General Relativity

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Contents

Conventions

1. Greek index (e.g. α, β, μ, ν) take value from $\{0,\ 1,\ 2,\ 3\}$.
2. $(x^0, x^1, x^2, x^3) \equiv (t, x, y, z) \equiv x^\alpha$.
3. Latin index (e.g. i, j, k) take value from $\{1,\ 2,\ 3\}$.
4. Natural units ($c = 1$).
5. Einstein summation convention.
$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu$$
6. Metric sign $(-, +, +, +)$.

1 Differential Geometry

1.1 Manifolds

Mathematically, spacetime is a **manifold**.
A manifold is a space that can be continuously (no hole, tear, etc.) covered by multiple coordinate charts. For example, think about a sphere, we can stretch a membrane with the grid to cover it except for one point, then we can use another membrane to cover that one point. The sphere is an example of a 2-D manifold, and the membrane is an example of a coordinate chart. Informally, a manifold locally looks like $\mathbb{R}^2$. For example, the surface of the Earth is a 2-sphere S^2, but look at the ground around you, it seems flat.

Definition 1.1.
An n-dimensional manifold is a mathematical structure consisting of

1. a set M of elements, called **points**.
2. a countable family of subsets of M, called **fundamental coordinate patches**, such that the union of all of them is M.
3. a reversible mapping of each patch onto the unit volume of $\mathbb{R}^n$. An n-tuple of numbers associated in this way are called the **coordinates** of each point.

More formally, an n-dimensional manifold is a topological space that is locally homeomorphic to the n-dimensional Euclidean space. See the discussion of topological space for the definition of homeomorphism.
If a manifold is differentiable for m times at each point, it is a **differentiable manifold**, denote by C^m. If a manifold is infinitely differentiable at each point, denote by C^∞, called a smooth manifold.
A coordinate system (also called a chart) is n labels uniquely with each point of an n-dimensional manifold through a one-to-one mapping from $\mathbb{R}^n$ to M.
Generally, more than one chart is required to cover the entire manifold, which is called **atlas**.

Differential geometry, as the name suggests, is geometry on differential manifolds.

Manifolds are very basic structures, but they support the following things on it.

1. saclar field: $\phi(x)$
2. curves: $x^\mu(\lambda)$
3. surfaces: $x^\mu(\lambda_1, \lambda_2)$
4. tangent vector and dual vector.
5. tangent space and cotangent space.

1.1.1 Vector and Dual Vector

At each point P of an n-dimensional differentiable manifold, there is an n-dimensional vector space which basis is defined by directional derivative at P for curves passing through P. This vector space is called **tangent space**. This space contains all **vectors** at point P. There is also another vector space whose basis is defined by evaluating the gradient of the scalar function at P. This space is called **cotangent space**, which contains all **dual vectors** at point P.
Vectors and dual vectors are local to a point.
Set of all tangent spaces in a manifold form a **tangent bundle**, and the set of all cotangent spaces on a manifold form a **cotangent bundle**. They are an example of **fiber bundle**.
A fiber bundle is a manifold that is locally the cartesian product of base space and fiber space but not globally.
There is no definite way to transport a vector from one point to another on a manifold because there is no relation between the tangent space at each point. To achieve this, we need to impose an additional structure into the manifold, which is called **connection**.

1.1.2 Maps Between Manifolds

@TODO pullback pushforward

1.2 Tensor

Tensor is a quantity that has the same form in all coordinate systems. Tensor does not have components naturally, but when we choose a specific coordinate system, we can write down its components. Tensor has **Covariance**, which means it follows a specific transformation law.

1.2.1 Tensor Notation

A tensor with k upper indices and l lower indices

$$T^{\mu^1\mu^2\mu^3\cdots\mu^k}_{\nu^1\nu^2\nu^2\ldots\nu^l}$$

is the cartesian product of k vectors and l dual vectors. Which map k dual vectors and l vectors to a real number. **Cartesian product** $X \times Y$ is a set of all possible ordered pairs of the element one from X and one from Y.

