6.9

$$V^\alpha{}_{;\alpha} = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}\,V^\alpha\right)_{,\alpha}$$ ① ↓ ?

$$V^\alpha{}_{;\alpha} = \frac{1}{r}\frac{\partial}{\partial r}\left(r V^\alpha\right) + \frac{\partial}{\partial \theta} V^\theta$$ ②

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$$g = \det(g_{\alpha\beta}) = r^2$$

$$V^\alpha{}_{;\alpha} = \frac{1}{\sqrt{|g|}}\left(\sqrt{|g|}\,V^\alpha\right)_{,\alpha}$$

$$= \frac{1}{\sqrt{|g|}}\left[\frac{\partial\sqrt{|g|}}{\partial x^\alpha}\cdot V^\alpha + \sqrt{|g|}\cdot\frac{\partial V^\alpha}{\partial x^\alpha}\right]$$

$$= \frac{1}{r}\left[V^r + r\cdot\left[\frac{\partial V^r}{\partial r} + \frac{\partial V^\theta}{\partial \theta}\right]\right]$$

$$= \frac{1}{r}V^r + \frac{\partial V^r}{\partial r} + \frac{\partial V^\theta}{\partial \theta}$$

$$= \frac{1}{r}\left(\frac{\partial}{\partial r}\left(r V^r\right)\right) + \frac{\partial V^\theta}{\partial \theta}$$

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

$$\det(g_{ij}) = r^2 \cdot r^2 \sin^2\theta - 0 + 0$$

$$= r^4 \sin^2\theta$$

$$V^\alpha{}_{;\alpha} = \frac{1}{\sqrt{-g}} \left(\sqrt{-g}\, V^\alpha\right)_{,\alpha}$$

$$= \frac{1}{r^2\sin\theta} \left(r^2\sin\theta\, V^\alpha \right)_{,\alpha}$$

$$= \frac{1}{r^2\sin\theta} \left(\partial_\alpha (r^2\sin\theta) \cdot V^\alpha + r^2\sin\theta\, \partial_\alpha V^\alpha \right)$$

$$2r\sin\theta \cdot V^r + r^2\cos\theta\, V^\theta$$

$$r^2\sin\theta \left(\partial_r V^r + \partial_\theta V^\theta + \partial_\phi V^\phi \right)$$

$$= \frac{2}{r} V^r + \frac{\cos\theta}{\sin\theta} V^\theta + \partial_r V^r + \partial_\theta V^\theta + \partial_\phi V^\phi$$

6.11

$$V^{\alpha}{}_{;\gamma\beta} = V^{\alpha}{}_{;\beta\gamma} \quad \left\{ V^{\alpha}{}_{,\beta} + \Gamma^{\alpha}{}_{\mu\beta} V^{\mu} = 0 \right.$$

$$V^{\alpha}{}_{,\beta\gamma} + \left(\Gamma^{\alpha}{}_{\mu\beta} V^{\mu}\right)_{,\gamma} = 0$$

$$V^{\alpha}{}_{,\beta\gamma} + \Gamma^{\alpha}{}_{\mu\beta,\gamma} V^{\mu} + \Gamma^{\alpha}{}_{\mu\beta} V^{\mu}{}_{,\gamma} = 0$$

$$V^{\alpha}{}_{,\gamma\beta} + \Gamma^{\alpha}{}_{\mu\gamma,\beta} V^{\mu} + \Gamma^{\alpha}{}_{\mu\gamma} V^{\mu}{}_{,\beta} = 0$$

$$\left(\Gamma^{\alpha}{}_{\mu\beta,\gamma} - \Gamma^{\alpha}{}_{\mu\gamma,\beta}\right) V^{\mu} + \left(\Gamma^{\alpha}{}_{\mu\beta} V^{\mu}{}_{,\gamma} - \Gamma^{\alpha}{}_{\mu\gamma} V^{\mu}{}_{,\beta}\right) = 0$$

$$\underline{V^{\mu}{}_{,\gamma} = -\Gamma^{\mu}{}_{\sigma\gamma} V^{\sigma}}$$

$$V^{\mu}{}_{,\beta} = -\Gamma^{\mu}{}_{\sigma\beta} V^{\sigma}$$

$$\left(\Gamma^{\alpha}{}_{\mu\beta,\gamma} - \Gamma^{\alpha}{}_{\mu\gamma,\beta}\right) V^{\mu} = \left(\Gamma^{\alpha}{}_{\mu\beta} \Gamma^{\mu}{}_{\sigma\gamma} - \Gamma^{\alpha}{}_{\mu\gamma} \Gamma^{\mu}{}_{\sigma\beta}\right) V^{\sigma}$$

