Gravitation

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Contents

Conventions

1. Greek index (e.g. α, β, μ, ν) take value from $\{0, 1, 2, 3\}$.
2. $(x^0, x^1, x^2, x^3) \equiv (t, x, y, z) \equiv x^\alpha$.
3. Latin index (e.g.i, j, k) take value from $\{1, 2, 3\}$.
4. Natural units ($c = 1$).
5. Einstein summation convention $ds^2 = g_{\mu\nu}dx^\mu dx^\nu = \sum_{\mu=0}^{3}\sum_{\nu=0}^{3} g_{\mu\nu}dx^\mu dx^\nu$.
6. Metric sign $(-, +, +, +)$.

1 Differential Geometry

1.1 Manifolds

Mathematically, specetime is a **manifold**.
Manifold is a space can be continuously (no hole, tear, etc.) covered by multiple coordinate charts. For example, think about a sphere, we can stretch a membrane with grid to cover it except one point, then we can use another membrane to cover that one point. Sphere is a example of 2-D manifold, and membrane is example of coordinate chart. Informally, manifold is locally looks like $\mathbb{R}^2$. For example, surface of the Earth is a 2-sphere S^2, but look at the ground around you, it seems flat.

Definition 1.1.
An n-dimensional manifold is is a mathematical structure consisting of

1. a set M of elements, called **points**.
2. a countable family of subsets of M, called **fundamental coordiante patches**, such that union of all of them is M.
3. a reversible mapping of each patch onto unit volumn of $\mathbb{R}^n$. An n-tuple of numbers associated in this way are called the **coordinates** of each point.

In a more formal way, a n-dimensional manifold is a topological space that locally homeomorphic to n-dimensional Eucildean space. See the discussion of topological space for definition of homeomorphism.
If a manifold is differentiable for m times at each point, it is a **differentiable manifold**, denote by C^m. If a manifold is inifinite differentiable at each point, denote by C^∞, called smooth manifold.
A coordinate system (also called chart) is n labels uniquely with each point of an n-dimensional manifold through a one-to-one mapping from $\mathbb{R}^n$ to M.
Generally, more than one charts are required to cover entire manifold, which called **atlas**.

Differential geometry, as the name suggests, is geometry on differential manifolds.

Manifolds are very bascis structure, but they support following things on it.

1. saclar field: $\phi(x)$
2. curves: $x^\mu(\lambda)$
3. surfaces: $x^\mu(\lambda_1, \lambda_2)$
4. tangent vector and dual vector.
5. tangent space and cotangent space.

1.1.1 Vector and Dual Vector

At each point P of a n-dimensional differentiable manifold, there is a n-dimensional vector space which basis is defined by directional derivative at P for curves passing through P. This vector space is called **tangent space**. This space contains all **vectors** at point P. There is also another vector space whose basis is defined by evaluating the gradients of curves passing through P at P. This space is called **cotangent space**, which contains all **dual vectors** at point P.
Vectors and dual vectors are local to a point.
Set of all tangent space in a manifold form a **tangent bundle**, and set of all cotangent space on a manifold form a **cotangent bundle**. They are example of **fiber bundle**.
Fiber bundle is a manifold which is locally the cartesian product of base space and fiber space, but not globally.
There is no definite way to transport a vector from one point to another on a manifold, because there is not relation between the tangent space at each point. To achecive this, we need impose an additional structure into manifold, which called connection.

1.1.2 Maps Between Manifolds

@TODO pullback pushforward

1.2 Tensor

Tensor is a quantity that have same form in all coordinate system. Tensor does not have components naturally, but when we choose specific coordinate system, we can write down its components. Tensor have **Covariance**, which mean it follow a specific transformation law.

1.2.1 Tensor Notation

A tensor with k upper indices and l lower indices

$$T^{\mu^1\mu^2\mu^3\cdots\mu^k}_{\nu^1\nu^2\nu^2\cdots\nu^l}$$

is the cartesian product of k vectors and l dual vector. Which map k dual vectors and l vectors to a real number. **Cartesian product** $X \times Y$ is set of all possible ordered pairs of element which one from X and one from Y.

1.2.2 Tensor Transformation Law

When we changing coordinate system, tensor components transform follow **tensor transformation law**.

