Quantum Field Theory in/of Curved Spacetime

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Contents

Conventions

1. Greek index (e.g. α, β, μ, ν) run over time and space.
2. Latin index (e.g. i, j, k) run over space.
3. Natural units ($c = 1$).
4. Einstein summation convention.

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \sum_{\mu=0}^{n-1} \sum_{\nu=0}^{n-1} g_{\mu\nu} dx^\mu dx^\nu$$

5. Metric signature $(+, -, -, -)$.

1 QFT in Flat Spacetime Revisit

In this chapter, we will review the quantum field theory in flat spacetime and carefully distinguish the concepts only valid in flat spacetime and the concepts that could be generalized to curved spacetime.

1.1 Scalar Field Construction

Consider a massive scalar field $\phi(t, x^i)$ defined in spacetime point (t, x^i) satisfying the **Klein-Gordon equation**:

$$(\Box + m^2)\phi = 0 \tag{1.1.1}$$

Remark.
Here we use the metric sign convention $(+,-,-,-)$; if we used other sign convention, the Klein-Gordon equation would read $(\Box - m^2)\phi = 0$.

where the d'Alembertian $\Box$ is defined as $\Box = g^{\mu\nu}\partial_\mu\partial_\nu$. In this chapter, we assume the spacetime is flat, so $g^{\mu\nu} = \eta^{\mu\nu}$.

This equation could be obtained from the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2)$$

by demanding the variations of the action

$$S = \int \mathcal{L}(x)\mathrm{d}^n x$$

vanish.

The solution of the Klein-Gordon equation satisfying

$$f_{\mathbf{k}}(t, \mathbf{x}) = Ae^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}$$

The dispersion relation is

$$\omega = \sqrt{(|\mathbf{k}|^2 + m^2)}.$$

We can rewrite the above mode functions use four-wave vector $k^\mu = (\omega, \mathbf{k})$:

$$f_{\mathbf{k}}(x^\mu) = Ae^{-i(k_\mu x^\mu)} \tag{1.1.2}$$

The above solution is very similar to the solution of harmonic oscillators. However, there is a significant difference: A harmonic oscillator only has one independent solution because it has a fixed, unique frequency. This feature no longer holds for fields theory because we have an infinite number solution for each value of k. Therefore, we should construct a general solution by constructing a complete, orthonormal set of modes that any solution can express as a linear combination of modes. To achieve this, first define the inner product of mode functions:

Definition 1.1.1 (Klein-Gordon inner product)**.**

$$(\phi_1, \phi_2) = -i\int_{\Sigma_t} \mathrm{d}^{n-1}x\ [\phi_1\partial_t\phi_2^* - \phi_2^*\partial_t\phi_1]$$

Which is integral over constant-time hypersurface Σ_t.

From generalized Stoke's theorem:

$$\int_M \mathrm{d}\omega = \int_{\partial M} \omega$$

it is easy to check that the inner product is independent of choose of the hypersurface. By explicitly calculating the inner product:

$$\begin{aligned}
(f_{\mathbf{k}}, f_{\mathbf{k}'}) &\propto \left(e^{-ik^\mu x_\mu}, e^{-ik'^\nu x_\nu}\right) && (1.1.3)\\
&= -i\int_{\Sigma_t} \mathrm{d}^{n-1}x \ \left[e^{-i\omega t+\mathbf{k}\cdot\mathbf{x}}\partial_t e^{i\omega' t-\mathbf{k}'\cdot\mathbf{x}} - e^{i\omega' t-\mathbf{k}'\cdot\mathbf{x}}\partial_t e^{-i\omega t+\mathbf{k}\cdot\mathbf{x}}\right] && (1.1.4)\\
&= -i\int_{\Sigma_t} \mathrm{d}^{n-1}x \ \left[e^{-i\omega t}e^{\mathbf{k}\cdot\mathbf{x}}\partial_t e^{i\omega' t}e^{-\mathbf{k}'\cdot\mathbf{x}} - e^{i\omega' t}e^{-\mathbf{k}'\cdot\mathbf{x}}\partial_t e^{-i\omega t}e^{\mathbf{k}\cdot\mathbf{x}}\right] && (1.1.5)\\
&= \int_{\Sigma_t} \mathrm{d}^{n-1}x \ \left[e^{-i\omega t}e^{\mathbf{k}\cdot\mathbf{x}}\omega' e^{i\omega' t}e^{-\mathbf{k}'\cdot\mathbf{x}} + e^{i\omega' t}e^{-\mathbf{k}'\cdot\mathbf{x}}\omega e^{-i\omega t}e^{\mathbf{k}\cdot\mathbf{x}}\right] && (1.1.6)\\
&= e^{-i\omega t}e^{i\omega' t}(\omega'+\omega)\int_{\Sigma_t} \mathrm{d}^{n-1}x \ \left[e^{\mathbf{k}\cdot\mathbf{x}}e^{-\mathbf{k}'\cdot\mathbf{x}}\right] && (1.1.7)\\
&= e^{i(\omega'-\omega)t}(\omega'+\omega)\int_{\Sigma_t} \mathrm{d}^{n-1}x \ \left[e^{(\mathbf{k}-\mathbf{k}')\cdot\mathbf{x}}\right] && (1.1.8)\\
&= e^{i(\omega'-\omega)t}(\omega'+\omega)(2\pi)^{n-1}\delta^{n-1}\left(\mathbf{k}-\mathbf{k}'\right) && (1.1.9)
\end{aligned}$$

we find that $(f_{\mathbf{k}}, f_{\mathbf{k}'}) = 0$ for $\mathbf{k} \neq \mathbf{k}'$. Furthermore, if we choose the normalization constant A in eq 1.1.2 as $\frac{1}{\sqrt{2\omega(2\pi)^{n-1}}}$, we find the mode function

$$f_{\mathbf{k}}(x^\mu) = \frac{e^{-ik_\mu x^\mu}}{\sqrt{2\omega(2\pi)^{n-1}}} \tag{1.1.10}$$

obey

$$(f_{\mathbf{k}}, f_{\mathbf{k}'}) = \delta^{(n-1)}(\mathbf{k}-\mathbf{k}'). \tag{1.1.11}$$

Given our dispersion relation, $\mathbf{k}$ only determines the absolute value of frequency. However, we can require that all mode functions have positive frequency and still give a complete set of mode functions by including complex conjugates $f^*_{\mathbf{k}}(x^\mu)$.

The positive frequency mode is defined as

$$\frac{\partial}{\partial t} f_{\mathbf{k}} = -i\omega f_{\mathbf{k}}.$$

And the mode with negative frequency is

$$\frac{\partial}{\partial t} f^*_{\mathbf{k}} = i\omega f^*_{\mathbf{k}}.$$

The negative frequency modes are orthogonal to the positive frequency modes:

$$(f_{\mathbf{k}}, f^*_{\mathbf{k}'}) = 0. \tag{1.1.12}$$

And they are orthonormal with each other with a negative norm:

$$(f^*_{\mathbf{k}}, f^*_{\mathbf{k}'}) = -\delta^{(n-1)}(\mathbf{k} - \mathbf{k}') \tag{1.1.13}$$

Hence, modes $f_{\mathbf{k}}$ and $f^*_{\mathbf{k}}$ form a complete set, which any possible solution of Klein-Gordon equation can be expressed in terms of them.

