⑦

$$\Lambda^{\alpha'}{}_{\beta} = \frac{\partial x^{\alpha'}}{\partial x^{\beta}} = \left(\begin{array}{c|c} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ \hline -\frac{y}{x^2}\cdot\frac{1}{1+\frac{y^2}{x^2}} & \frac{1}{x}\,\frac{1}{1+\frac{y^2}{x^2}} \end{array}\right)$$

$$= \left(\begin{array}{c|c} \frac{x}{r} & \frac{y}{r} \\ \hline \frac{-y}{r^2} & \frac{x}{r^2} \end{array}\right) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{pmatrix}$$

$$\Lambda^{\mu}{}_{\nu'} = \frac{\partial x^{\mu}}{\partial x^{\nu'}} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \cancel{-r\sin\theta}\ \sin\theta & r\cos\theta \end{pmatrix}$$

⑧

ⓐ

$$f = r^2\cos^2\theta + r^2\sin^2\theta + 2r^2\sin\theta\cos\theta$$
$$= r^2(1+\sin 2\theta)$$

$$V^{\alpha'} = \Lambda^{\alpha'}{}_{\beta} V^{\beta} \qquad V^{\beta} = \begin{pmatrix} r^2\cos^2\theta + 3r\sin\theta \\ r^2\sin^2\theta + 3r\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta\cdot V^0 + \sin\theta\cdot V^1 \\ -\frac{\sin\theta}{r}\cdot V^0 + \frac{\cos\theta}{r}\cdot V^1 \end{pmatrix}$$

$$= \begin{pmatrix} r^2\cos^3\theta + 3r\sin\theta\cos\theta + r^2\sin^3\theta + 3r\sin\theta\cos\theta \\ -r\sin\theta\cos^2\theta - 3\sin^2\theta + r\sin^2\theta\cos\theta + 3\cos^2\theta \end{pmatrix} = \begin{pmatrix} V^r \\ V^\theta \end{pmatrix}$$

$$= \begin{pmatrix} r^2(\sin^3\theta + \cos^3\theta) + 3r\sin(2\theta) \\ r(\sin^2\theta\cos\theta - \sin\theta\cos^2\theta) + 3(\cos^2\theta - \sin^2\theta) \end{pmatrix}$$

$$W^\beta = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$W^{\alpha'} = \Lambda^{\alpha'}{}_{\beta} W^\beta$$

$$= \begin{pmatrix} \sin\theta + \cos\theta \\ (\cos\theta - \sin\theta)/r \end{pmatrix}$$

(3)

$$\tilde{d}f \to \left(\frac{\partial f}{\partial r}, \frac{\partial f}{\partial \theta}\right) \qquad f = r^2(1 + \sin 2\theta)$$

$$\to \left(2r(1+\sin 2\theta), r^2 2\cos(2\theta)\right)$$

$$\tilde{d}f \to (2x+2y, 2x+2y)$$

$$\to \left((2x+2y)(\sin\theta + \cos\theta), (2x+2y)\cdot(\cos\theta - \sin\theta)\cdot r\right)$$

$$\left(2r(\cos\theta + \sin\theta)(\sin\theta + \cos\theta), 2r(\sin\theta + \cos\theta)(\cos\theta - \sin\theta)\cdot r\right)$$

$$\to \left\langle 2r(1+\sin 2\theta),\ 2r^2(\cos^2\theta-\sin^2\theta) \right\rangle$$

$$\downarrow$$

$$2r^2\cos 2\theta$$

(c) (i)

$$g_{\mu'\nu'} = \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\mu\nu} \quad A^{\mu}{}_{\nu'} = \begin{pmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta+\sin^2\theta & \underbrace{-r\sin\theta\cos\theta + r\cos\theta\sin\theta}_{0} \\ \underbrace{\cos\theta\cdot -r\sin\theta + \sin\theta\cdot r\cos\theta}_{0} & \underbrace{r^2\sin^2\theta + r^2\cos^2\theta}_{r^2} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$$

$$\tilde{V} = \begin{pmatrix} V_r \\ V_\theta \cdot r^2 \end{pmatrix} = \begin{pmatrix} V_x \Lambda^1{}_{1'} + V_y \Lambda^2{}_{1'} \\ V_x \Lambda^1{}_{2'} + V_y \Lambda^2{}_{2'} \end{pmatrix}$$

$$\tilde{W} = \begin{pmatrix} W_r \\ W_\theta \cdot r^2 \end{pmatrix} = \begin{pmatrix} W_x \Lambda^1{}_{1'} + W_y \Lambda^2{}_{1'} \\ W_x \Lambda^1{}_{2'} + W_y \Lambda^2{}_{2'} \end{pmatrix}$$

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$$V^{\alpha}{}_{,\beta} = \begin{pmatrix} 2x & 3 \\ 3 & 2y \end{pmatrix} = \begin{pmatrix} 2r\cos\theta & 3 \\ 3 & 2r\sin\theta \end{pmatrix}$$

