CS70 - Combinatorial Arguments

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Strategy

1. Define equivalent quantity Q
2. Count Q in a way to determine $LHS = Q$
3. Count Q in a way to determine $RHS = Q$
4. Conclude $LHS = RHS$
5. **Review: 2,3,4,7,8,9**

Hints

- $\binom{n}{k} \equiv \binom{n}{n-k}$: Include k out of n items = Do not include $n-k$ items
- 2^n= All possible ways to choose any number of n items. Counting Product Rule.
- $\sum \binom{n}{k} = \binom{n+1}{k+1} \rightarrow$ set one value to include and pick remaining.
- $\sum \binom{n}{k} = \binom{n+1}{k} \rightarrow$ set aside one value to not include and pick remaining.

Examples

$$2^n = \sum_{k=0}^{n} \binom{n}{k} \tag{1}$$

LHS: Using counting product rule, this is the number subsets of n decided by either choosing or not choosing each item. There are a total of 2^n possible subsets of size $k \in (0, n)$ from a total of n items.

RHS: The summation of all possible ways to choose $k \in (0, n)$ items from n, which is the same as left.

$$\sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1} \tag{2}$$

LHS: For each iteration of the summation, set aside a largest element l then choose the remaining k elements to obtain a subset of n that is size $k+1$. Such that you only pick items less than l, limiting you to m items to choose from. All combinations of this is the same as RHS.

RHS: All possible subsets of $\{1, \cdots, n+1\}$ that are size $k+1$.

$$\sum_{k=0}^{n} \binom{m+k}{k} = \binom{n+m+1}{n} \tag{3}$$

LHS: Specify the smallest element from $\{1, \cdots, n+m+1\}$ not in the selected subset. If it is 1, you must choose the remaining n items from $\{2, \cdots, n+m+1\}$, $\binom{m+n}{n}$, which is the $k=n$ term. If it is 2, 1 is included in the subset, so you only need to choose from $n-1$ remaining items from $\{2, \cdots, n+m+1\}$, $\binom{m+n-1}{n-1}$. This continues down to having the smallest number not in the set be $n+1$, which would cause, $\binom{m+0}{0}$. This equates to all possible ways to select select n items from $n+m+1$ choices

RHS: All possible ways to select select n items from $n+m+1$ choices.

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n} \tag{4}$$

LHS: Split $\{\pm 1, \cdots, \pm n\}$ into 2 sets, 1 of positive and 1 of negative elements. Choose k positive items and $n-k$ negative items (which is still equivalent to $\binom{n}{k}$). This is equivalent to choosing n items out of $\{\pm 1, \cdots, \pm n\}$.

RHS: All possible ways to directly choose n items out of $\{\pm 1, \cdots, \pm n\}$.

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} = \binom{n+m}{r} \tag{5}$$

LHS: Split the collection of books into a set of textbooks and one of comic books. Select k books from the set of textbooks and $r-k$ books from the set of m comic books. This is the same as selecting r total books from a collection of n textbooks and m comic books.

RHS: Choose r books from a collection of n textbooks and m comic books.

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \tag{6}$$

LHS: All possible ways to select k items from a collection of n choices.

RHS: Set aside one element. Assuming this item must be in the selected subset of k items, select the remaining $k-1$ items from the remaining $n-1$ choices $\binom{n-1}{k-1}$. Then assume the item set aside is not in the selected subset of k items. Select the rest of the k items from the remaining $n-1$ choices $\binom{n-1}{k}$. These two disjoint sets add up to the original selection of k items from n choices

$$\binom{2n}{2} = 2\binom{n}{2} + n^2 \tag{7}$$

LHS: All possible ways to choose 2 items from a set of $2n$ choices.

RHS: There are 3 ways to choose 2 items from a set of $2n$ choices if you split the $2n$ choices into 2 sets of n choices. You can choose both from the first set of n choices $\binom{n}{2}$, choose both from the second set of n choices $\binom{n}{2}$, or you can choose 1 from each set $\binom{n}{1}^2 = n^2$. Add these disjoint sets together to get LHS.

$$\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1} \tag{8}$$

LHS: Pick a team of size k, then pick the captain out of the k people on the team. All possible sizes of the team is $k = (0, n)$, so sum all those possibilities.

RHS: Pick one of n people to be a captain. For each rest of the $n - 1$ people, choose whether or not to include him/her on the team. (Counting)

$$\sum_{k=j}^{n} \binom{n}{k}\binom{k}{j} = 2^{n-j}\binom{n}{j} \tag{9}$$

LHS: Select a team of size k from n students, then choose j leaders from the k team members. Sum all the probabilities for teams of size j to n

RHS: Out of n students, choose j leaders, out of the remaining $n - j$ people, select whether or not each will be on the team.

$$\sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} = \binom{m+n}{k} \tag{10}$$

LHS: Split a collection of m fiction and n non-finction books into a set of m fiction books and n non-fiction books. Select k total books by selecting i books from the fiction pile and the remaining $k - i$ from the non-finction pile. This is the same as RHS.

RHS: All possible ways to directly choose k books from a collection of m fiction and n non-fiction books.