6.9

$$V^\alpha{}_{;\alpha} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,V^\alpha\right)_{,\alpha}$$ ① ↓ ?

$$V^\alpha{}_{;\alpha} = \frac{1}{r}\frac{\partial}{\partial r}(r V^\alpha) + \frac{\partial}{\partial \theta} V^\theta$$ ②

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$$g = \det(g_{\alpha\beta}) = r^2$$

$$V^\alpha{}_{;\alpha} = \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}\,V^\alpha\right)_{,\alpha}$$

$$= \frac{1}{\sqrt{|g|}}\left[\frac{\partial\sqrt{|g|}}{\partial x^\alpha}\cdot V^\alpha + \sqrt{|g|}\cdot\frac{\partial V^\alpha}{\partial x^\alpha}\right]$$

$$= \frac{1}{r}\left[V^r + r\cdot\left[\frac{\partial V^r}{\partial r} + \frac{\partial V^\theta}{\partial \theta}\right]\right]$$

$$= \frac{1}{r}V^r + \frac{\partial V^r}{\partial r} + \frac{\partial V^\theta}{\partial \theta}$$

$$= \frac{1}{r}\left(\frac{\partial}{\partial r}(r V^r)\right) + \frac{\partial V^\theta}{\partial \theta}$$

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$$\det(g_{ij}) = r^2 \cdot r^2\sin^2\theta - 0 + 0$$

$$= r^4 \sin^2\theta$$

$$V^\alpha{}_{;\alpha} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,V^\alpha\right)_{,\alpha}$$

$$= \frac{1}{r^2\sin\theta}\left(r^2\sin\theta\, V^\alpha\right)_{,\alpha}$$

$$= \frac{1}{r^2\sin\theta}\left(\partial_\alpha\left(r^2\sin\theta\right)\cdot V^\alpha + r^2\sin\theta\,\partial_\alpha V^\alpha\right)$$

$$2r\sin\theta \cdot V^r + r^2\cos\theta\, V^\theta$$

$$r^2\sin\theta\left(\partial_r V^r + \partial_\theta V^\theta + \partial_\phi V^\phi\right)$$

$$= \frac{2}{r}V^r + \frac{\cos\theta}{\sin\theta}V^\theta + \partial_r V^r + \partial_\theta V^\theta + \partial_\phi V^\phi$$

6.11

$$V^{\alpha}{}_{;\gamma\beta} = V^{\alpha}{}_{;\beta\gamma} \quad \left\{ V^{\alpha}{}_{;\beta} + \Gamma^{\alpha}{}_{\mu\beta} V^{\mu} = 0 \right.$$

$$V^{\alpha}{}_{,\beta\gamma} + \left(\Gamma^{\alpha}{}_{\mu\beta} V^{\mu}\right)_{,\gamma} = 0$$

$$V^{\alpha}{}_{,\beta\gamma} + \Gamma^{\alpha}{}_{\mu\beta,\gamma} V^{\mu} + \Gamma^{\alpha}{}_{\mu\beta} V^{\mu}{}_{,\gamma} = 0$$

$$V^{\alpha}{}_{,\gamma\beta} + \Gamma^{\alpha}{}_{\mu\gamma,\beta} V^{\mu} + \Gamma^{\alpha}{}_{\mu\gamma} V^{\mu}{}_{,\beta} = 0$$

$$\left(\Gamma^{\alpha}{}_{\mu\beta,\gamma} - \Gamma^{\alpha}{}_{\mu\gamma,\beta}\right) V^{\mu} + \left(\Gamma^{\alpha}{}_{\mu\beta} V^{\mu}{}_{,\gamma} - \Gamma^{\alpha}{}_{\mu\gamma} V^{\mu}{}_{,\beta}\right) = 0$$

$$\underline{V^{\mu}{}_{,\gamma} = -\Gamma^{\mu}{}_{\sigma\gamma} V^{\sigma}}$$

$$V^{\mu}{}_{,\beta} = -\Gamma^{\mu}{}_{\sigma\beta} V^{\sigma}$$

$$\left(\Gamma^{\alpha}{}_{\mu\beta,\gamma} - \Gamma^{\alpha}{}_{\mu\gamma,\beta}\right) V^{\mu} = \left(\Gamma^{\alpha}{}_{\mu\beta} \Gamma^{\mu}{}_{\sigma\gamma} - \Gamma^{\alpha}{}_{\mu\gamma} \Gamma^{\mu}{}_{\sigma\beta}\right) V^{\sigma}$$

$\square$

11b

$$\left(\Gamma^{\alpha}_{\mu\beta,\nu} - \Gamma^{\alpha}_{\mu\nu,\beta}\right)V^{\mu} = \left(\Gamma^{\alpha}_{\sigma\beta}\Gamma^{\sigma}_{\mu\nu} - \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\mu\beta}\right)V^{\mu}$$

$$\left(\Gamma^{\alpha}_{\mu\beta,\nu} - \Gamma^{\alpha}_{\mu\nu,\beta} - \Gamma^{\alpha}_{\sigma\beta}\Gamma^{\sigma}_{\mu\nu} + \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\mu\beta}\right)V^{\mu} = 0$$

