GR notes

Yucun Xie

August 7, 2022

Contents

Conventions

1. Greek index (e.g. α, β, μ, ν) take value from $\{0,\ 1,\ 2,\ 3\}$.
2. $(x^0, x^1, x^2, x^3) \equiv (t, x, y, z) \equiv x^\alpha$.
3. Latin index (e.g.i, j, k) take value from $\{1,\ 2,\ 3\}$.
4. Natural units ($c = 1$).
5. Einstein summation convention $ds^2 = g_{\mu\nu} x^\mu x^\nu = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu} x^\mu x^\nu$.
6. Metric sign $(-, +, +, +)$.

1 Differential Geometry

1.1 Manifolds

Mathematically, specetime is a **manifold**.

Definition 1.1. An n-dimensional manifold is a set that can be parameterized continuously by n independent real coordinates for each point. If a manifold is differentiable at each point, it is a **differentiable manifold**.

Definition 1.2. A coordinate system (also called chart) is n labels uniquely with each point of an n-dimensional manifold through a one-to-one mapping from $\mathbb{R}^n$ to M.
Generally, more than one charts are required to cover entire manifold, which called **atlas**.

Definition 1.3. Cartesian product $X \times Y$ is set of all possible ordered pairs of element which one from X and one from Y.

Subset of points within a manifold for curves and surfaces. Our spacetime is a 4-dimensianl **pseudo Riemannian manifold** which is a manifold with some additional structures.

Remark. Manifolds also have a important property which is locally **homeomorphism** to $\mathbb{R}^n$. See the discussion of topology for definition of homeomorphism.

1.2 Tensor

Tensor is a quantity that have same form in all coordinate system. Tensor does not have components naturally, but when we choose specific coordinate system, we can write down its components. Tensor have **Lorentz Covariance**, which mean it follow a specific transformation law.

1.3 Connection

Connection is an additional structure inposed into manifold. There is no naturally defined connection between tangent space at each point on a manifold, so we can define this additional structure. The manifold equip with a flat, torsion-free connection is called **affine manifold**.

Proof. Here is a proof shows that connection not a tensor by show connection does not obey tensor transformation law.

$$
\begin{aligned}
\nabla_{\beta'} e_{\alpha'} &= \Gamma^{\gamma'}_{\alpha'\beta'} e_{\gamma'} \\
&= \frac{\partial x^\lambda}{\partial x^{\beta'}} \nabla_\lambda \left(\frac{\partial x^\mu}{\partial x^{\alpha'}} e_\mu\right) \\
&= \frac{\partial x^\lambda}{\partial x^{\beta'}} \left(\frac{\partial}{\partial x^\lambda}\frac{\partial x^\mu}{\partial x^{\alpha'}} e_\mu + \frac{\partial x^\mu}{\partial x^{\alpha'}} \Gamma^\gamma_{\mu\lambda} e_\gamma\right) \\
&= \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial}{\partial x^\lambda}\frac{\partial x^\mu}{\partial x^{\alpha'}} e_\mu + \frac{\partial x^\lambda}{\partial x^{\beta'}}\frac{\partial x^\mu}{\partial x^{\alpha'}} \Gamma^\gamma_{\mu\lambda} e_\gamma \\
&= \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial}{\partial x^\lambda}\frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\mu} e_{\gamma'} + \frac{\partial x^\lambda}{\partial x^{\beta'}}\frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\gamma}\Gamma^\gamma_{\mu\lambda} e_{\gamma'}
\end{aligned}
$$

which yield

$$
\Gamma^{\gamma'}_{\alpha'\beta'} = \frac{\partial x^\lambda}{\partial x^{\beta'}} \frac{\partial}{\partial x^\lambda}\frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\mu} + \frac{\partial x^\lambda}{\partial x^{\beta'}}\frac{\partial x^\mu}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\gamma}\Gamma^\gamma_{\mu\lambda}
$$

There is an extra term in transformation of connection, so connection is not a tensor. □

1.4 Geodesics

1.5 Riemann Tensor

2 Gravitation

2.1 Equivalence Principle

2.2 General Covariance Principle

2.3 Einstein's Equation

3 Black Holes

3.1 Schwarzschild

3.2 Kerr

4 Gravitational Radiation

4.1 Linearized Gravity

When the gravitational field are weak, the metric take following form :

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

which we treat the gravitational field as a perturbation of flat spacetime metric.

