(7a)

$$\ell = \int_0^{2\pi} \sqrt{ds^2}$$

$$= \int_0^{2\pi} \sqrt{g_{tt}\, dt^2 + g_{\phi\phi}\, d\phi^2}$$

$$dt = \sqrt{\left(\frac{r^3}{M}\right)} \cdot d\phi$$

$$= \int_0^{2\pi} \sqrt{\left(g_{tt}\frac{r^3}{M} + g_{\phi\phi}\right) d\phi^2}$$

$$= \int_0^{2\pi} \sqrt{\left(g_{tt}\frac{r^3}{M} + g_{\phi\phi}\right)}\ d\phi$$

$$= 2\pi \sqrt{-\frac{4}{5} \cdot 10^3 \cdot M^2 + 10^2 M^2}$$

$$= 2\pi M \cdot \sqrt{1000 \cdot -\frac{4}{5} + 100}$$

$$= 2\pi M \cdot \sqrt{700}$$

$$= 20\sqrt{7}\pi M$$

(b)

$$t = 2\pi \cdot \sqrt{\frac{r^3}{M}} = 20\sqrt{10}\,\pi M$$

c

$$\Delta e = \int_0^{2\pi} \sqrt{-g_{tt} \frac{r^3}{M} \cdot d\phi^2}$$

$$= 2\pi \cdot 10 \cdot \sqrt{4}\ M$$

$$= 40\pi\sqrt{2}\ M$$

d

$$t = 2\pi\sqrt{\frac{(6M)^3}{M}}$$

$$= 12\pi M\sqrt{6}$$

e

$$\frac{\Delta e}{\Delta t} = \frac{2\pi\sqrt{\frac{r^3(1-3M/r)}{M}}}{2\pi\sqrt{\frac{r^3}{M}}}$$

$$= \sqrt{1-\frac{3M}{r}}$$

$$= \frac{\sqrt{2}}{2}$$

$$40\sqrt{2}$$

20a

follow the first derivative of metric tensor.
only non vanished are:

1. $g_{tt,r}$ 2. $g_{rr,r}$ 3. $g_{\theta\theta,r}$ 4. $g_{\phi\phi,r}$ 5. $g_{\phi\phi,\theta}$

So only non vanish Christoffel symbols are

Γ^0_{01} Γ^1_{00} Γ^1_{11} Γ^1_{22} Γ^1_{33} Γ^2_{12} Γ^2_{33} Γ^3_{13} Γ^3_{23}.

20b

for $R_{\mu\nu\alpha\beta}$, 6 pair $\mu \neq \nu$, 6 pair $\alpha \neq \beta$

So 36 independent compund

eliminates off diagonal, $5 \cdot \frac{6}{2} + 6 = 21$

by cyclic sum, eliminate one, so 20 independent component.

by this metric, only six non zero component.

R_{trtr} $R_{t\theta t\theta}$ $R_{t\phi t\phi}$ $R_{r\theta r\theta}$ $R_{r\phi r\phi}$ $R_{\theta\phi\theta\phi}$

agree with question.

21a

$$dr = \frac{dt\,\left(1-\frac{2M}{r}\right)\sqrt{\left(E^2-1+\frac{2M}{r}\right)}}{E}$$

$$\nabla\tau = \int_{3M}^{2M}\sqrt{-ds^2} = \int_{3M}^{2M}\sqrt{\left(1-\frac{2M}{r}\right)dt^2-\left(1-\frac{2M}{r}\right)^{-1}dr^2}$$

$$\frac{\cancel{\left(1-\frac{2M}{r}\right)}\,dr^2E^2}{\left(1-\frac{2M}{r}\right)^{\cancel{2}}\left(E^2-1+\frac{2M}{r}\right)} = \frac{E^2\,dr^2}{\left(E^2-1+\frac{2M}{r}\right)\left(1-\frac{2M}{r}\right)}$$

$$\left(\frac{E^2}{\left(E^2-1+\frac{2M}{r}\right)\left(1-\frac{2M}{r}\right)} - \frac{1}{1-\frac{2M}{r}}\right)^{\frac{1}{2}} dr$$

$$= \sqrt{\frac{\cancel{E^2-E^2}+\cancel{1-\frac{2M}{r}}}{\left(E^2-1+\frac{2M}{r}\right)\cancel{\left(1-\frac{2M}{r}\right)}}}\;dr$$

$$= \sqrt{\frac{1}{E^2-1+\frac{2M}{r}}}\;dr$$

$$= \left(E^2-1+\frac{2M}{r}\right)^{\frac{1}{2}} dr$$

$$\nabla\tau = \int_{3M}^{2M}\left(E^2-1+\frac{2M}{r}\right)^{-\frac{1}{2}} dr$$