Online Appendix for Aging and Sectoral Productivity Gap

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1 Appendix

A Proofs and Model Appendix

A.1 Derivation for Population Evolution

The evolution of population of the economy is characterized by the following transition parameter: (1) ω is the probability of remaining as a worker in the next period; (2) γ is the probability that a retiree will survive in the next period; (3) n is the population growth rate.

The average time for an individual to stay as a worker is given by

$$T_w = \sum_{t=0}^{\infty} (\omega)^t = \frac{1}{1-\omega}, \quad \omega \in (0,1).$$

The average retirement period for a retiree is given by

$$T_r = \sum_{t=0}^{\infty} (\gamma)^t = \frac{1}{1-\gamma}, \quad \gamma \in (0,1).$$

Let $\mathcal{R}_t$ be the total survival retirees in period t, recall the probability for a worker to become a retiree is $1-\omega$, so $\mathcal{R}_t$ is given by

$$\mathcal{R}_t = \sum_{j=1}^{\infty} \gamma^{j-1}(1-\omega)N_{t-j},$$

combine with population growth where

$$N_{t-j}(1+n)^j = N_t,$$

we obtain

$$\mathcal{R}_t = \sum_{j=1}^{\infty} \gamma^{j-1}(1-\omega)\frac{N_t}{(1+n)^j} = \left(\frac{1-\omega}{1+n}\right) N_t \sum_{t=0}^{\infty} \left(\frac{\gamma}{1+n}\right)^j,$$

thus $\mathcal{R}_t = N_t \left(\frac{1-\omega}{1+n-\omega}\right)$. The ratio of retirees to workers ψ is given by

$$\psi = \frac{\mathcal{R}_t}{N_t} = \frac{1-\omega}{1+n-\gamma}.$$

A.2 Proof for Proposition 1

Retiree's Marshallian demand function

$$c^r_{s,t} = \frac{\theta^r_s(e^r_t, p_{s,t})e^r_t}{p_{s,t}} = \left[\phi^r + \nu^r \left(\frac{p^{\phi^r}_{s,t}}{e^r_t}\right)^{\eta^r}\right] \cdot \frac{e^r_t}{p_{s,t}} = \phi^r \left(\frac{e^r_t}{p_{s,t}}\right) + \nu^r \frac{p^{\phi^r \eta^r - 1}_{s,t}}{(e^r_t)^{\eta^r - 1}}$$

and

$$c^r_{m,t} = (1-\theta^r_s)e^r_t = \left(1 - \phi^r - \nu^r \left(\frac{p^{\phi^r}_{s,t}}{e^r_t}\right)^{\eta^r}\right) e^r_t$$

Worker's Marshallian demand function

$$c^w_{i,s,t} = \frac{\theta^w_s(e^w_{i,t}, p_{s,t})e^w_{i,t}}{p_{s,t}} = \phi^w \left(\frac{e^w_{i,t}}{p_{s,t}}\right) + \nu^w \frac{p^{\phi^w \eta^w - 1}_{s,t}}{(e^w_{i,t})^{\eta^w - 1}}$$

and

$$c^w_{i,m,t} = (1-\theta^w_s)e^w_{i,t} = \left(1 - \phi^r - \nu^r \left(\frac{p^{\phi^w}_{s,t}}{e^w_{i,t}}\right)^{\eta^w}\right) e^w_{i,t}$$

Aggregate consumption

$$C_s(p_s, \psi) = \psi N \cdot c^r_s + N \int c^w_{i,s} dG = \psi N \cdot c^r_s + N\mathbb{E}[c^w_s]$$

and

$$C_m(p_s, \psi) = \psi N \cdot c^r_m + N \int c^w_{i,m} dG = \psi N \cdot c^r_m + N\mathbb{E}[c^w_m]$$

And efficient labor supply is given by

$$L_s(p_s) = \int_{\Omega_s(p_s)} z^i_s dG, \quad L_m(p_s) = \int_{\Omega_m(p_s)} z^i_m dG,$$

where $\Omega_s(p_s) = \{i : p_s z^i_s \geq z^i_m\}$, and $\Omega_m(p_s) = \{i : p_s z^i_s < z^i_m\}$

