Exercise on Convex Optimization

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Abstract

this material is personal answers for the exercise on convex optimization wirtten by stephen boyd and lieven vandenberghe. i only put the main skeletons here.

1 Convex sets

1.1 Definition of convexity

1. $\sum_{i=1}^{k}\theta_k x_k = (\sum_{i=1}^{k-1}\theta_i)\sum_{i=1}^{k-1}\frac{\theta_i}{\sum_{i=1}^{k-1}\theta_i}x_i + \theta_k x_k$ in which, $\sum_{i=1}^{k-1}\frac{\theta_i}{\sum_{i=1}^{k-1}\theta_i}x_i \in \rfloor$ acording to the mathmatical induction.

2. Both are direct, since θ x + (1-θ)y $\in$ a line.

3. We can construct a sequence by the binary search [1], then for any $\theta > 0$, θ x + (1 - θ) y is either in the sequence or the limitation of the sequence. Note a closed set in the Euclean Space is tight, which means the limitation of the sequence (all the points in the sequences belong to the closed set) is in the set.

4. The affine hull is the minumum affine set that contains $\mathcal{C}$, and the same as the intersection of all the affine sets that contain $\mathcal{C}$.

[1] The binary search means that for two given points x and y, if we want to find a point between them, we can firstly use their middle point. If the middle point is not, then the point must lie in either [x, middle point] or [middle point, y] (we assume that x $<$ y). Then we can follow the same procedure with x or y replaced with the middle point. Then the point will be in found by our procedure, or be the limitation of this procedure, which means we can approximate it as closely as we want.

1.2 Examples

1. |b$_1$ - b$_2$|

2. When $a = c\tilde{a}, c \neq 0, d \leq \tilde{d}$, the first halfspace belongs to the second one. When $(a, b) = c(\tilde{a}, \tilde{b}), c \neq 0$, they are equal.

3. $||x - a||_2 \leq ||x - b||_2$ equals to $x^T x - 2a^T x + a^T a \leq x^T x - 2b^T x + b^T b$. Then we can get $(b - a)^T x \leq \frac{b^T b - a^T a}{2}$

4. (a) When both are zeros, yes; when one is zero, then it's a line segment, yes; when both are non-zeros, it forms a parallelogram (specially, $a_1 \perp a_2$, it's a square), so yes.
   (b) Yes.
   $$S = \{x| - \mathcal{I}x \preceq 0, \begin{bmatrix} 1^T \\ a^T \\ a^{2T} \end{bmatrix} x = \begin{bmatrix} 1 \\ b_1 \\ b_2 \end{bmatrix}\}$$
   (c) Yes. $S = \{x | x \preceq 1, -x \preceq 0\}$. Note, y is on the unit ball. $x^T y$ is the line passing through origin with y as its normal. In fact, it involves the space.
   (d) Yes, the same as above.

5. (a) From Exercise 7, we know that V is the intersection of all the halfspaces, then it's polyhedron.
   (b) We need to choose a interior point from S (S has nonempty interior). Then for each halfspace expression, we can reversely construct the corresponding equations x$_i$ according to Exercise 7.
   (c) Since no specific rules for choosing x$_0$. We can then say V$_k$ is polyhedon, for k $\in$ 0, ..., K. For each pair of V$_i$ and V$_k$ (i is not equal to k), their intersection is the boundary between them. For any point in $\mathcal{R}$ it must be near to one point from x$_0$,..., x$_K$. Then the union of V$_k$ is the whole space. *The latter question.*

6. (a) If $A \succeq 0$, i.e., A is a symmetric positive semidefinite matrix. For any x$_1$, x$_2$, we have,
   $$(x_1 - x_2)^T A(x_1 - x_2) = x_1^T Ax_1 + x_2^T Ax_2 - 2x_1^T Ax_2^T \geq 0$$
   Then, we get
   $$2x_1^T Ax_2^T \leq x_1^T Ax_1 + x_2^T Ax_2$$
   Then for any convex combination [2] of x$_1$ and x$_2$, y, we have:
   $$\begin{aligned}
   & y^T y + b^T y + c \\
   & = \theta^2 x_1^T Ax_1 + (1-\theta)^2 x_2^T Ax_2 + 2\theta(1-\theta)x_1^T Ax_2 + b^T(\theta x_1 + (1-\theta)x_2) + (\theta + 1 - \theta)c \\
   & \leq \theta^2 x_1^T Ax_1 + (1-\theta)^2 x_2^T Ax_2 + \theta(1-\theta)(x_1^T Ax_1 + x_2^T Ax_2) + b^T(\theta x_1 + (1-\theta)x_2) + (\theta + 1 - \theta)c \\
   & \leq \theta(x_1^T Ax_1 + b^T x_1 + c) + (1-\theta)(x_2^T Ax_2 + b^T x_2 + c) \\
   & \leq 0
   \end{aligned}$$

[2] Here we define a convex combination is a mapping from the production of two same regular space to the same space with parameter $\theta \in [0, 1]$: $f(x_1, x_2) = \theta x_1 + (1 - \theta)x_2$