GR notes

Yucun Xie

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Contents

Conventions

1. Greek index (e.g. α, β, μ, ν) take value from $\{0, 1, 2, 3\}$.
2. $(x^0, x^1, x^2, x^3) \equiv (t, x, y, z) \equiv x^\alpha$.
3. Latin index (e.g.i, j, k) take value from $\{1, 2, 3\}$.
4. Natural units ($c = 1$).
5. Einstein summation convention $ds^2 = g_{\mu\nu} x^\mu x^\nu = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu} x^\mu x^\nu$.
6. Metric sign $(-, +, +, +)$.

1 Differential Geometry

1.1 Manifolds

Mathematically, specetime is a **manifold**.

Definition 1.1. An n-dimensional manifold is a set that can be parameterized continuously by n independent real coordinates for each point. If a manifold is differentiable at each point, it is a **differentiable manifold**.

Differential geometry, as the name suggests, is geometry on differential manifolds.

Definition 1.2. A coordinate system (also called chart) is n labels uniquely with each point of an n-dimensional manifold through a one-to-one mapping from $\mathbb{R}^n$ to M.
Generally, more than one charts are required to cover entire manifold, which called **atlas**.

Subset of points within a manifold form curves and surfaces. Our spacetime is a 4-dimensional **pseudo-Riemannian manifold** which is a differentiable manifold with some additional structures.

Remark. Manifolds also have a important property which a n-dimensional manifold locally **homeomorphism** to $\mathbb{R}^n$. See the discussion of topology for definition of homeomorphism. Basically this mean a small enough region on manifold is looks same as flat space. For example, surface of the Earth is a 2-sphere S^2, but look at the ground around you, it seems flat.

1.1.1 Maps Between Manifolds

pullback pushforward

1.2 Tensor

Tensor is a quantity that have same form in all coordinate system. Tensor does not have components naturally, but when we choose specific coordinate system, we can write down its components. Tensor have **Covariance**, which mean it follow a specific transformation law.

1.2.1 Vector and Dual Vector

At each point P of a n-dimensional differentiable manifold, there is a n-dimensional vector space which basis is defined by directional derivative at P for curves passing through P. This vector space is called **tangent space**. This space contains all **vectors** at point P. There is also another vector space whose basis is defined by evaluating the gradients of curves passing through P at P. This space is called **cotangent space**, which contains all **dual vectors** at point P.
Vectors and dual vectors are local to a point.
Set of all tangent space in a manifold form a **tangent bundle**, and set of all cotangent space on a manifold form a **cotangent bundle**. They are example of **fiber bundle**.

Definition 1.3. fiber bundle is a manifold which is locally the cartesian product of base space and fiber space, but not globally.

1.2.2 Tensor Notation

A tensor with k upper indices and l lower indices

$$T^{\mu^1\mu^2\mu^3\cdots\mu^k}_{\nu^1\nu^2\nu^2\cdots\nu^l}$$

is the cartesian product of k vectors and l dual vector. Which map k dual vectors and l vectors to a real number. **Cartesian product** $X \times Y$ is set of all possible ordered pairs of element which one from X and one from Y.

1.2.3 Tensor Transformation Law

When we changing coordinate system, tensor components transform follow **tensor transformation law**.

Definition 1.4. Tensor components in new coordinate system $(\alpha'\beta'\mu'\nu'\cdots)$ can be express as

$$T^{\alpha'\beta'\cdots}_{\mu'\nu'\cdots} = \frac{\partial x^{\mu}}{\partial x^{\mu'}}\frac{\partial x^{\nu}}{\partial x^{\nu'}}\frac{\partial x^{\alpha'}}{\partial x^{\alpha}}\frac{\partial x^{\beta'}}{\partial x^{\beta}}\cdots T^{\alpha\beta\cdots}_{\mu\nu\cdots}$$

Each upper indice is covariance with coordinate transform, each lower indice is contravariance with cooordiante transform. If some quantity obey tensor transformation law, it is a tensor. If a tensorial equation is hold in one coordinate system, it is hold in all coordinate system because both side are following same law to transform.

1.3 Affine Space

Connection is an additional structure equiped into manifold. There is no naturally defined connection between tangent space at each point on a manifold, so we can define this additional structure. The manifold equiped with a flat, torsion-free connection is called **affine manifold**. Connection coeffienct in a coordinate system is express as **Christoffel symbol** $\Gamma^{\lambda}_{\alpha\beta}$.

1.3.1 Covariant Derivative

Because partial differentiation is not covariance, we are looking for another kind of differentiation that is covariant. Here let me show that partial derivative not covariance and then construct a covariant one which we can apply to our curved spacetime.

$$\begin{aligned} A_{\mu,\nu} &= \frac{\partial A_\mu}{\partial x^\nu} = \frac{\partial}{\partial x^\nu}\left(A_\mu \frac{\partial x^\alpha}{\partial x^\mu}\right) \\ &= \frac{\partial A_\alpha}{\partial x^\mu}\frac{\partial x^\alpha}{\partial x^\mu} + \underbrace{A_\alpha \frac{\partial^2 x^\alpha}{\partial x^\mu \partial x^\nu}}_{\text{Not a tensor}} \end{aligned}$$

By using the connection coeffienct below, which also not a tensor, we can eliminate the none tensorial part of partial derivative. Which defined as follow.