1.2.2 Tensor Transformation Law

When we change the coordinate system, tensor components transform following **tensor transformation law**.

Definition 1.2.
Tensor components in new coordinate system $(\alpha'\beta'\mu'\nu'\cdots)$ can be express as

$$T^{\alpha'\beta'\cdots}_{\mu'\nu'\cdots} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\frac{\partial x^{\beta'}}{\partial x^{\beta}}\cdots T^{\alpha\beta\cdots}_{\mu\nu\cdots}$$

Each upper indices is covariance with coordinate transform, each lower indices is contravariance with coordinate transform. If some quantity obeys tensor transformation law, it is a tensor. If a tensorial equation is held in one coordinate system, it is held in all coordinate system because both sides are following the same law to transform.

1.2.3 Exterior Calculus

A covariant totally antisymmetric rank $(0, p)$ tensor is called differential p-form. This is the reason why a dual vector is sometimes called 1-form. We can generate new differential forms by taking the antisymmetric part of the tensor product, which is called **wedge product**.

Example.
The wedge product of two one form is

$$(A \wedge B)_{\mu\nu} = 2A_{[\mu}B_{\nu]}$$

The **exterior derivative** can generate $p+1$-form from p-form.

Definition 1.3.
Exterior derivative of a p-form A is defined as the following.

$$dA = (p+1)\partial_{[\mu_1}A_{\mu_2\mu_3\cdots\mu_{p+1}]}$$

A differential p-form A is **closed** if $dA = 0$, and **exact** if exist another differential form which $dB = A$. We see that definition of the exterior derivative does not require any additional structure. The reason of define differential form is that we can have derivative and integral without the help of additional structure on the manifold.

1.3 Affine Connection

Recall that we are unable to transform vectors from one point to another on a manifold because we missing the relation between tangent space at each point. So we introduce an additional structure called **connection**. Connection is an additional structure that connects the tangent space on each point of a manifold. There is no naturally defined connection on a manifold, we have to define one, therefore connection on a manifold is not unique. In general relativity, our spacetime manifold equipped with a metric compatible, torsion-free connection is called **affine connection**. Connection coeffienct in a coordinate system is express as **Christoffel symbol** $\Gamma^{\lambda}_{\alpha\beta}$.
In general, a manifold with connection is a manifold allowing **parallel transport** of vectors.

1.3.1 Covariant Derivative

The partial derivative is not covariance, we are looking for another kind of differentiation that is covariant. Here let me show that partial derivative, not covariance, and then construct a covariant one that we can apply to our curved spacetime.

$$\begin{aligned} A_{\mu,\nu} = \frac{\partial A_\mu}{\partial x^\nu} &= \frac{\partial}{\partial x^\nu}\left(A_\mu \frac{\partial x^\alpha}{\partial x^\mu}\right) \\ &= \frac{\partial A_\alpha}{\partial x^\nu}\frac{\partial x^\alpha}{\partial x^\mu} + A_\alpha \frac{\partial^2 x^\alpha}{\partial x^\mu \partial x^\nu} \\ &= A_{\alpha,\beta}\frac{\partial x^\beta}{\partial x^\nu}\frac{\partial x^\alpha}{\partial x^\mu} + \underbrace{A_\alpha \frac{\partial^2 x^\alpha}{\partial x^\mu \partial x^\nu}}_{\text{Not a tensor}} \end{aligned}$$

By using the connection coefficient below, which is also not a tensor, we can eliminate the none tensorial part of the partial derivative. Which is defined as follows.

Definition 1.4.
The covariant derivative of the dual vector.

$$A_{\mu;\nu} = A_{\mu,\nu} - \Gamma^{\lambda}_{\mu\nu} A_\lambda$$

This is called **covariant derivative**, two non-tensorial parts added together to form a tensor.

