CS70 - Lecture 20 Notes

Name: Felix Su SID: 25794773

Spring 2016 GSI: Gerald Zhang

Expectation

Expected Value Defintion:

$$E[X] := \sum_x x\Pr[X = x] = \sum_\omega X(\omega)\Pr[\omega] \tag{1}$$

Expectation of Function of RVs

$$E[g(X, Y)] = \sum_{x,y} g(x, y)\Pr[X = x, Y = y] = \sum_\omega g(X(\omega), Y(\omega))\Pr[\omega] \tag{2}$$

Linearity of Expectation

$$E[aX + bY + c] = aE[X] + bE[Y] + c \tag{3}$$

Uniform Distribution

1. RV X equally likely to take on any of its values
2. $E[X]$ of uniformly distributed X is avg of all outcomes

Uniform Distribution Example

$$\Pr[X = m] = \frac{1}{n} \text{ for } m = 1, 2, \cdots, n \tag{4}$$

Expectation Uniform Distribution

$$E[X] = \sum_{m=1}^{n} m\Pr[X = m] = \sum_{m=1}^{n} m\frac{1}{n} = \frac{n+1}{2} \tag{5}$$

Geometric Distribution

1. Let X = Number of flips of coin with $\Pr[H] = p$ until we get H, so $X(\omega_n) = n$
2. Higher $p \Rightarrow$ smaller expected X

Geometric Distribution

$$\Pr[X = n] = (1 - p)^{n-1}p, n \geq 1 \tag{6}$$

Use Geometric Sum

$$\sum_{n=1}^{\infty} \Pr[X_n] = \sum_{n=1}^{\infty}(1 - p)^{n-1}p = p\sum_{n=0}^{\infty}(1 - p)^n = \frac{p}{1 - (1 - p)} = 1 \tag{7}$$

Derive Geometric Sum

If $|a| < 1$ and $S := \sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$:

$$S = 1 + a + a^2 + a^3 + \cdots \tag{8}$$
$$aS = \quad a + a^2 + a^3 + \cdots \text{ (shifted right 1)} \tag{9}$$
$$(1-a)S = 1 + a - a + a^2 - a^2 + \cdots = 1 \text{ (subtract above two terms)} \tag{10}$$

Geometric Distribution Expectation

1. $X =_D G(p)$ where $_DG(p) \equiv \Pr[X = n] = (1-p)^{n-1}p, n \geq 1$

- $E[X] = \sum_{n=1}^{\infty} n\Pr[X = n] = \sum_{n=1}^{\infty} n(1-p)^{n-1}p$

Expectation of Geometric Distribution

$$E[X] = \sum_{n=1}^{\infty} n\Pr[X = n] = \sum_{n=1}^{\infty} n(1-p)^{n-1}p = \frac{1}{p} \tag{11}$$

Derive Expectation of Geometric Distribution

$$E[X] = p + 2(1-p)p + 3(1-p)^2p + 4(1-p)^3p + \cdots \tag{12}$$
$$(1-p)E[X] = \quad (1-p)p + 2(1-p)^2p + 3(1-p)^3p + \cdots \text{ (shifted right 1)} \tag{13}$$
$$pE[X] = p + (1-p)p + (1-p)^2p + (1-p)^3p + \cdots = \sum_{n=1}^{\infty} \Pr[X = n] = 1 \text{ (subtract above two terms)} \tag{14}$$

Time to Collect Coupons

- Note: $H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ (Harmonic Number)
- $H(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} \approx \int_1^n \frac{1}{x}dx = \ln(n)$
- $H(n) \approx \ln(n) + \gamma$ where $\gamma \approx 0.58$ (Euler-Mascheroni constant) to account for bars sticking above graph

X - time to get n coupons
$\Pr[\text{get ith coupon}|\text{got i-1 coupons}] = \frac{n-(i-1)}{n} = \frac{n-i+1}{n}$
$E[X_i] = \frac{1}{p} = \frac{n}{n-i+1}, i = 1, 2, \cdots, n$
$E[X] = \sum_{i=1}^{n} E[X_i] = \sum_{i=1}^{n} \frac{n}{n-i+1} = n(1 + \frac{1}{2} + \cdots + \frac{1}{n}) =: nH(n) \approx n(\ln n + \gamma)$

Stacking

- Cards have width 2
- Keep putting cards 1/2 to the right
- At center of mass, all mass on left = all mass on right

- So, if $x =$ dist. between C.O.M and right side of base card and n cards weigh n, $nx = 1 - x$ and $x = \frac{1}{(n+1)}$
- Induction shows that the C.O.M after n cards is $H(n)$ away from the rightmost edge of the bas card