Definition 1.2.
Tensor components in new coordinate system $(\alpha'\beta'\mu'\nu'\cdots)$ can be express as

$$T^{\alpha'\beta'\cdots}_{\mu'\nu'\cdots} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\frac{\partial x^{\beta'}}{\partial x^{\beta}}\cdots T^{\alpha\beta\cdots}_{\mu\nu\cdots}$$

Each upper indice is covariance with coordinate transform, each lower indice is contravariance with cooordiante transform. If some quantity obey tensor transformation law, it is a tensor. If a tensorial equation is hold in one coordinate system, it is hold in all coordinate system because both side are following same law to transform.

1.2.3 Exterior Calculus

A covariant totally antisymmetric rank $(0, p)$ tensor is called differential p-form. This is the reason why dual vector is sometime called 1-form. We can generate new differential forms by take antisymmetric part of tensor product, which called **wedge product**.

Example.
The wedge product of two one form is

$$(A \wedge B)_{\mu\nu} = 2A_{[\mu}B_{\nu]}$$

.

The **exterior derivative** can generate $p+1$-form from p-form.

Definition 1.3.
Exterior derivative of a p-form A is define as following.

$$dA = (p+1)\partial_{[\mu_1}A_{\mu_2\mu_3\cdots\mu_{p+1}]}$$

A differential p-form A is **closed** if $dA = 0$, and **exact** if exist another differential form which $dB = A$. We see that definition of exterior derivative does not require any additional structure. The reason of define differential form is that we can have derivative and integral without help of additional structure on manifold.

1.3 Affine Connection

Recall that we are unable to transform vector form one point to another on a manifold, because we missing realation between tangent space at each point. So we introduce an additional structure called **connection**. Connection is an additional structure that connect the tangent space on each point of a manifold. There is no naturally defined connection on a manifold, we have to define one, therefore connection on a manifold is not unique. In general relativity our spacetime manifold equiped with a metric compatible, torsion-free connection is called **affine connection**. Connection coeffienct in a coordinate system is express as **Christoffel symbol** $\Gamma^{\lambda}_{\alpha\beta}$.
In general, manifold with connection is a manifold allowing **parallel transport** of vectors.

1.3.1 Covariant Derivative

Partial eerivative is not covariance, we are looking for another kind of differentiation that is covariant. Here let me show that partial derivative not covariance and then construct a covariant one which we can apply to our curved spacetime.

$$\begin{aligned} A_{\mu,\nu} &= \frac{\partial A_\mu}{\partial x^\nu} = \frac{\partial}{\partial x^\nu}\left(A_\mu \frac{\partial x^\alpha}{\partial x^\mu}\right) \\ &= \frac{\partial A_\alpha}{\partial x^\mu}\frac{\partial x^\alpha}{\partial x^\mu} + \underbrace{A_\alpha \frac{\partial^2 x^\alpha}{\partial x^\mu \partial x^\nu}}_{\text{Not a tensor}} \end{aligned}$$

By using the connection coeffienct below, which also not a tensor, we can eliminate the none tensorial part of partial derivative. Which defined as follow.

Definition 1.4.
Covariant derivative of dual vector.

$$A_{\mu;\nu} = A_{\mu,\nu} - \Gamma^{\lambda}_{\mu\nu} A_\lambda$$

This is called **covariant derivative**, two non tensorial parts add together to form a tensor.

Proof. Here is a proof shows that connection not a tensor by show connection does not obey tensor transformation law.

$$\begin{aligned} \nabla_{\beta'} e_{\alpha'} &= \Gamma^{\gamma'}_{\alpha'\beta'} e_{\gamma'} \\ &= \frac{\partial x^\beta}{\partial x^{\beta'}} \nabla_\beta \left(\frac{\partial x^\alpha}{\partial x^{\alpha'}} e_\alpha\right) \\ &= \frac{\partial x^\beta}{\partial x^{\beta'}} \left(\frac{\partial}{\partial x^\beta}\frac{\partial x^\alpha}{\partial x^{\alpha'}} e_\alpha + \frac{\partial x^\alpha}{\partial x^{\alpha'}} \Gamma^{\gamma}_{\alpha\beta} e_\gamma\right) \\ &= \frac{\partial x^\beta}{\partial x^{\beta'}} \frac{\partial}{\partial x^\beta}\frac{\partial x^\alpha}{\partial x^{\alpha'}} e_\alpha + \frac{\partial x^\beta}{\partial x^{\beta'}} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \Gamma^{\gamma}_{\alpha\beta} e_\gamma \\ &= \frac{\partial x^\beta}{\partial x^{\beta'}} \frac{\partial}{\partial x^\beta}\frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\alpha} e_{\gamma'} + \frac{\partial x^\beta}{\partial x^{\beta'}} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\gamma} \Gamma^{\gamma}_{\alpha\beta} e_{\gamma'} \end{aligned}$$

which yield

$$\Gamma^{\gamma'}_{\alpha'\beta'} = \frac{\partial x^\beta}{\partial x^{\beta'}} \frac{\partial}{\partial x^\beta}\frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\alpha} + \frac{\partial x^\beta}{\partial x^{\beta'}} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\gamma} \Gamma^{\gamma}_{\alpha\beta}$$

