

GR notes

Yucun Xie

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Contents

1	Differential Geometry	3
1.1	Manifolds	3
1.2	Tensor	3
1.3	Connection	3
1.4	Geodesics	4
1.5	Riemann Tensor	4
2	Gravitation	4
2.1	Equivalence Principle	4
2.2	General Covariance Principle	4
2.3	Einstein's Equation	4
3	Black Holes	4
3.1	Schwarzschild	4
3.2	Kerr	4
4	Gravitational Radiation	4
4.1	Linearized Gravity	4
4.2	Effect of GW on matter	4
5	Cosmology	4
	Appendix	5
A	Special Relativity	5
A.1	Spacetime	5
A.2	Energy and Momentum	6
B	Topological Space	6
C	Property for some tensors	6

Conventions

1. Greek index (e.g. α, β, μ, ν) take value from $\{0, 1, 2, 3\}$.
2. Events denoted by cursive capitals (e.g. $\mathcal{A}, \mathcal{B}, \mathcal{E}$).
3. $(x^0, x^1, x^2, x^3) \equiv (t, x, y, z) \equiv x^\alpha$
4. Latin index (e.g. i, j, k) take value from $\{1, 2, 3\}$.
5. New unit that speed of light $c = 1$
6. Einstein summation convention $ds^2 = g_{\mu\nu} x^\mu x^\nu = \sum_{\mu=0}^3 \sum_{\nu=0}^3 g_{\mu\nu} x^\mu x^\nu$

1 Differential Geometry

1.1 Manifolds

Mathematically, spacetime is a **manifold**.

Definition 1.1. An n-dimensional manifold is a set that can be parameterized continuously by n independent real coordinates for each point. If a manifold is differentiable at each point, it is a **differentiable manifold**.

Definition 1.2. A coordinate system (also called chart) is n labels uniquely with each point of an n-dimensional manifold through a one-to-one mapping from \mathbb{R}^n to M . Generally, more than one charts are required to cover entire manifold, which called **atlas**.

Definition 1.3. Cartesian product $X \times Y$ is set of all possible ordered pairs of element which one from X and one from Y .

Subset of points within a manifold for curves and surfaces. Our spacetime is a 4-dimensional **pseudo Riemannian manifold** which is a manifold with some additional structures.

Remark. Manifolds also have a important property which is locally **homeomorphism** to \mathbb{R}^n . See the discussion of topology for definition of homeomorphism.

1.2 Tensor

Tensor is a quantity that have same form in all coordinate system. Tensor does not have components naturally, but when we choose specific coordinate system, we can write down its components. Tensor have **Lorentz Covariance**, which mean it follow a specific transformation law.

1.3 Connection

Connection is an additional structure inposed into manifold. There is no naturally defined connection between tangent space at each point on a manifold, so we can define this additional structure. The manifold equip with a flat, torsion-free connection is called **affine manifold**.

Proof. Here is a proof shows that connection not a tensor by show connection does not obey tensor transformation law.

$$\begin{aligned}\nabla_{\beta'} e_{\alpha'} &= \Gamma_{\alpha'\beta'}^{\gamma'} e_{\gamma'} \\ &= \frac{\partial x^\lambda}{\partial x^{\beta'}} \nabla_\lambda \left(\frac{\partial x^\mu}{\partial x^{\alpha'}} e_\mu \right) \\ &= \frac{\partial x^\lambda}{\partial x^{\beta'}} \left(\frac{\partial}{\partial x^\lambda} \frac{\partial x^\mu}{\partial x^{\alpha'}} e_\mu + \frac{\partial x^\mu}{\partial x^{\alpha'}} \Gamma_{\mu\lambda}^\gamma e_\gamma \right) \\ &= \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial}{\partial x^\lambda} \frac{\partial x^\mu}{\partial x^{\alpha'}} e_\mu + \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial x^\mu}{\partial x^{\alpha'}} \Gamma_{\mu\lambda}^\gamma e_\gamma \\ &= \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial}{\partial x^\lambda} \frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\mu} e_{\gamma'} + \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\gamma} \Gamma_{\mu\lambda}^\gamma e_{\gamma'}\end{aligned}$$

which yield

$$\Gamma_{\alpha'\beta'}^{\gamma'} = \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial}{\partial x^\lambda} \frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\mu} + \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\gamma} \Gamma_{\mu\lambda}^\gamma$$

