Some Review Topics on DEs for Final Exam in Calculus 1B

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1. Definitions and Basic Questions

Be able to **write** precise definitions for any of the following concepts (where appropriate: both in words and in symbols), to **give** examples of each definition, to **prove** that these definitions are satisfied in specific examples. Wherever appropriate, be able to **graph** examples for each definition and answer the posed questions.

(1) What is a *Differential Equation*? What is the *degree* of DE? What constitutes a *solution* to a DE?
(2) What is an *initial condition* of a DE? How does it affect the number of solutions to a DE? How many solutions of a 1st-order DE pass through a point (a, b)? Of a 2nd-order DE?
(3) What is *exponential growth* and *decay*, and what DEs model them?
(4) What is a *logistic DE*? What is its *carrying capacity* and how does it affect the solutions to the DE?
(5) What solutions do we know of the *second degree DE*: $x''(t) = -x(t)$? What is a *linear combination of solutions* to a DE?
(6) What is the *direction field* of a DE and to what order DEs is it applicable? What does it represent and how does it help us draw solutions to the DE? Can you give an example of a 1st-order DE where two solutions curves *intersect* in a point (a, b)? How about an example of a 2nd-order DE?
(7) What is an *equilibrium solution* to a DE and how do we find all such solutions? What are its *stable* and *unstable equilibria*?
(8) What is an *autonomous* DE? Why are its solutions *horizontal translates* of one another and how does this help us draw the direction field and sketch solutions? Is an autonomous equation also a *separable* equation and why?
(9) How do we solve DE's in which there is no dependent variable "on the RHS", e.g. $y' = f(x)$? Why are the solutions *vertical translates* of one another and how does this help us draw the direction field and sketch solutions? Is such a DE also a *separable* equation and why?
(10) What is a *separable* DE? How do we decide if a DE is separable?
(11) What is the *angle* between two curves at a point A and how do we compute this angle? What does it mean for two curves to be *perpendicular*? What is an *orthogonal trajectory*?
(12) What is a *tank problem* and how do we translate the given data into mathematical terms and equations?
(13) What is the *relative growth* of population $P(t)$? How do we calculate it? In what types of DE is the relative growth constant?
(14) What is the *half-life* of a population/substance? How can it be used to set up a mathematical equation?
(15) What does *Newton's Law of Cooling* say and how do we use it to set up DEs and solve problems?
(16) What does it mean to *compound interest annually, (monthly, daily), continuously*? What formulas for the amount of money in n years do we have in each case?
(17) In the *basic logistic model* with initial relative growth rate k and carrying capacity K:
  (a) How does the population $P(t)$ grow when its size is relatively small compared to K; why is the DE approximating the growth of $P(t)$ in such a case given by $dP/dt \approx kP(t)$?
  (b) How does the population $P(t)$ grow when its size is relatively close to K; why is the DE approximating the growth of $P(t)$ in such a case given by $dP/dt \approx 0$?
  (c) How does $P(t)$ behave between the two equilibria 0 and K, above K, and below 0? Why?
  (d) How do we find the *inflection points* of $P(t)$ in the logistic DE? Where are all these inflection points and why?
(18) How do the following *variations of the basic logistic model* modify the basic logistic equation:

(a) *harvesting* at a constant rate c?

(b) *minimal survival level* m for the population?

What are the equilibria in each case? Using only the given DEs, what are the direction fields and what do the solutions look like? How do we compute exactly all solutions, i.e. solve these DEs?

(19) What is a *first-order linear DE*? Is it necessarily separable? What is an *integrating factor*, how do we calculate it and how do we use it to solve the first-order linear DE?

(20) What is a *predator-prey system*? How do the two populations of predators and prey interact and how is this reflected in the corresponding system of DEs?

(21) What is the *phase DE* associated to a basic predator-prey DE system? What is the *phase plane*, a *phase trajectory* and a *phase portrait* of the DE-system?

(22) What are the *equilibrium points* of the basic predator-prey DE system and how do we find them?

(23) What are *typical phase trajectories* of the basic predator-prey model? What happens on each phase trajectory as one goes along it?

(24) In the basic predator-prey model, why is there a *phase-delay* in the predator function $W(t)$? How long is approximately such a phase delay and why?

(25) Why do we need to *modify the basic predator-prey model* in order to reflect reality? Why do we change the DE equation of the prey into a *logistic equation*? What is a *typical phase trajectory* for this *improved logistic/predator-prey model*? How many equilibrium points are there, which of them are *stable* and why?

(26) Looking at the DE system associated to *two co-inhabiting species*, how can we decide if the species are competing with each other for resources, or are in a predator-prey relationship, or are living together for mutual benefit? What would the corresponding phase trajectories and functions $R(t)$ and $W(t)$ look like in each case?

(27) What is a *2nd-order linear DE equation*? A 2nd-order linear DE equation *with constant coefficients*? A *homogeneous* 2nd order linear DE equation with constant coefficients?

(28) What is the *characteristic equation* associated to a homogeneous 2nd-order linear DE equation with constant coefficients? What is the significance of its roots?

(29) When do we say that two DE solutions are *linearly independent*? Why do we care about their linear independence?

(30) What is a *complex* number? What are the *real* and the *imaginary* parts of a complex number and how do we find them?

(31) What is i? What does i^n equal and how does this depend on the natural number n?

(32) What does e^{a+bi} equal to when $a, b \in \mathbb{R}$?

(33) What is an *initial-value problem* for a 2nd-order DE? What is the geometric interpretation of the initial-value conditions? How many initial values do we need to determine a unique solution to the DE? Are all initial-value problems for 2nd-order DEs solvable?

(34) What is a *boundary-value problem* for a 2nd-order DE? What is the geometric interperation of the boundary-value conditions? How many boundary values do we need to pose such a problem? (Bonus:) Give an example of a boundary-value problem which doesn't have a solution and explain why this is so.

(35) What is a *non-homogeneous* 2nd-order linear DE with constant coefficients? What is the *complementary equation* associated to a non-homogeneous equation and what is it used for?

(36) What is a *complementary solution* $y_c(x)$, a *particular solution* $y_p(x)$, and a *general solution* $y(x)$ to a *non-homogeneous* 2nd-order linear DE with constant coefficients? How do these solutions relate to each other?

