

Exercise on Convex Optimization

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Abstract

this material is personal answers for the exercise on convex optimization written by Stephen Boyd and Lieven Vandenbergh. I only put the main skeletons here.

1 Convex sets

1.1 Definition of convexity

1. $\sum_{i=1}^k \theta_i x_i = (\sum_{i=1}^{k-1} \theta_i) \sum_{i=1}^{k-1} \frac{\theta_i}{\sum_{i=1}^{k-1} \theta_i} x_i + \theta_k x_k$ in which, $\sum_{i=1}^{k-1} \frac{\theta_i}{\sum_{i=1}^{k-1} \theta_i} x_i \in \mathcal{C}$ according to the mathematical induction.
2. Both are direct, since $\theta x + (1-\theta)y \in \mathcal{C}$ is a line.
3. We can construct a sequence by the binary search¹, then for any $\theta > 0$, $\theta x + (1-\theta)y$ is either in the sequence or the limitation of the sequence. Note a closed set in the Euclidean Space is tight, which means the limitation of the sequence (all the points in the sequences belong to the closed set) is in the set.
4. The affine hull is the minimum affine set that contains \mathcal{C} , and the same as the intersection of all the affine sets that contain \mathcal{C} .

¹The binary search means that for two given points x and y , if we want to find a point between them, we can firstly use their middle point. If the middle point is not, then the point must lie in either $[x, \text{middle point}]$ or $[\text{middle point}, y]$ (we assume that $x < y$). Then we can follow the same procedure with x or y replaced with the middle point. Then the point will be found by our procedure, or be the limitation of this procedure, which means we can approximate it as closely as we want.

1.2 Examples

1. $|b_1 - b_2|$
2. When $a = c\tilde{a}, c \neq 0, d \leq \tilde{d}$, the first halfspace belongs to the second one. When $(a, b) = c(\tilde{a}, \tilde{b}), c \neq 0$, they are equal.
3. $\|x - a\|_2 \leq \|x - b\|_2$ equals to $x^T x - 2a^T x + a^T a \leq x^T x - 2b^T x + b^T b$. Then we can get $(b - a)^T x \leq \frac{b^T b - a^T a}{2}$
4. (a) When both are zeros, yes; when one is zero, then it's a line segment, yes; when both are non-zeros, it forms a parallelogram (specially, $a_1 \perp a_2$, it's a square), so yes.
(b) Yes.

$$S = \{x \mid -\mathcal{I}x \preceq 0, \begin{bmatrix} 1^T \\ a^T \\ a^{2T} \end{bmatrix} x = \begin{bmatrix} 1 \\ b_1 \\ b_2 \end{bmatrix}\}$$

- (c) Yes. $S = \{x \mid x \preceq 1, -x \preceq 0\}$. Note, y is on the unit ball. $x^T y$ is the line passing through origin with y as its normal. In fact, it involves the space.
- (d) Yes, the same as above.
5. (a) From Exercise 7, we know that V is the intersection of all the halfspaces, then it's polyhedron.
(b) We need to choose a interior point from S (S has nonempty interior). Then for each halfspace expression, we can reversely construct the corresponding equations x_i according to Exercise 7.
(c) Since no specific rules for choosing x_0 . We can then say V_k is polyhedon, for $k \in 0, \dots, K$. For each pair of V_i and V_k (i is not equal to k), their intersection is the boundary between them. For any point in \mathcal{R} it must be near to one point from x_0, \dots, x_K . Then the union of V_k is the whole space. *The latter question.*