
        w=w+h/4*(f(t,w)+3*f(t+2/3*h,w+2/3*h*f(t,w)))
        t=a+i*h
        k=k+0.2
        print("%.1f & %.8f & %.8f\\\\" %(k,w,abs(w-y(t))) )
def interpolation(t,a,b,c,d):
    return (d-c)/(b-a)*(t-a)+c
Heun(0,2,10,-2)
# interpolation approximation
y1=interpolation(1.3,1.2,1.4,-1.1699932308872074,-1.1183617793290332)
y2=interpolation(1.93,1.8,2,-1.0562505811390515,-1.0391938189655483)
print()
print('The result y(1.3) applying Heun method with h=0.2 is %.8f with error %.8f' ...
                                                %(y1,y(1.3)-y1)  )
print('The result y(1.93) applying Heun method with h=0.2 is %.8f with error %.8f' ...
                                                %(y2,y(1.93)-y2)  )
```

6.4 Heun's Method

```python
import math
# Mid_Point method
def f(t,y):
    return -(y+1)*(y+3)
def y(t):
    return -3+2*(1+math.exp(-2*t))**(-1)
def Mid_Point(a,b,N,alpha):
    h=(b-a)/N
    t=a
    w=alpha
    k=0
    for i in range(1,N+1):
        w=w+h*f(t+h/2,w+h/2*f(t,w))
        t=a+i*h
        k=k+0.2
        print("%.1f & %.8f & %.8f\\\\" %(k,w,abs(w-y(t))) )
def interpolation(t,a,b,c,d):
    return (d-c)/(b-a)*(t-a)+c
Mid_Point(0,2,10,-2)
# interpolation approximation
y1=interpolation(1.3,1.2,1.4,-1.1688975879735313,-1.1175164873711958)
y2=interpolation(1.93,1.8,2,-1.0557987215668139,-1.038222697151529)
print()
print('The result y(1.3) applying Mid_Point method with h=0.2 is %.8f with error %.8f' ...
                                                  %(y1,y(1.3)-y1)  )
print('The result y(1.93) applying Mid_Point method with h=0.2 is %.8f with error %.8f' ...
                                                  %(y2,y(1.93)-y2)  )
```

6.5 Midpoint Method


```python
import math
# Mid_Point method
def f(t,y):
    return -(y+1)*(y+3)
def y(t):
    return -3+2*(1+math.exp(-2*t))**(-1)
def Mid_Point(a,b,N,alpha):
    h=(b-a)/N
    t=a
    w=alpha
    k=0
    for i in range(1,N+1):
        w=w+h*f(t+h/2,w+h/2*f(t,w))
        t=a+i*h
        k=k+0.2
        print("%.1f & %.8f & %.8f\\\\" %(k,w,abs(w-y(t))) )
def interpolation(t,a,b,c,d):
    return (d-c)/(b-a)*(t-a)+c
Mid_Point(0,2,10,-2)
# interpolation approximation
y1=interpolation(1.3,1.2,1.4,-1.1688975879735313,-1.1175164873711958)
y2=interpolation(1.93,1.8,2,-1.0557987215668139,-1.038222697151529)
print()
print('The result y(1.3) applying Mid_Point method with h=0.2 is %.8f with error %.8f' ...
                                                  %(y1,y(1.3)-y1)  )
print('The result y(1.93) applying Mid_Point method with h=0.2 is %.8f with error %.8f' ...
                                                  %(y2,y(1.93)-y2)  )
```

6.6 Runge-Kutta's Method

```
import math
import numpy as np
from sympy import *
from matplotlib import pyplot as plt
# Runge-Kutta's method
def f(t,y):
    return -(y+1)*(y+3)
def y(t):
    return -3+2*(1+math.exp(-2*t))**(-1)
def Runge_Kutta(a,b,N,alpha):
    h=(b-a)/N
    t=a
    w=alpha
    k=0
    for i in range(1,N+1):
        K1=h*f(t,w)
        K2=h*f(t+h/2,w+K1/2)
        K3=h*f(t+h/2,w+K2/2)
        K4=h*f(t+h,w+K3)
        w=w+(K1+2*K2+2*K3+K4)/6
        t=a+i*h
        k=k+0.2
        print("%.1f & %.8f & %.8f\\\\" %(k,w,abs(w-y(t))) )
def hermite(p,x, y, dy):
    f = 0
    n = len(x)
    for i in range(n):
        la = 1
        lp = 0
        for j in range(n):
            if j != i:
                la = la*(p - x[j])/(x[i] - x[j])
                lp = lp + 1/(x[i] - x[j])
        temp1 = 1 - 2 * (p - x[i])*lp
        temp2 = y[i] * temp1 * la * la
        temp3 = dy[i] * (p - x[i]) * la * la
        f = f + temp2 + temp3
    return f
Runge_Kutta(0,2,20,-2)
# interpolation approximation
y1=hermite(1.3,[1.2,1.4],[-1.16637354,-1.11467694],[f(1.2,-1.16637354),f(1.4,-1.11467694)])
y2=hermite(1.93,[1.8,2],[-1.05321755,-1.03599222],[f(1.2,-1.05321755),f(1.4,-1.03599222)])
print()
print('The result y(1.3) applying Runge_Kutta method with h=0.2 is %.8f with error %.8f' ...
    %(y1,y(1.3)-y1)  )
print('The result y(1.93) applying Runge_Kutta method with h=0.2 is %.8f with error %.8f' ...
    %(y2,y(1.93)-y2)  )ult y(1.93) applying ...
    Mid_Point method with h=0.2 is %.8f with ...
    error %.8f' %(y2,y(1.93)-y2)  )
```

6.7 Adams Bashforth's method