(37) Given a non-homogeneous 2nd-order linear DE with constant coefficients $ay'' + by' + cy = G(x)$, what does it mean to make an *intelligent guess* for its particular solution $y_p(x)$? For which functions $G(x)$ does this "guessing" work well?
(38) What is the difference between the method of *undetermined coefficients* and the method of *variation of parameters*? To what type of DE equations is each applicable? What are the pros and cons of each method?
(39) What is a *spring constant* k and to what DEs is it associated? What is a *restoring force*?
(40) What is a *simple harmonic motion*? What are its *frequency*, *amplitude* and *phase angle*?
(41) What is a *damping constant* and *damping force* and to what DEs are they associated? What is *overdamping*, *critical damping* and *underdamping*, when does each occur and how does it affect the motion? What happens with the motion as $t \to \infty$?
(42) What is a *forced vibration* and an *external force*? What is *resonance* and when does it appear?
(43) What is the difference between *simple harmonic vibrations*, *damped vibrations* and *forced vibrations*? How are these differences reflected in the corresponding DEs and the graphical presentation of their solutions?
(44) (Bonus:) What is a *steady state solution* in a spring systems? How do we calculate it?
(45) (Bonus:) What is a *series solution* to a DE?
(46) (Bonus:) What does it mean *to shift the index* of a series, how and why do we change indices of series?
(47) (Bonus:) How do *recursive sequences* enter the process of solving DEs? What is the difference between recursive and direct formulas for sequences? Which of these two types of formulas is more useful for us when solving DEs and why?

2. Theorems

Be able to **write** what each of the following theorems (laws, propositions, corollaries, etc.) says. Be sure to understand, distinguish and **state** the conditions (hypothesis) of each theorem and its conclusion. Be prepared to **give** examples for each theorem, and most importantly, to **apply** each theorem appropriately in problems. The latter means: decide which theorem to use, check (in writing!) that all conditions of your theorem are satisfied in the problem in question, and then state (in writing!) the conclusion of the theorem using the specifics of your problem.

(1) **Exponential Growth/Decay.** The solutions to the DE: $P'(t) = kP(t)$ are given by $P(t) = Ce^{kt}$ for any $C \in \mathbb{R}$. If given the initial value $P(0)$, then $C = P(0)$.
(2) **Newton's Law of Cooling.** A body changes its temperature $f(t)$ to match the temperature T_s of the environment at a rate proportional to the difference of temperatures: $f'(t) = k(f(t) - T_s)$.
(3) **Limit Theorem.** $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e$.
(4) **Compounding Interest Theorems.** Given principal P_0 and yearly interest rate r (written as a decimal, e.g. $r = 0.06$), the amount of money accumulated in t years is given by
   (a) $P(t) = P_0 \left(1 + \frac{r}{n}\right)^{nt}$ if interest is compounded n times a year.
   (b) $P(t) = P_0 e^{rt}$ if interest is compounded continuously.
(5) **Logistic DE.** The solutions to the logistic DE: $\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right)$ are given by $P(t) = \frac{K}{1 + Ae^{-kt}}$. If given initial condition $P(0)$, then $P(0) = \frac{K}{1+A}$, from where $A = \frac{K}{P(0)} - 1$.

(6) **Predator-Prey DE Systems.** The prey population $R(t)$ and the predator population $W(t)$ interact according to the following system of DEs, where a, b, k, r are some positive constants:

$$\left|\begin{array}{l} \dfrac{dR}{dt} = kR(t) - aR(t)W(t) \\ \dfrac{dW}{dt} = -rW(t) + bR(t)W(t) \end{array}\right.$$

The solutions $(R(t), W(t))$ satisfy the following equation for some constant $c \geq 0$: $\dfrac{W^k R^r}{e^{aW} e^{bR}} = c$. The *phase DE* satisfied by $W(R)$ as a function of R is

$$\frac{dW}{dR} = \frac{(-r + bR)W}{(k - aW)R}.$$

Note that there is no time t in this phase DE.

(7) **Linear Combinations Theorem.** If $y_1(x)$ and $y_2(x)$ are two solutions of a homogeneous linear DE, then any linear combination of them $Ay_1(x) + By_2(x)$ for $A, B \in \mathbb{R}$ is also a solution to the DE.

(8) **Linear Independence Theorem.** If $y_1(x)$ and $y_2(x)$ are two linearly independent solutions of a homogeneous 2nd-order linear DE $P(x)y'' + Q(x)y' + R(x)y = 0$, then all solutions to the DE are given by the linear combinations of these two solutions: $y(x) = Ay_1(x) + By_2(x)$ for $A, B \in \mathbb{R}$.

(9) **Characteristic Equation Theorem.** Given a homogeneous linear DE with constant coefficients $a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$, all of its solutions of the form $y = e^{rx}$ correspond to the roots r of the associated characteristic equation $a_n r^n + a_{n-1}r^{n-1} + \cdots + a_1 r + a_0 = 0$.

(10) **Euler's Formula.** If $z = \alpha + i\beta \in \mathbb{C}$ is a complex number with real and imaginary parts α and β respectively, then $e^z = e^{\alpha+i\beta} = e^\alpha(\cos\beta + i\sin\beta)$.

(11) **General Solutions Homogeneous 2nd-Order Linear DE.** Let $ay'' + by' + cy = 0$ be a homogeneous 2nd-order linear DE with constant coefficients a, b, c; $a \neq 0$. Consider the associated characteristic equation $ar^2 + br + c = 0$ and its discriminant $D = b^2 - 4ac$.

  (a) If $ar^2 + br + c = 0$ has two different real roots $r_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (i.e. $D > 0$), then the general DE solution is given by $y(x) = Ae^{r_1 x} + Be^{r_2 x}$ for any $A, B \in \mathbb{R}$.

  (b) If $ar^2 + br + c = 0$ has one real root $r = \dfrac{-b}{2a}$ (i.e. $D = 0$), then the general DE solution is given by $y(x) = Ae^{rx} + Bxe^{rx}$ for any $A, B \in \mathbb{R}$.

  (c) If $ar^2 + br + c = 0$ doesn't have real roots (i.e. $D < 0$), then its complex roots given by the quadratic formula can be written in the form $z_{1,2} = \alpha \pm i\beta$ where $\alpha = \mathbb{R}\mathrm{e}(z)$ and $\beta = \mathbb{I}\mathrm{m}(z)$, and the general DE solution is given by $y(x) = Ae^{\alpha x}\cos(\beta x) + Be^{\alpha x}\sin(\beta x)$ for any $A, B \in \mathbb{R}$.