At Market Clearing we have

$$Y_s = C_s, \quad Y_m = C_m$$

So

$$D(p_s, \psi) := \frac{C_s(p_s, \psi)}{C_m(p_s, \psi)} - \frac{L_s(p_s)}{L_m(p_s)} = 0.$$

And IFT:
$$\frac{\partial p_s}{\partial \psi} = -\frac{D_\psi}{D_{p_s}}$$

We first calculate D_ψ

$$D_\psi = \frac{C_m \partial C_s / \partial \psi - C_s \partial C_m / \partial \psi}{C_m^2} = \frac{N}{C_m^2}\left(C_m c_s^r - C_s c_m^r\right)$$

Simplifying:
$$\begin{aligned} D_\psi &= \frac{N}{C_m^2}\left[(\psi N c_m^r + N\mathbb{E}[c_m^w])c_s^r - (\psi N c_s^r + N\mathbb{E}[c_s^w])c_m^r\right] \\ &= \frac{N^2}{C_m^2}\left[(\psi c_m^r + \mathbb{E}[c_m^w])c_s^r - (\psi c_s^r + \mathbb{E}[c_s^w])c_m^r\right] \\ &= \frac{N^2}{C_m^2}\underbrace{\left(c_s^r \mathbb{E}[c_m^w] - c_m^r \mathbb{E}[c_s^w]\right)}_{\Delta} \end{aligned}$$

So
$$\text{sign}(D_\psi) = \text{sign}\left(\frac{c_s^r}{c_m^r} - \frac{\mathbb{E}[c_s^w]}{\mathbb{E}[c_m^w]}\right)$$

It is convenient to see that $\frac{\partial}{\partial p_s}\left(\frac{C_s}{C_m}\right) < 0$, and from the previous cutoff rule, a increased p_s will lower the threshold $z_m/z_s \leq p_s$, more workers will enter service sector, thus $\frac{\partial}{\partial p_s}\left(\frac{L_s}{L_m}\right) > 0$, thus $D_p < 0$. By IFT,
$$\frac{dp_s}{d\psi} = -\frac{D_\psi}{D_p} > 0, \quad \text{iff} \quad \theta_s^r > \theta_s^w$$

Thus **Proposition 1** is proved.

A.3 Conditional Expectations

When $\alpha + \kappa = 0$, we have $\alpha = -\kappa$ ($\alpha < 0$), and

$$\ln z_s(q) = \bar{\alpha} - \kappa \ln q, \quad \ln z_m(q) = \bar{\alpha} - \kappa \ln(1-q).$$

For $q \sim U(0,1)$, assume q follows a unit uniform distribution,

$$E(\ln q | q \leq q_s) = \frac{1}{q_s}\int_0^{q_s} \ln q dq = \ln q_s - 1,$$

$$E(\ln(1-q_s)|q > q_s) = \frac{1}{1-q_s}\int_{q_s}^1 \ln(1-q)dq = \ln(1-q) - 1,$$

substitute the calculated expectations to the original expression of conditional expectations to obtain

$$\boxed{E(\ln z_s|q \leq q_s) = \bar{\alpha} - \kappa \ln q_s + \kappa, \quad E(\ln z_m|q > q_s) = \bar{\alpha} - \kappa \ln(1 - q_s) + \kappa}.$$

The derivation is similar for the case when $\alpha > 0$.

A.4 Steady State

At equilibrium, taking the exogenous population evolution process, economy-wide technology and individual productivity distribution $\{n, \omega, \psi, \gamma, \beta, \nu^w, \nu^r, \phi^r, \phi^w, \eta^r, \eta^w, R_t, X_t, G(z_s, z_m)\}$ as given, the sequence of endogenous state variables are$\{A_t^r, A_t^w, \mathbf{A}_t, N_t\}$ and control variables are$\{e_t^r, e_{i,t}^w, \mathbf{C}_{s,t}, \mathbf{C}_{m,t}, T_t, L_{s,t}, L_{m,t}, \mathbf{W}_t, \mathbf{Y}_{s,t}, \mathbf{Y}_{m,t}, p_{s,t}, N_{s,t}, N_{m,t}, \Omega_{s,t}, \Omega_{m,t}\}$.