Proof. Here is proof shows that connection is not a tensor by show connection does not obey tensor transformation law.

$$\begin{aligned}
\nabla_{\beta'} e_{\alpha'} &= \Gamma^{\gamma'}_{\alpha'\beta'} e_{\gamma'} \\
&= \frac{\partial x^\beta}{\partial x^{\beta'}} \nabla_\beta \left(\frac{\partial x^\alpha}{\partial x^{\alpha'}} e_\alpha \right) \\
&= \frac{\partial x^\beta}{\partial x^{\beta'}} \left(\frac{\partial}{\partial x^\beta} \frac{\partial x^\alpha}{\partial x^{\alpha'}} e_\alpha + \frac{\partial x^\alpha}{\partial x^{\alpha'}} \Gamma^\gamma_{\alpha\beta} e_\gamma \right) \\
&= \frac{\partial x^\beta}{\partial x^{\beta'}} \frac{\partial}{\partial x^\beta} \frac{\partial x^\alpha}{\partial x^{\alpha'}} e_\alpha + \frac{\partial x^\beta}{\partial x^{\beta'}} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \Gamma^\gamma_{\alpha\beta} e_\gamma \\
&= \frac{\partial x^\beta}{\partial x^{\beta'}} \frac{\partial}{\partial x^\beta} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\alpha} e_{\gamma'} + \frac{\partial x^\beta}{\partial x^{\beta'}} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\gamma} \Gamma^\gamma_{\alpha\beta} e_{\gamma'}
\end{aligned}$$

which yield

$$\Gamma^{\gamma'}_{\alpha'\beta'} = \frac{\partial x^\beta}{\partial x^{\beta'}} \frac{\partial}{\partial x^\beta} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\alpha} + \frac{\partial x^\beta}{\partial x^{\beta'}} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\gamma} \Gamma^\gamma_{\alpha\beta}$$

There is an extra term in the transformation of connection, so the connection is not a tensor. □

Our connection is metric compatible and torsion-free, which mean

$$g_{\mu\nu;\alpha} = 0$$

and

$$S_{\mu\nu} = \Gamma^\lambda_{[\mu\nu]} = 0$$

Where $S_{\mu\nu}$ is the torsion tensor.

1.3.2 Parallel Transport

Recall we defined connection to allow us to transport vectors on the manifold. For **parallel transport** a vector, we want each infinitesimal step of transport to maintain the vector parallel. Which required **intrinsic derivative** vanish.

$$\frac{DV^\mu}{D\lambda} = \frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\alpha\nu} V^\alpha \frac{dx^\nu}{d\lambda} = 0$$

λ is **affine parameter**, which is the parameter along the curve.

1.4 Metric Space

Now, in addition to affine connection, we want a structure allowing measurement of distance on the manifold. The **metric space** is a kind of space with the notion of distance.

Definition 1.5.
A Riemannian manifold is an manifold with connection and equipped with a symmetric **metric tensor** $g_{\mu\nu}$. which distance ds between two points with displacement dx^μ is defined as the following.

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu$$

The metric tensor is one of the most important tensors in general relativity, which directly lead to the all structures of spacetime.

Example.
The connection coefficient can be directly calculated from the metric tensor.

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$$

In general relativity, proper time is equivalence to distance in timelike direction $d\tau^2 = -ds^2$.

1.4.1 Riemann Tensor

Now, we have everything we need to define the **curvature**. Think about transporting a vector V^ρ around an infinitesimal loop on a curved manifold, the loop is defined by two vectors dA^μ and dB^ν. Then, the change on V^ρ can be express as

$$\delta V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma dA^\mu dB^\nu$$

Definition 1.6.
$R^\rho{}_{\sigma\mu\nu}$ is known as **Riemann curvature tensor**, or just Riemann tensor, can be express through connection coeffiecnt:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}$$

There is some property in the Riemann tensor.

$$
\begin{aligned}
R_{\alpha\beta\mu\nu} &= -R_{\beta\alpha\mu\nu} \\
R_{\alpha\beta\mu\nu} &= -R_{\alpha\beta\nu\mu} \\
R_{\alpha\beta\mu\nu} &= R_{\mu\nu\alpha\beta} \\
R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} &= 0 \\
\nabla_{[\lambda} R_{\alpha\beta]\mu\nu} &= 0
\end{aligned}
$$

The last line above is called **Bianchi identity**.