(12) **Initial-Value Problems.** Let $ay'' + by' + cy = 0$ be a homogeneous 2nd-order linear DE with constant coefficients a, b, c; $a \neq 0$. Then any initial-value problem $y(x_0) = y_0$ and $y'(x_0) = s$ for arbitrary fixed constants x_0, y_0, s, has a unique DE solution $y(x)$, i.e. the constants A, B from the general DE solution $y(x)$ can be calculated uniquely so that $y(x)$ satisfies the two given initial values.

(13) **Uniqueness of Boundary-Value Solutions (Bonus).** Let $ay'' + by' + cy = 0$ be a homogeneous 2nd-order linear DE with constant coefficients a, b, c, $a \neq 0$. A boundary-value problem $y(x_0) = y_0$ and $y(x_1) = y_1$ for arbitrary fixed constants x_0, y_0, x_1, y_1 will have a unique solution in all cases except when the characteristic equation has complex roots $z = \alpha + i\beta$ **and** $x_2 - x_1 = \dfrac{k\pi}{\beta}$ for some integer k. In the latter case, the boundary-value problem has infinitely many solutions if in addition $y_2 = (-1)^k e^{\frac{\alpha}{\beta}k\pi}$, or else it has no solutions. [1]

[1] Try to prove this theorem as a challenging bonus exercise. You will need to come up with a condition for a linear system of 2 equations with 2 unknowns to fail to have a unique solution. If and when you take linear algebra, you will associate this problem with a 2×2 matrix and will investigate if its determinant is 0 or not.

(14) **General Solutions to Non-Homogeneous 2nd-Order Linear DE.** Given the DE $ay'' + by' + cy = G(x)$, its general solution is given by $y(x) = y_p(x) + y_c(x)$ where $y_p(x)$ is a particular solution to the DE, and $y_c(x)$ is the general solution to the complementary homogeneous equation. Note that in this formula $y_p(x)$ is fixed and $y_c(x)$ varies over all complementary solutions.

(15) **Theorem on Method of Undetermined Coefficients: Intelligent Guessing for $y_p(x)$.** Consider the DE $ay''+by'+cy = G(x)$ where $G(x)$ equals polynomial $P(x)$, e^{rx}, $\sin(mx)$, $\cos(kx)$, or any function which can be obtained from these basic functions via addition, taking linear combinations, or multiplication. Then the following guesses for a particular solution $y_p(x)$ to the DE always work:

$G(x)$	guess for $y_p(x)$
1. poly $P(x)$ of degree n	same type as $G(x)$: poly $Q(x)$ of degree n, i.e. $y_p(x) = Q(x) = c_n x^n + c_{n-1}x^{n-1} + \cdots + c_2x^2 + c_1x + c_0$
2. e^{rx}	same type as $G(x)$: $y_p(x) = ce^{rx}$
3. $\sin(mx)$	$y_p(x) = c_1 \sin(mx) + c_2 \cos(mx)$
4. $\cos(kx)$	$y_p(x) = c_1 \sin(kx) + c_2 \cos(kx)$
5. $G_1(x) + G_2(x)$ e.g. $7e^{3x} - 2\sin(5x)$	sum of all guesses for $G_1(x)$ and for $G_2(x)$ e.g. $y_p(x) = c_1e^{3x} + c_2 \sin(5x) + c_3 \cos(5x)$
6. $G_1(x) \cdot G_2(x)$ e.g. $(x^2 + 1)\cos(2x)$	products of all guesses for $G_1(x)$ and for $G_2(x)$ e.g. $y_p(x) = (c_2x^2 + c_1x + c_0)(b_1 \sin(2x) + b_2 \cos(2x)) =$ $d_1x^2 \sin(2x) + d_2x \sin(2x) + d_3 \sin(2x) + d_4x^2 \cos(2x)$ $+ d_5x \cos(2x) + d_6 \cos(2x)$
7. some term of $y_p(x)$ participates in $y_c(x)$	multiply this term by x (or by x^2, if necessary) in order to avoid repetition with terms of $y_c(x)$.

(16) **Method of Variation of Parameters:** Consider the DE $ay'' + by' + cy = G(x)$ where $G(x)$ is any continuous function on some interval J. The following procedure always produces the general DE solutions, provided we can integrate at the end.[2]

(a) Find the complementary solution $y_c(x) = Ay_1(x) + By_2(x)$ for some linearly independent solutions $y_1(x)$ and $y_2(x)$, $A, B \in \mathbb{R}$.

(b) For the particular solution, set $y_p(x) = f(x)y_1(x) + g(x)y_2(x)$ for some unknown functions $f(x)$ and $g(x)$, continuous on J.

(c) Set up the following system with unknowns $f'(x)$ and $g'(x)$:

$$\left| \begin{array}{l} f'(x)y_1(x) + g'(x)y_2(x) = 0 \\ f'(x)y_1'(x) + g'(x)y_2'(x) = G(x) \end{array} \right.$$

(d) Solve the system for $f'(x)$ and $g'(x)$ via elimination of one of them. Thus, arrive at $f'(x) = S(x)$ and $g'(x) = Q(x)$ for some functions $S(x)$ and $Q(x)$, continuous on J.

(e) Integrate to solve for $f(x)$ and $g(x)$: $f(x) = \int S(x)dx$ and $g(x) = \int Q(x)dx$. (No constants C are necessary here.)

(f) Put together $y_p(x) = \left(\int S(x)dx\right) y_1(x) + \left(\int Q(x)dx\right) y_2(x)$.

(g) Put together $y(x) = y_p(x) + y_c(x)$.

[2] We may view the Method of Variation of Parameters as a 2-dimensional version of the method for solving Linear DE's via an integration factor. Further, Linear Algebra will put the system in part (c) into the matrix form:

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \cdot \begin{pmatrix} f' \\ g' \end{pmatrix} = \begin{pmatrix} 0 \\ G \end{pmatrix}$$

There are standard methods for solving such systems in Linear Algebra. As a bonus "food for thought": looking at the form of this matrix equation, can you guess the corresponding system for a non-homogeneous 3rd-order linear DE? nth-order?

(17) **Simple Harmonic Motion** is described via the DE $mx''(t) + kx(t) = 0$, where k is the spring constant, m is the mass of the spring, and $x(t)$ is the units of stretching/contracting of the spring relative to its natural state, at time t. The general solution of the DE is

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t) = A\cos(\omega t + \delta),$$

where $\omega = \sqrt{k/m}$ is the frequency, $A = \sqrt{c_1^2 + c_2^2}$ is the amplitude, and $\sin\delta = -c_2/A, \cos\delta = c_1/A$ give the phase angle δ.

