GR notes

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Contents

Conventions

1. Greek index (e.g. α, β, μ, ν) take value from $\{0, 1, 2, 3\}$.
2. Events denoted by cursive capitals (e.g. $\mathscr{A}, \mathscr{B}, \mathscr{E}$).
3. $(x^0, x^1, x^2, x^3) \equiv (t, x, y, z) \equiv x^\alpha$
4. Latin index (e.g.i, j, k) take value from $\{1, 2, 3\}$.
5. New unit that speed of light $c = 1$
6. Einstein summation convention $ds^2 = g_{\mu\nu} x^\mu x^\nu = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu} x^\mu x^\nu$

1 Differential Geometry

1.1 Connection

Proof. Here is a proof shows that connection not a tensor by show connection does not obey tensor transformation law.

$$\begin{aligned}
\nabla_{\beta'} e_{\alpha'} &= \Gamma^{\gamma'}_{\alpha'\beta'} e_{\gamma'} \\
&= \frac{\partial x^\lambda}{\partial x^{\beta'}} \nabla_\lambda \left(\frac{\partial x^\mu}{\partial x^{\alpha'}} e_\mu\right) \\
&= \frac{\partial x^\lambda}{\partial x^{\beta'}} \left(\frac{\partial}{\partial x^\lambda}\frac{\partial x^\mu}{\partial x^{\alpha'}} e_\mu + \frac{\partial x^\mu}{\partial x^{\alpha'}} \Gamma^\gamma_{\mu\lambda} e_\gamma\right) \\
&= \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial}{\partial x^\lambda}\frac{\partial x^\mu}{\partial x^{\alpha'}} e_\mu + \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial x^\mu}{\partial x^{\alpha'}} \Gamma^\gamma_{\mu\lambda} e_\gamma \\
&= \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial}{\partial x^\lambda}\frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\mu} e_{\gamma'} + \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\gamma} \Gamma^\gamma_{\mu\lambda} e_{\gamma'}
\end{aligned}$$

which yield

$$\Gamma^{\gamma'}_{\alpha'\beta'} = \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial}{\partial x^\lambda}\frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\mu} + \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\gamma} \Gamma^\gamma_{\mu\lambda}$$

There is an extra term in transformation of connection, so connection is not a tensor. □

1.2 Geodesics

1.3 Riemann Tensor

2 Gravitation

A Special Relativity

A.1 Spacetime

Definition A.1. Inertial coordinate
The coordinate system must satisfy three property to be consider inertial coordinat:

1. The distance between two points are independent of time.
2. The clocks at every points ticking off time coordinate t at same rate.
3. The geometry of space is always Euclidean (flat).

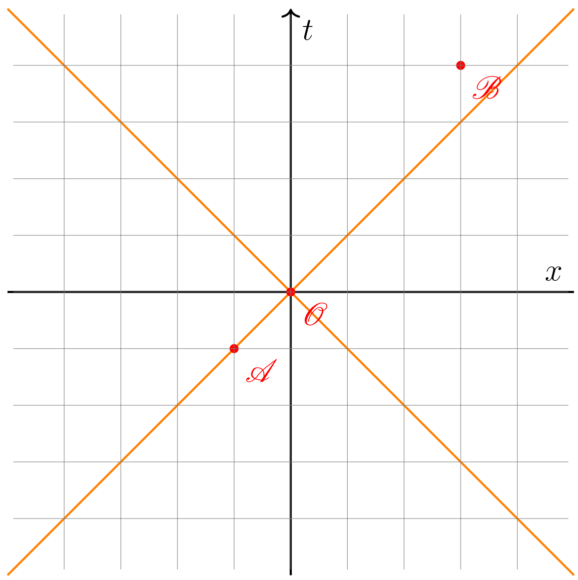

Figure 1: two events with coordinate $(-1,-1,0,0)$ and $(4,3,0,0)$. Orange line is light's worldline.

The event in 4-D spacetime is defined by a set of coordinate (t,x,y,z). For simplicity, we assume those events have $y=0, z=0$ so that we can draw a 2D graph to represent them.
Analog to Euclidean geometry, just like the euclidean distance $\Delta l^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$, we define the **spacetime interval** $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$.

Remark. There are a lot different conventions to define the sign of interval, here we just use the popular one $(-,+,+,+)$.

Example.
Interval for the two events in Figure 1 is $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -9$.

The universality speed of light means that $\frac{\Delta r}{\Delta t} = \frac{\sqrt{\Delta x^2+\Delta y^2+\Delta z^2}}{\Delta t} = 1$ are always hold, then we can then write the interval $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = 0$. This experimental fact yield all law of special relativity.
When the interval Δs^2 is less than 0, we call the seperation bewteen events is **timelike**; When the interval Δs^2 is equal to 0, we call it **lightlike** or null; When the interval Δs^2 is greater than 0, we call it **spacelike**.

A.2 Tensor

A.3 Energy and Momentum

B Topological Space

C Manifolds

D Property for some tensors

$$
\begin{aligned}
T_{ij} &= T_{ji} \\
g_{\mu\nu} &= g_{\nu\mu} \\
R_{\alpha\beta\mu\nu} &= -R_{\beta\alpha\mu\nu} \\
R_{\alpha\beta\mu\nu} &= -R_{\alpha\beta\nu\mu} \\
R_{\alpha\beta\mu\nu} &= R_{\mu\nu\alpha\beta} \\
R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} &= 0 \\
R_{\alpha\beta} &= R_{\beta\alpha}
\end{aligned}
$$