Definition 1.7.
From the Riemann tensor, we can construct a symmetric rank 2 tensor, **Ricci tensor** by contract first and the third indices of Riemann tensor.

$$R_{\mu\nu} = R_{\nu\mu} = R^{\lambda}{}_{\mu\lambda\nu}$$

Furthermore, we can take the trace of the Ricci tensor, to get **Ricci scalar** or scalar curvature.

$$R = g^{\mu\nu} R_{\mu\nu}$$

We can recover the Riemann tensor from scalar curvature through metric tensor.

$$R_{\alpha\beta\mu\nu} = \frac{R}{2}(g_{\alpha\mu}g_{\beta\nu} - g_{\beta\mu}g_{\alpha\nu})$$

From Bianchi identity $\nabla_{[\lambda} R_{\alpha\beta]\mu\nu} = 0$, we can find that

$$\nabla_{\mu}(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R) = 0$$

We see the divergence of this tensor vanish, therefore, we define a new tensor named **Einstein tensor**,

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R.$$

The trace-free part of the Riemann tensor, is called **Weyl tensor**, can be defined as follows.

Definition 1.8.
Weyl tensor. @TODO

$$C_{\rho\sigma\mu\nu} =$$

1.5 Geodesics

Geodesic is the generalization of the straight line in curved spacetime, which is the trajectory of the inertially moving particle. More specifically, geodesic is a path parallel-transport its own tangent vector. Recall our equation of parallel transport, $\frac{DV^\mu}{D\lambda} = \frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\alpha\nu} V^\alpha \frac{dx^\nu}{d\lambda} = 0$ and replace V by tangent vector of the path $\frac{dx^\mu}{d\lambda}$. We get

$$\frac{D}{D\lambda}\frac{dx^\mu}{d\lambda} = \frac{d}{d\lambda}\frac{dx^\mu}{d\lambda} + \Gamma^\mu_{\alpha\nu}\frac{dx^\alpha}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$$

This is famous **geodesic equation**, all path satisfy this equation is a geodesic.

Definition 1.9.
Geodesic equation.

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\nu}\frac{dx^\alpha}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$$

Geodesic always maximized the proper time. Think about the twin paradox in special relativity, the twin stays on Earth and approximately moves alone geodesic, but the twin travels away at the tuning point isn't.

1.6 Killing Vector Field

The symmetric property of spacetime is represented by **killing vector field**.

Definition 1.10.
A killing vector K satisfy the **killing equation**:

$$\mathcal{L}_K g_{\mu\nu} = 0$$

where $\mathcal{L}_K$ is the Lie derivative alone K. For the definition of Lie derivative, refer to the appendix.

1.7 Geodesic Deviation

2 Gravitation

Our spacetime is not a simply Riemannian manifold, but a **pseudo-Riemannian manifold**, which more general Riemannian manifold. Riemannian manifold requires metric tensor to be positive definite, which means all eigenvalue of the metric tensor is positive. For pseudo-Riemannian manifold, the metric eigenvalue can be either positive or negative, called indefinitely.

Remark.
The eigenvalue of the metric can not be zero, if an eigenvalue is zero, the metric will degenerate and have no inverse.

More specifically, our spacetime is **Lorentzian manifold**, which is a pseudo-Riemannian manifold with only one negative eigenvalue. The Lorentzian manifold allows us to classify tangent vectors into timelike, spacelike, and lightlike.

2.1 Equivalence Principle

Definition 2.1.

1. Inertial mass is defined through Newton's second law, $m = F/a$.
2. Gravitational mass is defined through Newton's law of gravity $m = Fr^2/GM$.