(18) **Damped Vibration** is described via the DE $mx''(t)+cx'(t)+kx(t) = 0$, where m is the mass of the spring, c and k are the damping and spring constants, and $x(t)$ is the units of stretching/contracting of the spring relative to its natural state, at time t. The general solution of the DE depends on the roots of the quadratic characteristic equation $mr^2 + cr + k = 0$: $r_{1,2} = \frac{-c+\sqrt{c^2-4mk}}{2m}$.

(a) If $D = c^2 - 4mk > 0$, the general solution to the DE is given by:

$$x(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

In this case, both roots $r_{1,2} < 0$ (why?), and hence $\lim_{t\to\infty} x(t) = 0$, i.e. the oscillations in the original harmonic motion die out fast and overdamping occurs.

(b) If $D = c^2 - 4mk = 0$, the general solution to the DE is given by:

$$x(t) = c_1 e^{rt} + c_2 t e^{rt},$$

where $r = r_1 = r_2 = -c/(2m) < 0$. Again, one can calculate that $\lim_{t\to\infty} x(t) = 0$ (why?), i.e. the oscillations of the original harmonic motion die over time but slower than in the previous case. The damping is just enough to surpress the oscillations: we say that critical damping occurs.

(c) If $D = c^2 - 4mk < 0$, the general solution to the DE is given by:

$$x(t) = c_1 e^{-c/2m} \cos\omega t + c_2 e^{-c/2m} \sin\omega t = Ae^{-c/2m}\cos(\omega t + \delta),$$

where the two roots are $r_{1,2} = -\frac{c}{2m} \pm \omega i$, and $\omega = \frac{\sqrt{4mk-c^2}}{2m}$. In this case $\lim_{t\to\infty} x(t) = 0$ too (why?), but the vibrations survive with smaller and smaller amplitudes: there are still well-defined oscillations no matter how much time has elapsed.

(19) **Forced Vibration** is described via the non-homogeneous DE $mx''(t)+cx'(t)+kx(t) = F(t)$, where m is the mass of the spring, c and k are the damping and spring constants, $F(t)$ is an external force, and $x(t)$ is the units of stretching/contracting of the spring relative to its natural state, at time t. The complementary solutions $x_c(t)$ of the DE depend on the roots of the quadratic characteristic equation $mr^2 + cr + k = 0$ (covered in the three cases above), and a particular solution $x_p(t)$ is obtained via the methods of undetermined coefficients or variation of parameters.

(20) **Forced Vibration without Damping** commonly occurs when the external force $F(t) = F_0\cos(\omega_0 t)$ is a continuous periodic function, and $c = 0$. Then the general solution to the DE

$$mx''(t) + kx(t) = F_0\cos(\omega_0 t)$$

is given by

$$x(t) = c_1\cos\omega t + c_2\sin\omega t + \frac{F_0\cos(\omega_0 t)}{m(\omega^2 - \omega_0^2)}.$$

Correspondingly, resonance occurs when $\omega_0 = \omega$, i.e. the original spring motion and the forced vibration have the same frequency.

3. Problem Solving Techniques

(1) **Without solving a 1st-order DE, how do we use the DE to analyze its solutions and draw sketches of these solutions?** A first-order DE is of the form: $y'(x) = f(x, y)$. We investigate the sign of the derivative y' by finding out for which x's and y's $f(x, y)$ is positive, zero or negative. Correspondingly, the solutions $y(x)$ will be increasing, leveling or decreasing at the points (x, y). This gives us a rough idea of the behavior of each solution and helps us sketch representative solutions. Try this on $y' = x - y$, on $y' = x - y^2$, and on the logistic equation $dP/dt = 2P(1 - P/800)$.

(2) **How do we use the 1st-order DE to draw its direction field?** Compared to the method above, this is a slightly more precise way of sketching the solutions to the DE. For various values of x and y, we calculate $RHS = f(x, y)$, and hence we have calculated the $LHS = y'$. These calculations can be recorded in a table:

x			$\cdots$
y			$\cdots$
y'			$\cdots$

For each of these points (x, y), we draw a small segment centered at (x, y) with slope $= f(x, y)$. The collection of these small segments comprise the direction field of the DE. By following the directions of the segments we can trace rough sketches of solutions to the DE.

(3) **How to choose the points (x, y) at which to calculate y' depends on the given DE.**

(a) If given **autonomous DE**: $y'(x) = f(y)$, we know that the solutions and their slopes do **not** depend on the variable x, i.e. the solutions are horizontal translates of each other, and the slopes are equal along the same horizontal line. Thus, it suffices to find the slopes only along one vertical line, say, $x = 1$. The slope calculations can be recorded in a table of the form:

x	$*$	$*$	$*\cdots$
y			$\cdots$
y'			$\cdots$

It remains to draw these slopes on the vertical line $x = 1$ and to draw the same slopes along all horizontal lines. Try this on $y' = y(1 - y/2)(1 + y/2)$.

(b) If given DE **without dependent variable on** RHS: $y'(x) = f(x)$, we know that all solutions are of the form $y(x) = \int f(x)dx$, thus, having found one solutions $y(x) = F(x)$, all other solutions are vertical translates of it: $y(x) = F(x) + C$. Thus, it suffices to find the slopes only along one horizontal line, say, $y = 1$. The slope calculations can be recorded in a table of the form:

x			$\cdots$
y	$*$	$*$	$\cdots$
y'			$\cdots$

It remains to draw these slopes on the horizontal line $y = 1$ and to draw the same slopes along all vertical lines. Try this on $y' = x^2$.

(c) If given a **general first-order DE**: there is no recipe that works in all cases. In class, we did two examples of this type: $y' = x - y$ and $y' = y + xy$. Several things that could be done are:

(i) look for the points where $y' = 0$ and solve for x and y. In the first example, $y' = 0$ happens along the line $x = y$, while in the second example: along the lines $y = 0$ and $x = -1$. (Why?)

(ii) look for equilibrium solutions: plug in $y = K$ and $y' = 0$ and solve for x. The first example doesn't have equilibrium solutions, but the second example has equilibrium $y = 0$.

(iii) if possible, find where $y' = 1, -1, 2, -2$ and some other convenient slopes. In the first example, we can actually solve for **all** wanted slopes $y' = c$: this happens along the line

$y = x - c$. In the second example, this happens along the curves $y = c/(1+x)$. (Why?) Draw a few representatives of such curves.