There is an extra term in transformation of connection, so connection is not a tensor. □

Our connection is metric compatible and torsion free, which mean

$$g_{\mu\nu;\alpha} = 0$$

and

$$S_{\mu\nu} = \Gamma^{\lambda}_{[\mu\nu]} = 0$$

Which $S_{\mu\nu}$ is torsion tensor.

1.3.2 Parallel Transport

Recall we defined connection allow us transport vectors on manifold. For **parallel transport** a vector, we want each infinitesimal step of transport maintain the vector parallel. Which required **intrinsic derivative** vanish.

$$\frac{DV^\mu}{D\lambda} = \frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\alpha\nu} V^\alpha \frac{dx^\nu}{d\lambda} = 0$$

λ is **affine parameter**, which is the parameter alone the curve.

1.4 Metric Space

Now, in addition to affine connection, we want a structure allowing measurement of distance on manifold. The **metric space** is a kind of space have notion of distance.

Definition 1.5.
A Riemannian manifold is an manifold with connection and equipped with a symmetric **metric tensor** $g_{\mu\nu}$. which distance ds between two point with displacement dx^μ is defined as following.

$$ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu$$

Metric tensor is one of the most important tensor in general relativity, which direct lead to the all structures of the spacetime.

Example.
The connection coeffienct can be direct calculate from metric tensor.

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$$

In general relativity, proper time is equivalence to distance in timelike direction $d\tau^2 = -ds^2$.

1.4.1 Riemann Tensor

Now, we have everything we need to define the **curvature**. Think about transport a vector V^ρ around a infinitesimal loop on a curved manifold, the loop is defined by two vector dA^μ and dB^ν. Then, the change on V^ρ can be express as

$$\delta V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma dA^\mu dB^\nu$$

Definition 1.6.
$R^\rho{}_{\sigma\mu\nu}$ is known as **Riemann curvature tensor**, or just Riemann tensor, can be express through connection coeffienct:

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

There are some property in Riemann tensor.

$$\begin{aligned} R_{\alpha\beta\mu\nu} &= -R_{\beta\alpha\mu\nu} \\ R_{\alpha\beta\mu\nu} &= -R_{\alpha\beta\nu\mu} \\ R_{\alpha\beta\mu\nu} &= R_{\mu\nu\alpha\beta} \\ R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} &= 0 \\ \nabla_{[\lambda} R_{\alpha\beta]\mu\nu} &= 0 \end{aligned}$$

The last line above is called **Bianchi identity**.

Definition 1.7.
From Riemann tensor, we can construct a symmetric rank 2 tensor, **Ricci tensor** by contract first and

third indice of Riemann tensor.

$$R_{\mu\nu} = R_{\nu\mu} = R^{\lambda}{}_{\mu\lambda\nu}$$

Furthermore, we can take the trace of Ricci tensor, to get **Ricci scalar** or scalar curvature.

$$R = g^{\mu\nu} R_{\mu\nu}$$

We can recover the Riemann tensor from scalar curvature through metric tensor.

$$R_{\alpha\beta\mu\nu} = \frac{R}{2}(g_{\alpha\mu}g_{\beta\nu} - g_{\beta\mu}g_{\alpha\nu})$$

From Bianchi identity $\nabla_{[\lambda} R_{\alpha\beta]\mu\nu} = 0$, we can find that

$$\nabla_{\mu}(R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R) = 0$$

We see divergence of this tensor vanish, therefore, we define a new tensor named **Einstein tensor**,

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R.$$

The trace-free part of Riemann tensor, is called **Weyl tensor**, can be define as following.

Definition 1.8.
Weyl tensor. @TODO

$$C_{\rho\sigma\mu\nu} =$$

2 Gravitation

Our spacetime not a simply Riemannian manifold, but a **pseudo-Riemannian manifold**, which more general Riemannian manifold. Riemannian manifold require metric tensor being positive definite, which mean all eigenvalue of metric tensor is positive. For pseudo-Riemannian manifold, the metric eigenvalue can be either positive or negative, called indefinite.

Remark.
The eigenvalue of metric can not be zero, if a eigenvalue is zero, the metric will be degenerate and have no inverse.