Geometric Distribution: Memoryless

> 1. If $X = G(p)$, probability of any X occurring is not dependent on previous events

- Let $X = G(p)$. For $n \geq 0$: $\Pr[X \geq n] = \Pr[\text{first n flips are T}] = (1-p)^n$
- **Thm**: $\Pr[X > n + m | X > n] = \Pr[X > m], m, n \geq 0$
- $\Pr[X > n + m | X > n] = \dfrac{\Pr[X > n + m \cap X > n]}{\Pr[X > n]} = \dfrac{\Pr[X > n + m]}{\Pr[X > n]} = \dfrac{(1-p)^{n+m}}{(1-p)^n} = (1-p)^m = \Pr[X > m]$

Memoryless Geometric Distribution Theorem

$$\Pr[X > n + m | X > n] = \Pr[X > m], m, n \geq 0 \tag{15}$$

Expectation of Natural Numbers (works for Geometric Distribution)

- **Thm:** For RV X that takes values $\{0, 1, 2, \cdots\}$: $E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i]$

For Natural Number RVs X

$$E[X] = \sum_{i=1}^{\infty} \Pr[X \geq i] \tag{16}$$

Poisson

> 1. For binomial distribution where $\Pr[H] = \lambda/n$
> 2. $X = P(\lambda) \iff \Pr[X = m] \approx \frac{\lambda^m}{m!} e^{-\lambda}$
> 3. Expecation of Poisson Distribution $= \lambda$

- Flip coin n times where $\Pr[H] = \lambda/n$
- RV $X =$ no. of heads (Binomial), thus $X = B(n, \lambda/n)$
- **Poisson Distribution** is the distribution of X "for large n" and λ is constant
- Binomial Representation: $\Pr[X = m] = \binom{n}{m} p^m (1-p)^{n-m}$, with $p = \lambda n$

Poisson Proof

- $\Pr[X = m] = \frac{n(n-1)\cdots)(n-m+1)}{m!} (\frac{\lambda}{n})^m (1 - \frac{\lambda}{n})^{n-m} = \frac{n(n-1)\cdots)(n-m+1)}{n^m} (\frac{\lambda^m}{m!})(1 - \frac{\lambda}{n})^{n-m}$
- $\approx (1)(\frac{\lambda^m}{m!})(1 - \frac{\lambda}{n})^{n-m} \approx (\frac{\lambda^m}{m!})(1 - \frac{\lambda}{n})^n$ (Because $n >> m$)
- $\approx \frac{\lambda^m}{m!} e^{-\lambda}$ (Because $(1 - \frac{a}{n})^n \approx e^{-a}$)

Poisson Expectation Proof

- $E[X] = \sum_{m=1}^{\infty} m \times \frac{\lambda^m}{m!} e^{-\lambda} = e^{-\lambda} \sum_{m=1}^{\infty} \frac{\lambda^m}{(m-1)!} = e^{-\lambda} \sum_{m=0}^{\infty} \frac{\lambda^{m+1}}{m!} = e^{-\lambda} \lambda \sum_{m=0}^{\infty} \frac{\lambda^m}{m!} = e^{-\lambda} \lambda e^{\lambda} = \lambda$

Poisson Distribution

$$\Pr[X = m] \approx \frac{\lambda^m}{m!} e^{-\lambda}, m \geq 0 \tag{17}$$

Expectation of Poisson

$$E[X] = \lambda \tag{18}$$

Distributions Summary

Uniform Distribution

$$U[1, \cdots, n] : \Pr[X = m] = \frac{1}{n}, m = 1, \cdots, n; E[X] = \frac{n+1}{2} \tag{19}$$

Binomial Distribution

$$B(n, p) : \Pr[X = m] = \binom{n}{m} p^m (1-p)^{n-m}, m = 0, \cdots, n; E[X] = np \tag{20}$$

Geometric Distribution

$$G(p) : \Pr[X = n] = (1-p)^{n-1} p, n = 1, 2, \cdots; E[x] = \frac{1}{p} \tag{21}$$

Poisson Distribution

$$P(\lambda) : \Pr[X = m] = \frac{\lambda^m}{m!} e^{-\lambda}, m \geq 0; E[X] = \lambda \tag{22}$$

Independent Random Variables

1. Two RVs X and Y are **independent** if and only if, all the corresponding events are independent.
2. Same as independence of events.
3. RVs are **mutually independent** if the product of their combined intersection (and) is the same as the product of their individual probabilities.
4. Events A, B, C are pairwise (resp. mutually) independent iff their Indicator RVs $1_A, 1_B, 1_C$ are also pairwise independent.