1.2 Canonical Quantization

The system could be quantized in the canonical quantization scheme by treating the field ϕ as an operator $\hat{\phi}$, then impose the canonical commutation relations on equal-time hypersurface:

Definition 1.2.1 (Canonical commutation relation)**.**

$$\begin{aligned}\left[\hat{\phi}(t,\mathbf{x}),\hat{\phi}(t,\mathbf{x}')\right] &= 0\\ \left[\hat{\pi}(t,\mathbf{x}),\hat{\pi}(t,\mathbf{x}')\right] &= 0\\ \left[\hat{\phi}(t,\mathbf{x}),\hat{\pi}(t,\mathbf{x}')\right] &= i\delta^{(n-1)}(\mathbf{x}-\mathbf{x}')\end{aligned}$$

The first two commutation relations come from the causality requirement, as those operators have spacelike separation. The delta function implies that field and momentum operators commute everywhere except the spacetime point they intersect. Just like the classical solution of the Klein-Gordon equation can be expanded in terms of mode, the field operator $\hat{\phi}$ also can be expanded in term mode function and have coefficients $\hat{a}_{\mathbf{k}}$ and $\hat{a}^{\dagger}_{\mathbf{k}}$ respectively as shown below:

1.2.1 (Mode expansion)**.**

$$\phi(t,\mathbf{x}) = \int \mathrm{d}^{n-1}k \left[\hat{a}_{\mathbf{k}} f_{\mathbf{k}}(t,\mathbf{x}) + \hat{a}^{\dagger}_{\mathbf{k}} f^{*}_{\mathbf{k}}(t,\mathbf{x})\right]$$

By using the commutation relation defined in 1.2.1, we can obtain the commutation relation of operator $\hat{a}_{\mathbf{k}}$ and $\hat{a}^{\dagger}_{\mathbf{k}}$:

$$\left[\hat{a}_{\mathbf{k}},\hat{a}_{\mathbf{k}'}\right] = 0 \tag{1.2.1}$$
$$\left[\hat{a}^{\dagger}_{\mathbf{k}},\hat{a}^{\dagger}_{\mathbf{k}'}\right] = 0 \tag{1.2.2}$$
$$\left[\hat{a}_{\mathbf{k}},\hat{a}^{\dagger}_{\mathbf{k}'}\right] = \delta^{(n-1)}(\mathbf{k}-\mathbf{k}') \tag{1.2.3}$$

Analog to harmonic oscillators, the operator $\hat{a}_{\mathbf{k}}$ and $\hat{a}^{\dagger}_{\mathbf{k}}$ are annihilation and creation operator respectively. The only difference is that we now have an infinite set of annihilation and creation operators corresponding to each spatial wave vector $\mathbf{k}$.

Remark.
The quantization process described above is sometimes referred to as **second quantization**. Historically, this name comes from the fact that we first treat the mode as discrete and then have an integer number of excitation of each mode. However, the name "second quantization" can be misleading because the discrete mode is a classical phenomenon. We quantized the field exactly once.

There is a single state $|0\rangle$ that would be anihilated by all $\hat{a}_{\mathbf{k}}$, called **vacuum**.

Definition 1.2.2 (Vacuum)**.**

$$\forall\, \mathbf{k},\ \hat{a}_{\mathbf{k}}\,|0\rangle = 0.$$

A state with n particles with identical momentum $\mathbf{k}$ can be constructed by repeat acting $\hat{a}^\dagger_{\mathbf{k}}$ on the vacuum:

$$|n_{\mathbf{k}}\rangle = \frac{1}{\sqrt{n_{\mathbf{k}}!}} \left(\hat{a}^\dagger_{\mathbf{k}}\right)^n |0\rangle \tag{1.2.4}$$

Similarly, we can construct a state with n_i particle for momentum $\mathbf{k}_i$:

$$|n_1, n_2, \cdots, n_j\rangle = \frac{1}{\sqrt{n_1! n_2! \cdots n_j!}} \left(\hat{a}^\dagger_{\mathbf{k}_1}\right)^{n_1} \left(\hat{a}^\dagger_{\mathbf{k}_2}\right)^{n_2} \cdots \left(\hat{a}^\dagger_{\mathbf{k}_j}\right)^{n_j} |0\rangle \tag{1.2.5}$$

We can create or annihilate particles with certain momentum:

Example 1.2.1.

$$\begin{aligned}
\hat{a}_{\mathbf{k}_i} |n_1, n_2, \cdots, n_i, \cdots, n_j\rangle &= \sqrt{n_i}\, |n_1, n_2, \cdots, n_i - 1, \cdots, n_j\rangle \\
\hat{a}^\dagger_{\mathbf{k}_i} |n_1, n_2, \cdots, n_i, \cdots, n_j\rangle &= \sqrt{n_i + 1}\, |n_1, n_2, \cdots, n_i + 1, \cdots, n_j\rangle
\end{aligned}$$

Furthermore, we can define **number operator**:

Definition 1.2.3 (Number operator)**.**

$$\hat{n}_{\mathbf{k}} = \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}}$$

Which obeys:

$$\hat{n}_{\mathbf{k}_i} |n_1, n_2, \cdots, n_i, \cdots, n_j\rangle = n_i |n_1, n_2, \cdots, n_i, \cdots, n_j\rangle \tag{1.2.6}$$

The eigenstates of the number operator form a basis span Hilbert space, known as **Fock basis**. The space span by this basis is called **Fock space**.

To illustrate the essential difference between flat and curved spacetime, we will examine how the Fock basis behaves under Lorentz transformation. Consider a boost along x direction; the time derivative in the boosted frame is:

$$\begin{aligned}
\partial_{t'} f_{\mathbf{k}} &= \frac{\partial x^\mu}{\partial t'} \partial_\mu f_{\mathbf{k}} && (1.2.7)\\
&= \gamma(-i\omega) f_{\mathbf{k}} + \gamma \mathbf{v} \cdot (i\mathbf{k}) f_{\mathbf{k}} && (1.2.8)\\
&= -i\omega' f_{\mathbf{k}} && (1.2.9)
\end{aligned}$$

where

$$\omega' = \gamma\omega - \gamma \mathbf{v} \cdot \mathbf{k}$$

. Hence, a state describing a collection of particles in the boosted frame would describe the identical particles with boosted momentum. Therefore, the number operator in the boosted frame is identical to

the original number operator; then, the vacuum will also be the same. This particular result is due to the existence of a timelike Killing vector in flat spacetime, a direct consequence of Poincare symmetry. With Poincare symmetry, our original choice of the inertial frame would be irrelevant. Therefore, all inertial observers agree with a unique, single vacuum state.

1.3 Vacuum Energy

The energy-momentum tensor of scalar field theory can be constructed in a standard manner:

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta(g^{\mu\nu})} \tag{1.3.1}$$

$$= \phi_{,\mu}\phi_{,\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\lambda\sigma}\phi_{,\lambda}\phi_{,\sigma} + \frac{1}{2}m^2\phi^2\eta_{\mu\nu} \tag{1.3.2}$$

The Hamiltonian operator can be obtained from the classical theory of field in the same manner. Recall the Hamiltonian of Klein-Gordon field is:

$$H = \int \mathrm{d}^3x\, \mathcal{H} = \int \mathrm{d}^3x\, T_{tt} = \int \mathrm{d}^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right] \tag{1.3.3}$$

By substituting the mode expansion of $\hat{\phi}$, we obtained the expression of Hamiltonian of quantized K-G field:

$$\hat{H} = \frac{1}{2}\int \mathrm{d}^{n-1}k \left[\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \hat{a}_{\mathbf{k}}\hat{a}^\dagger_{\mathbf{k}}\right]\omega \tag{1.3.4}$$

Use the commutation relation of $\hat{a}_{\mathbf{k}}$ and $\hat{a}^\dagger_{\mathbf{k}}$, we can further simplify the Hamiltonian operator:

$$\hat{H} = \int \mathrm{d}^{n-1}k \left[\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}} + \frac{1}{2}\delta^{(n-1)}(0)\right]\omega \tag{1.3.5}$$

$$= \int \mathrm{d}^{n-1}k \left[\hat{a}^\dagger_{\mathbf{k}}\hat{a}_{\mathbf{k}}\omega\right] + \int \mathrm{d}^{n-1}k \left[\frac{1}{2}\delta^{(n-1)}(0)\omega\right] \tag{1.3.6}$$

The problem has arisen: if we calculate the expectation value of Hamiltonian in the vacuum state, one would expect to get 0, however, we get infinite. The vacuum has infinite energy! The first reason we see infinite vacuum energy is that we are integral over all space. This is reasonable, analog to harmonic oscillator zero point energy, if we sum over infinite many ground state harmonic oscillators, we are expecting infinite energy. The divergences caused by infinitely large space are often referred to as **infrared divergences**. We can eliminate this kind of infinite through **regularization**. Let confine the field in a box with length L by imposing periodic boundary conditions, and rewrite the second term in 1.3.5 as:

$$\int \mathrm{d}^{n-1}k \left[\frac{1}{2}\delta^{(n-1)}(0)\omega\right] \to \frac{1}{2}\left[\frac{L}{2\pi}\right]^{n-1}\sum_{\mathbf{k}}\omega \tag{1.3.7}$$

We have used the Fourier transform of δ function.