4.2 Effect of GW on matter

5 Cosmology

A Special Relativity

A.1 Spacetime

Definition A.1. Inertial coordinate
The coordinate system must satisfy three property to be consider inertial coordinat:

1. The distance between two points are independent of time.
2. The clocks at every points ticking off time coordinate t at same rate.
3. The geometry of space is always flat.

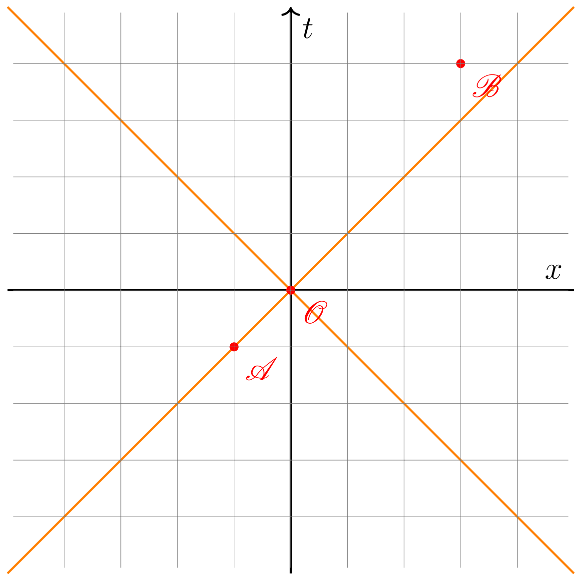

Figure 1: two events with coordinate $(-1, -1, 0, 0)$ and $(4, 3, 0, 0)$. Orange line is light's worldline.

The event in 4-D spacetime is defined by a set of coordinate (t, x, y, z). For simplicity, we assume those events have $y = 0, z = 0$ so that we can draw a 2D graph to represent them.
Analog to Euclidean geometry, just like the euclidean distance $\Delta l^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$, we define the **spacetime interval** $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$.

Remark. There are a lot different conventions to define the sign of interval, here we just use the popular one $(-, +, +, +)$.

Example.
Interval for the two events in Figure 1 is $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -9$.

The universality speed of light means that $\frac{\Delta r}{\Delta t} = \frac{\sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}}{\Delta t} = 1$ are always hold, then we can then write the interval $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = 0$. This experimental fact yield all laws of special relativity.
When the interval Δs^2 is less than 0, we call the seperation bewteen events is **timelike**; When the interval Δs^2 is equal to 0, we call it **lightlike** or null; When the interval Δs^2 is greater than 0, we call it **spacelike**.

A.2 Energy and Momentum

A.3 Fluid

B Topological Space

C Property for some tensors

$$
\begin{aligned}
F_{\mu\nu} &= -F_{\nu\mu} \\
T_{ij} &= T_{ji} \\
g_{\mu\nu} &= g_{\nu\mu} \\
\Gamma^{\lambda}_{\mu\nu} &= \Gamma^{\lambda}_{\nu\mu} \text{ (Torsion free)} \\
R_{\alpha\beta\mu\nu} &= -R_{\beta\alpha\mu\nu} \\
R_{\alpha\beta\mu\nu} &= -R_{\alpha\beta\nu\mu} \\
R_{\alpha\beta\mu\nu} &= R_{\mu\nu\alpha\beta} \\
R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} &= 0 \\
R_{\alpha\beta} &= R_{\beta\alpha}
\end{aligned}
$$