The per capita steady state system is given by the below conditions:

A.4.1 Production

$$Y_s = X\bar{L}_s \tag{1}$$

$$Y_m = X\bar{L}_m \tag{2}$$

A.4.2 Wage rate

$$w_s = p_s X \tag{3}$$

$$w_m = X \tag{4}$$

A.4.3 Labor allocation

$$\frac{z_s^i}{z_m^i} \leq p_s \tag{5}$$

$$\Omega_s = \left\{ i | \frac{z_m^i}{z_s^i} \leq p_s \right\} \tag{6}$$

$$\Omega_m = \left\{ i | \frac{z_m^i}{z_s^i} > p_s \right\} \tag{7}$$

$$\bar{L}_s = \frac{\int_{i \in \Omega_s} z_s^i dG}{N} \tag{8}$$

$$\bar{L}_m = \frac{\int_{i \in \Omega_m} z_m^i dG}{N} \tag{9}$$

A.4.4 Consumption:

$$\theta_s^r(e^r, p_s) = \phi^r + \nu^r \left(\frac{p_s^{\phi^r}}{e^r} \right)^{\eta^r} \tag{10}$$

$$\theta_s^w(e_i^w, p_s) = \phi^w + \nu^w \left(\frac{p_s^{\phi^w}}{e_i^w} \right)^{\eta^w} \tag{11}$$

$$\bar{C}_s = \psi \frac{\theta_s^r e^r}{p_s} + \frac{\mathbb{E}[\theta_s^w e^w]}{p_s} \tag{12}$$

$$\bar{C}_m = \psi(1 - \theta_w^r)e^r + \mathbb{E}[(1 - \theta_s^w)e^w] \tag{13}$$

A.4.5 Wealth

$$\bar{W} = p_s X \bar{L}_s + X \bar{L}_m \tag{14}$$

$$e_r = (R/\gamma - 1)A_r \tag{15}$$

$$e_{i,w} = (R - 1)A_r + W_i \tag{16}$$

$$A = \psi N A_r + N\mathbb{E}[A_w] \tag{17}$$

A.5 Model Extension with Skill Types and Automation

We now extend our baseline model to incorporate automation as a mechanism for labor replacement, and to allow for differentiated roles of high- and low-skill workers in production, following the framework of Buera et al. (2022). In the benchmark model, overall technological progress is introduced exogenously and lies outside the production function, implying that automation–labor substitution is neglected. While the benchmark specification shows that using GDP per capita to approximate technological progress absorbs much of the dynamic effect of aging, the extended model allows us to explicitly identify and analyze the automation channel through which aging may affect sectoral productivity and labor allocation. By doing so, we can amplify the selection effect of aging without additional modification to the skill distribution.

A.5.1 Production

In the extended model, we assume the production of service and manufacturing share similar CES form, but in manufacturing, the production is given by a mixture of machine and labor. The production technologies for service and manufacturing are give by

$$Y_s = \bar{A}\left[\alpha_s H_s^{\frac{\rho-1}{\rho}} + (1-\alpha_s)L_s^{\frac{\rho-1}{\rho}}\right]^{\frac{\rho}{\rho-1}}, \quad Y_m = \bar{A}\left[\alpha_m H_m^{\frac{\rho-1}{\rho}} + (1-\alpha_m)X_m^{\frac{\rho-1}{\rho}}\right]^{\frac{\rho}{\rho-1}}$$

and

$$X_m = \left[\mu L_m^{\frac{\xi-1}{\xi}} + (1-\mu)A_m^{\frac{\xi-1}{\xi}}\right]^{\frac{\xi}{\xi-1}},$$

where Y_s is the output of service sector and Y_m is the output of manufacturing sector, H_j is the aggregate efficient labor of high skill worker and L_j is the aggregate efficient labor of low skill worker. X_m is the manufacturing specific production task composite, which is intuitively relative codifiable thus can be completed by both low skill workers and machine. And the efficient labor aggregators are given by

