4.23

$T^{\mu\nu}{}_{,\nu} = 0$

(1) $\frac{\partial}{\partial t}\int d^3x\, T^{0\alpha} = 0$

$= \int d^3x\, T^{0\alpha}{}_{,t}$

$T^{\alpha 0}{}_{,t} = -T^{\alpha i}{}_{,i}$

$T^{\alpha 0}{}_{,t} + T^{\alpha 1}{}_{,x} + T^{\alpha 2}{}_{,y} + T^{\alpha 3}{}_{,z} = 0$

$\to -\int T^{i\alpha}{}_{,i}\, d^3x$ $\quad T^{0\alpha}{}_{,t} = -T^{i\alpha}{}_{,i}$

$= -\oint T^{i\alpha} n_i\, d^2S$

$= 0$

(b) $\frac{\partial^2}{\partial t^2}\int T^{00} x^i x^j d^3x = 2\int T^{ij} d^3x$

$\int T^{00}{}_{,tt}\, x^i x^j\, d^3x$

$= -\int T^{k0}{}_{,k0}\, x^i x^j\, d^3x$

$$= -\frac{\partial}{\partial t}\int T^{k0}{}_{,k}\, x^i x^j d^3x$$

$$= -\frac{\partial}{\partial t}\int \partial_k T^{k0}\, x^i x^j d^3x$$

$$= -\frac{\partial}{\partial t}\left(T^{k0} x^i x^j - \int \partial_k T^{k0} dx^i dx^j \right)$$

$$= \frac{\partial}{\partial t}\int \partial_k T^{k0} dx^i dx^j$$

$$= \frac{\partial}{\partial t}\int T^{k0} \frac{\partial (x^i x^j)}{\partial x^k}$$

$$= \frac{\partial}{\partial t}\int T^{k0} \left(\frac{\partial x^i}{\partial x^k} x^j + \frac{\partial x^j}{\partial x^k} x^i \right)$$

$$= \frac{\partial}{\partial t}\int T^{k0} \left(\delta_{ik} x^j + \delta_{jk} x^i \right)$$

$$= \frac{\partial}{\partial t}\int \left(T^{i0} x_j + T^{j0} x^i \right)$$

$$= \int T^{i0}{}_{,0}\, x_j + T^{j0}{}_{,0}\, x^i$$

$$= -\int T^{ik}{}_{,k} x^j + T^{jk}{}_{,k} x^i$$

$$= \int T^{ik} \frac{\partial x^j}{\partial x^k} + \int T^{jk} \frac{\partial x^i}{\partial x^k}$$

$$= \int T^{ij} + \int T^{ji}$$

$$= 2 \int T^{ij} d^3x$$

(c) $$\frac{\partial^2}{\partial t^2} \int T^{00} (x^i x_i)^2 d^3x$$

$$= 4 \int T^i{}_i x^j x_j d^3x + 8 \int T^{ij} x_i x_j d^3x$$

$$= \frac{\partial}{\partial t} \int \frac{\partial T^{00}}{\partial t} (x^i x_i)^2 d^3x + \frac{\partial}{\partial t} \int T^{00} \cdot \frac{\partial (x^i x_i)^2}{\partial t} d^3x$$

$$= \frac{\partial}{\partial t} \int -T^{k0}{}_{,k} (x^i x_i)^2 d^3x - \int T^{j0}{}_{,j} \frac{\partial (x^i x_i)^2}{\partial t} d^3x$$

$$+ \int T^{00} \cdot \frac{\partial^2 (x^i x_i)^2}{\partial t^2} d^3x$$