PHYS1110D – Engineering Physics: Mechanics and Thermodynamics

Tutorial Problems for Week 9: A Systematic Summary on Rotations and Rigid Body

*Disclaimer: complicated situations (including but not limited to precession, highly unsymmetric rigid bodies) will not be considered here.*

**Topic 1 – Rigid Body Kinetics**

To describe rigid body motion, a good choice of the coordinate system is crucial. It is convenient to use 2 sets of coordinates:

* The usual *static* system where we live;
* The *moving* system with its origin *fixed* at some point (let’s call it the *center of rotation[[1]](#footnote-2)* (CR)) on the rigid body. In this system, the rigid body is *purely rotating with no translational motion* (why?).

*Example: Spinning Angular Speed = 10 Orbital Angular Speed*

The CR *can be anywhere in the rigid body*, but the convenient choice depends on the problem. The **spinning angular velocity** is the *same* for all choices of CR *(please think about it)*. The most common choice of the CR is the **center of mass (CM)** of the rigid body.

As an example, consider a spinning square, whose CM is also rotating around the origin of the system. For an arbitrarily chosen CR, the path of this CR (red dashed curve) can be complicated (left); when CR is chosen at CM, the path is simply a circle around the origin (right). However, in *both* cases, the motion of the square in the system is *pure* rotation. The only difference is the position of the rotation axis.

Assume that the trajectory of CR in the system is given by . Using the two sets of coordinate systems, the velocity on any point of the rigid body can be written as

i.e. the superposition of the velocity of CR and the velocity of the point *relative to the CR (in the system)*.

To get the acceleration, we only need to differentiate again:

But you already know what is: it is the spinning velocity in the system

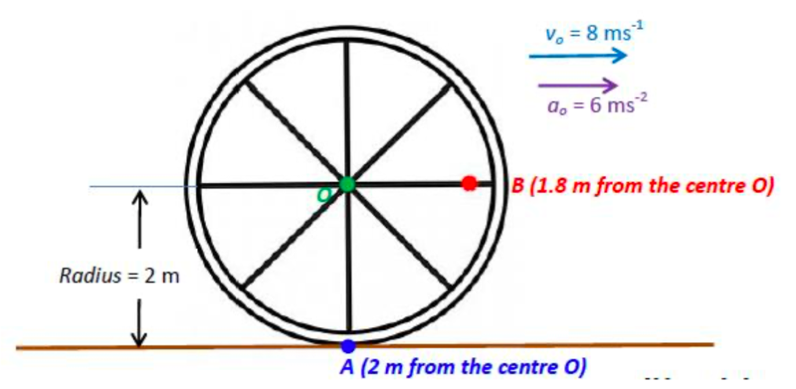
Therefore, the final expression for the acceleration is

Equations (1) and (2) are in *vector form*, and thus have a much wider applicability than those you might have learned in high school.

*Remark:*  is the *vector form* of the **centripetal acceleration**.

*Exercise: Rotation Kinetics*

A wheel is rolling on the plane. At a given instant, the center of the wheel has a velocity of 8 m/s and an acceleration of 6 m/s2. Assuming that the wheel does not slip, calculate the velocity and acceleration of points A and B at the instant shown in the figure.



**Solution 1:**

Let us choose the CR to be the center of the wheel (which also happens to be the CM). In this problem,

The wheel radius is . Because the wheel is purely rolling, the angular speed and the angular acceleration of the wheel are related to by

Now we can apply equations (1) and (2) to find the quantities at .

* Point A:
* Point B:

**Solution 2:**

You can also express in terms of its Cartesian components, and directly do the differentiations to find the velocity and the acceleration. As an example, for point A:



First, the position of the center of mass is at

By the pure rolling condition, the rotation angle is

Now we can write down:

* Position
* Velocity
* Acceleration

In the problem,

The question asks for and . Notice that . We obtain the velocity:

The acceleration is

*Remark:* The trajectory of the point A is called a **cycloid**, a curve famous for its various interesting properties. Visit Wikipedia to learn more about them.

**Topic 2 – The Rotational Kinetic Energy**

* *Separation of the Rotational Kinetic Energy and the Center-of-Mass Kinetic Energy*

Consider a rigid body moving in the space with CM velocity . The whole body is rotating around an axis through the CM with angular speed . For convenience, we choose the -axis to be parallel to the angular velocity, so that (where ).