$$H_j = \int z_j^i dG(z_{H,s}, z_{H,m}), \quad j \in (s,m)$$

and

$$L_j = \int z_j^i dG(z_{L,s}, z_{L,s}), \quad j \in (s,m).$$

A.5.2 Labor Force

Compared to the benchmark, now the workers are classified into high skill and low skill type to specify the contribution of different types of worker in the production. Their skills distribution is given by two component, general skill and sector specific skill:

$$z_j^i = g a_j^i, \quad j \in \{s,m\}$$

Let $g_L = 1$ and $g_H = \chi$ with $\chi > 1$, we have

$$z_{H,j}^i = \chi a_j^i \quad \text{and} \quad z_{L,j}^i = a_j^i, \quad j \in \{s,m\}$$

where a_j^i is drawn from a joint Fréchet distribution $G(a_s, a_m)$ controlled by Frank Copula which is same as the benchmark setting. Whether a worker has high or low general skills is decided by a binary distribution:

$$p(j = H) = \pi, \quad p(j = L) = 1 - \pi.$$

Thus the skill-specific labor force supply is pinned down by

$$N_L = (1-\psi)\cdot\pi\cdot N \quad \text{and} \quad N_H = (1-\psi)\cdot(1-\pi)\cdot N,$$

where N is the total population.

A.5.3 Automation

Manufacturing firms choose the level of R&D investment I to endogenize the input share of machine $(1-\mu(I))$. Following task-based models of automation (Acemoglu and Restrepo, 2019, 2022; Acemoglu et al., 2024), firms can spend resources to reassign an increasing set of tasks from labor to capital. In those models, automation expands along a "task frontier": firms first automate the cheapest, most codifiable tasks and as wages rise and adoption costs are paid, extend automation to progressively more complex tasks. As a result, the share of non-automated tasks falls with automation investment but with diminishing marginal returns. To capture this diminishing marginal returns and let $\mu(I)$ take the form of

$$\mu(I) = \bar{\mu}\exp(-\kappa I),$$

where $\bar{\mu}$ is the employed share of low skill worker without the process of automation and κ is the marginal diminishing scale parameter.

A.5.4 Equilibrium

Normalize the price of manufacturing goods as 1, and manufacturing firms solve the following profit maximization problem.

$$\max_{H_m,A_m,L_m,I} \quad \Pi_m = Y_m - w_{m,H}H_m - p_A A_m - w_{m,L}L_m - I. \quad \text{s.t.} \quad X_m \le F(H,A)$$

From the FOCs we have

$$w_{m,H} = \bar{A}\left[\alpha_m H_m^{\frac{\rho-1}{\rho}} + (1-\alpha_m)X_m^{\frac{\rho-1}{\rho}}\right]^{\frac{1}{\rho-1}} \alpha_m H_m^{-\frac{1}{\rho}},$$

$$w_{m,L} = \bar{A}\left[\alpha_m H_m^{\frac{\rho-1}{\rho}} + (1-\alpha_m)X_m^{\frac{\rho-1}{\rho}}\right]^{\frac{1}{\rho-1}} \cdot (1-\alpha_m)\,\mu X_m^{\frac{1}{\xi}-\frac{1}{\rho}} \cdot L_m^{-\frac{1}{\xi}}$$

and

$$p_A = \bar{A}\left[\alpha_m H_m^{\frac{\rho-1}{\rho}} + (1-\alpha_m)X_m^{\frac{\rho-1}{\rho}}\right]^{\frac{1}{\rho-1}} \cdot (1-\alpha_m)\,(1-\mu)X_m^{\frac{1}{\xi}-\frac{1}{\rho}} \cdot A_m^{-\frac{1}{\xi}}.$$

Rearranging the FOCs to obtain the optimized input ratio:

$$\frac{A_m}{L_m} = \left(\frac{1-\mu}{\mu}\frac{w_{m,L}}{p_A}\right)^{\xi}.$$

Since the choice of automation degree is also endogenized, we obtain the following implicit equation for R&D investment I:

$$\kappa\mu_0 \exp(-\kappa I^\star) \cdot \frac{\xi}{\xi - 1} \cdot \bar{A}(1-\alpha_m)\left(\alpha_m H_m^{\frac{\rho-1}{\rho}} + (1-\alpha_m) X_m^{\frac{\rho-1}{\rho}}\right)^{\frac{1}{\rho-1}} \cdot X_m^{\frac{1}{\xi}-\frac{1}{\rho}} \cdot \left(A_m^{\frac{\xi-1}{\xi}} - L_m^{\frac{\xi-1}{\xi}}\right) = 1.$$

