CS70 - Lecture 18 Notes

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Random Variables

- Random variable is a known, deterministic, function that maps outcome to a number (onto, but not necessarily one-to-one)
- Random Variable X for an experiment with sample space Ω is a function $X : \Omega \to \mathbb{R}$
  - X assigns $X(\omega) \in \mathbb{R}$ to each $\omega \in \Omega$
  - X is not random, nor a variable, it is called a random variable, because its outcome depends on the intial probability/experiment (varies from experiment to experiment)
  - After deriving X and knowing the initial probabilities, can determine the likelihood of each outcome of X
- $X^{-1}(A) = \{\omega | X(\omega) = A\}$: Inverse image of value A
  - Set of outcomes that map to A

Distribution

- The probability of X taking on a value A.
- Definition: The distribution of a random variable X is $\{(a, \Pr[X = a]) : a \in \mathbb{A}\}$ where $\mathbb{A}$ is the range of X
- $\Pr[X = A] = \Pr[X^{-1}(A)]$

Random Variable Method

- Examine the elements of the Sample Space (ex. $\{HHH, THH, HTH, TTH, HHT, THT, HTT, TTT\}$
- Define the Random Variables by given params ex. +1 on heads, -1 on tails $\Rightarrow \{3, 1, 1, -1, 1, -1, -1, -3\}$
- Determine the probability of any generalized event (ex. getting i heads and $n - i$ tails if the prob. of getting heads is p: $p^i(1-p)^{n-i}$)
- Multiply the above probability by the amount of times it occurs (summation) (ex. All ways to get i heads out of n flips: $\binom{n}{i}$)
- This determines the probability of each element in the Random Variable: the distribution of outcomes. (ex. Binomial Distribution)

Binomial Distribution

$$B(n,p) : \Pr[X = i] = \binom{n}{i} p^i (1-p)^{n-i}, i \in \{0, \cdots, n\} \tag{1}$$

- Flip n coins with probability p to get heads, Random var: No. of heads
- Ways to choose i heads out of n flips: $\binom{n}{i}$
- Determine probability of $\omega = i$ heads (probability of heads in any position is p): Pr of i heads and $n - 1$ tails is p^i and $(1-p)^{n-i}$ respectively $\therefore \Pr[\omega] = p^i(1-p)^{n-i}$
- Probability of $X = i$ is the sum of all $\Pr[\omega]$ where ω contains i heads: $\binom{n}{i}$

Error Channel

- Apply Binomial Distribution
- Packet is corrupted with probability p
- Send $n+2k$ packets
- Find probability of at most k corruptions
- $\sum\limits_{i \leq k} \binom{n+2k}{i} p^i (1-p)^{n+2k-i}$: Sum gets total probability of all i corruptions s.t. $i \leq k$
- For RS Code, choose k s.t. the above probability is large

Combining Random Variables

- Let X and Y be two Random Vars in the same Probability space
- Then, $X+Y$ is an RV that assigns value $X(\omega)+Y(\omega)$ to ω
- General Case: $g(X,Y,Z)$ assigns value $g(X(\omega),Y(\omega),Z(\omega))$ to ω

Expectation

- $E[X] = \sum\limits_a a \times \Pr[X=A] \approx \frac{X_1+\cdots+X_n}{N}$
- Random variable X, X has a possible values (gains). Multiply each possible gain a by the probability of RV $X=a$ and sum over all RVs to get expected value.
- Average $= E(X)$ holds for uniform probability
- Expectation is linear

Expectation Theorem

$$E[X] = \sum_{\omega} X(\omega) \times \Pr[\omega] \tag{2}$$

- Sum of all products of X_i and the probability of getting that X_i (also called the mean by frequentist interpretation): **Law of Large Numbers**

Expectation Derivation Methods

- Two ways to compute the mean value
  - Given: Distribution of X (set of values a and their probabilities)
    - $E[X] = \sum\limits_a a \times \Pr[X=A] \approx \frac{X_1+\cdots+X_n}{N}$
  - Given: Probability Space
    - Sum over all ω's in probability space
    - $E[X] = \sum\limits_{\omega} X(\omega) \times \Pr[\omega]$

Example of Both Derivation Methods

- Flip fair coin 3 times
- $\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$
- $X =$ number of H's $: \{3,2,2,2,1,1,1,0\}$
- Method 1: $E[X] = \sum\limits_a a \times \Pr[X=A] = 3(\frac{1}{8}) + 2(\frac{3}{8}) + 1(\frac{3}{8}) + 0(\frac{1}{8}) = \frac{3}{2}$
- Method 2: $E[X] = \sum\limits_{\omega} X(\omega) \times \Pr[\omega] = (3+2+2+2+1+1+1+0)(\frac{1}{8})$