(iv) analyse when $y' > 0$, < 0. In the first example, all points above the line $x = y$ correspond to negative slopes, and all points below the line $x = y$ correspond to positive slopes. Thus, solutions above $x = y$ are decreasing, and below $x = y$ are increasing. In the second example, if we form a coordinate system with axis the usual x-axis and $x = -1$, then the slopes are positive and solutions are increasing in the "first" and "third quadrants", and the slopes are negative and solutions are decreasing in the "second" and "forth quadrants".

(v) All of the above observations can be summarized in tables, the directions fields can be drawn, and representative solutions sketched.

(4) **How do we solve separable DE?**

(a) The first step of turning a DE into the form of a separable DE is sometimes the toughest to see. If you believe that a DE is separable, put all terms involving y' to LHS, all other terms to the RHS. Factor out y' on the LHS: $\square_1 \cdot y' = \square_2$, and solve for y': $y' = f(x,y)$. Now comes the moment of truth: in a separable DE one can factor $f(x,y)$ as a product of a function of x and another function of y: $f(x,y) = g(x) \cdot h(y)$. If this is not possible, then your DE is **not** separable, and you must try another method.

(b) If it is possible to separate the x and the y's on RHS, then we proceed in the standard way:

$$\frac{dy}{dx} = g(x)h(y) \;\Rightarrow\; \frac{dy}{h(y)} = g(x)dx \;\Rightarrow\; \int \frac{1}{h(y)}dy = \int g(x)dx.$$

After integrating wrt to y on LHS, and wrt x on RHS (i.e. here all integration methods we learned this semester kick in), we get rid of all integrals (and derivatives) to arrive at an equation:

$$F(y) = G(x) + C \text{ for all } C \in \mathbb{R}.$$

(c) Solving for y is the next moderately difficult step here, for it may involve raising e to both sides, or taking ln on both sides, or taking some radical on both sides, or some other algebraic technique. If $F(y)$ happens to be a quadratic polynomial, then the quadratic formula will give the solutions for y. Many calculational mistakes are possible here, especially forgetting absolute values, $+/-$ signs, forgetting about the constant C on RHS, etc.

(d) After the dust settles down, we have $y =$ some function or functions of x. We analyse here the involved constant and simplify the expression that involves this constant. A common situation is $y = \pm e^C \cdot H(x) + Q(x)$, which can be rewritten as $y = BH(x) + Q(x)$ for all $B \neq 0$. However, we divided upstairs by all sorts of functions, so we may have lost a solution that corresponds to $B = 0$ in it. One needs to substitute $y = 0 + Q(x) = Q(x)$ and $y' = Q'(x)$ in the original DE to check if this is indeed a solution, and if indeed we can write $y = BH(x) + Q(x)$ for all $B \in \mathbb{R}$.

Another common situation is $y = \pm\sqrt{H(x) + C}$, and a common error is to pull the C from under $\sqrt{\;\;}$: $y = \pm\sqrt{H(x)} \pm C = \pm\sqrt{H(x)} + B$ is **FALSE**! One cannot split square roots this way since $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$. Instead, we leave $y = \pm\sqrt{H(x) + C}$, and comment that $H(x) + C \geq 0$, i.e. $C \geq -H(x)$.

(5) **How do we find all orthogonal trajectories to a family $\mathcal{F}$ of curves?**

(a) Such a family of curves is given usually via an equation $y = f(x,k)$ where for each fixed k we get a different curve of the family. The class example was $y = 1/(x+k)$ for all $k \in \mathbb{R}$. However, the family $\mathcal{F}$ may be described in words, e.g. all lines through (1,-4), or all circles centered at (-3,7), all lines parallel to $y = 2x + 9$, etc. Here one needs to come up with the defining equation for the curves in $\mathcal{F}$. In class, we discussed all circles through the origin: $x^2 + y^2 = k$ for $k \geq 0$, and all lines through the origin: $y = kx$ for all k. Try to come up with the equations for $\mathcal{F}$ in the other examples above.

(b) We create a first-order DE from the given equations for $\mathcal{F}$ by differentiating wrt x. We obtain something like $\dfrac{dy}{dx} = g(x, k)$. In the class example of $y = 1/(x + k)$, the DE we obtained was: $\dfrac{dy}{dx} = \dfrac{-1}{(x + k)^2}$. For the family of circles through the origin, one needs to differentiate *implicitly*: $2x + 2y \cdot y' = 0$ so that $y' = -x/y$.

(c) Next we reciprocate and negate the RHS of the DE in order to find the slope of the orthogonal trajectory: $\dfrac{dy}{dx} = \dfrac{1}{g(x, k)}$. In the class example, we obtained $\dfrac{dy}{dx} = (x + k)^2$. For the family of circles through the origin, one will get $dy/dx = y/x$.

(d) Solve for k from the original equation for $\mathcal{F}$ and substitute into the new DE in order to get rid of k. In the class example, $k = \frac{1}{y} - x$, so the new DE becomes after several algebraic manipulations: $\dfrac{dy}{dx} = y^2$. For the family of circles through the origin, we don't have any k in the new DE, so we are happy about it.

(e) Solve the new DE using any methods we learned. As we see, inevitably, a constant C will arise in the solutions, so we will indeed get a **family** $\mathcal{G}$ of curves, each of which is orthogonal to each of the original curves; in rather illuminating notation: $\mathcal{G} \perp \mathcal{F}$.
In the class example, we arrived at $y = \sqrt[3]{3x + 3C}$, which we relabeled $y = \sqrt[3]{3x + B}$ for any $B \in \mathbb{R}$. For drawing purposes, we solved here for $x = y^3/3 - C$ for all C, drew sketches of curves in the original family $\mathcal{F}$ and in the new family $\mathcal{G}$, and verified that the curves from the two families were indeed orthogonal wherever they intersected.
For the circles through the origin, check that the orthogonal trajectories are given by $y = Bx$ for all B, i.e. these are, as expected, all lines through the origin.

(6) **How do we solve "word problems"?** There are several parts in solving word problems:

(a) Translate the word problem into a mathematical set up. This is the hardest part! First one has to decide a rough outline of what mathematical problem this translates into, e.g. DE with exponential growth/decay, logistic DE, some other DE (linear first-order, second order, something else), cooling/warming problem, variation on logistic DE (harvesting, minimal survival level, etc), predator-prey system, some other system of DEs, etc... Even if you guessed the wrong type of mathematical problem from the beginning, you'll have some frame for action in your mind and you will be able to start the translation: later on, after you have input all data you will correct yourself if indeed you guessed wrong. Decide which function(s) are the most important in the problem: which function(s) describes the process you are given, and give a name to this function(s). Write out all given data into mathematical terminology.