There is an extra term in transformation of connection, so connection is not a tensor. □

1.4 Geodesics

1.5 Riemann Tensor

2 Gravitation

2.1 Equivalence Principle

2.2 General Covariance Principle

2.3 Einstein's Equation

3 Black Holes

3.1 Schwarzschild

3.2 Kerr

4 Gravitational Radiation

4.1 Linearized Gravity

4.2 Effect of GW on matter

5 Cosmology

A Special Relativity

A.1 Spacetime

Definition A.1. Inertial coordinate

The coordinate system must satisfy three property to be consider inertial coordinat:

1. The distance between two points are independent of time.
2. The clocks at every points ticking off time coordinate t at same rate.
3. The geometry of space is always flat.

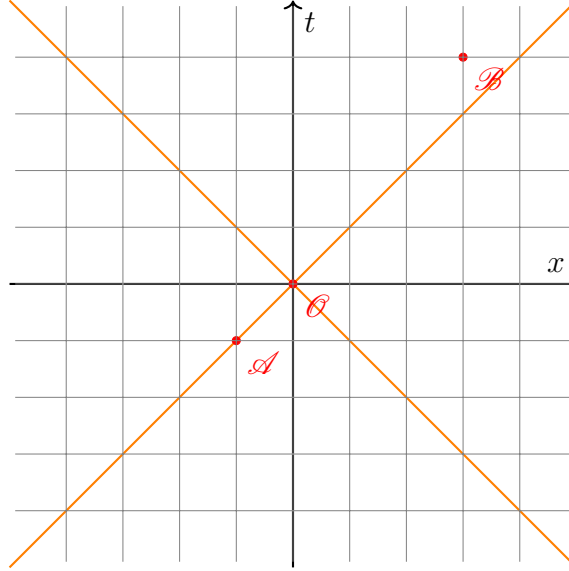


Figure 1: two events with coordinate $(-1, -1, 0, 0)$ and $(4, 3, 0, 0)$. Orange line is light's worldline.

The event in 4-D spacetime is defined by a set of coordinate (t, x, y, z) . For simplicity, we assume those events have $y = 0, z = 0$ so that we can draw a 2D graph to represent them.

Analog to Euclidean geometry, just like the euclidean distance $\Delta l^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$, we define the **spacetime interval** $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$.

Remark. There are a lot different conventions to define the sign of interval, here we just use the popular one $(-, +, +, +)$.

Example.

Interval for the two events in Figure 1 is $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -9$.

The universality speed of light means that $\frac{\Delta r}{\Delta t} = \frac{\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}}{\Delta t} = 1$ are always hold, then we can then write the interval $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = 0$. This experimental fact yield all laws of special relativity.

When the interval Δs^2 is less than 0, we call the seperation bewteen events is **timelike**; When the interval Δs^2 is equal to 0, we call it **lightlike** or null; When the interval Δs^2 is greater than 0, we call it **spacelike**.

A.2 Energy and Momentum

B Topological Space

C Property for some tensors

$$F_{\mu\nu} = -F_{\nu\mu}$$

$$T_{ij} = T_{ji}$$

$$g_{\mu\nu} = g_{\nu\mu}$$

$$\Gamma_{\mu\nu}^{\lambda} = \Gamma_{\nu\mu}^{\lambda} \text{ (Torsion free connection)}$$

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu}$$

$$R_{\alpha\beta\mu\nu} = -R_{\alpha\beta\nu\mu}$$

$$R_{\alpha\beta\mu\nu} = R_{\mu\nu\alpha\beta}$$

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0$$

$$R_{\alpha\beta} = R_{\beta\alpha}$$