For service firms we have similar optimized wage rate condition:

$$w_{s,H} = p_s \bar{A}\left[\alpha_s H_s^{\frac{\rho-1}{\rho}} + (1-\alpha_s) L_s^{\frac{\rho-1}{\rho}}\right]^{\frac{1}{\rho-1}} \alpha_s H_s^{-\frac{1}{\rho}}$$

and

$$w_{s,L} = p_s \bar{A}\left[\alpha_s H_s^{\frac{\rho-1}{\rho}} + (1-\alpha_s) L_s^{\frac{\rho-1}{\rho}}\right]^{\frac{1}{\rho-1}} (1-\alpha_s) L_s^{-\frac{1}{\rho}}.$$

Workers choose the sector with highest pay and enter service sector if

$$\frac{z^i_{m,H}}{z^i_{s,H}} \leq \frac{w_{s,H}}{w_{m,H}} = p_s \frac{MPH_s}{MPH_m}$$

and for the low skill workers

$$\frac{z^i_{m,L}}{z^i_{s,L}} \leq \frac{w_{s,L}}{w_{m,L}} = p_s \frac{MPL_s}{MPL_m}.$$

A.5.5 Calibration

A.5.6 Results

B Graphs and Figures

B.1 Tables

Table 1: Expenditure shares by sector and period (%)

	Including housing				Excluding housing			
	82-91	92-01	02-11	12-23	82-91	92-01	02-11	12-23
Goods	**43.5**	**40.6**	**37.3**	**35.5**	**55.0**	**53.1**	**50.3**	**47.4**
Food at home	12.2	11.6	10.9	11.2	15.4	15.2	14.8	15.0
Vehicle	11.0	11.7	9.8	9.5	13.9	15.3	13.2	12.8
Gas	4.6	3.5	4.9	4.4	5.8	4.5	6.6	5.9
Entertainment equipment	3.8	3.9	4.0	3.6	4.7	5.1	5.4	4.8
Appliances	2.2	2.2	1.8	1.9	2.8	2.9	2.4	2.5
Clothing	3.1	2.4	1.5	0.9	4.0	3.1	2.1	1.2
Furniture & fixtures	2.1	1.6	1.4	1.4	2.6	2.2	1.8	1.9
Alcoholic beverages	1.3	1.0	0.9	1.0	1.7	1.3	1.2	1.3
Shoes & other apparel	1.3	1.0	0.7	0.6	1.7	1.3	1.0	0.7
Tobacco	1.1	0.9	0.8	0.6	1.3	1.2	1.0	0.8
Children's clothing	0.8	0.8	0.6	0.4	1.1	1.0	0.8	0.5
Personal care	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Services	**56.4**	**59.6**	**62.8**	**64.6**	**45.0**	**46.8**	**49.6**	**52.7**
Health	5.0	5.6	6.5	8.6	6.3	7.4	8.7	11.6
Utilities	7.9	7.7	8.2	7.8	10.0	10.1	11.1	10.4
Cash contributions	2.6	2.7	3.2	3.3	3.2	3.5	4.3	4.4
Car maintenance	4.3	4.5	3.9	4.0	5.5	5.8	5.2	5.4
Food away from home	5.0	4.5	4.8	5.1	6.3	5.8	6.5	6.8
Domestic services	3.2	3.2	3.3	3.7	4.0	4.3	4.5	5.0
Education	1.6	1.8	2.3	2.3	2.0	2.3	3.1	3.2
Entertainment	2.3	2.2	1.9	1.5	2.9	2.9	2.5	2.1
Public transport	1.3	1.4	1.2	1.3	1.7	1.8	1.6	1.8
Personal care	1.0	0.9	0.7	0.7	1.3	1.2	0.9	0.9
Housing	20.8	23.8	25.9	25.5	.	.	.	.
Personal insurance	1.4	1.3	0.9	0.8	1.8	1.7	1.2	1.1

Notes: Calculations are based on the classification rule of Aguiar and Bils (2015).