- Independence Thm Proof:
- 'if' left direction: if you choose $A = \{a\}, B = \{b\}$, then the thm eq. is the same as:
- $\Pr[X = a \cap Y = b] = \Pr[X = a]\Pr[Y = b], \forall a, b$
- 'only if' right direction: sum over all possible pairs in A and B of the probability that X is in A and Y is in B
- $\overbrace{\sum_{a \in A} \sum_{b \in B}}^{\Pr[X \in A \cap Y \in B]} \Pr[X = a \cap Y = b] = \sum_{a \in A} \sum_{b \in B} \Pr[X = a]\Pr[Y = b]$
- $= \sum_{a \in A} \Pr[X = a] \sum_{b \in B} \Pr[Y = b] = \Pr[X \in A]\Pr[Y \in B]$

Functions of Independent RVs

> 1. Functions of independent RVs are independent
> 2. If X, Y, Z are pairwise independent, but not mutually independent, it may indicate that $f(x)$ and $g(Y, Z)$ are not independent
> 3. Functions of disjoint collections of mutually independent RVs are also mutually independent

- Independent functions of inpdendent RVs Proof:
  - Definition of Inverse Image: $h(z) \in C \iff z \in h^{-1}(C) := \{z | h(z) \in C\}$
  - $\Pr[f(X) \in A, g(Y) \in B] = \Pr[X \in f^{-1}(A) \cap Y \in g^{-1}(B)]$, by def. of Inv. Im.
  - $= \Pr[X \in f^{-1}(A)]\Pr[Y \in g^{-1}(B)]$, because X, Y ind.
  - $= \Pr[f(x) \in A]\Pr[g(Y) \in B]$
- Functions of disjoint collections of mutually independent RVs are also mutually independent
  - Let $\{X_n, n \geq 1\}$ be mutually independent. And $Y_1 := f(X_1, X_2), Y_2 := f(X_3, X_4), Y_3 := f(X_5, X_6)$
  - Then, Y_1, Y_2, Y_3 are mutually independent

Mean($E[X]$) of product of Ind. RVs

> 1. Expectation of XY is equal to exp. of X times exp. of Y

- Proof:
- $E[g(x, y, z)] = \sum_{x,y} g(x, y)\Pr[X = x \cap Y = y]$, so, $E[XY] = \sum_{x,y} xy\Pr[X = x \cap Y = y] = \sum_{x,y} xy\Pr[X = x]\Pr[Y = y]$
- $= \sum_x x\Pr[X = x] \sum_y y\Pr[Y = y] = E[X]E[Y]$

Independence of X and Y

$$\Pr[Y = b | X = a] = \Pr[Y = b], \forall a, b \tag{23}$$

$$\Pr[X = a \cap Y = b] = \Pr[X = a]\Pr[Y = b], \forall a, b \tag{24}$$

Thm: X and Y are independent iff

$$\Pr[X \in A \cap Y \in B] = \Pr[X \in A]\Pr[Y \in B], \forall A, B \subset \mathbb{R} \tag{25}$$

Independent Functions

$$\text{If } X, Y \text{ are independent RVs} \implies f(X), g(Y) \text{ are independent } \forall f(), g() \tag{26}$$

Mean of product of Independent RV (Only for Independent RVs)

$$\text{If } X, Y \text{ are ind. RVs} \implies E[XY] = E[X]E[Y] \tag{27}$$

X, Y, Z are Mutually Independent If

$$\Pr[X = x, Y = y, Z = z] = \Pr[X = x]\Pr[Y = y]\Pr[Z = z], \forall x, y, z \tag{28}$$

Independent RV Examples
X, Y, Z are pairwise independent and $U[1, 2, \cdots, n]$
Let $E[X] = E[Y] = E[Z] = 0, E[X^2] = E[Y^2] = E[Z^2] = 1$

- $E[(X+2Y+3Z)^2] = E[X^2+4Y^2+9Z^2+4XY+12YZ+6XZ]$
- $= 1+4+9+(4)(0)(0)+12(0)(0)+6(0)(0) = 14$ (Because Independent RV product and Lin. of Exp.)

- $E[(X-Y)^2] = E[X^2+Y^2-2XY] = 2E[X^2]-2E[X]^2$
- $= \frac{1+3n+2n^2}{3} - \frac{(n+1)^2}{2}$