However, after we restrict the vacuum in a finite region, the expression in 1.3.7 is still divergent. Since the value of $\omega = \sqrt{|\mathbf{k}|^2 + m^2}$ can be arbitrarily large. This infinite arises because we assumed quantum field theory is valid for arbitrarily high frequency/energy which corresponds to arbitrarily short distance. We expect to see new physics at that energy scale! The divergences caused by infinitely high frequency are often referred to as **ultraviolet divergences**. We can eliminate this kind of infinite through **renormalization**.

The simplified idea is just substrating off infinite from our expression. This is valid because what we can measure in the experiment is the energy difference, we can simply rescale the zero point of energy and not affect the observable.

1.4 Green Function

Vacuum expectation values of various combinations of field operators can be treated with various Green functions of the Klein-Gordon equation.

For the K-G field, the expectation value of the commutator of the field operator can be represented by **Schwinger function** $G(x^\mu, x'^\mu)$:

$$iG(x^\mu, x'^\mu) = \langle 0|\Big[\hat{\phi}(x^\mu), \hat{\phi}(x'^\mu)\Big]|0\rangle \tag{1.4.1}$$

The anticommutator of field operator is defined as:

Definition 1.4.1 (Anticommutator)**.**

$$\Big\{\hat{\phi}(x^\mu), \hat{\phi}(x'^\mu)\Big\} = \hat{\phi}(x^\mu)\hat{\phi}(x'^\mu) + \hat{\phi}(x'^\mu)\hat{\phi}(x^\mu)$$

The expectation value of anticommutator can be represented by **Hadamard's elementary function** $G^{(1)}(x^\mu, x'^\mu)$:

$$G^{(1)}(x^\mu, x'^\mu) = \langle 0|\Big\{\hat{\phi}(x^\mu), \hat{\phi}(x'^\mu)\Big\}|0\rangle \tag{1.4.2}$$

These Green functions can be split into positive and negative frequency parts:

$$iG(x^\mu, x'^\mu) = G^+(x^\mu, x'^\mu) - G^-(x^\mu, x'^\mu) \tag{1.4.3}$$
$$G^{(1)}(x^\mu, x'^\mu) = G^+(x^\mu, x'^\mu) + G^-(x^\mu, x'^\mu) \tag{1.4.4}$$

where $G^\pm$ are **Wightman functions**, given by

1.4.1 (Wightman functions)**.**

$$G^+(x^\mu, x'^\mu) = \langle 0|\hat{\phi}(x^\mu)\hat{\phi}(x'^\mu)|0\rangle$$
$$G^-(x^\mu, x'^\mu) = \langle 0|\hat{\phi}(x'^\mu)\hat{\phi}(x^\mu)|0\rangle\,.$$

Furthermore, the **Feynman propagator** G_F can be defined as the time-ordered product of fields operators:

1.4.2 (Feynman propagator)**.**

$$\begin{aligned} iG_F(x^\mu, x'^\mu) &= \langle 0|T(\hat{\phi}(x^\mu)\hat{\phi}(x'^\mu))|0\rangle \\ &= \theta(t-t')G^+(x^\mu, x'^\mu) + \theta(t'-t)G^-(x^\mu, x'^\mu) \end{aligned}$$

Where $\theta(t)$ is the **Heaviside step function**, defined as

Definition 1.4.2 (Heaviside step function)**.**

$$\theta(t) = \begin{cases} 1 & \text{if } t > 0 \\ 0 & \text{if } t < 0. \end{cases}$$

Using the step function, we can further define retarded and advanced Green functions:

1.4.3 (Retarded/Advanced Green functions)**.**

$$G_R(x^\mu, x'^\mu) = -\theta(t - t')G(x^\mu, x'^\mu)$$
$$G_A(x^\mu, x'^\mu) = \theta(t - t')G(x^\mu, x'^\mu)$$

The average of retarded and advanced Green functions is denoted as

$$\bar{G}(x^\mu, x'^\mu) = \frac{1}{2}\left[G_R(x^\mu, x'^\mu) + G_A(x^\mu, x'^\mu)\right]. \tag{1.4.5}$$

Then the Feynman propagator can be written as

$$G_F(x^\mu, x'^\mu) = -\left[\bar{G}(x^\mu, x'^\mu) + \frac{1}{2}iG^{(1)}(x^\mu, x'^\mu)\right]. \tag{1.4.6}$$

The following relation holds for those Green functions:

$$(\Box + m^2)\begin{bmatrix} G(x^\mu, x'^\mu) \\ G^{(1)}(x^\mu, x'^\mu) \\ G^{\pm}(x^\mu, x'^\mu) \end{bmatrix} = 0 \tag{1.4.7}$$

$$(\Box + m^2)\begin{bmatrix} G_F(x^\mu, x'^\mu) \\ G_R(x^\mu, x'^\mu) \\ G_A(x^\mu, x'^\mu) \end{bmatrix} = \begin{bmatrix} -\delta^n(x^\mu - x'^\mu) \\ \delta^n(x^\mu - x'^\mu) \\ \delta^n(x^\mu - x'^\mu) \end{bmatrix} \tag{1.4.8}$$

All of those Green functions can be obtained by evaluating the contour integral:

$$G_{all} = (2\pi)^{-n}\int_C \mathrm{d}^n k\, \frac{e^{i\left[\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}') - k^0(t-t')\right]}}{(k^0)^2 - |\mathbf{k}|^2 - m^2}. \tag{1.4.9}$$

This contour integral has poles at $k^0 = \pm\sqrt{(|\mathbf{k}|^2) + m^2}$. The different Green functions take different contours, the contour for each Green function is shown in figure 1.

The opened contours in the figure should be closed by infinite semicircles in the lower/upper complex plane depending on boundary conditions. For example, in the case that $t' > t$, the contour for $G_R(x^\mu, x'^\mu)$ is closed by a semicircle in the upper complex plane. By residue theorem, this contour integral yields zero, which is expected.

The contour integral can be explicitly performed.

Example 1.4.1 (Feynman propagator)**.**
The Feynman propagator can be evaluated as

$$G_F(x^\mu, x'^\mu) = \frac{-\pi i}{(4\pi i)^{\frac{n}{2}}} \left[\frac{2m^2}{-\sigma + i\epsilon}\right]^{\frac{(n-2)}{4}} H^{(2)}_{\frac{1}{2}n-1}\left(\sqrt{2m^2(\sigma - i\epsilon)}\right).$$

Where $\sigma = \frac{1}{2}(\Delta s)^2 = \frac{1}{2}\eta_{\mu\nu}(x^\mu - x'^\mu)(x^\nu - x'^\nu)$ and $H^{(2)}$ is the Hankel function of the second kind. The $-i\epsilon$ is added to push the contour slightly down to the lower half complex plane, where the function is analytic.

We denote the massless limit case Green function by $\mathcal{G}$ instead of G.

