CS70 - Lecture 24 Notes

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Two State Markov Chain

> 1. Describes a random motion in $\{0, 1\}$

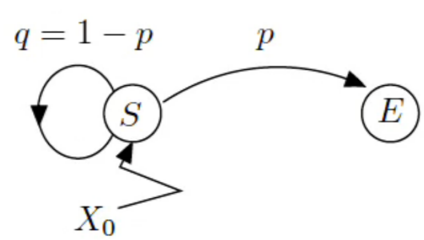
Finite Markov Chain

> 1. What happens in the future only depends on the current state (amnesic, but successive states are dependent on previous value)
> 2. A finite set of states: $\mathscr{X} = \{1, 2, \ldots, K\}$
> 3. A probability distribution π_0 on $\mathscr{X} : \pi_0(i) \geq 0, \sum_i \pi_0(i) = 1$
> 4. Transition probabilities: $P(i, j)$ for $i, j \in \mathscr{X}$
>   - $P(i, j) \geq 0, \forall j; jP(i, j) = 1, \forall i$
> 5. $\{X_n, n \geq 0\}$ is defined so that
>   - X_n = state at time n from time $0, 1, \ldots$
>   - Define how you start: $\Pr[X_0 = i] = \pi_0(i), i \in \mathscr{X}$ (initial distribution)
>   - Define how you move: $\Pr[X_{n+1} = j | X_0, \ldots, X_n = i] = P(i, j), i, j \in \mathscr{X}$
>   - $P(i, j)$ does not depend on what happened in the past or time.

Markov Chain Calculations

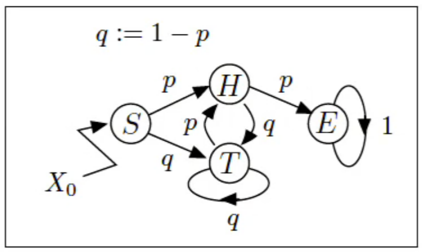
First Passage Time: Example 1

- Flip a coin with $\Pr[H] = p$ until we get H (Use Markov Chain to determine why it will take $\frac{1}{p}$ flips on average ($G(p)$)
- Define a Markov Chain:
  - $X_0 = S$ (start)
  - $X_n = S$ for $n \geq 1$ if the last flip was T w. no H yet
  - $X_n = E$ for $n \geq 1$, if we already got H (end)



- Let $\beta(S)$= the avg. time until we reach E, starting from S, then...
- **Claim**: $\beta(S) = 1 + q\beta(S) + p0$ decomposes into:
  - First step (1)
  - Returns to S: still need $\beta(S)$ steps to get to E w. prob. q ($q\beta(S)$)
  - Got to E (Found heads, needs 0 steps to get to E w. prob p ($p0$)
- Subtract $q\beta(S)$ from both sides to get $\beta(S) = \frac{1}{p}$
- Time until E is $G(p)$, so the mean of $G(p)$ is $\frac{1}{p}$

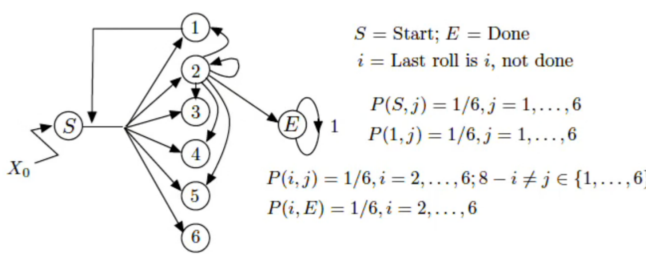
First Passage Time: Example 2



S: Start
H: Last flip $= H$
T: Last flip $= T$
E: Done

- Let $\beta(i)$= avg. time from state i until E (end)
- **First Step Equations**
  - $\beta(S) = 1 + p\beta(H) + q\beta(T)$
  - $\beta(H) = 1 + p0 + q\beta(T)$
  - $\beta(T) = 1 + p\beta(H) + q\beta(T)$
- **Solve:** $\beta(S) = 2 + 3qp^{-1} + q^2p^{-2}$ (E.g., $\beta(S) = 6$ if $p = 1/2$)

First Passage Time: Example 3



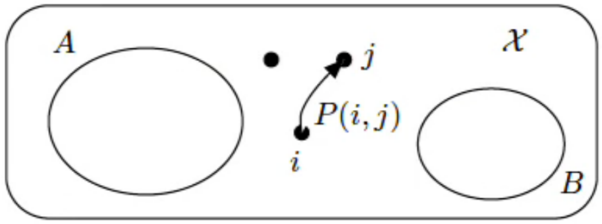
The arrows out of $3, \ldots, 6$ (not shown) are similar to those out of 2.

- $\beta(S) = 1 + \frac{1}{6} \sum_{j=1}^{6} \beta(j)$
- $\beta(1) = 1 + \frac{1}{6} \sum_{j=1}^{6} \beta(j)$
- $\beta(i) = 1 + \frac{1}{6} \sum_{j=1,\ldots,6; j \neq 8i} \beta(j), i = 2, \ldots, 6$
- Symmetry: $\beta(2) = \cdots = \beta(6) = \gamma$. Also, $\beta(1) = \beta(S)$.
  - Thus, $\beta(S) = 1 + (5/6)\gamma + \beta(S)/6; \gamma = 1 + (4/6)\gamma + (1/6)\beta(S)$
  - $\implies \cdots \beta(S) = 8.4$

Summary:

First Step Equations

- Given X_n is a Markov Chain on $\mathscr{X}$ and $A, B \subset \mathscr{X}$ with $A \cap B = \emptyset$



- Define $T_A = min\{n \geq 0 | X_n \in A\}$ and $T_B = min\{n \geq 0 | X_n \in B\}$
- Let $\beta(i) = E[T_A | X_0 = i]$ and $\alpha(i) = \Pr[T_A < T_B | X_0 = i], i \in X$
- $\beta(i)$ denotes a timestep so it adds 1
  - $\beta(i) = 0, i \in A$
  - $\beta(i) = 1 + \sum_j P(i,j)\beta(j), i \notin A$
- $\alpha(i)$ denotes probabilities, so there is no 1
  - $\alpha(i) = 1, i \in A$
  - $\alpha(i) = 0, i \in B$
  - $\alpha(i) = \sum_j P(i,j)\alpha(j), i \notin A \cup B$