PHYS1110D – Engineering Physics: Mechanics and Thermodynamics

Tutorial Problems for Week 2: Basic Concepts of Functions and Derivatives

**Problem 1 – Translation of Function Graphs**

1. Sketch the curves of the following functions on the same graph:
2. How does the curve change when is replaced by ( where is a positive constant?

**Solution:**



1. : Moving towards left; : Moving towards right.

This conclusion can be obtained by fixing

We see that for the same argument of the function , the required becomes

Corresponding to smaller/larger .

**Problem 2 – Derivatives and Extremum of Functions**

Given a function , where and are some constants.

1. Find the velocity and the acceleration .
2. At which value of does the function reaches its extremum? Is it a local maximum or minimum?

**Solution:**

1. By direct differentiation, we can find

*Remark*: We can think as the displacement of an object. Then and are velocity and acceleration respectively. Therefore, the above describes the motion of an object under constant acceleration.

1. The function assumes its local extremum when

To check whether it is a local maximum or minimum, we calculate

Thus, it is a local maximum.

**Problem 3 – Second Derivative and Convexity of Functions**

Complete the below table with .

1. In the first and second row (except the grey cells), determine if the derivative is positive or negative;
2. In the last row (except the grey cells), draw the tendency (monotonicity and convexity) of the curve;
3. In the grey cells, fill in the corresponding values.

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Then, sketch the curve of by using the above table.

*Hints*:

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**Solution:**

We first calculate the derivatives

Hence:

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|  |  | 0 |  | 4 |  | 2 |  | 0 |  |

Sketch of the curve:

*Remark:* The point is special: *vanishes* at this point but takes *different sign* on its two sides. It is then called an **inflection point**of the function , for the function *changes its convexity* there.

**Problem 4 – Application of Differentiation in Kinetics**

The displacement of a ball is recorded as

where is in meters, and is in seconds.

Find the velocity and acceleration of the ball for .

**Solution:**

By definition of velocity and acceleration

**Problem 5 – The Centripetal Acceleration**

For a “reasonable” function , if is again a function of another variable (i.e. ), and we want to calculate the derivative of with respect to , the **chain rule** states that:

To physicists’ level of rigor, the rule is almost self-evident (you must be tempted to cross out the two ’s). For example, if , then

As one of its simple applications, assume the position of a particle in the -plane is given by

Now, provided that *(you’d better memorize these as well)*

Find the velocity and the acceleration as a function of time. Do you recover something familiar?

**Solution:**

Because the basis vectors in the -coordinate system does not change with time, we can leave them alone, and differentiate the vector components directly. Therefore

How to deal with ? Well, using the chain rule, we get

Similarly

Therefore

Does it look familiar to you? We see that is perpendicular to , as it should be. And the magnitude of velocity is

Familiar! Finding the acceleration is now easy:

This is in the *opposite* direction of : it is exactly the *centripetal acceleration*! Its magnitude is

which is also a result you already know.

**Problem 6 – Euler’s Formula**

1. Using Euler’s Formula

And the well-known property of exponential function

Show that:

1. Consider the uniform circular motion of a point on the 2D plane, with the position

Suppose that we pack the coordinates of the point into a single complex number:

* 1. Rewrite in the exponential notation (with help of Euler’s formula)
  2. Directly calculate the derivatives (do not use the format of complex numbers)

What do they mean?

**Solution**:

1. We combine the two questions to a single complex number:

The left-hand side is

By comparing the real part and the imaginary part, we have proved the two angle sum/difference identities at once!

1. The first question is quite straightforward:

Then

which is just , where is the linear velocity of the point. Similarly

which is just , where is the centripetal acceleration.

First look at differential equation: for an object connected to a spring

A nice property of exponential function is that differentiation reduces to multiplication by a constant:

Try : substitute it into

General Solution to the differential equation:

The constant coefficients can be determined by the initial conditions.