Example 1.4.2 (4-dimensional massless Feynman propagator)**.**

$$\mathcal{G}_F(x^\mu, x^{\mu'}) = \frac{i}{8\pi^2\sigma} - \frac{\delta(\sigma)}{8\pi}$$

Example 1.4.3 (4-deimensional massless Hadamard's elementary function)**.**

$$\mathcal{G}^{(1)}(x^\mu, x^{\mu'}) = -\frac{1}{4\pi^2\sigma}$$

All the Green functions above are calculated as expectation values of field operators in a pure state, which are suitable for describing systems at zero temperature. However, the system at non-zero temperature is described by a state that is statistically distributed over all pure states. The Green functions for a non-zero temperature system are given by the average of all pure states of the expectation value of the products of field operators in those pure states.

Suppose $|\psi_i\rangle$ is a pure state and eigenstate of the Hamiltonian operator, then it will also be an eigenstate of the number operator. Because both numbers of particles and energy are variable, an equilibrium system at temperature T should be described by a grand canonical ensemble of states. The probability of the system in the state $|\psi_i\rangle$ is given by grand partition function:

1.4.4.

$$\rho_i = \frac{e^{-\beta(E_i - \mu n_i)}}{Z}$$

where $\beta = \frac{1}{k_B T}$, μ is chemical potential, $Z = \sum_j e^{-\beta(E_j - \mu n_j)} = e^{-\beta\Omega}$ is grand partition function, and Ω is thermodynamic potential.

The **ensemble average** at a temperature given by β pf arbitrary operator $\hat{\mathcal{O}}$ is given by

$$\left\langle \hat{\mathcal{O}} \right\rangle_\beta = \sum_i \rho_i \langle \psi_i | \hat{\mathcal{O}} | \psi_i \rangle . \tag{1.4.10}$$

Define quantum density operator:

Definition 1.4.3 (Density operator)**.**

$$\rho = e^{\beta(\Omega+\mu N-H)}$$

then the probability of the system in the state $|\psi_i\rangle$ is given by matrix element:

$$\rho_i = \langle\psi_i|\rho|\psi_i\rangle \tag{1.4.11}$$

The probability should be normalized to 1:

$$\operatorname{tr}\rho = \sum_i \langle\psi_i|\rho|\psi_i\rangle = 1 \tag{1.4.12}$$

eq 1.4.10 is reduced to

$$\left\langle\hat{\mathcal{O}}\right\rangle_\beta = \operatorname{tr}\left(\rho\hat{\mathcal{O}}\right) \tag{1.4.13}$$

Now we can define **thermal Green functions** by replacing the vacuum expectation values with the ensemble average.

Example 1.4.4 (Thermal Wightman functions)**.**

$$G^+_\beta(x^\mu, x'^\mu) = \left\langle\hat{\phi}(x^\mu)\hat{\phi}(x'^\mu)\right\rangle_\beta$$
$$G^-_\beta(x^\mu, x'^\mu) = \left\langle\hat{\phi}(x'^\mu)\hat{\phi}(x^\mu)\right\rangle_\beta .$$

When the chemical potential vanishes, we find an important property of thermal Green functions:

$$G^+_\beta(t,\mathbf{x};t',\mathbf{x}') = \frac{\operatorname{tr}\left[e^{-\beta H}\hat{\phi}(t,\mathbf{x})\hat{\phi}(t',\mathbf{x}')\right]}{\operatorname{tr}\left(e^{-\beta H}\right)} \tag{1.4.14}$$
$$= \frac{\operatorname{tr}\left[e^{-\beta H}\hat{\phi}(t,\mathbf{x})e^{\beta H}e^{-\beta H}\hat{\phi}(t',\mathbf{x}')\right]}{\operatorname{tr}\left(e^{-\beta H}\right)} \tag{1.4.15}$$
$$= \frac{\operatorname{tr}\left[\hat{\phi}(t+i\beta,\mathbf{x})e^{-\beta H}\hat{\phi}(t',\mathbf{x}')\right]}{\operatorname{tr}\left(e^{-\beta H}\right)} \tag{1.4.16}$$
$$= \frac{\operatorname{tr}\left[e^{-\beta H}\hat{\phi}(t',\mathbf{x}')\hat{\phi}(t+i\beta,\mathbf{x})\right]}{\operatorname{tr}\left(e^{-\beta H}\right)} \tag{1.4.17}$$
$$= G^-_\beta(t+i\beta,\mathbf{x};t',\mathbf{x}') \tag{1.4.18}$$

In step 1.4.16, we have to use Heisenberg equations of motion

$$\hat{\phi}(t,\mathbf{x}) = e^{iH(t-t_0)}\hat{\phi}(t_0,\mathbf{x})e^{-iH(t-t_0)}, \tag{1.4.19}$$

and property of the trace

$$\mathrm{tr}(AB) = \mathrm{tr}(BA) \tag{1.4.20}$$

A similar property also holds for other Green functions.

Example 1.4.5 (Wightman functions)**.**

$$G^{\pm}_{\beta}(t, \mathbf{x}; t', \mathbf{x}') = G^{\mp}_{\beta}(t + i\beta, \mathbf{x}; t', \mathbf{x}')$$

Example 1.4.6 (Hadamard elementary function)**.**

$$G^{(1)}_{\beta}(t, \mathbf{x}; t', \mathbf{x}') = G^{(1)}_{\beta}(t + i\beta, \mathbf{x}; t', \mathbf{x}')$$

However, we should note that

$$iG_{\beta}(t, \mathbf{x}; t', \mathbf{x}') = iG(t, \mathbf{x}; t', \mathbf{x}') = \left[\hat{\phi}(x^{\mu}), \hat{\phi}(x'^{\mu})\right]. \tag{1.4.21}$$

This property is due to the commutator of the free scalar field is not an operator but a number. This does not usually hold for the interacting field.

