

Midterm 2

6:00-8:00pm, 16 April

*Notes: There are **five** questions on this midterm. Answer each question part in the space below it, using the back of the sheet to continue your answer if necessary. If you need more space, use the blank sheet at the end. In both cases, be sure to clearly label your answers! **None of the questions requires a very long answer, so avoid writing too much! Unclear or long-winded solutions may be penalized.** The approximate credit for each question part is shown in the margin (total 60 points). Points are not necessarily an indication of difficulty!*

Your Name:

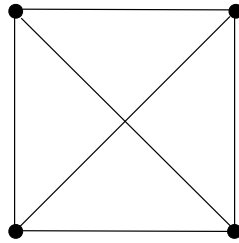
Your Section:

For official use; please do not write below this line!

Q1	
Q2	
Q3	
Q4	
Q5	
Total	

1. Coloring Hypercubes

Let $G = (V, E)$ be an undirected graph. G is said to be k -vertex-colorable if it is possible to assign one of k colors to each vertex of G so that no two adjacent vertices receive the same color. G is k -edge-colorable if it is possible to assign one of k colors to each edge of G so that no two edges incident on the same vertex receive the same color. For example, the following graph is 4-vertex-colorable (but not 3-vertex-colorable) and 3-edge-colorable (but not 2-edge-colorable). Check that you understand these definitions before proceeding!



- (a) Show that the n -dimensional hypercube is 2-vertex-colorable for every n .

5pts

-
- (b) Show that the n -dimensional hypercube is n -edge-colorable for every n .

5pts

2. Random Graphs

A *random undirected graph* G with n vertices $\{1, 2, \dots, n\}$ is formed by including each possible edge $\{u, v\}$ with probability p , independently of all other edges. Answer the following questions, giving each answer in terms of n and p .

- (a) A vertex is “isolated” if it is not connected by an edge to any other vertex. What is the expected number of isolated vertices in G ? 4pts

-
- (b) What is the expected number of vertices with degree exactly 3 in G ?

4pts

-
- (c) A “square” is a simple cycle of length four with no diagonals present. (E.g., the set of vertices $\{1, 2, 3, 4\}$ forms a square if G contains edges $\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}$, and does not contain edges $\{1, 3\}$ or $\{2, 4\}$.) What is the expected number of squares in G ?

4pts

3. Counting

For each of the following problems, simplify your answer as much as possible and **circle your final answer**. In case the answer is an expression involving factorials or binomial coefficients, you *do not* need to simplify it further. In each case, you should give a **brief** explanation of your answer.

- (a) In Morse code, letters are represented by a sequence of dots and dashes. How many different letters can be formed by using *at most* 10 symbols? 3pts

-
- (b) An undirected graph on n labeled vertices is described by specifying, for each unordered pair of vertices $\{u, v\}$, whether the graph contains an edge between u and v . How many different undirected graphs are there on n vertices? 3pts

-
- (c) How many orderings are there of the numbers from 1 to $2n$, in which the numbers $1, \dots, n$ occur in increasing order (but not necessarily contiguously)? 3pts

-
- (d) Two committees (say, the Education Committee and the Arts Committee) are to be formed from a group of n citizens. In how many ways can these committees be formed, so that each person serves on *at most one* committee, and each committee contains *at least one* member. [HINT: Think about how to solve the problem if we remove the restriction that each committee must contain at least one member.] 3pts

-
- (e) A Social Security Number is any sequence of nine digits. How many Social Security Numbers have at least eight different digits? 3pts
-

4. Colorful Jelly Beans

A candy factory has an endless supply of red, orange and yellow jelly beans. The factory packages the jelly beans into jars of 100 jelly beans each, with each possible combination of colors in the jar being equally likely. (One possible color combination, for example, is a jar of 56 red, 22 orange, and 22 yellow jelly beans.) As a marketing gimmick, the factory claims that the number of possible combinations is so large that there is negligible probability of finding two jars with the same color combination. You are skeptical about this claim and decide to do some calculations to test it.

- (a) Find N , the number of different possible color combinations of jelly beans in a single jar. *3pts*

-
- (b) In terms of N , write down the probability that two jars of jelly beans have different color combinations. *2pts*

-
- (c) Again in terms of N , write down the probability that m jars of jelly beans all have different color combinations. [NOTE: You do not need to simplify your expression.] *3pts*

-
- (d) Approximately how many jars of jelly beans would you have to buy until the probability of seeing two jars with the same color combination is at least $\frac{1}{2}$? [NOTE: You should state your answer only as an order of magnitude (i.e., 10, 100, 1000, . . .). You may appeal to any result from class in order to derive your estimate; it should not be necessary to perform a detailed calculation.] *3pts*
-

5. Learning and Betting

You are given two biased coins, one of which has probability 0.55 of coming up Heads, and the other of which has probability 0.2 of coming up Heads. However, you don't know which is which.

- (a) You randomly pick one coin and flip it. If the coin comes up Heads you win \$1, and if it comes up Tails you lose \$1. What is the probability of winning? Would you play this game? *4pts*

-
- (b) Now suppose you get to randomly pick a coin and flip it for free. Then you get to choose a coin for the second flip. Again, you win or lose \$1 according to whether the second flip is Heads or Tails. Given that you see Heads on the first flip, which coin would you choose for the second flip? Assuming this strategy, would you bet the \$1 on the second flip? Justify your answer with a calculation. *4pts*

-
- (c) Now suppose you get to randomly pick a coin and flip it (the same coin) for free *twice*. Then you get to choose a coin for the third flip. Suppose you get Heads on both of the first two flips. Would you bet the \$1 on the next flip? Again, justify your answer. *4pts*
-

[The end]