$$T^{\alpha\beta}{}_{;\beta} = 0$$

$$v_{,t} + (v\cdot\nabla)v + \nabla p/\rho + \nabla\phi = 0$$

$$T^{\alpha\beta} = (\rho+p)\,U^\alpha U^\beta + p\,g^{\alpha\beta}$$

$$T^{\alpha\beta}{}_{;\beta} = \nabla_\beta\,(\rho+p)\,U^\alpha U^\beta + \nabla_\beta\left(p\,g^{\alpha\beta}\right)$$

$$= \nabla_\beta\left[(\rho+p)\,U^\alpha U^\beta\right] + \nabla_\beta p\cdot g^{\alpha\beta} + \cancel{p\,\nabla_\beta g^{\alpha\beta}}$$

$$= \nabla_\beta\left[(\rho+p)\,U^\alpha U^\beta\right] + p_{,\beta}\cdot g^{\alpha\beta}$$

$$= \nabla_\beta(\rho+p)\cdot U^\alpha U^\beta + (\rho+p)\left[U^\alpha{}_{;\beta}U^\beta + U^\alpha U^\beta{}_{;\beta}\right] + \partial_\beta p\cdot g^{\alpha\beta}$$

$$= \nabla_\beta\rho\,U^\alpha U^\beta + \nabla_\beta p\,U^\alpha U^\beta + (\rho+p)\left[U^\alpha{}_{;\beta}U^\beta + U^\alpha U^\beta{}_{;\beta}\right] + \partial_\beta p\cdot g^{\alpha\beta}$$

$$= U^\alpha\left(\rho\,U^\beta{}_{;\beta} + p\,U^\beta{}_{;\beta}\right) + (\rho+p)\left(U^\alpha U^\beta\right)_{;\beta} + p_{,\beta}\cdot g^{\alpha\beta}$$

$$= p\,U^\alpha U^\beta{}_{;\beta} + (\rho+\cancel{p})\left(U^\alpha U^\beta\right)_{;\beta} + p_{,\beta}\cdot g^{\alpha\beta}$$

$$= p\,U^\alpha U^\beta{}_{,\beta} + \rho_0\,U^\beta U^\alpha{}_{;\beta} + p_{,\beta}\,g^{\alpha\beta}$$

$$= \left(g^{\alpha\beta} + U^\alpha U^\beta\right)p_{,\beta} + \rho_0\,U^\alpha U^\beta{}_{;\beta}$$

$$= (g^{\alpha\beta} + U^\alpha U^\beta) P_{,\beta} + \rho_0 U^\beta (U^\alpha{}_{,\beta} + \Gamma^\alpha_{\lambda\beta} U^\lambda) = T^{\alpha\beta}{}_{;\beta}$$

replace $\alpha \to i$

$$T^{i\beta}{}_{;\beta} = \rho_0 U^\beta U^i{}_{,\beta} + \rho_0 U^\beta \Gamma^i_{\sigma\beta} U^\sigma + P_{,\beta}(g^{i\beta} + U^i U^\beta)$$

$$= \rho_0 \frac{\partial v}{\partial t} + \rho_0 (v\cdot\nabla) v = -\rho_0 \nabla\phi - \nabla p$$

7.5C

$$T^{\mu\nu}{}_{;\mu} = U^\mu (\rho U^\nu)_{;\nu} + (\rho+p) U^\nu U^\mu{}_{;\nu} + p_{,\nu} g^{\mu\nu}$$

$$(\rho+p) U^0 U^\mu{}_{;0} \qquad\qquad p_{,i}\, g^{\mu i}$$

$$= [(\rho+p)U^0](U^\mu{}_{,0} + \Gamma^\mu_{\alpha 0} U^\alpha)$$

$$= (\rho+p) U^0 \cdot \Gamma^\mu_{00} U^0$$

$$U^\mu (\rho U^\nu)_{;\nu} = U^\mu (\rho_{,\nu} U^\nu + \rho U^\nu{}_{;\nu})$$

$$= U^\mu \rho U^\nu{}_{;\nu}$$

$$= U^\mu \rho \left(U^\nu{}_{,\nu} + \frac{1}{\sqrt{-g}} U^\nu (\sqrt{-g})_{,\nu}\right)$$

$$= U^0 p\left(\cancel{U^0_{,0}} + \frac{1}{\sqrt{-g}} U^0 (\sqrt{-g})\right)_{,t}$$

$$= 0$$

$$0 = (\rho + p) U^0 \cdot \Gamma^\mu_{00} U^0 \quad + \quad p_{,i}\, g^{\mu i}$$

$$\Gamma^\mu_{00} = \frac{1}{2} g^{\mu\sigma}\left(g_{\sigma 0,0} + g_{\sigma 0,0} - g_{00,\sigma}\right)$$