$\square$

11b

$$\left(\Gamma^{\alpha}_{\mu\beta,\nu}-\Gamma^{\alpha}_{\mu\nu,\beta}\right)V^{\mu}=\left(\Gamma^{\alpha}_{\sigma\beta}\Gamma^{\sigma}_{\mu\nu}-\Gamma^{\alpha}_{\sigma\nu}\Gamma^{\sigma}_{\mu\beta}\right)V^{\mu}$$

$$\left(\Gamma^{\alpha}_{\mu\beta,\nu}-\Gamma^{\alpha}_{\mu\nu,\beta}-\Gamma^{\alpha}_{\sigma\beta}\Gamma^{\sigma}_{\mu\nu}+\Gamma^{\alpha}_{\sigma\gamma}\Gamma^{\sigma}_{\mu\beta}\right)V^{\mu}=0$$

6.29

$g_{\alpha\beta}=\mathrm{diag}(r^2, r^2\sin^2\theta)$ $\quad$ $R_{\theta\phi\theta\phi}$

$$\begin{vmatrix} 0 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin\theta \end{vmatrix}$$

$$R^{\theta}{}_{\phi\theta\phi}=\Gamma^{\theta}_{\phi\phi,\theta}-\cancel{\Gamma^{\theta}_{\phi\theta,\phi}}+\cancel{\Gamma^{\theta}_{\sigma\theta}\Gamma^{\sigma}_{\phi\phi}}-\Gamma^{\theta}_{\sigma\phi}\Gamma^{\sigma}_{\phi\theta}$$

$$\Gamma^{\theta}_{\phi\phi}=-\cos 2\theta+\cos^2\theta$$
$$=\sin^2\theta\cancel{-\cos^2\theta+\cos^2\theta}$$

$\cancel{\Gamma^{\theta}_{\phi\theta}}$

$$\Gamma^{\theta}_{\sigma\theta}\Gamma^{\sigma}_{\phi\phi}=\cancel{\Gamma^{\theta}_{r\theta}\Gamma^{r}_{\phi\phi}}+\cancel{\Gamma^{\theta}_{\theta\theta}\Gamma^{\theta}_{\phi\phi}}+\cancel{\Gamma^{\theta}_{\phi\theta}\Gamma^{\phi}_{\phi\phi}}$$

$$\Gamma^{\theta}_{\sigma\phi}\Gamma^{\sigma}_{\phi\theta}=\cancel{\Gamma^{\theta}_{r\phi}\Gamma^{r}_{\phi\theta}}+\cancel{\Gamma^{\theta}_{\theta\phi}\Gamma^{\theta}_{\phi\theta}}+\Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\phi\theta}$$

$$\| \quad -\sin\theta \quad -\cos^2\theta$$

$$\Gamma^{\alpha}_{\mu\nu}=\frac{1}{2}g^{\alpha\beta}(g_{\beta\mu,\nu}+g_{\beta\nu,\mu}-g_{\mu\nu,\beta})$$

$$\Gamma^{\theta}_{\phi\phi}=\frac{1}{2}g^{\theta\theta}(\cancel{g_{\theta\phi,\phi}}+\cancel{g_{\theta\phi,\phi}}-g_{\phi\phi,\theta})=\frac{1}{2}g^{\theta\theta}(-g_{\phi\phi,\theta})=-\sin\theta\cdot\cos\theta$$

$$\Gamma^{\theta}_{\phi\theta}=\frac{1}{2}g^{\theta\theta}(\cancel{g_{\theta\phi,\theta}}+\cancel{g_{\theta\theta,\phi}}-\cancel{g_{\phi\theta,\beta}})$$

$$\Gamma^{\phi}_{\phi\theta}=\frac{1}{2}g^{\phi\phi}(g_{\phi\phi,\theta}+\cancel{g_{\phi\theta,\phi}}-\cancel{g_{\phi\theta,\phi}})=\frac{1}{2}r^{-2}\sin^{-2}\theta\cdot 2r^2\sin\theta\cos\theta=\frac{\cos\theta}{\sin\theta}$$