Definition 1.5. Covariant derivative of dual vector.

$$A_{\mu;\nu} = A_{\mu,\nu} - \Gamma^{\lambda}_{\mu\nu}A_\lambda$$

This is called **covariant derivative**, two non tensorial parts add together to form a tensor.

Proof. Here is a proof shows that connection not a tensor by show connection does not obey tensor transformation law.

$$\begin{aligned} \nabla_{\beta'} e_{\alpha'} &= \Gamma^{\gamma'}_{\alpha'\beta'} e_{\gamma'} \\ &= \frac{\partial x^\beta}{\partial x^{\beta'}} \nabla_\beta \left(\frac{\partial x^\alpha}{\partial x^{\alpha'}} e_\alpha\right) \\ &= \frac{\partial x^\beta}{\partial x^{\beta'}} \left(\frac{\partial}{\partial x^\beta}\frac{\partial x^\alpha}{\partial x^{\alpha'}} e_\alpha + \frac{\partial x^\alpha}{\partial x^{\alpha'}}\Gamma^{\gamma}_{\alpha\beta} e_\gamma\right) \\ &= \frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial}{\partial x^\beta}\frac{\partial x^\alpha}{\partial x^{\alpha'}} e_\alpha + \frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial x^\alpha}{\partial x^{\alpha'}}\Gamma^{\gamma}_{\alpha\beta} e_\gamma \\ &= \frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial}{\partial x^\beta}\frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial x^{\gamma'}}{\partial x^\alpha} e_{\gamma'} + \frac{\partial x^\beta}{\partial x^{\beta'}}\frac{\partial x^\alpha}{\partial x^{\alpha'}}\frac{\partial x^{\gamma'}}{\partial x^\gamma}\Gamma^{\gamma}_{\alpha\beta} e_{\gamma'} \end{aligned}$$

which yield

$$\Gamma^{\gamma'}_{\alpha'\beta'} = \frac{\partial x^\beta}{\partial x^{\beta'}} \frac{\partial}{\partial x^\beta} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\alpha} + \frac{\partial x^\beta}{\partial x^{\beta'}} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^{\gamma'}}{\partial x^\gamma} \Gamma^\gamma_{\alpha\beta}$$

There is an extra term in transformation of connection, so connection is not a tensor. □

Our connection is flat and torsion free, which mean

$$g_{\mu\nu;\alpha} = 0$$

and

$$S_{\mu\nu} = \Gamma^\lambda_{[\mu\nu]} = 0$$

Which $S_{\mu\nu}$ is torsion tensor.

1.3.2 Intrinsic Derivative

1.3.3 Lie Derivative

1.4 Parallel Transport

In a curved manifold, there is no natural way to transport a vector form one point to another. Recall that vector is defined in the tangent space and each point on the manifold has a different tangent space. Then we need connection between tangent space to transport vectors in curved manifold. For **parallel transport** a vector, we want each infinitesimal step of transport maintain the vector parallel. Which required intrinsic derivative vanish.

$$\frac{DV^\mu}{D\lambda} = \frac{dV^\mu}{d\lambda} + \Gamma^\mu_{\alpha\nu} V^\alpha \frac{dx^\nu}{d\lambda} = 0$$

λ is **affine parameter**, which is the parameter alone the curve.

1.5 Metric Space

Metric tensor is one of the most important tensor in general relativity, which direct lead to the all structures of the spacetime.

Example. The connection coeffienct can be direct calculate from metric tensor.

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$$

The **metric space** is a kind of space have notion of distance, our Riemannian manifold is an example of metric space. Distance is define through metric tensor.

$$ds^2 = g_{\mu\nu} x^\mu x^\nu$$

Which also lead to proper time through this path $d\tau^2 = -ds^2$.

1.5.1 Riemann Tensor

2 Gravitation

2.1 Equivalence Principle

Definition 2.1.
1. Inertial mass is define through Newton's second law, $m = F/a$.
2. Gravitational mass is defined through Newton's aw of gravity $m = Fr^2/GM$.

There are two equivalence principle, weak equivalence principle and strong equivalence principle (or Einstein equivalence principle).

Definition 2.2. Weak Principle of Equivalence state that inertial mass are always equals to gravitational mass.

Einstein extended this idea to a stronger statement, which is **Strong Equivalence Principle**. Which state that observer is unable to distinguish acceleration and gravitational field by local experiment. This lead the idea of **local inertial frame**, which is correspondence to the property of manifold (local flatness). With equivalence principle, we can explain deflection of light and gravitational redshift in gravitational field.

General Covariance Principle state that if a tensorial equation hold in a gravitiational field yield that

1. hold in absence of gravity.
2. hold under coordinate transformation.

