GR notes

Yucun Xie

July 9, 2022

Contents

Conventions

1. Greek index (e.g. α, β, μ, ν) take value from {0, 1, 2, 3}.
2. Events denoted by cursive capitals (e.g. $\mathscr{A}, \mathscr{B}, \mathscr{E}$).
3. $(x^0, x^1, x^2, x^3) \equiv (t, x, y, z) \equiv x^\alpha$
4. Latin index (e.g.i, j, k) take value from {1, 2, 3}.
5. New unit that speed of light $c = 1$
6. Einstein summation convention $ds^2 = g_{\mu\nu} x^\mu x^\nu = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu} x^\mu x^\nu$

1 Special Relativity

1.1 4-Dimensional Spacetime

Definition 1.1. Inertial coordinate
The coordinate system must satisfy three property to be consider inertial coordinat:

1. The distance between two points are independent of time.
2. The clocks at every points ticking off time coordinate t at same rate.
3. The geometry of space is always Euclidean (flat).

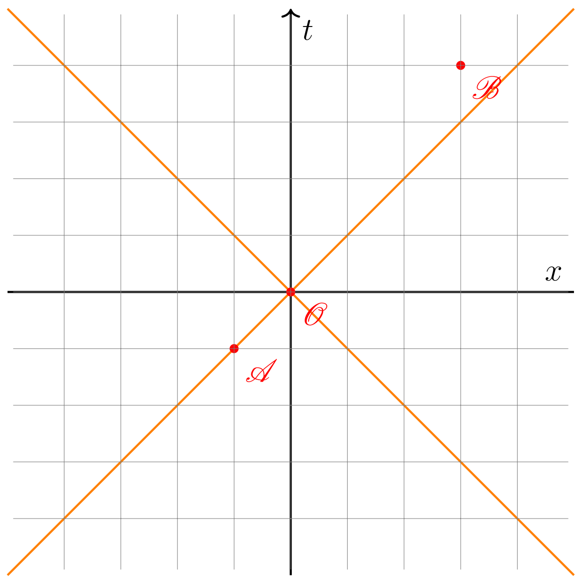

Figure 1: two events with coordinate $(-1,-1,0,0)$ and $(4,3,0,0)$. Orange line is light's worldline.

The event in 4-D spacetime is defined by a set of coordinate (t,x,y,z). For simplicity, we assume those events have $y=0, z=0$ so that we can draw a 2D graph to represent them.
Analog to Euclidean geometry, just like the euclidean distance $\Delta l^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$, we define the **spacetime interval** $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$.

Remark. There are a lot different conventions to define the sign of interval, here we just use the popular one $(-,+,+,+)$.

Example.
Interval for the two events in Figure 1 is $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -9$.

Due to universal speed of light, interval is invariant change of inertial coordinate, this means that $\Delta s^2 = \Delta \bar{s}^2$ When the interval Δs^2 is less than 0, we call it **timelike**; When the interval Δs^2 is equal to 0, we call it **lightlike** or null; When the interval Δs^2 is greater than 0, we call it **spacelike**. The $x^\mu = \{x^0, x^1, x^2, x^3\} = \{t,x,y,z\}$ is a set of coordinate.

2 Differential Geometry

2.1 Connection

Proof. Here is a proof shows that connection not a tensor by show connection does not obey tensor transformation law.

$$
\begin{aligned}
\nabla_{\beta'} e_{\alpha'} &= \Gamma^{\gamma'}_{\alpha'\beta'} e_{\gamma'} \\
&= \frac{\partial x^{\lambda}}{\partial x^{\beta'}} \nabla_{\lambda} \left(\frac{\partial x^{\mu}}{\partial x^{\alpha'}} e_{\mu} \right) \\
&= \frac{\partial x^{\lambda}}{\partial x^{\beta'}} \left(\frac{\partial}{\partial x^{\lambda}} \frac{\partial x^{\mu}}{\partial x^{\alpha'}} e_{\mu} + \frac{\partial x^{\mu}}{\partial x^{\alpha'}} \Gamma^{\gamma}_{\mu\lambda} e_{\gamma} \right) \\
&= \frac{\partial x^{\lambda}}{\partial x^{\beta'}} \frac{\partial}{\partial x^{\lambda}} \frac{\partial x^{\mu}}{\partial x^{\alpha'}} e_{\mu} + \frac{\partial x^{\lambda}}{\partial x^{\beta'}} \frac{\partial x^{\mu}}{\partial x^{\alpha'}} \Gamma^{\gamma}_{\mu\lambda} e_{\gamma} \\
&= \frac{\partial x^{\lambda}}{\partial x^{\beta'}} \frac{\partial}{\partial x^{\lambda}} \frac{\partial x^{\mu}}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^{\mu}} e_{\gamma'} + \frac{\partial x^{\lambda}}{\partial x^{\beta'}} \frac{\partial x^{\mu}}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^{\gamma}} \Gamma^{\gamma}_{\mu\lambda} e_{\gamma'}
\end{aligned}
$$

which yield

$$\Gamma^{\gamma'}_{\alpha'\beta'} = \frac{\partial x^{\lambda}}{\partial x^{\beta'}} \frac{\partial}{\partial x^{\lambda}} \frac{\partial x^{\mu}}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^{\mu}} + \frac{\partial x^{\lambda}}{\partial x^{\beta'}} \frac{\partial x^{\mu}}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^{\gamma}} \Gamma^{\gamma}_{\mu\lambda}$$

There is an extra term in transformation of connection, so connection is not a tensor. □