$R^{\theta}{}_{\phi\theta\phi} = \sin^2\theta$

$R_{\theta\phi\theta\phi} = g_{\alpha\theta} R^{\alpha}{}_{\phi\theta\phi} = g_{\theta\theta} R^{\theta}{}_{\phi\theta\phi} = r^2\sin^2\theta$

$R_{\phi\theta\theta\phi} = -r^2\sin^2\theta$

$R_{\theta\phi\phi\theta} = -r^2\sin^2\theta$

Q.E.D

6.30

expand cylinder to flat sheet with one side connected, then use cartesian coordinate.

6.32 (a)

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = g$$

$$g \cdot \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = \lambda \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}$$

$$\det(g - \lambda I) = 0$$

$$\begin{pmatrix} -\lambda & 1 & 0 & 0 \\ 1 & -\lambda & 0 & 0 \\ 0 & 0 & 1-\lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{pmatrix}$$

$a_2 = \lambda a_1$

$a_1 = \lambda a_2$

$a_3 = \lambda a_3$

$a_4 = \lambda a_4$

$\lambda = 1$

$(1-\lambda)^2(\lambda^2 - 1) = 0$

$(1-\lambda)^2(\lambda+1)(\lambda-1) = 0$

$1-\lambda = 0 \quad \lambda+1 = 0 \quad \lambda - 1 = 0$

$1 \quad 1 \quad\quad -1 \quad\quad 1$

$1 \Rightarrow (0\ 0\ 1\ 0)$

$(0\ 0\ 0\ 1)$

$(\frac{1}{\sqrt{2}}\ \frac{1}{\sqrt{2}}\ 0\ 0)$

$-1 \Rightarrow (\frac{1}{\sqrt{2}}\ -\frac{1}{\sqrt{2}}\ 0\ 0)$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = S$$

$A = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$ $SAS^{-1} = g$

signature$(g) = +2$ $A = S^{-1} g S$

(b)

(6.33) $w = r\cos(\chi)$ $z = r\sin(\chi)\cos(\theta)$

$y = r\sin(\chi)\sin(\theta)\sin(\phi)$ $x = r\sin(\chi)\sin\theta\cos\phi$

$$r^2\cos^2(\chi) + r^2\sin^2(\chi)\cos^2\theta + r^2\sin^2\chi\sin^2\theta\sin^2\phi + r^2\sin^2\chi\sin^2\theta\cos^2\phi$$

$$\cancel{r^2}\left(\cos^2\chi + \sin^2\chi\cos^2\theta + \sin^2\chi\sin^2\theta\sin^2\phi + \sin^2\chi\sin^2\theta\cos^2\phi\right) = \cancel{r^2}$$

$$= 1$$

(b)

$$g_{\mu\nu} = \Lambda^\alpha{}_\mu \Lambda^\beta{}_\nu\, g_{\alpha\beta} \qquad g_{\alpha\beta} = \mathrm{diag}(1,1,1,1)$$

$$g_{\theta\theta} = \Lambda^\alpha{}_\theta \Lambda^\alpha{}_\theta\, g_{\alpha\alpha}$$ ~~because $\alpha = \beta$~~

$$= 1\cdot\left(\frac{\partial x}{\partial\theta}\right)^2 + 1\cdot\left(\frac{\partial y}{\partial\theta}\right)^2 + \left(\frac{\partial z}{\partial\theta}\right)^2 + \left(\frac{\partial w}{\partial\theta}\right)^2$$

$= r^2\sin^2\chi\cos^2\theta\cos^2\phi + r^2\sin^2\chi\cos^2\theta\sin^2\phi$

$+ r^2\sin^2\chi\sin^2\theta$

$= r^2\sin^2\chi\cos^2\theta + r^2\sin^2\chi\sin^2\theta$

$= r^2\sin^2\chi(\cos^2\theta + \sin^2\theta)$

$= r^2\sin^2\chi$

$g_{\theta\phi} = \frac{\partial x}{\partial\theta}\cdot\frac{\partial x}{\partial\phi} + \frac{\partial y^2}{\partial\theta\partial\phi} + \cancel{\frac{\partial z^2}{\partial\theta\partial\phi}} + \cancel{\frac{\partial w^2}{\partial\theta\partial\phi}}$

$= -r\sin\chi\cos\theta\cos\phi\cdot r\sin\chi\sin\theta\sin\phi$

$+ r\sin\chi\cos\theta\sin\phi\cdot r\sin\chi\sin\theta\cos\phi$

$= r^2\sin^2\chi\sin\theta\cos\theta\sin\phi\cos\phi - r^2\sin^2\chi\sin\theta\cos\theta\sin\phi\cos\phi$

$= 0$

follow same way for rest component.