```
import math
import numpy as np
from sympy import *
```

```
from matplotlib import pyplot as plt
# Adams_Bashforth's method
def f(t,y):
    return -(y+1)*(y+3)
def y(t):
    return -3+2*(1+math.exp(-2*t))**(-1)
def Adams_Bashforth(a,b,N,alpha):
    w = [[0.5]*(N+10) for a in range(6)]
    w=np.mat(w)
    h=(b-a)/N
    t=a
    w[2,0]=alpha
    w[2,1]=-1.90033209
    w[3,0]=alpha
    w[3,1]=-1.90033209
    w[3,2]=-1.80262486
    w[4,0]=alpha
    w[4,1]=-1.90033209
    w[4,2]=-1.80262486
    w[4,3]=-1.70868768
    w[5,0]=alpha
    w[5,1]=-1.90033209
    w[5,2]=-1.80262486
    w[5,3]=-1.70868768
    w[5,4]=-1.62005146
    k=0
    for i in range(1,N):
        w[2,i+1]=w[2,i]+h/2*(3*f(t,w[2,i])-f(t-h,w[2,i-1]) )
        w[3,i+2]=w[3,i+1]+h/12*(23*f(t,w[3,i+1])-16*f(t-h,w[3,i])+5*f(t-2*h,w[3,i-1]) )
        w[4,i+3]=w[4,i+2]+h/24*(55*f(t,w[4,i+2])-59*f(t-h,w[4,i+1])
        +37*f(t-2*h,w[4,i])-9*f(t-3*h,w[4,i-1]) )
        w[5,i+4]=w[5,i+3]+h/720*(1901*f(t,w[5,i+3])-2774*f(t-h,w[5,i+2])
        +2616*f(t-2*h,w[5,i+1])-1274*f(t-3*h,w[5,i])+251*f(t-4*h,w[5,i-1]))
        t=a+i*h
        k=k+0.1
        print("%.1f & %.8f &%.8f &%.8f & %.8f & %.8f\\\\" ...
                                                            %(k,y(k),w[2,i],w[3,i],w[4,i],w[5,i]) ...
                                                            )
Adams_Bashforth(0,2,20,-2)
```

6.8 Runge-Kutta Method for Systems of Differential Equations

```
import math
import numpy as np
from sympy import *
from matplotlib import pyplot as plt
# initialization
w=[0.5,0.5,0.5,0.5]
k=[[0.5]*5 for i in '12345']
k=np.matrix(k)
def y(t):
    return 43/36*math.exp(t)+1/4*math.exp(-t)-4/9*math.exp(-2*t)+1/6*t*math.exp(t)
def f1(t,w1,w2,w3):
    return w2
def f2(t,w1,w2,w3):
    return w3
def f3(t,w1,w2,w3):
    return 2*w1+w2-2*w3+math.exp(t)
# Runge-Kutta Method for Systems of Differential Equations
```

```
def Runge_Kutta_Systems(a,b,N,alpha):
    h=(b-a)/N
    t=a
    for j in range(1,4):
        w[j]=alpha[j]
    print("%.1f & %.8f & %.8f & %.8f & %.8f & %.8f\\\\" %(t,y(t),w[1],abs(y(t)-w[1]),w[2],w[3]))
    for i in range(1,N+1):
        # 1
        k[1,1]=h*f1(t,w[1],w[2],w[3])
        k[1,2]=h*f2(t,w[1],w[2],w[3])
        k[1,3]=h*f3(t,w[1],w[2],w[3])
        # 2
        k[2,1]=h*f1(t+h/2,w[1]+k[1,1]/2,w[2]+k[1,2]/2,w[3]+k[1,3]/2)
        k[2,2]=h*f2(t+h/2,w[1]+k[1,1]/2,w[2]+k[1,2]/2,w[3]+k[1,3]/2)
        k[2,3]=h*f3(t+h/2,w[1]+k[1,1]/2,w[2]+k[1,2]/2,w[3]+k[1,3]/2)
        # 3
        k[3,1]=h*f1(t+h/2,w[1]+k[2,1]/2,w[2]+k[2,2]/2,w[3]+k[2,3]/2)
        k[3,2]=h*f2(t+h/2,w[1]+k[2,1]/2,w[2]+k[2,2]/2,w[3]+k[2,3]/2)
        k[3,3]=h*f3(t+h/2,w[1]+k[2,1]/2,w[2]+k[2,2]/2,w[3]+k[2,3]/2)
        # 4
        k[4,1]=h*f1(t+h,w[1]+k[3,1],w[2]+k[3,2],w[3]+k[3,3])
        k[4,2]=h*f2(t+h,w[1]+k[3,1],w[2]+k[3,2],w[3]+k[3,3])
        k[4,3]=h*f3(t+h,w[1]+k[3,1],w[2]+k[3,2],w[3]+k[3,3])
        for j in range(1,4):
            w[j]=w[j]+(k[1,j]+2*k[2,j]+2*k[3,j]+k[4,j])/6
        t=a+i*h
        print("%.1f & %.8f & %.8f & %.8f & %.8f & %.8f\\\\" %(t,y(t),w[1],abs(y(t)-w[1]),w[2],w[3]))
Runge_Kutta_Systems(0,3,15,[0,1,2,0])
```