

6.9

$$V^{\alpha}_{;\alpha} = \frac{1}{\sqrt{-g}} (\sqrt{-g} V^{\alpha})_{,\alpha} \quad \textcircled{1} \quad \downarrow ?$$

$$V^{\alpha}_{;\alpha} = \frac{1}{r} \frac{\partial}{\partial r} (r V^r) + \frac{\partial}{\partial \theta} V^{\theta} \quad \textcircled{2}$$

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}$$

$$g = \det(g_{\alpha\beta}) = r^4$$

$$V^{\alpha}_{;\alpha} = \frac{1}{\sqrt{|g|}} (\sqrt{|g|} V^{\alpha})_{,\alpha}$$

$$= \frac{1}{\sqrt{|g|}} \left[\frac{\partial \sqrt{|g|}}{\partial x^{\alpha}} V^{\alpha} + \sqrt{|g|} \cdot \frac{\partial V^{\alpha}}{\partial x^{\alpha}} \right]$$
$$= \frac{1}{r} \left[V^r + r \cdot \left[\frac{\partial V^r}{\partial r} + \frac{\partial V^{\theta}}{\partial \theta} \right] \right]$$

$$= \frac{1}{r} V^r + \frac{\partial V^r}{\partial r} + \frac{\partial V^{\theta}}{\partial \theta}$$

$$= \frac{1}{r} \left(\frac{\partial}{\partial r} (r V^r) \right) + \frac{\partial V^{\theta}}{\partial \theta}$$



$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2 \theta \end{pmatrix}$$

$$\det(g_{ij}) = r^2 \cdot r^2 \sin^2 \theta - 0 + 0 \\ = r^4 \sin^2 \theta$$

$$V^\alpha_{; \alpha} = \frac{1}{\sqrt{-g}} (\sqrt{-g} V^\alpha)_{, \alpha}$$

$$= \frac{1}{r^2 \sin \theta} (r^2 \sin \theta V^\alpha)_{, \alpha}$$

$$= \frac{1}{r^2 \sin \theta} \left(\partial_\alpha (r^2 \sin \theta) \cdot V^\alpha + r^2 \sin \theta \partial_\alpha V^\alpha \right)$$

$$\downarrow$$

$$2r \sin \theta \cdot V^r + r^2 \cos \theta V^\theta$$

$$\downarrow$$

$$r^2 \sin \theta (\partial_r V^r + \partial_\theta V^\theta + \partial_\phi V^\phi)$$

$$= \frac{2}{r} V^r + \frac{\cos \theta}{\sin \theta} V^\theta + \partial_r V^r + \partial_\theta V^\theta + \partial_\phi V^\phi$$

6.11

$$V^{\alpha}_{,\beta\gamma} = V^{\alpha}_{,\beta\gamma} \left\{ \begin{array}{l} V^{\alpha}_{,\beta} + \Gamma^{\alpha}_{\mu\beta} V^{\mu} = 0 \end{array} \right.$$

$$V^{\alpha}_{,\beta\gamma} + \left(\Gamma^{\alpha}_{\mu\beta} V^{\mu} \right)_{,\gamma} = 0$$

$$V^{\alpha}_{,\beta\gamma} + \Gamma^{\alpha}_{\mu\beta,\gamma} V^{\mu} + \Gamma^{\alpha}_{\mu\beta} V^{\mu}_{,\gamma} = 0$$

$$V^{\alpha}_{,\gamma\beta} + \Gamma^{\alpha}_{\mu\gamma,\beta} V^{\mu} + \Gamma^{\alpha}_{\mu\gamma} V^{\mu}_{,\beta} = 0$$

$$\left(\Gamma^{\alpha}_{\mu\beta,\gamma} - \Gamma^{\alpha}_{\mu\gamma,\beta} \right) V^{\mu} + \left(\Gamma^{\alpha}_{\mu\beta} V^{\mu}_{,\gamma} - \Gamma^{\alpha}_{\mu\gamma} V^{\mu}_{,\beta} \right) = 0$$

$$\underline{V^{\mu}_{,\gamma} = -\Gamma^{\mu}_{\sigma\gamma} V^{\sigma}} \quad \nearrow$$

$$V^{\mu}_{,\beta} = -\Gamma^{\mu}_{\sigma\beta} V^{\sigma}$$

$$\left(\Gamma^{\alpha}_{\mu\beta,\gamma} - \Gamma^{\alpha}_{\mu\gamma,\beta} \right) V^{\mu} = \left(\Gamma^{\alpha}_{\mu\beta} \Gamma^{\mu}_{\sigma\gamma} - \Gamma^{\alpha}_{\mu\gamma} \Gamma^{\mu}_{\sigma\beta} \right) V^{\sigma}$$