More specifically, our spacetime is **Lorentzian manifold**, which is a pseudo-Riemannian manifold with only one negative eigenvalue. Lorentzian manifold allow us to classify tangent vectors into timelike, spacelike, and lightlike.

2.1 Equivalence Principle

Definition 2.1.

1. Inertial mass is defined through Newton's second law, $m = F/a$.
2. Gravitational mass is defined through Newton's aw of gravity $m = Fr^2/GM$.

There are two equivalence principle, weak equivalence principle and strong equivalence principle (or Einstein equivalence principle). Weak Principle of Equivalence state that inertial mass are always equals to gravitational mass. Einstein extended this idea to a stronger statement, which is **Strong Equivalence Principle**. Which state that observer is unable to distinguish acceleration and gravitational field by local experiment. This lead the idea of **local inertial frame**, which is correspondence to the property of manifold (local flatness). With equivalence principle, we can explain deflection of light and gravitational redshift in gravitational field.

General Covariance Principle state that if a tensorial equation hold in a gravitiational field yield that

1. hold in absence of gravity.

2. hold under coordinate transformation.

Therefore, we can write our classical equations in tensorial form, then they will hold in general relativity.

Example.
Tensorial Maxwell's equations:

$$\partial_\mu F^{\nu\mu} = J^\nu$$
$$\partial_{[\mu} F_{\nu\lambda]} = 0$$

In general, we can just replace the partial derivative by covariant derivative, and replace flat spactime metric by $g_{\mu\nu}$, then it will valid in curved spacetime.

2.2 Geodesics

Geodesic is generalization of straight line in curved spacetime, which is the trajectory of inertial moving particle. More specifically, geodesic is a path parallel-transport its own tangent vector. Recall our equation of parallel transport, $\frac{DV^\mu}{D\lambda} = \frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\alpha\nu} V^\alpha \frac{dx^\nu}{d\lambda} = 0$ and replace V by tangent vector of the path $\frac{dx^\mu}{d\lambda}$. We get

$$\frac{D}{D\lambda}\frac{dx^\mu}{d\lambda} = \frac{d}{d\lambda}\frac{dx^\mu}{d\lambda} + \Gamma^\mu_{\alpha\nu}\frac{dx^\alpha}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$$

This is famous **geodesic equation**, all path satisfy this equation is a geodesic.

Definition 2.2.
Geodesic equation.

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\nu}\frac{dx^\alpha}{d\lambda}\frac{dx^\nu}{d\lambda} = 0$$

Geodesic always maximized the proper time. Think about twin paradox in special relativity, the twin stay on Earth approximately move alone geodesic, but the twin travel away at tuning point isn't.

2.3 Killing Vector Field

2.4 Einstein's Equation

@TODO

3 Relativistic Star

3.1 Schwarzschild Solution

The first exact solution to Einstein's equation by Karl Schwarzschild is named **Schwarzschild solution**, which is the exterior solution to spherical star. This solution is solved by three boundary conditions.

1. Static, which mean completely independent of time. More formally, metric tensor satisfy time translational symmetry and time reversal symmetry.
2. Vacuum, which mean outside the star, implying vanish of energy-momentum tensor. This mean the solution is not vaild in the interior of star.
3. Spherical symmetry, which mean for given radius, the spacetime should be the same.

The line element take form

$$ds^2 = -\left(1 - \frac{r_\mathrm{s}}{r}\right) dt^2 + \left(1 - \frac{r_\mathrm{s}}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

which $r_\mathrm{s} = 2GM$ and $d\Omega^2 = (d\theta^2 + \sin^2\theta\, d\phi^2)$.

3.1.1 Interior Solution

3.1.2 Black hole

3.1.3 Maximal Extension

3.2 Kerr Solution

3.2.1 Dragging of Inertial Frame

3.2.2 Ergosphere

3.2.3 Penrose Process

4 Gravitational Radiation

4.1 Linearized Gravity

When the gravitational field are weak, the metric take following form :

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

which we treat the gravitational field as a perturbation of flat spacetime metric.

4.2 Gravitational Wave

4.2.1 Effect of Gravitational Wave on matter

4.2.2 Source of Gravitational Wave

5 Cosmology

5.1 Roberson-Walker Metric

5.2 Friedmann Equation

5.3 Inflation

A Special Relativity

A.1 Spacetime

In spacial relativity, we discard the absolute concept of time, in contrast to Newton, there is no preferd reference frame and time is one of the coordinate. Now we have a 4-dimensional **spacetime**. Our discussion is focus on inertial coordinate system.