(b) Solve the mathematical problem using the methods we learned. This is a relatively easy part, as long as the problem is translated correctly.

(c) Translate back your mathematical results into the set up of the original problem. This part is the easiest, but usually forgotten by happy people who already successfully went through the two parts above.

(7) **How do we set up and solve exponential growth/decay problems?** The problem that will eventually translate into a basic exponential growth/decay set up involves only one population $P(t)$, with some characteristics that can be encoded in various ways, e.g. "$P(t)$ grows proportionately to its size", "half-life of $P(t)$ is 5 days", "the relative growth of $P(t)$ is constant", "interest rate has been accumulating continuously", etc. In all such cases, start with the basic DE $dP/dt = kP(t)$. You have essentially two ways of proceeding:

(a) input the given data into the equation now (e.g. $P(0) = k$, or $P'(5) = 4$, or $P(5) = 2P_0$, or something else) and solve for k or whatever else left;
(b) or solve the given equation: $P(t) = Ce^{kt}$, then input the given data, and finally solve for k, C or something else that is left.

Finally, look at the questions they are asking you, plug into your resulting equation $P(t) = Ce^{kt}$, or into the original equation $dP/dt = kP(t)$ and solve for whatever you need to.

(8) **How do we set up and solve "tank problems"?** One needs to recognize the basic situation of a solution, say, water mixed with salt, where the concentration of salt changes due to incoming water/salt and outgoing water/salt. The basic assumption is that the amount of incoming solution equals the amount of outgoing solution per unit of time, so that the tank (lake, swamp, container, etc.) has the same amount of solution at any time. Another fairly standard assumption is that the solution in the tank is being stirred constantly (say, a waterfall!) so that the concentration of salt everywhere in the tank is the same.
(a) Depending on the problem, one usually (but not always) labels $y(t)$ to be the total amount of salt in the tank at time t.
(b) Set up the rate of change DE: $dy/dt =$ (rate-in)-(rate-out).
(c) Come up with formulas for the rate-in and the rate-out of the salt. Usually, but not always,

$$\text{(rate-in)} = \begin{pmatrix} \text{concentration of salt in} \\ \text{incoming solution at time t} \end{pmatrix} \cdot \begin{pmatrix} \text{amount of incoming solution} \\ \text{per unit time} \end{pmatrix}$$

$$\text{(rate-out)} = \begin{pmatrix} \text{concentration of salt} \\ \text{in tank at time t} \end{pmatrix} \cdot \begin{pmatrix} \text{amount of outgoing solution} \\ \text{per unit time} \end{pmatrix}$$

Usually, the problems supplies data to figure out all 4 quantities above.
(d) The resulting DE is usually of the form: $dy/dt = A - By(t)$ for some constants A and B. In such a case, one can solve the DE either as a separable or a 1st-order linear equation. You should try doing it both ways for the class example $dy/dt = 7 - 0.07y(t)$ and compare your answers.
In more complex problems, the resulting DE will have *functions* $A(t)$ and $B(t)$: $dy/dt = A(t) - B(t)y(t)$, in which case one solves the DE as a 1st-order linear equation.
(e) Now that we have come up a formula for $y(t)$, one needs to input any remaining data and answer the given questions. Note that occasionally it is easier to input the data into the DE to answer some of the questions, but, by-and-large, you will be answering the questions using the resulting formula for $y(t)$.

(9) **How do we set up and solve "cooling/warming" problems?** One needs to recognize the basic situation of a body cooling down or warming up due to being placed in a colder/warmer environment. Start from the basic equation

$$f'(t) = k(f(t) - T_s),$$

and substitute the given data: this could be T_s, or k, or something else. Then we solve the DE by following the steps below:
(a) Substitute $y(t) = f(t) - T_s$ in order to turn the equation into a well-known exponential growth/decay DE: $y'(t) = f'(t)$, so that $y'(t) = ky(t)$.
(b) Apply the formula for exponential growth/decay: $y(t) = Ce^{kt}$.
(c) Substitute back to find $f(t) = y(t) + T_s = Ce^{kt} + T_s$.
(d) Finally, use the remaining data in the problem to find anything else that has remained unknown here: C, k, T_s.
(e) Use your newly found formula for $f(t)$ to answer the questions posed in the problem.

Note that, if we draw representative solutions, we will notice a picture similar to exponential growth or decay, except shifted vertically by T_s, and that the equilibrium solution here is $y = T_s$. Why?

(10) **How do we use the formula $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ to find limits in similar problems?** The idea is to make a suitable substitution and reduce all other problems to this basic one. Below is one such example, which by no means exhausts all possible situations:

$$\lim_{n\to\infty}\left(1+\frac{4}{7n}\right)^{-3n} = \lim_{n\to\infty}\left(\left(1+\frac{1}{\frac{7n}{4}}\right)^{\frac{7n}{4}}\right)^{-\frac{3\cdot 4}{7}} = \lim_{m\to\infty}\left(\left(1+\frac{1}{m}\right)^{m}\right)^{-\frac{12}{7}} = e^{-\frac{12}{7}}$$

Here we substituted $m = 7n/4$.

(11) **How can we solve problems about doubling money and interest rates?** On the surface, some problems may seem to give you insufficient data for solution. For example,

"How long will it take an investment to double in value if the interest rate is 10% *compounded continuously?"*

We don't know the principal P_0, but we can still use P_0 in letter notation in our equations. So, set up the basic continuous compounding equation: $P(t) = P_0e^{0.1t} = 2P_0$. The latter $2P_0$ simply means that at time t we will have twice the principal amount, as required in the problem. From here P_0 cancels, and we solve for $t = (\ln 2)/0.1 = 10\ln 2 \approx 6.9$ years.

"What is the equivalent annual interest rate?" In other words, if the interest rate is accumulated annually, what interest rate r would insure that the money will double in $10\ln 2$ years?

