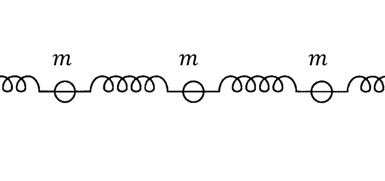
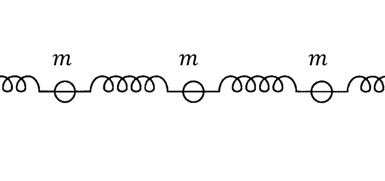
PHYS1110D – Engineering Physics: Mechanics and Thermodynamics

Tutorial Problems for Week 13: Oscillation and Waves

**Problem 1 – Oscillation on A Lattice: Introduction to Discrete Fourier Transform**



Consider small balls (all of mass ; neglect their size) connected by springs (all of the same stiffness and equilibrium length ). The balls are labeled from 0 to , and their displacements away from the equilibrium position are denoted by .

We also assume that the first ball is connected to the last ball by the same spring (so that the balls are on a closed loop), then we can set the counting convention (called the **periodic boundary condition**):

The equation of motion (Newton’s Second Law) for each ball can be easily found:

1. Verify that the expression

is a solution of the equation of motion for arbitrary real coefficients and ; the relation between and is called the **dispersion relation**.

*Hint: You may need the* ***angle sum and difference identities***

1. Use the periodic boundary condition to show that only a limited number of is allowed:
2. Because of the principle of superposition, we can of course add the solution for all allowed to get the general form of :

Now we define two time-dependent coefficients

Then (substituting in the expression of )

The procedure of finding and is called the **real discrete Fourier transform (DFT)**.

Please show that

*Hint: You need the following set of* ***orthogonal relations*** *of trigonometric functions*

1. The coefficients and are called the **Fourier coefficients**, or **normal modes** of the system. To see the origin of the latter name, please verify that:
   1. The equations of motion of and are:

They are much simpler than the original : just like the motion of a single particle of mass connected to a spring of stiffness ;

* 1. \* The energy (K.E. of the balls + P.E. in the springs) of the system can be put in a “diagonal” form:

just like the sum of many simple oscillators, without any “cross terms”.

*Hint: You will need (usually used in* ***discrete sine/cosine transforms [DST/DCT])****:*

*And the sum-to-product identities:*

**Solution:**

1. Directly substituting the expression to the equation of motion, we get (omitting the common factor on both sides:

To simplify the right-hand side, we notice that (using the angle sum and difference identities)

Then the right-hand side equals

This is already ensured by the relation

1. Since for all integers , we must have

Other integer values of are not necessary. Can you see why?

1. To get the coefficient , we multiply by :

Similarly

The desired results follow immediately.

* 1. Recall that the definition of and that of are respectively

Note that and are constants. Then the equation of motions can be directly verified.

* 1. We first write down the energy expression using the original variables :

This expression involves “cross terms” resulted from

In the second equality we used the sum-to-product identities. Then

Now you should have some feeling about the power of orthogonal relations. Besides, we also have for the potential energy:

Don’t forget that

Therefore

**Problem 2 - Continuum Lattice: The Wave Equation**

Let us play a trick that we used when learning the Principle of Least Action: taking the **continuum limit** of the lattice in Problem 1. We fix the three quantities and defined by

And introduce a two-variable function :

Now we take the limit .

1. \* Show that the equation of motion (eq. 1) becomes

This is the well-known **wave equation**. (How does it get its name? See the following question.)

1. Verify that the familiar sine and cosine waves

are solutions of the wave equation. (What is the physical meaning of ?)

1. Can you derive directly from by taking the limit?

**Solution:**

1. We rewrite the discrete equation of motion using the function :

The right-hand side (RHS) should remind you of the definition of derivatives. In fact, as

Therefore

1. For , straightforward calculation gives

(Don’t tell me you forget the chain rule) Since , the wave equation is of course satisfied by the sine wave. Similarly, we can show that the cosine wave is also a solution of the wave equation.

1. For any finite , since

We obtain