6.29

$$g_{\alpha\beta} = \mathrm{diag}(r^2, r^2\sin^2\theta) \qquad \underline{\underline{R_{\theta\phi\theta\phi}}}$$

$$\begin{vmatrix} 0 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin\theta \end{vmatrix}$$

$$R^{\theta}{}_{\phi\theta\phi} = \Gamma^{\theta}_{\phi\phi,\theta} - \cancel{\Gamma^{\theta}_{\phi\theta,\phi}} + \cancel{\Gamma^{\theta}_{\sigma\theta}\Gamma^{\sigma}_{\phi\phi}} - \Gamma^{\theta}_{\sigma\phi}\Gamma^{\sigma}_{\phi\theta}$$

$$\Gamma^{\theta}_{\phi\phi} = -\cos 2\theta + \cos^2\theta$$
$$= \sin^2\theta \cancel{- \cos^2\theta + \cos^2\theta}$$

$$\cancel{\Gamma^{\theta}_{\phi\theta}}$$

$$\Gamma^{\theta}_{\sigma\theta}\Gamma^{\sigma}_{\phi\phi} = \cancel{\Gamma^{\theta}_{r\theta}\Gamma^{r}_{\phi\phi}} + \cancel{\Gamma^{\theta}_{\theta\theta}\Gamma^{\theta}_{\phi\phi}} + \cancel{\Gamma^{\theta}_{\phi\theta}\Gamma^{\phi}_{\phi\phi}}$$

$$\Gamma^{\theta}_{\sigma\phi}\Gamma^{\sigma}_{\phi\theta} = \cancel{\Gamma^{\theta}_{r\phi}\Gamma^{r}_{\phi\theta}} + \cancel{\Gamma^{\theta}_{\theta\phi}\Gamma^{\theta}_{\phi\theta}} + \underbrace{\Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\phi\theta}}_{= -\cancel{\sin\theta} \; -\cos^2\theta}$$

$$\Gamma^{\alpha}_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta})$$

$$\Gamma^{\theta}_{\phi\phi} = \frac{1}{2}g^{\theta\theta}(\cancel{g_{\theta\phi,\phi}} + \cancel{g_{\theta\phi,\phi}} - g_{\phi\phi,\theta}) = \frac{1}{2}g^{\theta\theta}(-g_{\phi\phi,\theta}) = -\sin\theta\cdot\cos\theta$$

$$\Gamma^{\theta}_{\phi\theta} = \frac{1}{2}g^{\theta\theta}(\cancel{g_{\theta\phi,\theta}} + \cancel{g_{\theta\theta,\phi}} - \cancel{g_{\phi\theta,\theta}})$$

$$\Gamma^{\phi}_{\phi\theta} = \frac{1}{2}g^{\phi\phi}(g_{\phi\phi,\theta} + \cancel{g_{\phi\theta,\phi}} - \cancel{g_{\phi\theta,\phi}}) = \frac{1}{2}r^{-2}\sin^{-2}\theta\cdot 2r^2\sin\theta\cos\theta = \frac{\cos\theta}{\sin\theta}$$

$R^{\theta}{}_{\phi\theta\phi} = \sin^2\theta$

$R_{\theta\phi\theta\phi} = g_{\alpha\theta} R^{\alpha}{}_{\phi\theta\phi} = g_{\theta\theta} R^{\theta}{}_{\phi\theta\phi} = r^2 \sin^2\theta$

$R_{\phi\theta\theta\phi} = -r^2 \sin^2\theta$

$R_{\theta\phi\phi\theta} = -r^2 \sin^2\theta$

Q.E.D

6.30

expand cylinder to flat sheet with one side connected, then use cartesian coordinate.

6.32 a

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = g \qquad g \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \lambda \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

$$\det(g - \lambda I) = 0$$

$$\begin{pmatrix} -\lambda & 1 & 0 & 0 \\ 1 & -\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix}$$

$$(1-\lambda)^2(\lambda^2 - 1) = 0$$

$$(1-\lambda)^2(\lambda+1)(\lambda-1) = 0$$

$1-\lambda = 0 \quad \lambda+1 = 0 \quad \lambda - 1 = 0$

$1 \quad 1 \quad -1 \quad 1$

$a_2 = \lambda a_1$

$a_1 = \lambda a_2$

$a_3 = \lambda a_3$

$a_4 = \lambda a_4$

$\lambda = 1$

$1 \Rightarrow (0\ 0\ 1\ 0)$ $\quad -1 \Rightarrow (\tfrac{1}{\sqrt{2}}\ -\tfrac{1}{\sqrt{2}}\ 0\ 0)$

$(0\ 0\ 0\ 1)$

$(\tfrac{1}{\sqrt{2}}\ \tfrac{1}{\sqrt{2}}\ 0\ 0)$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = S$$

$$A = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \qquad SAS^{-1} = g$$

$\mathrm{signature}(g) = +2 \qquad A = S^{-1} g S$

(b)