□

11b

$$(\Gamma_{\mu\beta,\nu}^{\alpha} - \Gamma_{\mu\nu,\beta}^{\alpha}) V^{\mu} = (\Gamma_{\beta\beta,\mu\nu}^{\alpha} - \Gamma_{\beta\nu,\mu\beta}^{\alpha}) V^{\mu}$$

$$(\Gamma_{\mu\beta,\nu}^{\alpha} - \Gamma_{\mu\nu,\beta}^{\alpha} - \Gamma_{\beta\beta,\mu\nu}^{\alpha} + \Gamma_{\beta\nu,\mu\beta}^{\alpha}) V^{\mu} = 0$$

6.29

$$g_{\alpha\beta} = \text{diag}(r^2, r^2 \sin^2 \theta) \quad \underline{R_{\theta\phi\theta\phi}}$$

$$R_{\theta\phi\theta\phi} = \Gamma_{\phi\phi,\theta}^{\theta} - \cancel{\Gamma_{\phi\theta,\phi}^{\theta}} + \cancel{\Gamma_{\theta\theta,\phi\phi}^{\theta}} - \Gamma_{\theta\phi,\phi\theta}^{\theta}$$

$$\Gamma_{\phi\phi}^{\theta} = -\cos 2\theta + \cos^2 \theta$$

$$= \sin^2 \theta - \cos^2 \theta + \cos^2 \theta$$

$$\cancel{\Gamma_{\phi\theta}^{\theta}}$$

$$\Gamma_{\theta\theta,\phi\phi}^{\theta} = \cancel{\Gamma_{r\theta,\phi\phi}^{\theta}} + \cancel{\Gamma_{\theta\theta,\phi\phi}^{\theta}} + \cancel{\Gamma_{\phi\theta,\phi\phi}^{\theta}}$$

$$\Gamma_{\theta\phi,\phi\theta}^{\theta} = \cancel{\Gamma_{r\phi,\phi\theta}^{\theta}} + \cancel{\Gamma_{\theta\phi,\phi\theta}^{\theta}} + \cancel{\Gamma_{\phi\phi,\phi\theta}^{\theta}}$$

$$\Gamma_{\mu\nu}^{\alpha} = \frac{1}{2} g^{\alpha\beta} (g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta})$$

$$\frac{1}{2} \sin^2 \theta$$

$$\Gamma_{\phi\phi}^{\theta} = \frac{1}{2} g^{\theta\theta} (\cancel{g_{\theta\phi,\phi}} + \cancel{g_{\phi\theta,\phi}} - g_{\phi\phi,\theta}) = \frac{1}{2} g^{\theta\theta} (-g_{\phi\phi,\theta}) = -\sin \theta \cdot \cos \theta$$

$$\Gamma_{\phi\theta}^{\theta} = \frac{1}{2} g^{\theta\theta} (\cancel{g_{\theta\phi,\theta}} + \cancel{g_{\theta\theta,\phi}} - \cancel{g_{\phi\theta,\theta}})$$

$$\Gamma_{\phi\theta}^{\phi} = \frac{1}{2} g^{\phi\phi} (g_{\phi\phi,\theta} + \cancel{g_{\phi\theta,\phi}} - \cancel{g_{\theta\phi,\phi}}) = \frac{1}{2} (r^2 \sin^2 \theta \cdot \frac{1}{r^2} + 2 \sin \theta \cos \theta) = \frac{\cos \theta}{\sin \theta}$$

$$R^{\theta}_{\phi\theta\phi} = \sin^2 \theta$$

$$R_{\theta\phi\theta\phi} = g_{\alpha\theta} R^{\alpha}_{\phi\theta\phi} = g_{\theta\theta} R^{\theta}_{\phi\theta\phi} = r^2 \sin^2 \theta$$

$$R_{\phi\theta\theta\phi} = -r^2 \sin^2 \theta$$

$$R_{\theta\phi\phi\theta} = -r^2 \sin^2 \theta$$

Q.E.D

6.30

expand cylinder to flat sheet with one side connected, then use cartesian coordinate.