There are two equivalence principles, the weak equivalence principle, and the strong equivalence principle (or Einstein's equivalence principle). Weak Principle of Equivalence state that inertial mass is always equal to gravitational mass. Einstein extended this idea to a stronger statement, which is **Strong Equivalence Principle**. Which state that observer is unable to distinguish acceleration and gravitational field by local experiment. This leads to the idea of **local inertial frame**, which is correspondence to the property of manifold (local flatness). With the equivalence principle, we can explain the deflection of light and gravitational redshift in the gravitational field.
General Covariance Principle state that if a tensorial equation hold in a gravitational field yield that

1. hold in absence of gravity.
2. holds under a coordinate transformation.

Therefore, we can write our classical equations in tensorial form, then they will hold in general relativity.

Example.

Tensorial Maxwell's equations:

$$\nabla_\mu F^{\nu\mu} = J^\nu$$
$$\nabla_{[\mu} F_{\nu\lambda]} = 0$$

In general, we can just replace the partial derivative with the covariant derivative, and replace the flat spacetime metric by $g_{\mu\nu}$, then it will valid in curved spacetime.

2.2 Einstein's Equation

From the Poisson equation for Newtonian potential:

$$\nabla^2 \phi = 4\pi G \rho$$

To make a relativistic version of this equation, we can replace the gravitational potential with the metric and replace the mass density with the energy-momentum tensor.

$$\nabla^2 g_{\mu\nu} = cT_{\mu\nu}$$

But the second derivative of the metric vanished by metric compatibility of connection, therefore we want something constructed from the second derivative of the metric which does not vanishes. Riemann tensor fits this requirement but does not have the right number of indices, so we can use the Ricci tensor.

$$R_{\mu\nu} = cT_{\mu\nu}$$

But, this still not right, recall the conservation of energy, $\nabla^\mu T_{\mu\nu} = 0$, Ricci not fit this requirement. Now, we need a rank 2 tensor that conserved and fit the requirement above. We do have one, the Einstein tensor:

$$\nabla^\mu G_{\mu\nu} = 0$$

Then our equation read

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = cT_{\mu\nu}$$

This is the famous Einstein field equation. We can recover the constant by requiring the equation back to the Newtonian limit.

To recover Newtonian gravity, impose the following boundary conditions to the field equation:

1. Weak field ($g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ which $|h_{\mu\nu}| \ll 1$)
2. Stationary (time translational symmetry).
3. Low speed (reduce special relativity to Newtonian mechanics).

In this limit, the energy-momentum tensor simply $T_{00} = \rho$ and all other

components vanish. Rearrange Einstein equation, we get $R_{\mu\nu} = c\left(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu}\right)$, plug $T_{00} = \rho$ in to field equation, we get $R_{00} = \frac{1}{2}c\rho$.
From Riemann tensor in week field limit, we can see that $R_{00} = -\frac{1}{2}\nabla^2 h_{00}$; compare to the Poisson equation of Newtonian gravity, we know that $c = 8\pi G$.

> Einstein field equation.
> $$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}$$

3 Relativistic Star

3.1 Schwarzschild Solution

The first exact solution to Einstein's equation by Karl Schwarzschild is named **Schwarzschild solution**, which is the exterior solution to a spherical star. This solution is solved by three boundary conditions.

1. Static, which means completely independent of time. More formally, the metric tensor satisfies time translational symmetry and time reversal symmetry.
2. Vacuum, which means outside the star, implying vanish of the energy-momentum tensor. This means the solution is not valid in the interior of the star.
3. Spherical symmetry, which means for, given a radius, the spacetime should be the same.

> The line element takes the form
> $$ds^2 = -\left(1 - \frac{r_\mathrm{s}}{r}\right) dt^2 + \left(1 - \frac{r_\mathrm{s}}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$
> which $r_\mathrm{s} = 2GM$ and $d\Omega^2 = (d\theta^2 + \sin^2\theta\, d\phi^2)$.

From this line element, we can notice a few properties of the Schwarzschild metric.

1. Singularity at $r = 2GM$ and $r = 0$.
2. Change of sign of g_{tt} and g_{rr} at $r = 2GM$.
3. Recover to Newtonian gravity at the Newtonian limit.
4. Asymptotic flatness.