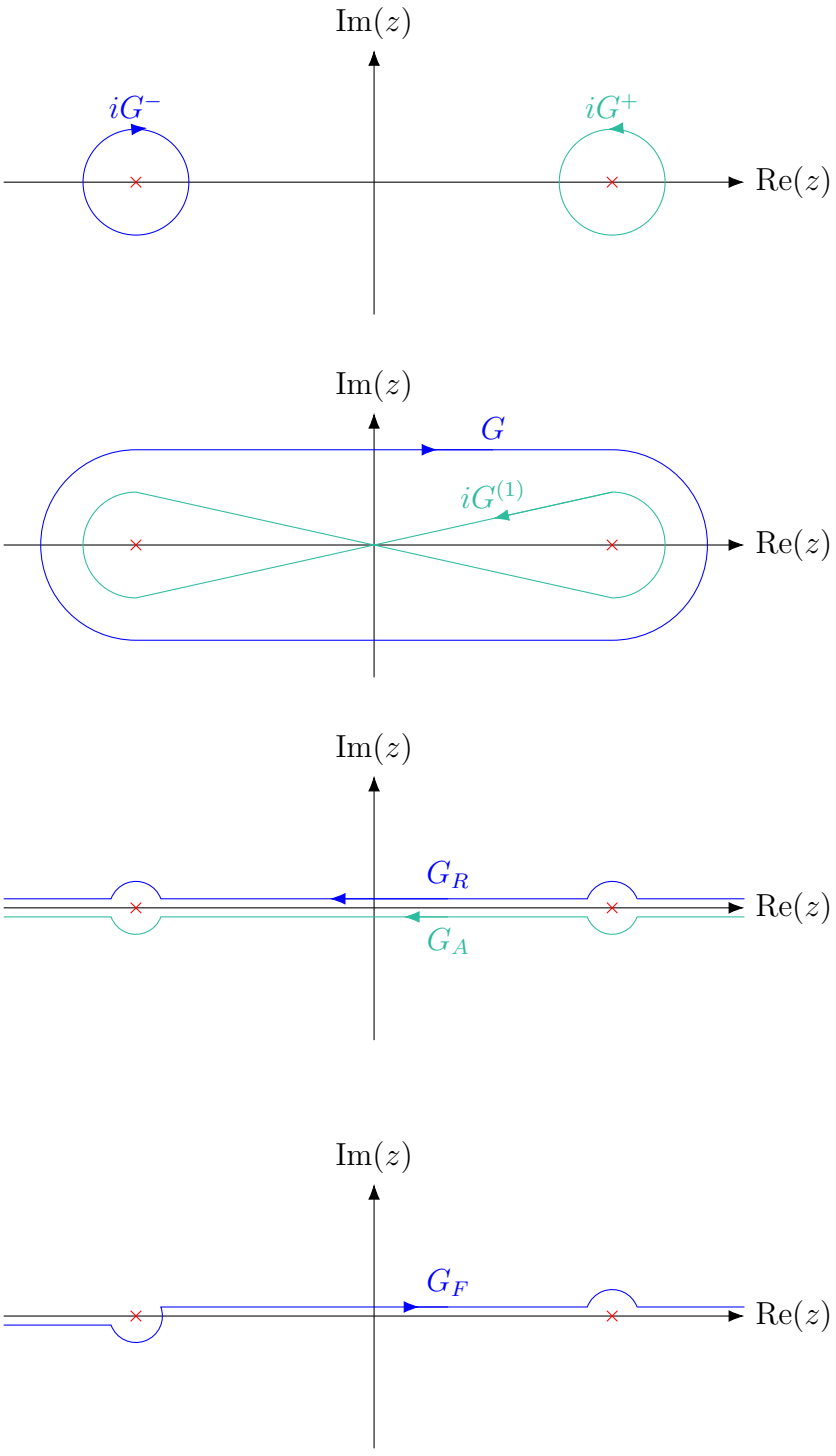

Figure 1: Contour of different Green functions.

2 QFT in Curved Spacetime

In the Minkowski spacetime, we have a "privileged" set of basis related to Poincare symmetry which allows us to construct our mode function. However, in general spacetime, the Poincare group no longer is the symmetry group, we do not have a set of mode functions that are preferred. The consequence of no preferred mode functions is that the concepts of vacuum and particle are ill-defined. The treatment in this chapter assumed that spacetime is globally hyperbolic.

2.1 Construct QFT in Curved Spacetime

Let us start with the Lagrangian density of a scalar field:

$$\mathcal{L} = \frac{1}{2}(\eta^{\mu\nu}\phi_{,\mu}\phi_{,\nu} - m^2\phi^2)$$

To obtain the curved spacetime expression, simply replace the Minkowski metric by $g^{\mu\nu}$ and replace the partial derivative with the covariant derivative. To involve the coupling to curved spacetime background, we also introduce a coupling term

$$\mathcal{L} = \frac{1}{2}\sqrt{-g}(g^{\mu\nu}\phi_{;\mu}\phi_{;\nu} - m^2\phi^2 - \xi R\phi^2) \tag{2.1.1}$$

Where the R is the Ricci scalar and ξ is the coupling constant.

There are two favorite choices of coupling constant: **minimal coupling** which simply set $\xi = 0$ which turn off the coupling, while **conformal coupling** sets

$$\xi = \frac{(n-2)}{4(n-1)}$$

which makes the theory invariant under conformal transformation when $m = 0$.

The equation of motion obtained from the Euler-Lagrangian equation is

$$\left[\Box + m^2 + \xi R\right]\hat{\phi} = 0 \tag{2.1.2}$$

The Klein-Gordon inner product 1.1.1 in curved spacetime is generalized as

Definition 2.1.1 (Curved spacetime Klein-Gordon inner product)**.**

$$(\phi_1, \phi_2) = -i\int_\Sigma \mathrm{d}^{n-1}x\ \sqrt{\gamma}n^\mu\left[\phi_1\nabla_\mu\phi_2^* - \phi_2^*\nabla_\mu\phi_1\right],$$

where $\gamma \equiv \det(\gamma_{ij})$ is the determinant of induced metric of hypersurface Σ, and n^μ is the unit normal vector of Σ.

We can proceed with the canonical quantization as in the previous chapter. The conjugate momentum is

$$\pi = \frac{\partial\mathcal{L}}{\partial(\nabla_0\phi)} = \sqrt{-g}\nabla_0\phi. \tag{2.1.3}$$

Impose canonical commutation relation:

2.1.1 (Curved spacetime canonical commutation relation)**.**

$$\begin{aligned}\left[\hat{\phi}(t,\mathbf{x}),\hat{\phi}(t,\mathbf{x}')\right]&=0\\ \left[\hat{\pi}(t,\mathbf{x}),\hat{\pi}(t,\mathbf{x}')\right]&=0\\ \left[\hat{\phi}(t,\mathbf{x}),\hat{\pi}(t,\mathbf{x}')\right]&=\frac{i}{\sqrt{g}}\delta^{(n-1)}(\mathbf{x}-\mathbf{x}').\end{aligned}$$

We can just continue the procedure of constructing flat spacetime quantum field.
There exists a complete orthonormal set of modes, satisfying

$$(f_i,f_j)=\delta_{ij} \tag{2.1.4}$$
$$\left(f_i,f_j^*\right)=0 \tag{2.1.5}$$
$$\left(f_i^*,f_j^*\right)=-\delta_{ij}. \tag{2.1.6}$$

The index ij represents some quantities that label the modes. Because the set of modes is complete, we can expand our field as

2.1.2 (Mode expansion in curved spacetime)**.**

$$\hat{\phi}=\sum_i\left[\hat{a}_i f_i+\hat{a}_i^\dagger f_i^*\right].$$

The coefficients of mode expansion have commutation relations

$$\left[\hat{a}_i,\hat{a}_j\right]=0 \tag{2.1.7}$$
$$\left[\hat{a}_i^\dagger,\hat{a}_j^\dagger\right]=0 \tag{2.1.8}$$
$$\left[\hat{a}_i,\hat{a}_j^\dagger\right]=\delta_{ij} \tag{2.1.9}$$

We interpret those coefficients as creation and annihilation operator, which defines a new vacuum state:

2.1.3 (f_i vacuum)**.**

$$\forall\, i,\ \hat{a}_i\left|0_f\right\rangle=0$$

Just like before, from this vacuum state, we can construct the Fock space. A state with n_i excitations could be create by repeated action of $\hat{a}_i^\dagger$,

$$\left|n_i\right\rangle=\frac{1}{\sqrt{n_i!}}\left(\hat{a}_i^\dagger\left|0_f\right\rangle\right), \tag{2.1.10}$$

and the number operator is

2.1.4 (Number operator of f_i)**.**

$$\hat{n}_{f_i} = \hat{a}_i^\dagger \hat{a}_i.$$

All the above constructions of Fock space and vacuum state are built for the set of modes f_i.

2.2 Vacuum Ambiguity

We should notice that our choice of mode function f_i is non-unique, and we can always construct another set of modes function g_i and follow the procedure in the previous chapter, to construct the Fock space and vacuum state for the mode g_i.

Before we discuss the issue of this ambiguity of mode in curved spacetime, we first think about how this problem was avoided in Minkowski spacetime.

In Minkowski spacetime, the natural set of modes is associated with the Cartesian coordinate system (t, x, y, z). These coordinates are associated with the Killing vectors and Poincare group. We have shown that the Fock space and vacuum state are invariant under the action of the Poincare group (c.f 1.2.7).

More specifically, the positive frequency mode in Minkowski spacetime is defined through the Lie derivative along the timelike Killing vector, $\mathcal{L}_{\partial t} f_{\mathbf{k}} = -i\omega f_{\mathbf{k}}$, and this timelike Killing vector are associated with the Poincare group, so that we are always able to distinguish positive frequency and negative frequency modes in flat spacetime.

