Quantum Field Theory

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Contents

Conventions

1. Greek index (e.g. α, β, μ, ν) take value from $\{0, 1, 2, 3\}$.
2. $(x^0, x^1, x^2, x^3) \equiv (t, x, y, z) \equiv x^\alpha$.
3. Latin index (e.g. i, j, k) take value from $\{1, 2, 3\}$.
4. Natural units $(c = 1)$.
5. Einstein summation convention.

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = \sum_{\mu=0}^{3} \sum_{\nu=0}^{3} g_{\mu\nu} dx^\mu dx^\nu$$

6. Metric signature $(+, -, -, -)$.

1 Scalar Field Theory

1.1 Canonical Quantization

2 Quantum Electrodynamics

A Classical Field Theory

Classical field theory is something you are supposed to know for QFT but is never formally taught, so I give a brief overview here. Field theory is very similar to Lagrangian mechanics; instead of the usual system, we now have a spacetime-dependent **fields** $\Phi^i(x^\mu)$ (the i here is not tensor index), and the action becomes a **functional** of these fields. A functional is a "function of function", which map a function to a number, note that the functional is not simply a composition function, which maps number to number.

In field theory, the Lagrangian is usually expressed as an integral over **Lagrange density**, which are a function of the fields and their derivative.

A.0.1.

$$L = \int d^3x \, \mathcal{L}(\Phi^i, \partial_\mu \Phi^i)$$

Then the action

A.0.2.

$$S = \int dt \, L = \int d^4x \, \mathcal{L}(\Phi^i, \partial_\mu \Phi^i)$$

By varying the action, just like what we did in classical mechanics, we can obtain the equation of motion of field:

A.0.3.

$$\delta S = \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \Phi^i} \delta \Phi^i + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)} \partial_\mu(\delta \Phi^i) \right]$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \Phi^i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)} \right] \delta \Phi^i$$

We have obtained the field version of **Euler-Lagrange equation**

A.0.4.

$$\frac{\partial \mathcal{L}}{\partial \Phi^i} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Phi^i)} = 0$$

Let's consider a simple example, the real scalar field

$$\phi: \ x^\mu \to \mathbb{R}$$

. The contribution of action are

1. Kinetic term: $\frac{1}{2}\dot{\phi}^2$
2. Gradient term: $\frac{1}{2}(\nabla\phi)^2$
3. potential term: $V(\phi)$

We could combine them into a Lorentz-invariant Lagrange density:

A.0.5.

$$\mathcal{L} = \frac{1}{2}\eta^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - V(\phi)$$

Apply the Euler-Lagrange equation; we then get the equation of motion:

A.0.6.

$$\eta^{\mu\nu}\partial_\mu\partial_\nu\phi + \frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi}$$
$$= \Box\phi + \frac{\mathrm{d}V(\phi)}{\mathrm{d}\phi}$$
$$= 0$$

Where $\Box$ is **d'Alembertian**.

A.1 Klein-Gordon Field

If the scalar field is massive, the potential $V(\phi) = \frac{1}{2}m^2\phi^2$, we obtained the **Klein-Gordon equation**:

A.1.1.

$$(\Box + m^2)\phi = 0$$

You will see this equation again and again in quantum field theory.

Remark.
The Klein-Gordon field could be analog to infinite many coupled infinitesimal harmonic oscillators; each "mass" is affected by neighboring springs and has its kinetic energy.

A.2 Hamiltonian Field Theory

The Lagrangian field theory is naturally Lorentz invariant. However, we also need Hamiltonian formalism for the field theory because it is easier to transition from quantum mechanics. Let's first consider the classical definition of Hamiltonian,

$$H = \sum p\dot{q} - L$$

. From the definition of Lagrangian, we can derive the conjugate momentum as follow:

A.2.1.

$$p(x^i) \equiv \frac{\partial L}{\partial\dot{\phi}(x^i)} = \frac{\partial}{\partial\dot{\phi}(x^i)}\int \mathrm{d}^3x'\mathcal{L}$$
$$\propto \pi(x^i)\mathrm{d}^3x$$

Where

$$\pi \equiv \frac{\partial\mathcal{L}}{\partial\dot{\phi}}$$

> is called **momentum density**

Therefore, the Hamiltonian for field theory is:

> **Definition A.2.1.**
>
> $$H = \sum p(x^i)\phi(x^i) - L$$
> $$= \int \mathrm{d}^3x \left[\pi\dot\phi - \mathcal{L} \right]$$
> $$= \int \mathrm{d}^3x \, \mathcal{H}$$
>
> Where $\mathcal{H}$ is called **Hamiltonian density**.

> **Example A.2.1.**
> As a simple example, the Hamiltonian for K-G field is:
> $$H = \int \mathrm{d}^3x \, \mathcal{H} = \int \mathrm{d}^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right]$$

A.3 Noether's Theorem

A Lagrangian may be invariant under some special type of transformation. For example, a Lagrangian for a complex scalar field
$$\mathcal{L} = |\partial_\mu\phi|^2 - m^2|\phi|^2$$
is invariant under transformation $\phi \to e^{i\alpha}\phi$. We call this transformation a **symmetry** of the Lagrangian. When the parameter of transformation (α for this case) can be taken infinitesimal, we say this symmetry is **continuous**. We can explicit see that $\frac{\delta\mathcal{L}}{\delta\alpha} = 0$.

By applying the Euler-Lagrange equation, we can deduce that $\partial_\mu J_\mu = 0$, where J_μ is defined as follow:

> **Definition A.3.1.**
>
> $$J_\mu = \sum_n \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)} \frac{\delta\phi_n}{\delta\alpha}$$
>
> J_μ is known as **Noether current** or **conserved current**

> **Example A.3.1.**
> For the above complex scalar field, we can calculate the conserved current:
>
> $$J_\mu = \sum_n \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_n)} \frac{\delta\phi_n}{\delta\alpha}$$
> $$= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \frac{\delta\phi}{\delta\alpha} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^*)} \frac{\delta\phi^*}{\delta\alpha}$$
> $$= -i \left[\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi \right]$$

We can check that $\partial_\mu J_\mu = 0$ explicitly:

$$\partial_\mu J_\mu = -i\left[\phi \Box \phi^* - \phi^* \Box \phi\right]$$
$$= 0$$

Where we applied the equation of motion in the last step.

We call J_μ conserved current because we can find conserved quantity by integral over its 0-component:

Definition A.3.2.

$$Q = \int d^3x \ J_0$$

Where the Q is called **conserved charge** or **Noether's charge**

[Conserved charge] We say Q is conserved because

A.3.1.

$$\partial_t Q = \int d^3x \ \partial_t J_0 = \int d^3x \ \vec{\nabla} \cdot \vec{J} = 0.$$

The above argument is called **Noether's theorem**.

Theorem A.3.3 (Noether's theorem)**.**
If a Lagrangian has a continuous symmetry, then exists a current associated with that symmetry that is conserved when the equations of motion are satisfied.

A.4 Energy-momentum Tensor

The physics at spacetime point x should have same form at spacetime point y, this argument arises a symmetry called **specetime translational symmetry**. By Noether's theorem, we can find the four conserved currents for this symmetry for an infinitesimal spacetime translation ξ^μ:

A.4.1.

$$T_{\mu\nu} = \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \partial_\nu \phi_n - g_{\mu\nu} \mathcal{L}$$

$T_{\mu\nu}$ is called **energy-momentum tensor**.

Four conserved currents correspond to values of ν, and four conserved charges are energy and momentum. We can see that the energy density is the 00-component of $T_{\mu\nu}$.