EECS 70 Discrete Mathematics and Probability Theory
Fall 2015 Jean Walrand

Midterm 3

PRINT Your Name: ______________________________, ______________________________
(last) (first)

SIGN Your Name: ______________________________

PRINT Your Student ID: ______________________________

CIRCLE your exam room: 2040 VLSB 2060 VLSB 145 Dwinelle 155 Dwinelle 10 Evans OTHER

Name of the person sitting to your left: ______________________________

Name of the person sitting to your right: ______________________________

- After the exam starts, please write your student ID (or name) on every odd page (we will remove the staple when scanning your exam).
- We will not grade anything outside of the space provided for a problem. Please use scratch paper as necessary and clearly indicate your answer.
- For questions 1(a)-(e). You need only circle True or False.
- For questions 2 (a)-(g), only provide the requested answer (e.g., probability value, one or more events). There is no need to justify your answer.
- For questions 3 (a)-(d), write clearly your answer in the space provided. There is no need to justify your answer.
- For questions 4 (a)-(h), you should indicate clearly your derivation in the space provided.
- You may not look at books, notes, etc. Calculators, phones, and computers are not permitted.
- There are 9 pages on the exam, including this first page. Notify a proctor immediately if a page is missing.
- **You may, without proof, use theorems and facts that were proven in the notes and/or in lecture.**
- **You have 105 minutes; there are 24 parts on this exam.**

Do not turn this page until your instructor tells you to do so.

1. **True or False. No justification needed. 15 points. 3/3/3/3/3.**

   Clearly indicate your correctly formatted answer: this is what is to be graded. No need to justify!

   (a) Disjoint events with a positive probability cannot be independent. (True or False.)

   (b) We can find events A and B with $Pr[A|B] > Pr[A]$ and $Pr[B|A] < Pr[B]$. (True or False.)

   (c) If $Pr[A|B] = Pr[B]$, then A and B are independent. (True or False.)

   (d) For a random variable X, it is always the case that $E[X^2 - X] \geq -1$. (True or False)

   (e) If $Pr[A] > Pr[\bar{A}]$, then $Pr[A|B] \geq Pr[\bar{A}|B]$. (True or False)

2. **Short Answer: Probability Space. 31 points: 4/4/4/5/5/4/5**

   Clearly indicate your correctly formatted answer: this is what is to be graded. No need to justify!

   (a) You flip a biased coin (such that $Pr[H] = p$) until you accumulate two Hs (not necessarily consecutive). What is the probability space? That is, what is Ω and what is $Pr[\omega]$ for each $\omega \in \Omega$?

(b) Let $\Omega = \{1,2,3,4\}$ be a uniform probability space. Let also $A = \{1,2,3\}$. Produce an event B such that $Pr[B] > 0$ and A and B are independent.

(c) Let $\Omega = \{1,2,3,4\}$ be a uniform probability space. Produce three events A, B, C that are pairwise independent but not mutually independent.

(d) You are dealt two cards from a deck of 52 cards. What is the probability that the value of the first card is strictly larger than that of the second? [In this question, the values are 1 for an ace, 2 through 10 for the number cards, then 11 for a Jack, 12 for a queen, 13 for a king.]

(e) You roll a balanced 6-sided die twice. What is the probability that the total number of pips is less than 10 given that it is larger than 7?

(f) With probability $1/2$, one rolls a die with four equally likely outcomes $\{1,2,3,4\}$ and with probability $1/2$ one rolls a balanced die with six equally likely outcomes $\{1,2,\ldots,6\}$. Given that the outcome is 4, what is the likelihood that the coin was four-sided?

(g) A coin is equally likely to be fair or such that $Pr[H] = 0.6$. You flip the coin 10 times and get 10 heads. What is the probability that the next coin flip yields heads?

3. Short Answers: Random Variables and Expectation. 14 points. 3/3/4/4

Clearly indicate your correctly formatted answer: this is what is to be graded. No need to justify!

(a) Define a 'random variable' in a short sentence.

(b) Let $\Omega = \{1,2\}$ be a uniform probability space. Produce a random variable that has mean zero and variance 1.

$X : \Omega \to R$

$X(1) = a$

$X(2) = -a$

$Var(X) = E(X^2)$

$\frac{1}{2} \cdot a^2 + \frac{1}{2}(-a)^2 = 1$

$a^2 = 1 \Rightarrow a = 1$

$\boxed{\begin{matrix} X(1) = 1 \\ X(2) = -1 \end{matrix}}$ $\boxed{\begin{matrix} X(1) = -1 \\ X(2) = 1 \end{matrix}}$

(c) Let $\Omega = \{1,2,3,4\}$ be a uniform probability space. Define two random variables X and Y such that $E[XY] = E[X]E[Y]$ even though the random variables are not independent.

(d) You roll a die twice. Let X be the maximum of the number of pips of the two rolls. What is $E[X]$. (You may leave the answer as a sum.)

$$Var(X) = E(X^2) - E(X)^2$$
$$E(X^2) = Var(X) + E(X)^2$$

4. Short Problems. 40 points: 5/5/5/5/5/5/5/5

Clearly indicate your answer and your derivation.