6.39

(a)

$$[\vec{U}, \vec{V}]^\alpha = U^\beta \nabla_\beta V^\alpha - V^\beta \nabla_\beta U^\alpha$$

$$= U^\beta\left(\partial_\beta V^\alpha + \Gamma^\alpha_{\beta\lambda} V^\lambda\right) - V^\beta\left(\partial_\beta U^\alpha + \Gamma^\alpha_{\beta\lambda} U^\lambda\right)$$

$$= U^\beta \partial_\beta V^\alpha + \cancel{U^\beta \Gamma^\alpha_{\beta\lambda} V^\lambda} - V^\beta \partial_\beta U^\alpha - \cancel{V^\beta \Gamma^\alpha_{\beta\lambda} U^\alpha}$$

$$= U^\beta \partial_\beta V^\alpha - V^\beta \partial_\beta U^\alpha \quad \checkmark$$

$$= -\left(V^\beta \partial_\beta U^\alpha - U^\beta \partial_\beta V^\alpha\right) \quad \checkmark$$

(b)

$$[\vec{U}, f\vec{V}] = U^\beta \partial_\beta (f V^\alpha) - V^\beta \partial_\beta (U^\alpha)$$

$$= U^\beta\left[(\partial_\beta f) V^\alpha + f(\partial_\beta V^\alpha)\right] - f V^\beta \partial_\beta U^\alpha$$

$$= (\partial_\beta f) U^\beta V^\alpha + f U^\beta \partial_\beta V^\alpha - f V^\beta \partial_\beta V^\alpha$$

$$= \partial_\beta f\, U^\beta V^\alpha + f[U, V]$$

$$= \left([\nabla f](\vec{U})\right)\vec{V} + f[\vec{U}, \vec{V}]$$

(c) $(\mathcal{L}_{\vec{v}}\tilde{w})[\vec{V}] = \mathcal{L}_{\vec{v}}[\tilde{w}(\vec{V})] - \tilde{w}[\mathcal{L}_{\vec{v}}\vec{V}]$

$$\mathcal{L}_{\vec{v}}[\tilde{w}(\vec{V})] = \vec{v}\cdot\nabla[\tilde{w}(\vec{V})]$$
$$= v^\alpha(w_{\beta,\alpha}V^\beta + w_\beta V^\beta_{,\alpha})$$
$$= v^\alpha w_{\beta,\alpha}V^\beta + v^\alpha w_\beta V^\beta_{,\alpha}$$

$$-\tilde{w}(\mathcal{L}_{\vec{v}}\vec{V}) = -\tilde{w}(v^\alpha V^\beta_{,\alpha} - V^\alpha v^\beta_{,\alpha})$$
$$= -\tilde{w}_\beta v^\alpha V^\beta_{,\alpha} + \tilde{w}_\beta V^\alpha v^\beta_{,\alpha}$$

$$\mathcal{L}_{\vec{v}}[\tilde{w}(\vec{V})] - \tilde{w}(\mathcal{L}_{\vec{v}}\vec{V})$$
$$= v^\alpha w_{\beta,\alpha}V^\beta + \cancel{v^\alpha w_\beta V^\beta_{,\alpha}} - \cancel{\tilde{w}_\beta v^\alpha V^\beta_{,\alpha}} + \tilde{w}_\beta V^\alpha v^\beta_{,\alpha}$$
$$= V^\beta(v^\alpha w_{\beta,\alpha} + w_\alpha v^\alpha_{,\beta})$$

$$\mathcal{L}_{\vec{v}}\tilde{w} = v^\alpha w_{\beta,\alpha} + w_\alpha v^\alpha_{,\beta}$$