Let's consider another set of modes g_i, we can expand the field operator in term of g_i:

2.2.1 (Alternative mode expansion)**.**

$$\hat{\phi} = \sum_i \left[\hat{\bar{a}}_i g_i + \hat{\bar{a}}_i^\dagger g_i^*\right].$$

Follow all procedures for mode f_i, the commutation relation is imposed:

$$\left[\hat{\bar{a}}_i, \hat{\bar{a}}_j\right] = 0 \tag{2.2.1}$$
$$\left[\hat{\bar{a}}_i^\dagger, \hat{\bar{a}}_j^\dagger\right] = 0 \tag{2.2.2}$$
$$\left[\hat{\bar{a}}_i, \hat{\bar{a}}_j^\dagger\right] = \delta_{ij} \tag{2.2.3}$$

The vacuum state of this set of annihilation operators is

2.2.2 (g_i vacuum)**.**

$$\forall\ i,\ \hat{\bar{a}}_i \left|0_g\right\rangle = 0$$

The Fock basis is constructed by repeated action of creation operators on this vacuum, and the number operator is defined as

2.2.3 (Number operator of g_i)**.**

$$\hat{n}_{g_i} = \hat{\bar{a}}_i^\dagger \hat{\bar{a}}_i.$$

Both sets of modes are constructed in an equal way, and we don't know which one defined a vacuum that is more close to the physical vacuum.

Now, since both sets are complete, we can express one in terms of another, known as **Bogolubov transformation**.

Definition 2.2.1 (Bogolubov transformation)**.**

$$f_i = \sum_j \left[\alpha^*_{ji} g_j - \beta_{ji} g^*_j\right]$$
$$g_i = \sum_j \left[\alpha_{ij} f_j + \beta_{ij} f^*_j\right]$$

The matrix element α_{ij} and β_{ij} are called **Bogolubov coefficients**

Using the orthogonality of mode functions, 2.1.4, we can compute the Bogolubov coefficients:

$$\alpha_{ij} = (g_i, f_j) \tag{2.2.4}$$
$$\beta_{ij} = -(g_i, f^*_j) \tag{2.2.5}$$

To make a physical mode function, the Bogolubov coefficients should be normalized:

$$\sum_j \left[\alpha_{ik}\alpha^*_{jk} - \beta_{ik}\beta^*_{jk}\right] = \delta_{ij} \tag{2.2.6}$$
$$\sum_j \left[\alpha_{ik}\beta_{jk} - \beta_{ik}\alpha_{jk}\right] = 0 \tag{2.2.7}$$

To obtain the relation of creation/annihilation operators of set f_i and g_i, we first equating the mode expansions 2.1.2 and 2.2.1, and use Bogolubov transformation 2.2.1.

$$\sum_i \left[\hat{a}_i f_i + \hat{a}^\dagger_i f^*_i\right] = \sum_i \left[\hat{\tilde{a}}_i g_i + \hat{\tilde{a}}^\dagger_i g^*_i\right] \tag{2.2.8}$$

We obtain

$$\hat{a}_i = \sum_j \left[\alpha_{ji}\hat{\tilde{a}}_j + \beta^*_{ji}\hat{\tilde{a}}^\dagger_j\right] \tag{2.2.9}$$
$$\hat{\tilde{a}}_i = \sum_j \left[\alpha^*_{ij}\hat{a}_j - \beta^*_{ij}\hat{a}^\dagger_j\right] \tag{2.2.10}$$

Now imagine an observer in the vacuum state defined by mode f_i, we want to know how many particles are there if the state is defined by g_i mode. We should exam the expectation value of number operator of g_i, $\hat{n}_{g_i}$ in f_i-vacuum:

$$
\begin{aligned}
\langle 0_{f_i}|\hat{n}_{g_i}|0_{f_i}\rangle &= \langle 0_{f_i}|\hat{\hat{a}}_i^\dagger \hat{\hat{a}}_i|0_{f_i}\rangle && (2.2.11)\\
&= \langle 0_{f_i}|\sum_{jk}\left(\alpha_{ij}\hat{a}_j^\dagger - \beta_{ij}\hat{a}_j\right)\left(\alpha_{ik}^*\hat{a}_k - \beta_{ik}^*\hat{a}_k^\dagger\right)|0_{f_i}\rangle && (2.2.12)\\
&= \sum_{jk}\left(-\beta_{ij}\right)\left(-\beta_{ik}^*\right)\,\langle 0_{f_i}|\hat{a}_j\hat{a}_k^\dagger|0_{f_i}\rangle && (2.2.13)\\
&= \sum_{jk}\left(\beta_{ij}\beta_{ik}^*\right)\,\langle 0_{f_i}|\hat{a}_k^\dagger\hat{a}_j + \delta_{jk}|0_{f_i}\rangle && (2.2.14)\\
&= \sum_{jk}\left(\beta_{ij}\beta_{ik}^*\right)\delta_{jk}\,\langle 0_{f_i}|0_{f_i}\rangle && (2.2.15)\\
&= \sum_{j}\beta_{ij}\beta_{ij}^* && (2.2.16)\\
&= \sum_{j}|\beta_{ij}|^2 && (2.2.17)
\end{aligned}
$$

Hence, the number of particle in mode g_i in the f_i-vacuum is equal to $\sum_j |\beta_{ij}|^2$.

Note that in some static spacetime, exist a timelike Killing vector η, and if the modes f_i are positive frequency modes satisfying

$$
\mathcal{L}_\eta f_i = -i\omega f_i, \quad \omega > 0 \tag{2.2.18}
$$

and Bogolubov coefficients β_{ij} vanish, i.e. the modes g_i are linear combination of f_i alone, two sets of modes share a common vacuum state.

2.3 Particle Detector

To see what is the meaning of particles in curved spacetime, we consider the model of particle detector given by Unruh and Dewitt.

The **Unruh-Dewitt detector** is an idealized particle detector, consisting of a point particle with internal energy level E, coupled via a monopole interaction with a scalar field ϕ. Suppose the particle detector moves along the worldline described by parameterized function $x^\mu(\tau)$, where τ is the detector's proper time. The detector is point-like so the interaction only takes place at a point along the trajectory.

The detector-field interaction is described by the interaction Lagrangian:

$$\mathcal{L} = \mathfrak{c}\hat{m}(\tau)\phi\left(x^\mu(\tau)\right) \tag{2.3.1}$$

Where the $\mathfrak{c}$ is the coupling constant and $\hat{m}$ is the monopole moment of the detector. In the Heisenberg picture, the time evolution of the monopole moment operator is governed by the Hamiltonian operator of the detector

$$\hat{m}(\tau) = e^{i\hat{H}_d\tau}\hat{m}(0)e^{-i\hat{H}_d\tau}. \tag{2.3.2}$$

where $\hat{H}_d\left|E\right\rangle = E\left|E\right\rangle$.

Suppose the field ϕ is in the Minkowski vacuum $\left|0\right\rangle$, defined by $a_{\mathbf{p}}\left|0\right\rangle = 0$. For the detector moves in a general trajectory, it will not remain in the ground state E_0 but will transition to an excited state E_n for $n > 0$, while the field will transition to an excited state $\left|\psi\right\rangle$.

We assume that at initial time τ_0 the detector and field are in the product state $\left|0, E_0\right\rangle = \left|0\right\rangle \otimes \left|E_0\right\rangle$. In the interaction picture, the time evolution of this product state $\left|\psi\right\rangle$ is governed by interaction Hamiltonian:

$$i\frac{\mathrm{d}}{\mathrm{d}\tau}\left|\psi(\tau)\right\rangle = H_{int}\left|\psi(\tau)\right\rangle \tag{2.3.3}$$

$$i\frac{\mathrm{d}}{\mathrm{d}\tau}\left|\psi(\tau)\right\rangle = -\mathcal{L}_{int}\left|\psi(\tau)\right\rangle \tag{2.3.4}$$

$$i\frac{\mathrm{d}}{\mathrm{d}\tau}\left|\psi(\tau)\right\rangle = -\mathfrak{c}\hat{m}(\tau)\phi\left(x^\mu(\tau)\right)\left|\psi(\tau)\right\rangle \tag{2.3.5}$$

The transition amplitude from state $\left|0, E_0\right\rangle$ at $\tau = \tau_0$ to state $\left|\psi, E_1\right\rangle$ at $\tau = \tau_1$ could be given by

$$\left\langle\psi, E_1|0, E_0\right\rangle = \left\langle\psi, E_1\right|\hat{T}\left[e^{-i\int_{\tau_0}^{\tau_1}\mathrm{d}\tau\, H_{int}(\tau)}\right]\left|0, E_0\right\rangle \tag{2.3.6}$$

where $\hat{T}$ is the time order operator.