B

$$\Lambda^{\mu'}{}_{\alpha} \Lambda^{\beta}{}_{\nu'} V^{\alpha}{}_{,\beta}$$

$$V^{r}{}_{,r} = \Lambda_{\alpha} \Lambda^{\beta}{}_{1'} V_{,\beta}$$

$$= \Lambda^{1'}{}_{1} \Lambda^{1}{}_{1'} V^{1}{}_{;1} + \Lambda^{1'}{}_{1} \Lambda^{2}{}_{1'} V^{1}{}_{;2} + \Lambda^{1'}{}_{2} \Lambda^{1}{}_{1'} V^{2}{}_{;1} + \Lambda^{1'}{}_{2} \Lambda^{2}{}_{1'} V^{2}{}_{;2}$$

$$= 2r\cos^3\theta + \cos\theta\sin\theta \cdot 3 + \sin\theta\cos\theta \cdot 3 + \sin\theta \cdot \sin\theta \, 2r\sin\theta$$

$$= 2r\cos^3\theta + 6\sin\theta \cdot \cos\theta + 2r\sin^3\theta$$

$$= 2r(\sin^3\theta + \cos^3\theta) + 6\sin\theta \cdot \cos\theta$$

C

$$\begin{pmatrix} 2r\cos\theta & 3 \\ 3 & 2r\sin\theta \end{pmatrix}$$

(d)

$$V^\alpha{}_{,\alpha} = 2\alpha(\sin\theta + \cos\theta)$$

(f)

$$V^\alpha{}_{;\alpha} = \frac{1}{r}\frac{\partial}{\partial r}(rV^r) + \frac{\partial}{\partial \theta}V^\theta$$

$$\begin{pmatrix} V^r \\ \\ V^\theta \end{pmatrix} = \begin{pmatrix} r^2(\sin^3\theta + \cos^3\theta) + 3r\sin(2\theta) \\ \\ r(\sin^2\theta\cos\theta - \sin\theta\cos^2\theta) + 3(\cos^2\theta - \sin^2\theta) \end{pmatrix}$$

$$V^\alpha{}_{;\alpha} = \frac{1}{r}\left[3r^2(\cos^3\theta + \sin^3\theta) + 6r\sin(2\theta)\right]$$

$$+ r\left[2\overset{\sin\theta}{\cos\theta}\cdot\cos\theta - \sin^2\theta\sin\theta - \cos\theta\cos^2\theta + 2\sin\theta\cdot\cos\theta\cdot\sin\theta\right]$$

$$- 3\cdot 2\cos\theta\cdot\sin\theta - 3\cdot 2\sin\theta\cdot\cos\theta$$

$$= 3r(\cos^3\theta + \sin^3\theta) + \cancel{6\sin(2\theta)} + r\left[2\sin\theta\cos^2\theta - \sin^3\theta - \cos^3\theta + 2\sin^2\theta\cos\theta\right]$$

$$\cancel{-6\sin2\theta}$$

$$= 2r\left(\cos^3\theta + \sin^3\theta + \sin\theta\cos^2\theta + \sin^2\theta\cos\theta\right)$$

$$= 2r\left(\sin\theta(\sin^2\theta + \cos^2\theta) + \cos\theta(\cos^2\theta + \sin^2\theta)\right)$$

$$= 2r(\sin\theta + \cos\theta)$$

(12) (a)

$$P_{\alpha,\beta} = \begin{pmatrix} 2x & 3 \\ 3 & 2y \end{pmatrix} = \begin{pmatrix} 2r\cos\theta & 3 \\ 3 & 2r\sin\theta \end{pmatrix}$$

$$\Lambda^{\alpha}{}_{\mu'} \Lambda^{\beta}{}_{\nu'} P_{\alpha,\beta}$$

(b)

$$P_{r;r} = \Lambda^{\alpha}{}_{1'} \Lambda^{\beta}{}_{1'} P_{\alpha,\beta}$$

$$= 2r(\cos^3\theta + \sin^3\theta) + 6\sin\theta\cos\theta$$

(c) $P_{\alpha;\beta} = P_{\alpha,\beta} - P_{\mu}\Gamma^{\mu}{}_{\alpha\beta}$

$$P_{r;r} = P_{r,r} - P_{\alpha}\Gamma^{\alpha}{}_{rr}$$

$$= 2r(\cos^3\theta + \sin^3\theta) + 6\sin\theta\cos\theta$$

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$V^\alpha{}_{;\mu;\nu}$ $\qquad V^r = 1 \quad V^\theta = 0$

$$V^\alpha{}_{;\mu} = V^\alpha{}_{,\mu} + V^\gamma \Gamma^\alpha_{\gamma\mu} = B^\alpha{}_\mu$$

$$B^\alpha{}_{\mu;\nu} = \left(B^\alpha{}_{\mu,\nu}\right) + \left(B^\beta{}_\mu \Gamma^\alpha_{\beta\nu}\right) - \left(B^\alpha{}_\beta \Gamma^\beta_{\mu\nu}\right)$$

$$\begin{pmatrix} V^\alpha \Gamma^r_{\alpha r} & V^\alpha \Gamma^r_{\alpha\theta} \\ V^\alpha \Gamma^\theta_{\alpha r} & V^\alpha \Gamma^\theta_{\alpha\theta} \end{pmatrix} = \begin{pmatrix} 0+0 & 0+0 \\ 0+0 & \frac{1}{r}+0 \end{pmatrix}$$

$$= V^\theta{}_{;\theta;r} = -\frac{1}{r^2}$$

$$\cancel{B^r{}_\mu \Gamma^\alpha_{r\nu}} + B^\theta{}_\mu \Gamma^\alpha_{\theta\nu} - \cancel{B^\alpha{}_r \Gamma^r_{\mu\nu}} + B^\alpha{}_\theta \Gamma^\theta_{\mu\nu}$$

$$V^r{}_{;\theta;\theta} = -1$$

$$V^\theta{}_{;r;\theta} = \frac{1}{r}$$