GR notes

Yucun Xie

August 7, 2022

Contents

Conventions

1. Greek index (e.g. α, β, μ, ν) take value from $\{0, 1, 2, 3\}$.
2. Events denoted by cursive capitals (e.g. $\mathscr{A}, \mathscr{B}, \mathscr{E}$).
3. $(x^0, x^1, x^2, x^3) \equiv (t, x, y, z) \equiv x^\alpha$
4. Latin index (e.g.i, j, k) take value from $\{1, 2, 3\}$.
5. New unit that speed of light $c = 1$
6. Einstein summation convention $ds^2 = g_{\mu\nu}x^\mu x^\nu = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}x^\mu x^\nu$

1 Differential Geometry

1.1 Tensor

Tensor is a quantity that have same form in all coordinate system. Tensor does not have components naturally, but when we choose specific coordinate system, we can write down its components.

1.2 Connection

Proof. Here is a proof shows that connection not a tensor by show connection does not obey tensor transformation law.

$$\begin{aligned}
\nabla_{\beta'} e_{\alpha'} &= \Gamma^{\gamma'}_{\alpha'\beta'} e_{\gamma'} \\
&= \frac{\partial x^\lambda}{\partial x^{\beta'}} \nabla_\lambda \left(\frac{\partial x^\mu}{\partial x^{\alpha'}} e_\mu\right) \\
&= \frac{\partial x^\lambda}{\partial x^{\beta'}} \left(\frac{\partial}{\partial x^\lambda} \frac{\partial x^\mu}{\partial x^{\alpha'}} e_\mu + \frac{\partial x^\mu}{\partial x^{\alpha'}} \Gamma^\gamma_{\mu\lambda} e_\gamma\right) \\
&= \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial}{\partial x^\lambda} \frac{\partial x^\mu}{\partial x^{\alpha'}} e_\mu + \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial x^\mu}{\partial x^{\alpha'}} \Gamma^\gamma_{\mu\lambda} e_\gamma \\
&= \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial}{\partial x^\lambda} \frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\mu} e_{\gamma'} + \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\gamma} \Gamma^\gamma_{\mu\lambda} e_{\gamma'}
\end{aligned}$$

which yield

$$\Gamma^{\gamma'}_{\alpha'\beta'} = \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial}{\partial x^\lambda} \frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\mu} + \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\gamma} \Gamma^\gamma_{\mu\lambda}$$

There is an extra term in transformation of connection, so connection is not a tensor. □

1.3 Geodesics

1.4 Riemann Tensor

2 Gravitation

A Special Relativity

A.1 Spacetime

Definition A.1. Inertial coordinate
The coordinate system must satisfy three property to be consider inertial coordinat:

1. The distance between two points are independent of time.
2. The clocks at every points ticking off time coordinate t at same rate.
3. The geometry of space is always flat.

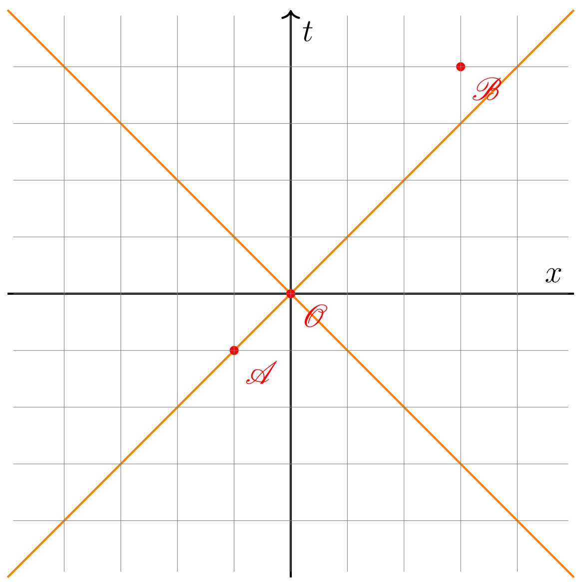

Figure 1: two events with coordinate $(-1,-1,0,0)$ and $(4,3,0,0)$. Orange line is light's worldline.

The event in 4-D spacetime is defined by a set of coordinate (t,x,y,z). For simplicity, we assume those events have $y=0, z=0$ so that we can draw a 2D graph to represent them.
Analog to Euclidean geometry, just like the euclidean distance $\Delta l^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$, we define the **spacetime interval** $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$.

Remark. There are a lot different conventions to define the sign of interval, here we just use the popular one $(-,+,+,+)$.

Example.
Interval for the two events in Figure 1 is $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -9$.

The universality speed of light means that $\frac{\Delta r}{\Delta t} = \frac{\sqrt{\Delta x^2+\Delta y^2+\Delta z^2}}{\Delta t} = 1$ are always hold, then we can then write the interval $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = 0$. This experimental fact yield all law of special relativity.
When the interval Δs^2 is less than 0, we call the seperation bewteen events is **timelike**; When the interval Δs^2 is equal to 0, we call it **lightlike** or null; When the interval Δs^2 is greater than 0, we call it **spacelike**.

A.2 Energy and Momentum

B Topological Space

C Manifolds

Mathematically, specetime is a **manifold**.

Definition C.1. An n-dimensional manifold is a set that can be parameterized continuously by n independent real coordinates for each point. If a manifold is differentiable at each point, it is a **differentiable manifold**.

Definition C.2. A coordinate system (also called chart) is n labels uniquely with each point of an n-dimensional manifold through a one-to-one mapping from $\mathbb{R}^n$ to M.
Generally, more than one charts are required to cover entire manifold, which called **atlas**.

Definition C.3. Cartesian product $X \times Y$ is set of all possible ordered pairs of element which one from X and one from Y.

Subset of points within a manifold for curves and surfaces. Our spacetime is a 4-dimensianl **pseudo Riemannian manifold** which is a manifold with some additional property and structures.

Remark. Manifolds also have a important property which is locally **homeomorphism** to $\mathbb{R}^n$. See the discussion of topology for definition of homeomorphism.

D Property for some tensors

$$
\begin{aligned}
F_{\mu\nu} &= -F_{\nu\mu} \\
T_{ij} &= T_{ji} \\
g_{\mu\nu} &= g_{\nu\mu} \\
\Gamma^{\lambda}_{\mu\nu} &= \Gamma^{\lambda}_{\nu\mu} \\
R_{\alpha\beta\mu\nu} &= -R_{\beta\alpha\mu\nu} \\
R_{\alpha\beta\mu\nu} &= -R_{\alpha\beta\nu\mu} \\
R_{\alpha\beta\mu\nu} &= R_{\mu\nu\alpha\beta} \\
R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} &= 0 \\
R_{\alpha\beta} &= R_{\beta\alpha}
\end{aligned}
$$