If the coupling constant $\mathfrak{c}$ is sufficiently small, the transition amplitude can be given by first-order perturbation theory:

$$\left\langle\psi, E_1|0, E_0\right\rangle = i\mathfrak{c}\left\langle\psi, E_1\right|\int_{\tau_0}^{\tau_1} d\tau\; \hat{m}(\tau)\phi\left(x^\mu(\tau)\right)\left|0, E_0\right\rangle \tag{2.3.7}$$

$$= i\mathfrak{c}\left\langle E_1\right| m(0)\left|E_0\right\rangle\int_{\tau_0}^{\tau_1}\mathrm{d}\tau\; e^{i(E_1-E_0)\tau}\left\langle\psi\right|\phi(x^\mu(\tau))\left|0\right\rangle. \tag{2.3.8}$$

If we want to calculate the transition probability to all possible states, we could square the expression 2.3.7 and summing over the complete set of $|\psi_i\rangle$, with resulting

$$\sum_j |\langle \psi_j, E_1|0, E_0\rangle|^2 \tag{2.3.9}$$

$$= \mathfrak{c}^2 |\langle E_1| m(0) |E_0\rangle|^2 \int_{\tau_0}^{\tau_1} \mathrm{d}\tau \int_{\tau_0}^{\tau_1} \mathrm{d}\tau' e^{-i(E_1-E_0)(\tau-\tau')} \sum_j \langle 0|\hat{\phi}(x^\mu(\tau))|\psi_j\rangle \langle\psi_j|\hat{\phi}(x^\mu(\tau'))|0\rangle \tag{2.3.10}$$

$$= \mathfrak{c}^2 |\langle E_1| m(0) |E_0\rangle|^2 \int_{\tau_0}^{\tau_1} \mathrm{d}\tau \int_{\tau_0}^{\tau_1} \mathrm{d}\tau' e^{-i(E_1-E_0)(\tau-\tau')} \langle 0|\hat{\phi}(x^\mu(\tau))\hat{\phi}(x^\mu(\tau'))|0\rangle \tag{2.3.11}$$

$$= \underbrace{\mathfrak{c}^2 |\langle E_1| m(0) |E_0\rangle|^2}_{\text{sensitivity}} \underbrace{\int_{\tau_0}^{\tau_1} \mathrm{d}\tau \int_{\tau_0}^{\tau_1} \mathrm{d}\tau' e^{-i(E_1-E_0)(\tau-\tau')} G^+(x^\mu(\tau), x^\mu(\tau'))}_{\text{response function}} \tag{2.3.12}$$

The equation 2.3.12 contains two parts, sensitivity and response function. The sensitivity depends on the internal detail of the detector, and the response function does not. Therefore the response function is a general property for all such detectors.

Rewrite the $E_1 - E_0$ as ω, we may express the response function as

2.3.1 (Response function)**.**

$$\mathcal{F}(\omega) = \int_{\tau_0}^{\tau_1} \mathrm{d}\tau \int_{\tau_0}^{\tau_1} \mathrm{d}\tau' e^{-i\omega(\tau-\tau')} G^+(x^\mu(\tau), x^\mu(\tau'))$$

In the special case in which the system is invariant under time translation in the detector's frame, the Wightman function could be written as

$$G^+(x^\mu(\tau), x^\mu(\tau')) = G^+(\tau-\tau') = G^+(\Delta\tau) \tag{2.3.13}$$

This corresponds to the equilibrium between the detector and the field. In this case, the number of quanta absorbed or emitted by the detector per unit of proper time is constant. If these rates are nonzero, and we integrate over an infinite time interval, the transition probability will diverge. Therefore, we should consider the transition probability per unit of proper time, given by

$$\mathfrak{c}^2 |\langle E_1| m(0) |E_0\rangle|^2 \int_{-\infty}^{\infty} \mathrm{d}\Delta\tau e^{-i\omega\Delta\tau} G^+(\Delta\tau). \tag{2.3.14}$$

Let us now consider a few examples that the transition probability can be exactly evaluated.

For an inertial detector in n-dimensional Minkowski spacetime, only the last term of xxx contributes to the detector response function:

$$\frac{F(E)}{T} = (2\pi)^{1-n} \int_{-\infty}^{\infty} \mathrm{d}(\Delta\tau)\, e^{-iE\Delta\tau} \int \frac{\mathrm{d}^{n-1}k}{2\omega} e^{i(\omega-\mathbf{k}\cdot\mathbf{v})\frac{\Delta\tau}{\sqrt{1-v^2}}} n_{\mathbf{k}} \tag{2.3.15}$$

Where T is the duration of the detector switched on. If $\mathbf{v} = 0$ we can further perform the integration:

$$\frac{\mathcal{F}(E)}{T} = (2\pi)^{1-n}\int_{-\infty}^{\infty}\mathrm{d}(\Delta\tau)\, e^{-i\Delta\tau E}\int\frac{\mathrm{d}^{n-1}k}{2\omega}e^{i\Delta\tau\frac{(\omega-\mathbf{k}\cdot\mathbf{v})}{\sqrt{1-v^2}}}n_{\mathbf{k}} \tag{2.3.16}$$

$$= (2\pi)^{1-n}\int\frac{\mathrm{d}^{n-1}k}{2\omega}\int_{-\infty}^{\infty}\mathrm{d}(\Delta\tau)\, e^{-i\Delta\tau E}e^{i\Delta\tau\frac{(\omega-\mathbf{k}\cdot\mathbf{v})}{\sqrt{1-v^2}}}n_{\mathbf{k}} \tag{2.3.17}$$

$$= (2\pi)^{1-n}\int\frac{\mathrm{d}^{n-1}k}{2\omega}\int_{-\infty}^{\infty}\mathrm{d}(\Delta\tau)\, e^{i\Delta\tau\frac{(\omega-\mathbf{k}\cdot\mathbf{v})}{\sqrt{1-v^2}}-i\Delta\tau E}n_{\mathbf{k}} \tag{2.3.18}$$

$$= (2\pi)^{1-n}\int\frac{\mathrm{d}^{n-1}k}{2\omega}\int_{-\infty}^{\infty}\mathrm{d}(\Delta\tau)\, e^{i\Delta\tau\left[\frac{(\omega-\mathbf{k}\cdot\mathbf{v})}{\sqrt{1-v^2}}-E\right]}n_{\mathbf{k}} \tag{2.3.19}$$

$$= (2\pi)^{1-n}(2\pi)\int\frac{\mathrm{d}^{n-1}k}{2\omega}\,\delta\left(\frac{(\omega-\mathbf{k}\cdot\mathbf{v})}{\sqrt{1-v^2}}-E\right)n_{\mathbf{k}} \tag{2.3.20}$$

$$= (2\pi)^{1-n}(2\pi)\frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}\int\frac{k^{n-2}\mathrm{d}k}{2\omega}\,\delta\left(\omega-E\right)n_{\mathbf{k}} \tag{2.3.21}$$

$$= \frac{2^{2-n}\pi^{\frac{3-n}{2}}}{\Gamma(\frac{n-1}{2})}\int\frac{k^{n-2}\mathrm{d}k}{\omega}\,\delta\left(\omega-E\right)n_{\mathbf{k}} \tag{2.3.22}$$

$$= \frac{2^{2-n}\pi^{\frac{3-n}{2}}}{\Gamma(\frac{n-1}{2})}\int\frac{k^{n-2}\mathrm{d}k}{\sqrt{k^2+m^2}}\,\delta\left(\sqrt{k^2+m^2}-E\right)n_{\sqrt{E^2-m^2}} \tag{2.3.23}$$

$$= \frac{2^{2-n}\pi^{\frac{3-n}{2}}}{\Gamma(\frac{n-1}{2})}\left(E^2-m^2\right)^{\frac{n-3}{2}}\,\theta\left(E-m\right)n_{\sqrt{E^2-m^2}} \tag{2.3.24}$$

3 Quantum Phenomena in Spacetime

3.1 Cosmological Particle Creation

3.2 Moving Mirror

3.3 Unruh Effect Revisit

3.4 Hawking Radiation

A Classical Field Theory

Classical field theory is something students are supposed to know for QFT but is never formally taught, so we give a brief overview here. Field theory is very similar to Lagrangian mechanics; instead of the usual system, we now have a spacetime-dependent **fields** $\Phi^i(x^\mu)$ (the i here is not tensor index), and the action becomes a **functional** of these fields. A functional is a "function of function", which map a function to a number, note that the functional is not simply a composition function, which maps number to number.