(a) Let X be a random variable with mean 1. Show that $E[2+3X+3X^2] \geq 8$.

$$E(2) + 3E(X) + 3E(X^2)$$
$$= 2 + 3 + 3(Var(X) + E(X)^2)$$
$$= 5 + 3 + 3Var(X)$$
$$= 8 + \cancel{3}\, 3\underbrace{Var(X)}_{\geq 0} \geq 8$$

(b) Let X be geometrically distributed with parameter p. Recall that this means that $Pr[X=n] = (1-p)^{n-1}p$ for $n \geq 1$. Find $E[X|X>n]$. Do not leave the answer as an infinite sum.

$$Pr(X > s+t \mid X \geq s) = Pr(X > t)$$

Let $Y = X - n \sim Geom(p)$

$$E(Y) = \frac{1}{p}$$
$$E(Y) = E(X) - n$$
$$E(X \mid X > n) = \boxed{n + \frac{1}{p}}$$

(c) Roll a die n times. Let X_n be the average number of pips per roll. What is $var[X_n]$? You may leave the answer as a sum.

Z_i = # pips roll i

$E(Z_1) = 3.5$

$$X_n = \frac{Z_1 + \cdots + Z_n}{n}$$

$$Var(Z_1) = E\left((Z_1 - E(Z_1))^2\right) = \sum_{i=1}^{6} (i-3.5)^2 \underbrace{Pr(Z_1 = i)}_{\frac{1}{6}}$$

$$Var(X_n) = \frac{1}{n^2} Var(Z_1 + \cdots + Z_n) = \frac{1}{n^2}\left(Var(Z_1) + \cdots + Var(Z_n)\right) = \frac{1}{n} Var(Z_1)$$

$$\boxed{Var(X_n) = \frac{1}{6n} \sum_{i=1}^{6} (i-3.5)^2}$$

(d) Let X and Y be independent with $X = G(p)$ and $Y = G(q)$. What is $Pr[X \le Y]$? Do not leave the answer as an infinite sum.

$Pr(X \le Y) = Pr(\text{you win before/during your opp. win})$

$= Pr(\text{you win} \mid \text{you win or opp. wins})$

$$= \frac{Pr(\text{you win})}{Pr(\text{someone wins})}$$

$Pr(\text{someone wins}) = 1 - Pr(\text{no one wins})$

$$= \boxed{\frac{p}{1-(1-p)(1-q)}}$$

(e) You roll a balanced die five times. Let X be the total number of pips you got and Y the total number of pips on the last two rolls. What is $E[X|Y=4]$? What is $E[Y|X=15]$?

$X_1+X_2+X_3+\underbrace{X_4+X_5}_{4}$

$E(X_1+X_2+X_3)=3\cdot\frac{7}{2}$

$\boxed{E(X|Y=4)=4+3\cdot\frac{7}{2}}$

3 3 3 Y

$X_1+X_2+X_3+X_4+X_5$

15 3 3

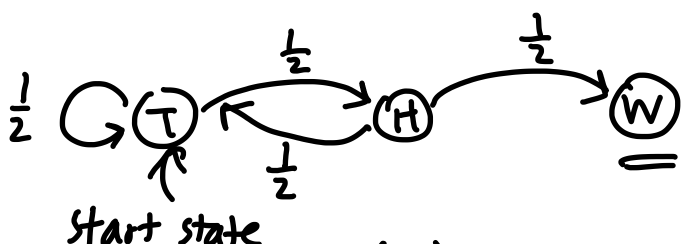
$\boxed{E(Y|X=15)=6}$

symmetry

(f) How many times do you have to flip a fair coin, on average, until you get two consecutive H's? [Hint: condition on the outcome of the last flip.]

W = win state
H = last flip H
T = last flip T

$E(T)$



$E(W)=0$

$E(H)=1+\frac{1}{2}E(T)+\cancel{\frac{1}{2}E(W)}$

$E(T)=1+\frac{1}{2}E(T)+\frac{1}{2}E(H)$

$\frac{1}{2}E(T)=1+\frac{1}{2}E(H)$

$\frac{1}{2}E(T)=1+\frac{1}{2}+\frac{1}{4}E(T)$

$\frac{1}{4}E(T)=\frac{3}{2} \Rightarrow E(T)=\boxed{6}$

$$E(X_i) = \frac{1}{p}$$

(g) Let $\{X_n, n \geq 1\}$ be independent and geometrically distributed with parameter p. Recall that $var[X] = (1-p)/p^2$. Provide an upper bound on

$$Pr[|\underbrace{\frac{X_1 + \cdots + X_n}{n}}_{\bar{X}} - \overset{\frac{1}{p}}{p}| \geq a]$$

using Chebyshev's inequality.

$$\bar{X} - \frac{1}{p}$$

$$\bar{X} = \frac{X_1 + \cdots + X_n}{n}$$

$$E(\bar{X}) = \frac{n \cdot \frac{1}{p}}{n} = \frac{1}{p}$$

$$Var(\bar{X}) = \frac{1}{n^2} Var(X_1 + \cdots + X_n)$$

$$= \frac{1}{n^2}(Var(X_1) + \cdots + Var(X_n))$$

$$= \frac{1}{n} Var(X_i) = \frac{1-p}{np^2}$$

$$Pr(|\bar{X} - E(\bar{X})| \geq a) \leq \frac{Var(\bar{X})}{a^2}$$

$$Pr(|\bar{X} - \frac{1}{p}| \geq a) \leq \boxed{\frac{1-p}{na^2p^2}}$$

(h) There are two envelopes. One contains checks with $\{1,3,5,6,7\}$ dollars. The other contains checks with $\{4,5,5,7\}$ dollars. You choose one of the two envelopes at random and pick one of the checks at random in the envelope. That check happens to be for 5 dollars. You are given the option to keep all the money in that envelope, including the check for 5 dollars, or to switch to the other envelope. What should you do?