Definition A.1.
Inertial coordinate
The coordinate system must satisfy three property to be consider inertial coordinate:

1. The distance between two points are independent of time.
2. The clocks at every points ticking off time coordinate t at same rate.
3. The geometry of space is always flat.

Basically, it is a coordinate system without acceleration.

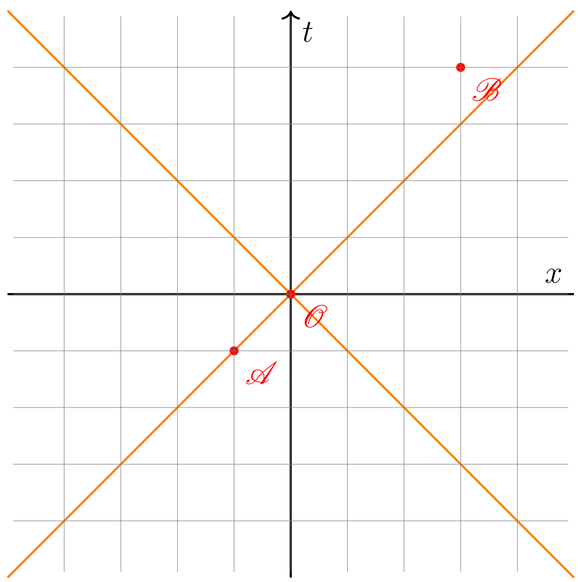

Figure 1: two events with coordinate $(-1, -1, 0, 0)$ and $(4, 3, 0, 0)$. Orange line is light's worldline.

The event in 4-D spacetime is defined by a set of coordinate (t, x, y, z). For simplicity, we assume those events have $y = 0, z = 0$ so that we can draw a 2D graph to represent them.
Analog to Euclidean geometry, just like the euclidean distance $\Delta l^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$, we define the **spacetime interval** $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$.

Remark.
There are a lot different conventions to define the sign of interval, here we just use the popular one $(-, +, +, +)$.

Example.

Interval for the two events in Figure 1 is $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -9$.

The universality speed of light means that $\frac{\Delta r}{\Delta t} = \frac{\sqrt{\Delta x^2+\Delta y^2+\Delta z^2}}{\Delta t} = 1$ are always hold, then we can then write the interval $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = 0$. This experimental fact yield all laws of special

relativity.
When the interval Δs^2 is less than 0, we call the seperation bewteen events is **timelike**; When the interval Δs^2 is equal to 0, we call it **lightlike** or null; When the interval Δs^2 is greater than 0, we call it **spacelike**. If there is another coordinate system, which move with speed v alone x direction of original frame, we can draw this fram like the Figure 2 below.

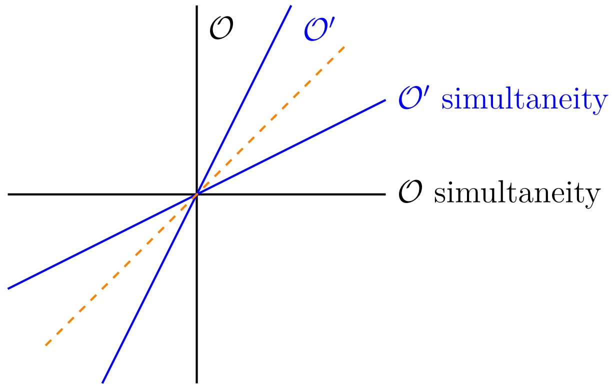

Figure 2: Frame $\mathcal{O}'$ move alone x direction of $\mathcal{O}$

A.2 Energy and Momentum

A.3 Fluid

B Topological Space

C Lie Algebra

C.1 Lie Derivative

D Penrose Diagram

E Summary of Formula

$$
\begin{aligned}
F_{\mu\nu} &= -F_{\nu\mu} \\
T_{ij} &= T_{ji} \\
g_{\mu\nu} &= g_{\nu\mu} \\
\Gamma^{\lambda}_{\mu\nu} &= \Gamma^{\lambda}_{\nu\mu} \text{ (Torsion free)} \\
R_{\alpha\beta\mu\nu} &= -R_{\beta\alpha\mu\nu} \\
R_{\alpha\beta\mu\nu} &= -R_{\alpha\beta\nu\mu} \\
R_{\alpha\beta\mu\nu} &= R_{\mu\nu\alpha\beta} \\
R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} &= 0 \\
R_{\alpha\beta} &= R_{\beta\alpha}
\end{aligned}
$$