PHYS1110D – Engineering Physics: Mechanics and Thermodynamics

Tutorial Problems for Week 10: Rotation and Kepler’s Second Law

**Problem 1 – Integrated Application of Conservation Laws**

Diagram

Description automatically generated

A uniform rod (mass , length ) is at rest on a frictionless horizontal plane. A small ball of mass with initial velocity towards right hits the rod at distance to the rod center (see the figure). No energy is lost in the collision. As a result, the rod starts moving to the right, and rotating around its center. Please find the following quantities after the collision:

1. The velocity of the ball;
2. The velocity of the center of mass (CM) of the rod;
3. The angular velocity of the rotation of the rod around its center.

**Solution:**

We write down all the conserved quantities of the system.

* Energy
* Momentum (the useful component is along the -direction)

*(Although we draw in the -direction, we write . If the answer turns out to be negative, then our drawing is correct.)*

* Angular momentum (*Coordinate origin is chosen at where the rod center was before collision*. The useful component is along the -direction; angular momentum of the rod center of mass is zero)

We now have enough equations to solve for and . The answers are

*Remark: When solving problems related to rotations, you should be very careful about the choice of the coordinate origin. If you are sloppy on this issue, you are likely to derive ridiculous results.*

A picture containing chart

Description automatically generated**Problem 2 – Direction of the Angular Momentum**

The angular velocity of a rotating body is along the rotation axis, but this is *not* true for the angular momentum, as you can show in this problem.

Consider a uniform rod (mass , length , negligible thickness) rotating at angular velocity about an axis through its center (chosen as the coordinate origin). However, the angle between the rod and the axis is some acute angle (see the figure). Please find:

1. The moment of inertia of the rod around the axis;
2. The angular momentum vector of the rod as a function of time.

**Solution:**

1. By definition, you can easily obtain
2. A picture containing object, clock

   Description automatically generatedWe first find the angular momentum of a small line segment of length , at distance (with a slight abuse of notations) from the origin. (If , then this segment is to the left of the rod). Suppose that at , the rod is in the -plane. After time , the rod has rotated an angle . So, the position of this line segment is at

(as you can see in the spherical coordinates). The velocity is

You can obtain this result geometrically or using the chain rule of differentiation. The mass of this line segment is (we use the letter to remind us that we are dealing with something that we will make it tend to zero)

The angular momentum of this small segment is

Finally, since mass is distributed continuously on the rod, we replace the sum by integration:

Here we used . We see that is definitely *not* in the same direction as the angular velocity vector for all .

*Remark:* For , we obtain the result

We warn you that this *vector* equation is generally *not* true.

**Problem 3 – Instant Center of Rotation in Rolling Problems**



A round uniform plate (radius ) in the -plane is rolling on a flat surface. At , the plate center is on the axis, at . The center of mass of the plate has a constant velocity .

We focus on a special point P on the plate and watch its trajectory. Let the position, the velocity and the acceleration of P be . Initially, P is at .

1. Let the position of P *relative* to the contact point be . What is the value of ?
2. Calculate . Here we promote all 2D vectors to 3D, and .

**Solution:**

1. The position of the color point is at

And the relative position is

Then

1. By direct calculation

You should recognize that this is equal to . Now you have confirmed that *the plate can be regarded as rotating around the contact point at every instant*. The corresponding angular speed is also . Because of this reason, the contact point is called the **instant center of rotation** of the rolling plate.

*Remark 1:*

Diagram

Description automatically generated

We remind you that the plate is not *really* rotating around the contact point, and you should *not* use this to calculate the *acceleration* of points on the plate. This is because the picture of instant rotation only gives the correct motion up to the *first order* of time (in the Taylor series of the position function), but the acceleration is a *second-order* derivative.

*Remark 2:* The kinetic energy of the rolling wheel is given by

However, we have another way to get the same answer: since at any instant , the wheel can be regarded as rotating around its lowest point with angular speed , we can use the moment of inertia with respect to the axis through this contact point. By the parallel axis theorem

Then

Now I hope that you have a better understanding of this table:

Table

Description automatically generated

**Problem 4 – Kepler’s Second Law and Conservation of Angular Momentum**

Diagram

Description automatically generated

In high school, you may have leanred about Kepler’s Second Law of planetary motion:

*A line segment joining a planet and the Sun sweeps out equal areas during equal intervals of time.*

A picture containing text

Description automatically generatedThe area swept per unit time by this line segment is called the **areal velocity**. We now show that *this law is a direct consequence of the conservation of angular momentum*.

To set up the stage, we put the Sun at the origin of our coordinate system. The mass of the sun is so large compared to the planet, that its position can be regarded as fixed. Let the plane in which the planet is moving be the -plane. The position of the planet at time is given by the vector . The area swept by in the interval is denoted by .

1. Prove that (up to first-order terms, i.e. proportional to )

*Hint 1: You may use the geometrical meaning of matrix determinant.*

*Hint 2: For any “reasonable” function of time, we can write for small*

1. What is the relationship between the areal velocity and the angular momentum of the planet? (let the mass of the planet be );
2. Why is the angular momentum of the planet conserved? Then show that Kepler’s Second Law follows immediately.

**Solution:**

1. This problem can be solved quickly using the determinant:
2. Recall that

If we choose the plane in which the planet is moving as the -plane, then all the time, and the vector reduces to

This -component is the same as times the areal velocity:

1. The only force on the planet is the gravitational force pointing from the planet to the sun. Since we choose the origin to be at the sun, this force does not produce any torque. Thus, the angular momentum of the planet must be a constant.

From the conclusion of question 2, this leads to a constant areal velocity, which is exactly Kepler’s Second Law.