

# GR notes

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## Conventions

1. Greek index (e.g.  $\alpha, \beta, \mu, \nu$ ) take value from  $\{0, 1, 2, 3\}$ .
2. Events denoted by cursive capitals (e.g.  $\mathcal{A}, \mathcal{B}, \mathcal{E}$ ).
3.  $(x^0, x^1, x^2, x^3) \equiv (t, x, y, z) \equiv x^\alpha$
4. Latin index (e.g.  $i, j, k$ ) take value from  $\{1, 2, 3\}$ .
5. New unit that speed of light  $c = 1$
6. Einstein summation convention  $ds^2 = g_{\mu\nu}x^\mu x^\nu = \sum_{\mu=0}^3 \sum_{\nu=0}^3 g_{\mu\nu}x^\mu x^\nu$

# 1 Differential Geometry

## 1.1 Tensor

Tensor is a quantity that have same form in all coordinate system. Tensor does not have components naturally, but when we choose specific coordinate system, we can write down its components.

## 1.2 Connection

*Proof.* Here is a proof shows that connection not a tensor by show connection does not obey tensor transformation law.

$$\begin{aligned}\nabla_{\beta'} e_{\alpha'} &= \Gamma_{\alpha'\beta'}^{\gamma'} e_{\gamma'} \\ &= \frac{\partial x^\lambda}{\partial x^{\beta'}} \nabla_\lambda \left( \frac{\partial x^\mu}{\partial x^{\alpha'}} e_\mu \right) \\ &= \frac{\partial x^\lambda}{\partial x^{\beta'}} \left( \frac{\partial}{\partial x^\lambda} \frac{\partial x^\mu}{\partial x^{\alpha'}} e_\mu + \frac{\partial x^\mu}{\partial x^{\alpha'}} \Gamma_{\mu\lambda}^\gamma e_\gamma \right) \\ &= \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial}{\partial x^\lambda} \frac{\partial x^\mu}{\partial x^{\alpha'}} e_\mu + \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial x^\mu}{\partial x^{\alpha'}} \Gamma_{\mu\lambda}^\gamma e_\gamma \\ &= \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial}{\partial x^\lambda} \frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\mu} e_{\gamma'} + \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\gamma} \Gamma_{\mu\lambda}^\gamma e_{\gamma'}\end{aligned}$$

which yield

$$\Gamma_{\alpha'\beta'}^{\gamma'} = \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial}{\partial x^\lambda} \frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\mu} + \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\gamma} \Gamma_{\mu\lambda}^\gamma$$

There is an extra term in transformation of connection, so connection is not a tensor. □

## 1.3 Geodesics

## 1.4 Riemann Tensor

# 2 Gravitation

# A Special Relativity

## A.1 Spacetime

### Definition A.1. Inertial coordinate

The coordinate system must satisfy three property to be consider inertial coordinat:

1. The distance between two points are independent of time.
2. The clocks at every points ticking off time coordinate  $t$  at same rate.
3. The geometry of space is always flat.

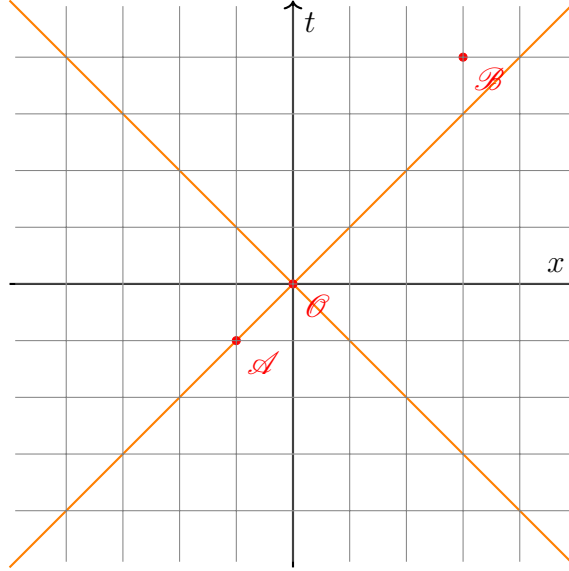


Figure 1: two events with coordinate  $(-1, -1, 0, 0)$  and  $(4, 3, 0, 0)$ . Orange line is light's worldline.

The event in 4-D spacetime is defined by a set of coordinate  $(t, x, y, z)$ . For simplicity, we assume those events have  $y = 0, z = 0$  so that we can draw a 2D graph to represent them.

Analog to Euclidean geometry, just like the euclidean distance  $\Delta l^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$ , we define the **spacetime interval**  $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$ .

**Remark.** There are a lot different conventions to define the sign of interval, here we just use the popular one  $(-, +, +, +)$ .

### Example.

Interval for the two events in Figure 1 is  $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -9$ .

The universality speed of light means that  $\frac{\Delta r}{\Delta t} = \frac{\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}}{\Delta t} = 1$  are always hold, then we can then write the interval  $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = 0$ . This experimental fact yield all laws of special relativity.

When the interval  $\Delta s^2$  is less than 0, we call the seperation bewteen events is **timelike**; When the interval  $\Delta s^2$  is equal to 0, we call it **lightlike** or null; When the interval  $\Delta s^2$  is greater than 0, we call it **spacelike**.

## A.2 Energy and Momentum

# B Topological Space

# C Manifolds

Mathematically, spacetime is a **manifold**.

**Definition C.1.** An  $n$ -dimensional manifold is a set that can be parameterized continuously by  $n$  independent real coordinates for each point. If a manifold is differentiable at each point, it is a **differentiable manifold**.

**Definition C.2.** A coordinate system (also called chart) is  $n$  labels uniquely with each point of an  $n$ -dimensional manifold through a one-to-one mapping from  $\mathbb{R}^n$  to  $M$ . Generally, more than one charts are required to cover entire manifold, which called **atlas**.

**Definition C.3.** Cartesian product  $X \times Y$  is set of all possible ordered pairs of element which one from  $X$  and one from  $Y$ .

Subset of points within a manifold for curves and surfaces. Our spacetime is a 4-dimensional **pseudo Riemannian manifold** which is a manifold with some additional property and structures.

**Remark.** Manifolds also have a important property which is locally **homeomorphism** to  $\mathbb{R}^n$ . See the discussion of topology for definition of homeomorphism.

# D Property for some tensors

$$F_{\mu\nu} = -F_{\nu\mu}$$

$$T_{ij} = T_{ji}$$

$$g_{\mu\nu} = g_{\nu\mu}$$

$$\Gamma_{\mu\nu}^{\lambda} = \Gamma_{\nu\mu}^{\lambda}$$

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}$$

$$R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu}$$

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$$

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0$$

$$R_{\alpha\beta} = R_{\beta\alpha}$$