6.32 (a)

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = g$$

$$\det(g - \lambda I) = 0$$

$$\begin{pmatrix} -\lambda & 1 & 0 & 0 \\ 1 & -\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix}$$

$$(1-\lambda)^2 (\lambda^2 - 1) = 0$$

$$(1-\lambda)^2 (\lambda+1)(\lambda-1) = 0$$

$$1-\lambda=0 \quad \lambda+1=0 \quad \lambda-1=0$$

$$1 \quad 1 \quad -1 \quad 1$$

$$g \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \lambda \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

$$a_2 = \lambda a_1$$

$$a_1 = \lambda a_2$$

$$a_3 = \lambda a_3$$

$$a_4 = \lambda a_4$$

$$\lambda = 1$$

$$1 \Rightarrow (0 \ 0 \ 1 \ 0)$$

$$(0 \ 0 \ 0 \ 1)$$

$$\left(\frac{1}{\sqrt{2}} \ \frac{1}{\sqrt{2}} \ 0 \ 0 \right)$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = S$$

$$A = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \quad SAS^{-1} = g$$

$$\text{signature}(g) = +2 \quad A = S^{-1}gS$$

⑥

6.33

$$w = r \cos(\chi) \quad z = r \sin(\chi) \cos(\theta)$$

$$y = r \sin(\chi) \sin(\theta) \sin(\phi) \quad x = r \sin(\chi) \sin(\theta) \cos(\phi)$$

$$r^2 \cos^2(\chi) + r^2 \sin^2(\chi) \cos^2(\theta) + r^2 \sin^2(\chi) \sin^2(\theta) \sin^2(\phi) + r^2 \sin^2(\chi) \sin^2(\theta) \cos^2(\phi)$$

$$r^2 (\cos^2 \chi + \sin^2 \chi \cos^2 \theta + \sin^2 \chi \sin^2 \theta \sin^2 \phi + \sin^2 \chi \sin^2 \theta \cos^2 \phi) = r^2$$

$$= 1$$

⑤

$$g_{\mu\nu} = \Lambda^\alpha_\mu \Lambda^\beta_\nu g_{\alpha\beta} \quad g_{\alpha\beta} = \text{diag}(1, 1, 1, 1)$$

$$g_{\theta\theta} = \Lambda^\alpha_\theta \Lambda^\alpha_\theta g_{\alpha\alpha} \quad \text{because } \alpha = \beta$$

$$= 1 \cdot \left(\frac{\partial x}{\partial \theta} \right)^2 + 1 \cdot \left(\frac{\partial y}{\partial \theta} \right)^2 + \left(\frac{\partial z}{\partial \theta} \right)^2 + \left(\frac{\partial w}{\partial \theta} \right)^2$$

$$= r^2 \sin^2 \chi \cos^2 \theta \cos^2 \phi + r^2 \sin^2 \chi \cos^2 \theta \sin^2 \phi$$

$$+ r^2 \sin^2 \chi \sin^2 \theta$$

$$= r^2 \sin^2 \chi \cos^2 \theta + r^2 \sin^2 \chi \sin^2 \theta$$

$$= r^2 \sin^2 \chi (\cos^2 \theta + \sin^2 \theta)$$

$$= r^2 \sin^2 \chi$$

$$g_{\theta\phi} = \frac{\partial x}{\partial \theta} \cdot \frac{\partial x}{\partial \phi} + \frac{\partial y^2}{\partial \theta \partial \phi} + \cancel{\frac{\partial z^2}{\partial \theta \partial \phi}} + \cancel{\frac{\partial w^2}{\partial \theta \partial \phi}}$$

$$= -r \sin \chi \cos \theta \cos \phi \cdot r \sin \chi \sin \theta \sin \phi$$

$$+ r \sin \chi \cos \theta \sin \phi \cdot r \sin \chi \sin \theta \cos \phi$$

$$= r^2 \sin^2 \chi \sin \theta \cos \theta \sin \phi \cos \phi - r^2 \sin^2 \chi \sin \theta \cos \theta \sin \phi \cos \phi$$

$$= 0$$

follow same way for rest component.