The surface at $r = 2GM$ is known as **Schwarzschild horizon**, this radius is known as the Schwarzschild radius. This led to the idea of the black hole, in which nothing can escape from the horizon (at least in the classical limit, we will see later discussion of Hawking radiation).
We can calculate the timelike geodesic from the metric, which implies planetary motion in general relativity, leading to the famous result of the precession of the perihelion of Mercury. We are also able to calculate the deflection of light by calculating lightlike geodesic from metric, this leads to the effect of **gravitational lensing**.

3.1.1 Interior Solution

Recall the Schwarzschild solution is only valid outside the spherical star, not the interior, so there is no singularity and horizon in our Earth. The interior solution can be solved by approximate the star by energy-momentum tensor of perfect fluid $T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu}$, and match the exterior solution at the boundary. The interior solution of spherical star read

$$ds^2 = -\frac{1}{4}\left(3\sqrt{1-\frac{r_s}{r_g}} - \sqrt{1-\frac{r^2 r_s}{r_g^3}}\right)^2 dt^2 + \left(1-\frac{r^2 r_s}{r_g^3}\right)^{-1} dr^2 + r^2 \mathrm{d}\Omega^2$$

Where r_g is the radius of the surface.

3.1.2 Black hole

3.1.3 Maximal Extension

3.2 Kerr Solution

Kerr solution is the vacuum solution to the axisymmetry rotating star, by applying the following conditions to the Einstein equation.

1. Stationary, which metric tensor satisfies time translational symmetry.
2. Vacuum, which means outside the star, implying vanish of the energy-momentum tensor. This means the solution is not valid in the interior of the star.
3. Axissymmetry, which means for given r and θ, metric components are the same.

Kerr metric can be expressed in Boyer-Lindquist coordinates.

$$\begin{aligned} ds^2 = -\left(1 - \frac{2GM\ r}{\Sigma}\right) dt^2 + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 \\ + \left(r^2 + a^2 + \frac{2GM\ ra^2}{\Sigma} \sin^2\theta\right) \sin^2\theta\ d\phi^2 \\ - \frac{4GM\ ra \sin^2\theta}{\Sigma} dt\, d\phi \end{aligned}$$

Where $a = \frac{J}{Mc}$, $\Sigma = r^2 + a^2 \cos^2\theta$, and $\Delta = r^2 - r_s r + a^2$.

3.2.1 Dragging of Inertial Frame

3.2.2 Ergosphere

3.2.3 Penrose Process

4 Gravitational Radiation

4.1 Linearized Gravity

When the gravitational field are weak, the metric take following form :

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

which we treat the gravitational field as a perturbation of flat spacetime metric.

4.2 Gravitational Wave

4.2.1 Effect of Gravitational Wave on matter

4.2.2 Source of Gravitational Wave

5 Cosmology

5.1 Roberson-Walker Metric

5.2 Friedmann Equation

5.3 Inflation

A Special Relativity

A.1 Spacetime

In special relativity, we discard the absolute concept of time, in contrast to Newton, there is no preferred reference frame and time is one of the coordinates. Now we have a 4-dimensional **spacetime**. Our discussion is focused on the inertial coordinate system.

Definition A.1.
Inertial coordinate
The coordinate system must satisfy three properties to be considered inertial coordinate:

1. The distance between two points is independent of time.
2. The clocks at every points ticking off time coordinate t at the same rate.
3. The geometry of space is always flat.

It is a coordinate system without acceleration.

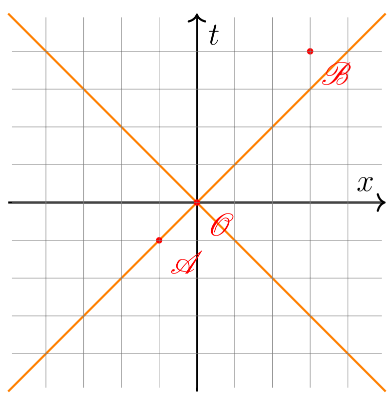

Figure 1: two events with coordinate $(-1, -1, 0, 0)$ and $(4, 3, 0, 0)$. The orange line is light's worldline.