2.2 Geodesics

2.3 Einstein's Equation

3 Black Holes

3.1 Schwarzschild

3.2 Kerr

4 Gravitational Radiation

4.1 Linearized Gravity

When the gravitational field are weak, the metric take following form :

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$

which we treat the gravitational field as a perturbation of flat spacetime metric.

4.2 Effect of GW on matter

5 Cosmology

A Special Relativity

A.1 Spacetime

In spacial relativity, we discard the absolute concept of time, in contrast to Newton, there is no preferd reference frame and time is one of the coordinate. Now we have a 4-dimensional **spacetime**. Our discussion is focus on inertial coordinate system.

Definition A.1. Inertial coordinate
The coordinate system must satisfy three property to be consider inertial coordinate:

1. The distance between two points are independent of time.
2. The clocks at every points ticking off time coordinate t at same rate.
3. The geometry of space is always flat.

Basically, it is a coordinate system without acceleration.

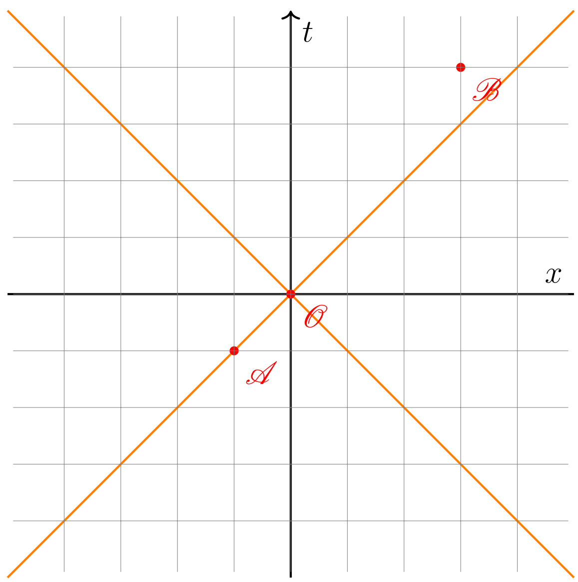

Figure 1: two events with coordinate $(-1, -1, 0, 0)$ and $(4, 3, 0, 0)$. Orange line is light's worldline.

The event in 4-D spacetime is defined by a set of coordinate (t, x, y, z). For simplicity, we assume those events have $y = 0, z = 0$ so that we can draw a 2D graph to represent them.
Analog to Euclidean geometry, just like the euclidean distance $\Delta l^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$, we define the **spacetime interval** $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2$.

Remark. There are a lot different conventions to define the sign of interval, here we just use the popular one $(-, +, +, +)$.

Example.
Interval for the two events in Figure 1 is $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = -9$.

The universality speed of light means that $\frac{\Delta r}{\Delta t} = \frac{\sqrt{\Delta x^2+\Delta y^2+\Delta z^2}}{\Delta t} = 1$ are always hold, then we can then write the interval $\Delta s^2 = -\Delta t^2 + \Delta x^2 + \Delta y^2 + \Delta z^2 = 0$. This experimental fact yield all laws of special relativity.
When the interval Δs^2 is less than 0, we call the seperation bewteen events is **timelike**; When the interval Δs^2 is equal to 0, we call it **lightlike** or null; When the interval Δs^2 is greater than 0, we call it **spacelike**.

If there is another coordinate system, which move with speed v alone x direction of original frame, we can draw this fram like the Figure 2 below.

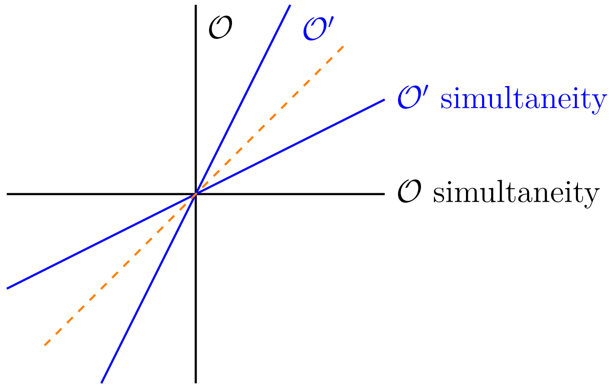

Figure 2: Frame $\mathcal{O}'$ move alone x direction of $\mathcal{O}$

A.2 Energy and Momentum

A.3 Fluid

B Topological Space

C Lie Algebra

D Property of some tensors

$$
\begin{aligned}
F_{\mu\nu} &= -F_{\nu\mu} \\
T_{ij} &= T_{ji} \\
g_{\mu\nu} &= g_{\nu\mu} \\
\Gamma^{\lambda}_{\mu\nu} &= \Gamma^{\lambda}_{\nu\mu} \text{ (Torsion free)} \\
R_{\alpha\beta\mu\nu} &= -R_{\beta\alpha\mu\nu} \\
R_{\alpha\beta\mu\nu} &= -R_{\alpha\beta\nu\mu} \\
R_{\alpha\beta\mu\nu} &= R_{\mu\nu\alpha\beta} \\
R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} &= 0 \\
R_{\alpha\beta} &= R_{\beta\alpha}
\end{aligned}
$$