$$= \frac{1}{2} g^{\mu\sigma}\left(-g_{00,\sigma}\right)$$

$$= \frac{1}{2} g^{\mu i}\left(-g_{00,i}\right)$$

$$\cancel{(\rho + p)\, \frac{1}{2} g^{\mu i} g_{00,i} + p_{,i}\, g^{\mu i}}$$

$$= \quad g^{\mu i}\left((\rho + p)\frac{1}{2} g_{00,i} \overset{\wedge}{\underset{\frac{1}{g_{00}}}{}} + p_{,i}\right)$$

$$= g^{\mu i}\left[(\rho + p)\frac{1}{2} g_{00,i} \cdot \frac{1}{g_{00}} + p_{,i}\right]$$

$$= (\rho + p)\, \frac{1}{2} \ln(-g_{00})_{,i} + p_{,i}$$

7a

i $g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$\frac{d g_{\mu\nu}}{d x^\alpha} = 0$ for all α, all p_α conserved

ii $g_{\mu\nu} = \begin{pmatrix} -(1-\frac{2M}{r}) & & & \\ & (1-\frac{2M}{r})^{-1} & & \\ & & r^2 & \\ & & & \sin^2\theta \end{pmatrix}$

$\frac{d g_{\mu\nu}}{d x^\alpha} = 0$ for $\alpha = 0, 3$ (t, ϕ)

iv only ϕ.

7.10

a $\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0$

$\nabla_\alpha \xi_\beta = - \nabla_\beta \xi_\alpha$

$K_{\alpha\beta} = -K_{\beta\alpha}$

$\frac{D}{D\tau}\left(p^\alpha \xi_\alpha\right) = m \frac{D}{D\tau}\left(U^\alpha \xi_\alpha\right)$

$$= m\cancel{\left(\frac{D}{D\tau}U^\alpha\right)\cdot\xi_\alpha} + mU^\alpha \frac{D}{D\tau}\xi_\alpha$$

$$= mU^\alpha U^\beta \nabla_\beta \xi_\alpha$$

$$= 0$$

(b) in Minkowski spacetime

$$\partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha = 0$$

$\xi = \vec{e}_\alpha$ or any constant vector field

4 out of 10

$\xi = (-x, t, 0, 0)$ ✓ $\quad \partial_\alpha \xi_\beta + \partial_\beta \xi_\alpha = 0$

$\xi = (-y, 0, t, 0)$

$\xi = (-z, 0, 0, t)$

$\xi = (0, y, -x, 0)$ $\quad$ 6 out of 10

$\xi = (0, 0, z, -y)$

$\xi = (0, z, 0, -x)$

(e)

for Schwarzschild

$\vec{p_t}$ and $\vec{p_\phi}$ is inveriant

$\vec{Q} = \vec{e_t} \quad \vec{R} = \vec{e_\phi}$

$$\vec{S} = \left[r\cos\theta \cdot \frac{1}{r}\cos\theta\cos\phi - r\sin\theta\cos\phi \cdot -\frac{1}{r}\sin\theta\right]\vec{e_\theta}$$
$$+ \left[r\cos\theta \cdot -\frac{1}{r}\frac{\sin\phi}{\sin\theta}\right]\vec{e_\phi}$$
$$= \cos\phi\,\vec{e_\theta} - \cot\theta\sin\phi\,\vec{e_\phi}$$

$$\vec{R} = \left[r\sin\theta\sin\phi \cdot \frac{1}{r}\cos\theta\cos\phi + -r\sin\theta\cos\phi\,\frac{1}{r}\cos\theta\sin\phi\right]\vec{e_\theta}$$
$$+ \left[r\sin\theta\sin\phi \cdot -\frac{1}{r}\frac{\sin\phi}{\sin\theta} + -r\sin\theta\cos\phi \cdot \frac{1}{r}\frac{\cos\phi}{\sin\theta}\right]\vec{e_\phi}$$
$$= (\sin\theta\cos\theta\sin\phi\cos\phi - \sin\theta\cos\theta\sin\phi\cos\phi)\,\vec{e_\theta}$$
$$+ (-\sin^2\phi - \cos^2\phi)\,\vec{e_\phi}$$

$$\vec{T} = z\vec{e_y} - y\vec{e_z} = \left(z \cdot \frac{\partial\theta}{\partial y} - y \cdot \frac{\partial\theta}{\partial z}\right)\vec{e_\theta} + \left(z \cdot \frac{\partial\phi}{\partial y} - y\frac{\partial\phi}{\partial z}\right)\vec{e_\phi}$$
$$= \left[r\cos\theta \cdot \frac{1}{r}\cos\theta\sin\phi - \left(r\sin\theta\sin\phi \cdot -\frac{1}{r}\sin\theta\right)\right]\vec{e_\theta}$$
$$+ \left(r\cos\theta \cdot \frac{1}{r}\frac{\cos\phi}{\sin\theta} - r\sin\theta\sin\phi \cdot 0\right)\vec{e_\phi}$$
$$= (\cos^2\theta\sin\phi + \sin^2\theta\sin\phi)\,\vec{e_\theta} + \cot\theta\cos\phi\,\vec{e_\phi}$$

$$= \sin\phi \, \vec{e}_\theta + \cot\theta \cos\phi \, \vec{e}_\phi$$