The event in 4-D spacetime is defined by a set of coordinates (t, x, y, z). For simplicity, we assume those events have $y = 0, z = 0$ so that we can draw a 2D graph to represent them.

Analog to Euclidean geometry, just like the euclidean distance $\Delta l^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$, we define the **spacetime interval** $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$.

Remark.
There are a lot of different conventions to define the sign of interval, here we just use the popular one $(-, +, +, +)$.

Example.

Interval for the two events in Figure 1 is $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -9$.

The universality speed of light means that $\frac{\Delta r}{\Delta t} = \frac{\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}}{\Delta t} = 1$ are always hold, then we can then write the interval $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = 0$. This experimental fact yield all laws of special relativity.
When the interval Δs^2 is less than 0, we call the separation between events is **timelike**; When the interval Δs^2 is equal to 0, we call it **lightlike** or null; When the interval Δs^2 is greater than 0, we call it **spacelike**. If there is another coordinate system, which moves with speed v alone x direction of the original frame, we can draw this frame like Figure 2 below.

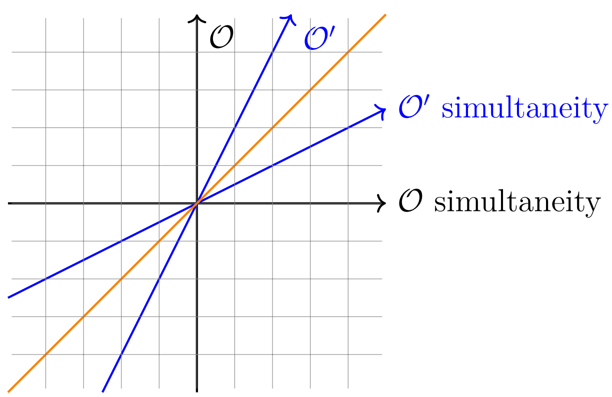

Figure 2: Frame $\mathcal{O}'$ move alone x direction of $\mathcal{O}$

A.2 Energy and Momentum

A.3 Fluid

B Topological Space

C Lie Algebra

C.1 Lie Derivative

Lie derivative is a more “fundamental” way to generalize derivative to a manifold compared to covariant derivative.
To see why it is more fundamental, consider the **Lie bracket**

Definition C.1.
Lie bracket

$$[X, Y]^\mu = X^\lambda \nabla_\lambda Y^\mu - Y^\lambda \nabla_\lambda X^\mu$$

It is easy to check that the above expression is equivalent to $[X, Y]^\mu = X^\lambda \partial_\lambda Y^\mu - Y^\lambda \partial_\lambda X^\mu$.

Lie bracket is a special kind of **Lie derivative**. The above expression is equivalent to $\mathcal{L}_X Y^\mu$. We see that the Lie derivative does not rely on affine connection.

The basic idea of Lie derivative is to use derivatives of an auxiliary vector field to cancel the non-tensorial part of the partial derivative.

Let's go step by step to see how the Lie derivative work.

Consider a tensor on a manifold defined at point P, it can be “dragged” to a nearby point Q, alone a curve of the congruence that generates by a vector field.

This step is called **Lie dragging**.

D Penrose Diagram

E Summary of Formula

$$
\begin{aligned}
F_{\mu\nu} &= -F_{\nu\mu} \\
T_{\mu\nu} &= T_{\nu\mu} \\
g_{\mu\nu} &= g_{\nu\mu} \\
\Gamma^{\lambda}_{\mu\nu} &= \Gamma^{\lambda}_{\nu\mu} \text{ (Torsion free)} \\
R_{\alpha\beta\mu\nu} &= -R_{\beta\alpha\mu\nu} \\
R_{\alpha\beta\mu\nu} &= -R_{\alpha\beta\nu\mu} \\
R_{\alpha\beta\mu\nu} &= R_{\mu\nu\alpha\beta} \\
R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} &= 0 \\
R_{\alpha\beta} &= R_{\beta\alpha}
\end{aligned}
$$