Set the equation for annual compounding $P(t) = P_0(1+r)^t = 2P_0$, plug $t = 10\ln 2$, and solve for r:

$$(1+r)^{10\ln 2} = 2 \;\Rightarrow\; \ln\left((1+r)^{10\ln 2}\right) = \ln 2 \;\Rightarrow\; 10\ln 2\ln(1+r) = \ln 2 \;\Rightarrow\; \ln(1+r) = 0.1$$

$$\Rightarrow\; 1+r = e^{0.1} \;\Rightarrow\; r = e^{0.1}-1 \approx 0.105, \text{ i.e. } r \approx 10.5\%.$$

(12) **How do we translate and solve problems in the logistic model set up?** The Logistic set up requires a population $P(t)$ whose growth is approximately exponential $P'(t) \approx kP(t)$ when $P(t)$ is small, but is restricted by a carrying capacity of K when $P(t)$ is big: $P'(t) \approx 0$. Once we have identified this set up, we use the basic logistic equation

$$\frac{dP}{dt} = kP\left(1-\frac{P}{K}\right).$$

One can either solve it from scratch as a separable equation, or use the corresponding answer formula for $P(t)$ from the Theorem section.

(13) **How do we set up and solve "minimal survival level" problems within the logistic set up?** In addition to the basic logistic set up, one also needs a minimal survival level m for the populaton $P(t)$. The corresponding DE is $\frac{dP}{dt} = kP\left(1-\frac{P}{K}\right)\left(1-\frac{m}{P}\right)$, which, after multiplication of the first and last term, becomes

$$\frac{dP}{dt} = k\left(1-\frac{P}{K}\right)(P-m).$$

This is a separable DE, which has to be solved from scratch. Note that the answer will involve two equilibria: $P(t) \equiv K$(stable) and $P(t) \equiv m$ (unstable).

(14) **How do we set up and solve "harvesting" problems within the logistic set up?** Such problems are a modification of the basic logistic set up. We have the basic logistic situation of a population $P(t)$ with carrying capacity K, initial relative growth rate k, and the modifying additional harvesting rate of c per unit time:

$$\frac{dP}{dt} = kP(1-\frac{P}{K}) - c$$

(a) Put the harvesting rate c to LHS and subsume it into the derivative:
$$\frac{dP}{dt} + c = kP(1 - \frac{P}{K}) \Leftrightarrow \frac{d(P+c)}{dt} = kP(1 - \frac{P}{K}).$$
(b) It is time to substitute in order to reduce to a simpler logistic model. Thus, set $y(t) = P(t) + c$ so that $P = y - c$ and substitute. The DE becomes
$$\frac{dy}{dt} = k(y-c)(1 - \frac{y-c}{K}) \Leftrightarrow \frac{dy}{dt} = \frac{k}{K}(y-c)(K+c-y).$$
(c) Looking closely into this equation, we see that it is nothing else but a "minimal survival level" problem, where the carrying capacity is $K + c$, and the minimal survival level is c. The last makes total sense since $y(t) = P(t) + c \geq 0 + c = c$. Thus, we proceed from here as in "minimal survival level problems" above: we solve the DE as a separable equation.
(d) Finally, we have found a formula for the solution $y(t)$. However, this is **not** the original population. We substitute back to find $P(t) = y(t) + c$ and answer all given questions.

(15) **How do we solve first-order linear DE?** The solutions to $y' + P(x)y = Q(x)$ where $P(x)$ and $Q(x)$ are continuous functions on some interval J can be found via
(a) calculating one integration factor $I(x) = e^{\int P(x)dx}$;
(b) multiplying both sides of the DE by $I(x)$: $I(x)y' + I(x)P(x)y = I(x)Q(x)$;
(c) recognizing the LHS as the derivative of a product: $(I(x)y)' = I(x)Q(x)$;
(d) integrating both sides wrt x: $I(x)y = \int I(x)Q(x)dx$;
(e) solving for $y = \frac{1}{I(x)} \int I(x)Q(x)dx$.
The only formula here to remember and use in problems is for the integrating factor $I(x)$ in (a). The rest should be viewed as an algorithm and you should go carefully through each of the above steps in solving every 1st-order linear DE.

(16) **How to solve DE with the method of undetermined coefficients?** Consider $ay'' + by' + cy = G(x)$ where $G(x)$ is equals $P(x)$ (polynomial), e^{rx}, $\sin(mx)$,$\cos(kx)$, or any function which can be obtained from these basic functions via addition, taking linear combinations, or multiplication. To solve the DE, we proceed as follows:
(a) Find first $y_c(x)$.
(b) Make an intelligent guess for $y_p(x)$ (cf. Theorem on Method of Undetermined Coefficients) and write out $y_p(x)$ as a linear combination of your guessed terms with unknown coefficients.
(c) Substitute $y_p(x)$ into the DE, perform differentiation and algebraic operations, simplify, and group alike terms on LHS. Multiply out and group similarly $G(x)$ on RHS.
(d) Equate coefficients on both sides to obtain a system of linear equations.
(e) Solve the system for your unknown coefficients.
(f) Write out a particular solution $y_p(x)$ using your guess and your newly-found coefficients.
(g) Write out the general solution $y(x) = y_p(x) + y_c(x)$.

(17) **(Bonus) How to find solutions to DE via series?** Given a DE, suppose we want to find a solution $y(x)$ close to $x = 0$.
(a) Set up a power series centered at $a = 0$ to represent the wanted solution: $y(x) = \sum_{n=0}^{\infty} c_n x^n$.
(b) TT$'$ several times to find power expression for $y'(x) = \sum_{n=1}^{\infty} nc_n x^{n-1}$, $y''(x) = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$, etc. Note that it is silly (but possible and sometimes necessary) to start the series for y' at $n = 0$ and the series for y'' at $n = 0, 1$, because the corresponding first terms are 0.
(c) Substitute your power series for y, y', y'', etc. into the DE, and perform the corresponding algebraic manipulations.

(d) You will end up the sum of several power series written, possibly, in various powers of x, e.g. x^n, x^{n-1}, x^{n+1}, etc. You will need to make all powers of x the same, i.e. choose an "anchor" power, e.g. x^n, and shift the index of any series so as to match the power x^n. For example, $\sum_{n=6}^{\infty} c_{n-2}(n+1)nx^{n-3}$ would require shift of powers $n-3 \mapsto n$, i.e. the shift of index $n \mapsto n+3$; Thus, substitute in this series $n+3$ everywhere for n, and note that the starting index is not anymore $n=6$ but $n+3=6$, i.e $n=3$: $\sum_{n=3}^{\infty} c_{n+1}(n+4)(n+3)x^n$. It is always a sign of good upbringing to check that the first terms or the original and of the new series coincide after the index shift: for $n=6$ the original series produces the term $c_4 \cdot 7 \cdot 6 \cdot x^3$, and for $n=3$ the new series produces the same term $c_4 \cdot 7 \cdot 6 \cdot x^3$.