Table 2: CPS Sectoral Classification

2 digit Code	Industry
Manufacturing	
2	Mining
3	Construction
4	Durable goods manufacturing
5	Non-durable goods manufacturing
Service	
6	Wholesale trade
7	Retail trade
8	Transportation and warehousing
9	Utilities
10	Information
11	Finance and insurance
12	Real estate and rental and leasing
13	Professional, scientific, and technical services
14	Management, administrative support, and waste management
15	Educational services
16	Health care and social assistance
17	Arts, entertainment, and recreation
18	Accommodation and food services
19	Private households
20	Other services, except private households
21	Public administration

Table 3: Annualized Productivity and Employment Share Change of U.S. (%)

2000-2017	Productivity Gap	Emp. Share	Va. Share
Prediction:	-2.62	0.20	0.22
Actual:	-6.15	0.37	0.21

B.2 Graphs

B.2.1 Using US Real GDP Per Capita at 2017 as Benchmark (=1)

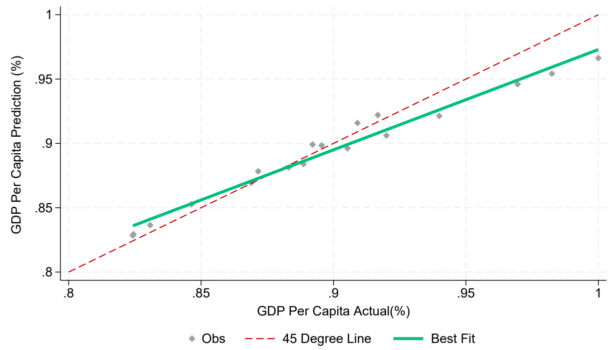

Figure 1: U.S. GDP Per Capita: Prediction vs Actual

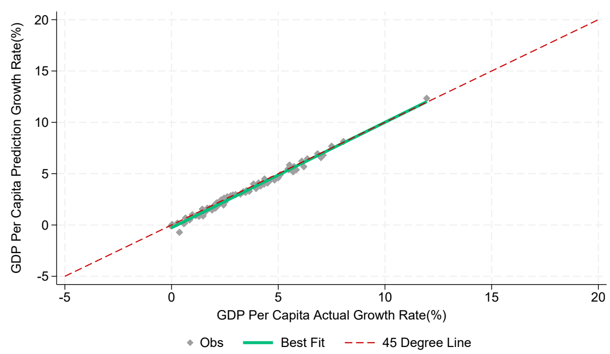

Figure 2: Cross Country GDP Per Capita: Prediction vs Actual

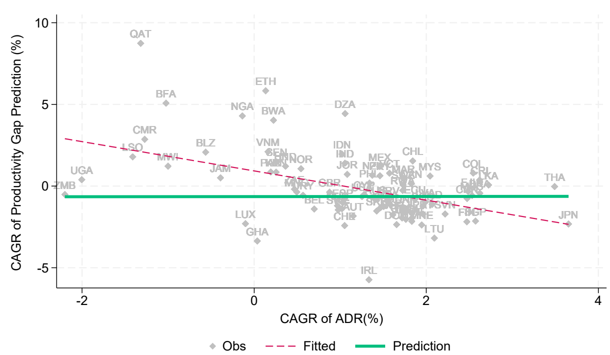

Figure 3: Dynamics: Productivity Gap vs ADR

References

Daron Acemoglu and Pascual Restrepo. Automation and new tasks: How technology displaces and reinstates labor. *Journal of economic perspectives*, 33(2):3–30, 2019.

Daron Acemoglu and Pascual Restrepo. Demographics and automation. *The Review of Economic Studies*, 89(1):1–44, 2022.

Daron Acemoglu, Fredric Kong, and Pascual Restrepo. Tasks at work: comparative advantage, technology and labor demand. 2024.

Mark Aguiar and Mark Bils. Has consumption inequality mirrored income inequality? *American Economic Review*, 105(9):2725–2756, 2015.

Francisco J Buera, Joseph P Kaboski, Richard Rogerson, and Juan I Vizcaino. Skill-biased structural change. *The Review of Economic Studies*, 89(2):592–625, 2022.