In field theory, the Lagrangian is usually expressed as an integral over **Lagrange density**, which are a function of the fields and their derivative.

$$L = \int d^3x \; \mathcal{L}(\Phi^i, \partial_\mu \Phi^i) \tag{A.0.1}$$

Then the action

$$S = \int dt \; L = \int d^4x \; \mathcal{L}(\Phi^i, \partial_\mu \Phi^i) \tag{A.0.2}$$

By varying the action, just like what we did in classical mechanics, we can obtain the equation of motion of field:

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \Phi^i} \delta \Phi^i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)} \partial_\mu (\delta \Phi^i) \right] \tag{A.0.3}$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \Phi^i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)} \right] \delta \Phi^i \tag{A.0.4}$$

We have obtained the field version of **Euler-Lagrange equation**

Definition A.0.1 (Euler-Lagrange equation)**.**

$$\frac{\partial \mathcal{L}}{\partial \Phi^i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)} = 0$$

Let's consider a simple example, the real scalar field

$$\phi : \; x^\mu \to \mathbb{R}$$

. The contribution of action are

1. Kinetic term: $\frac{1}{2}\dot{\phi}^2$
2. Gradient term: $\frac{1}{2}(\nabla\phi)^2$
3. potential term: $V(\phi)$

We could combine them into a Lorentz-invariant Lagrange density:

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi) - V(\phi) \tag{A.0.5}$$

Apply the Euler-Lagrange equation; we then get the equation of motion:

A.0.1.

$$\begin{aligned} &\eta^{\mu\nu}\partial_\mu\partial_\nu\phi + \frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi} \\ =& \Box\phi + \frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi} \\ =& 0 \end{aligned}$$

Where $\Box$ is **d'Alembertian**.

A.1 Klein-Gordon Field

If the scalar field is massive, the potential $V(\phi) = \frac{1}{2}m^2\phi^2$, we obtained the **Klein-Gordon equation**:

Definition A.1.1 (Klein-Gordon equation)**.**

$$(\Box + m^2)\phi = 0$$

You will see this equation again and again in quantum field theory.

Remark.
The Klein-Gordon field could be analog to infinite many coupled infinitesimal harmonic oscillators; each "mass" is affected by neighboring springs and has its kinetic energy. It is an idealized model used to study the massive scalar particle.

A.2 Hamiltonian Field Theory

The Lagrangian field theory is naturally Lorentz invariant. However, we also need Hamiltonian formalism for the field theory because it is easier to transition from quantum mechanics. Let's first consider the classical definition of Hamiltonian,

$$H = \sum p\dot{q} - L$$

. From the definition of Lagrangian, we can derive the conjugate momentum as follow:

A.2.1.

$$\begin{aligned} p(x^i) \equiv \frac{\partial L}{\partial\dot{\phi}(x^i)} &= \frac{\partial}{\partial\dot{\phi}(x^i)}\int \mathrm{d}^3x'\mathcal{L} \\ &\propto \pi(x^i)\mathrm{d}^3x \end{aligned}$$

Where

$$\pi \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}}$$

is called **momentum density**

Therefore, the Hamiltonian for field theory is:

Definition A.2.1 (Hamiltonian)**.**

$$\begin{aligned} H =& \sum p(x^i)\phi(x^i) - L \\ =& \int \mathrm{d}^3x \left[\pi\dot{\phi} - \mathcal{L}\right] \\ =& \int \mathrm{d}^3x\ \mathcal{H} \end{aligned}$$

Where $\mathcal{H}$ is called **Hamiltonian density**.

Example A.2.1.
As a simple example, the Hamiltonian for the K-G field is:

$$H = \int \mathrm{d}^3x\ \mathcal{H} = \int \mathrm{d}^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right]$$

A.3 Noether's Theorem

A Lagrangian may be invariant under some special type of transformation. For example, a Lagrangian for a complex scalar field

$$\mathcal{L} = |\partial_\mu\phi|^2 - m^2|\phi|^2$$

is invariant under transformation $\phi \to e^{i\alpha}\phi$. We call this transformation a **symmetry** of the Lagrangian. When the parameter of transformation (α for this case) can be taken infinitesimal, we say this symmetry is **continuous**. We can explicit see that $\frac{\delta\mathcal{L}}{\delta\alpha} = 0$.

By applying the Euler-Lagrange equation, we can deduce that $\partial_\mu J_\mu = 0$, where J_μ is defined as follow:

Definition A.3.1.

$$J_\mu = \sum_n \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\frac{\delta\phi_n}{\delta\alpha}$$

J_μ is known as **Noether current** or **conserved current**

Example A.3.1.
For the above complex scalar field, we can calculate the conserved current:

$$\begin{aligned} J_\mu =& \sum_n \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)}\frac{\delta\phi_n}{\delta\alpha} \\ =& \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\frac{\delta\phi}{\delta\alpha} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)}\frac{\delta\phi^*}{\delta\alpha} \\ =& -i\left[\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi\right] \end{aligned}$$

We can check that $\partial_\mu J_\mu = 0$ explicitly:

$$\begin{aligned} \partial_\mu J_\mu =& -i\left[\phi\Box\phi^* - \phi^*\Box\phi\right] \\ =& 0 \end{aligned}$$

Where we applied the equation of motion in the last step.

We call J_μ conserved current because we can find conserved quantity by integral over its 0-component:

Definition A.3.2 (Conserved charge)**.**

$$Q = \int d^3x \; J_0$$

Where the Q is called **conserved charge** or **Noether's charge**

We say Q is conserved because

$$\partial_t Q = \int d^3x \; \partial_t J_0 = \int d^3x \; \vec{\nabla} \cdot \vec{J} = 0. \tag{A.3.1}$$

The above argument is called **Noether's theorem**.

Theorem A.3.3 (Noether's theorem)**.**
If a Lagrangian has a continuous symmetry, then exists a current associated with that symmetry that is conserved when the equations of motion are satisfied.

A.4 Energy-momentum Tensor

The physics at spacetime point x should have same form at spacetime point y, this argument arises a symmetry called **specetime translational symmetry**. By Noether's theorem, we can find the four conserved currents for this symmetry for an infinitesimal spacetime translation ξ^μ:

Definition A.4.1 (Energy-momentum tensor)**.**

$$T_{\mu\nu} = \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \partial_\nu \phi_n - g_{\mu\nu} \mathcal{L}$$

$T_{\mu\nu}$ is called **energy-momentum tensor**.

Four conserved currents correspond to values of ν, and four conserved charges are energy and momentum. We can see that the energy density is the 00-component of $T_{\mu\nu}$.

Remark.
In fact, the conservation of energy is a direct consequence of **homogeneity of time**, or in the language of general relativity, the existence of the timelike Killing vector. We can see that in Roberson-Walker spacetime, in which the time is not homogeneous, the energy is no longer conserved.
The conservation of linear momentum and angular momentum are consequences of homogeneity and isotropic of space respectively.