(e) Now you are ready to add up and put together all resulting series: they are all written in the same power of x, say, x^n. The only trouble may come from different starting indices of the series: these all need to be matched, however, we cannot shift indices anymore (or else we will mess up our powers of x). Note that some series may be started earlier or later since some of their first terms are 0, e.g. $\sum_{n=1}^{\infty} nc_nx^n = \sum_{n=0}^{\infty} nc_nx^n$, and $\sum_{n=2}^{\infty} n(n-1)c_nx^n = \sum_{n=1}^{\infty} n(n-1)c_nx^n = \sum_{n=0}^{\infty} n(n-1)c_nx^n$.

However, sometimes this trick won't be sufficient to equalize the starting indices. So, take the highest starting index, e.g. $n=3$, peel off for all your series the first terms for $n<3$, and start all series at $n=3$. For example,

$$\begin{aligned}
& \sum_{n=1}^{\infty} nc_nx^n - \sum_{n=2}^{\infty} n(n+1)c_{n-2}x^n + \sum_{n=3}^{\infty} c_{n+1}(n+4)(n+3)x^n \\
= \quad & c_1x + 2c_2x^2 + \sum_{n=3}^{\infty} nc_nx^n - 6c_0x^2 - \sum_{n=3}^{\infty} n(n+1)c_{n-2}x^n + \sum_{n=3}^{\infty} c_{n+1}(n+4)(n+3)x^n \\
= \quad & c_1x + (2c_2 - 6c_0)x^2 + \sum_{n=3}^{\infty} \big[nc_n - n(n+1)c_{n-2} + c_{n+1}(n+4)(n+3)\big]x^n
\end{aligned}$$

(f) Equate the coefficients of x^n on both sides of the DE, not forgetting about any special "peeled off" first terms as in the previous step. For example, if the last sum above were supposed to equal x, then $c_1 = 1$, $2c_2 - 6c_0 = 0$, and $nc_n - n(n+1)c_{n-2} + c_{n+1}(n+4)(n+3) = 0$ for all $n \geq 3$. Solve each equation for the term with highest index, e.g.

$$c_2 = 3c_0, \; c_{n+1} = \frac{n}{(n+4)(n+3)}\big[-c_n + (n+1)c_{n-2}\big] \text{ for all } n \geq 3.$$

(g) If the problem gives you initial values, this is the time to input this extra data. For example, if in the above problem we had $y(0)=0$, this means $y(0) = c_0 = 0$, so we have $c_1 = 1, c_0 = c_2 = 0$. We need to roll up the recursive formula for various n's to find patterns for the remaining coefficients:

$$\begin{aligned}
n=2: \quad & c_3 = 0 \\
n=3: \quad & c_4 = \frac{3}{7\cdot 6}\big(-c_3 + 4c_1\big) = \frac{2}{7} \\
n=4: \quad & c_5 = \frac{4}{8\cdot 7}\big(-c_4 + 4c_2\big) = -\frac{1}{49} \\
n=5: \quad & c_6 = \frac{5}{9\cdot 8}\big(-c_5 + 4c_3\big) = \frac{5}{9\cdot 8\cdot 7^2}
\end{aligned}$$

In this particular problem, I don't see a ready nice pattern developing (but check it out!) The important part of this is to learn how to manipulate power series in order to find solutions to DEs.

4. Summary

Note that series solutions to DEs interlink the following basic concepts which we studied this semester:

(1) Differential equations;
(2) Power series;
(3) Term-by-term differentiation of power series;
(4) Change of indices of series;
(5) Equating coefficients in polynomial or power series equations;
(6) Recursive sequences and recursive formulas;
(7) Pattern chasing in recursive sequences;
(8) Direct formulas for recursive sequences;
(9) Recognition of functions given by their Taylor series as solutions to DEs.

5. Useful Formulas and Miscellaneous Facts

(1) $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.
(2) $c_1\cos\alpha + c_2\sin\alpha = A\cos(\alpha+\delta)$, where $A = \sqrt{c_1^2 + c_2}$, $A = \sqrt{c_1^2 + c_2^2}$, $\sin\delta = -c_2/A, \cos\delta = c_1/A$.
(3) $ax^2 + bx + c = 0$:
   (a) $x_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.
   (b) If $b^2 - 4ac = 0$, $x_1 = x_2 = \dfrac{-b}{2a}$.
   (c) If $b^2 - 4ac < 0$, $x_{1,2} = \alpha \pm \beta i$ for $\alpha = -b/2a, \beta = \dfrac{\sqrt{4ac - b^2}}{2a}$.
(4) $(n-1)!\, n = n!$, $n!\,(n+1)(n+2) = (n+2)!$
(5) $1 \cdot 3 \cdot 5 \cdots (2n+1) = \dfrac{(2n+1)!}{2 \cdot 4 \cdot 6 \cdots (2n)} = \dfrac{(2n+1)!}{2^n n!}$.
(6) $\ln(e^x) = x$ and $e^{\ln y} = y$ for all $x \in \mathbb{R}$ and all $y \geq 0$.
(7) $\lim_{n\to\infty}\left(1 + \dfrac{r}{n}\right)^n = e^r$.
(8) $(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + \sin n\alpha (= (e^{\alpha i})^n = e^{n\alpha i})$.
(9) $e^{\pi} + 1 = 0$.

6. Review Problems

All HW problems, all Quiz problems, all classnotes, and all Review Sections in textbook.

7. Cheat Sheet and Studying for the Exam

For the exam, you are allowed to have a "cheat sheet" - *two pages* of one regular 8×11 sheet. You can write whatever you wish there, under the following conditions:

- The whole cheat sheet must be **handwritten by your own hand**! No xeroxing, no copying, (and for that matter, no tearing pages from the textbook and pasting them onto your cheat sheet.)
- Any violation of these rules will disqualify your cheat sheet and may end in your own disqualification from the midterm. I may decide to check some cheat sheets, so let's play it fair and square. :)
- Don't be a **freakasaurus**! Start studying for the exam several days in advance, and prepare your cheat sheet at least 2 days in advance. This will give you enough time to become familiar with your cheat sheet and be able to use it more efficiently on the exam.
- **Do NOT overstudy on the day of the exam!! More than 3 hours of math study on the day before the final exam is counterproductive! No kidding!**