To do the separation, we use the relation just proved (the CR is chosen to be the CM of the body; the subscript labels different points in the body):

Here is the position of relative to the center of mass. With our choice of the coordinates,

We obtain

The first term is just the center-of-mass translational K.E:

The second term *vanishes*, because (*Exercise: prove this relation by yourself*)

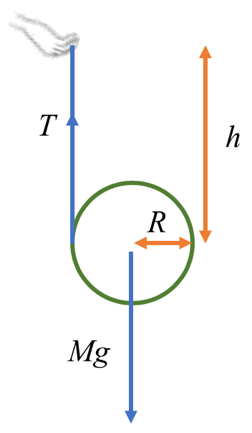
This shows why the separation is possible *only with respect to the CM*. The third term *motivates* the definition of : we want something that looks like , say

Therefore, we *define* the moment of inertia around the rotation axis *through CM* to be

The kinetic energy of a rigid body now separates into two terms:

*Remark:* You can think about the modifications we need to make if is along the or direction. However, the real challenge is the general case whenis *not along any coordinate axis*. In this case, the moment of inertia will become a **matrix** (or, using the more formal but accurate name, **tensor**), and the rotational K.E has the form

Try finding out the matrix elements .

*Exercise: Rotational Kinetic Energy*

A yo-yo of mass and radius is released from hand. No sliding occurs between the yo-yo and the rope. Neglecting the mass of the rope:

1. Find the tension on the string (the moment of inertia of the yo-yo about the perpendicular rotation axis through CM is ).
2. Find the acceleration of the center of mass of the yo-yo.
3. Find the speed of the center of mass in terms of through which it has descended.

**Solution:**

Instead of drawing the free body diagram and get all the forces, we start directly from energy conservation (with the condition of pure rolling )

(We have set the initial height of the yo-yo to correspond to zero potential energy.) Initially, the yo-yo has zero total energy. Therefore

Taking the time derivative of the energy:

Thus, we get the acceleration of the yo-yo:

Finally, the tension in the rope can be found from

* *The Parallel-Axis Theorem*

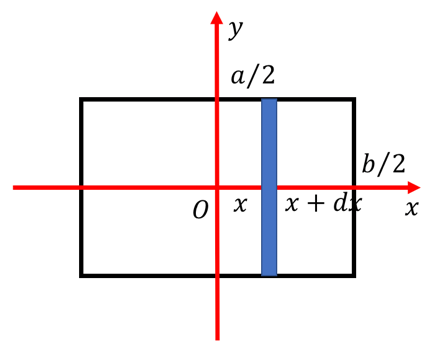
In this case, the center of mass velocity can also be expressed by the angular velocity vector:

Again, for convenience, we choose the rotation axis to be parallel to the -axis. Here is the position of the CR relative to the CM. Then the kinetic energy is

Now, if we *define* the moment of inertia around this rotation axis to be

Then the expression simplifies further:

Notice that is the *distance* between the CM and the rotation axis (because, by definition, the CR is on this axis, which is parallel to -axis). Eq. (6) is the well-known **Parallel Axis Theorem**.

*Exercise: Parallel Axis Theorem*

The standard method of finding the moment of inertia is multi-variable integration, which you have not yet learned. However, with the help of the parallel axis theorem, you can already tackle some objects with simple shapes. Based on the hints shown on the figure, show that the moment of inertia of a uniform rectangle (edge length ) around the axis through its center and perpendicular to it is .

*For your reference: The moment of inertia of a uniform rod (mass , length , with negligible thickness) is with respect to a perpendicular axis through its center.*

**Solution:**

Let the surface mass density be . Then by the Parallel-axis Theorem (take the -axis to be the rotation axis)

**Topic 3 – Linear Momentum and Angular Momentum of Rigid Bodies**

By choosing the CM as the origin of the system (i.e. the CR), we can simplify the expressions of the linear and the angular momentum of a rigid body.

* *The linear momentum*

By definition, we obtain ( is the position *relative to the CM*)

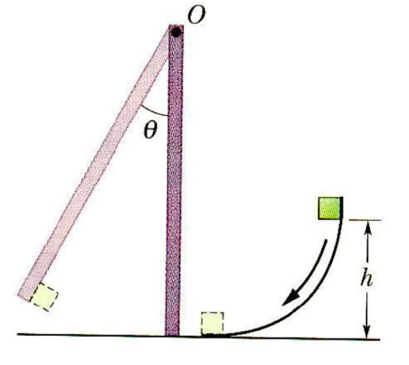
We meet the expression again, which is identically zero. Therefore

i.e. *the linear momentum of a rigid body is equal to the linear momentum of the CM*.

* *The angular momentum*

Again, by definition

i.e. *the angular momentum is the sum of the CM angular momentum and the spinning angular momentum in the system*. Notice that we *did not use* , thus this separation is *valid for anything, including non-rigid bodies*.

*Exercise: Application of Conservation Laws*

In the following figure, a small 50g block slides down a frictionless surface through height = 20 cm and then sticks to a uniform rod of mass 100 g and length 40 cm. The rod then (together with the block) swings about point O. Find the maximum angle of swing that the rod can reach.

**Solution:**

The most difficult part is finding the angular velocity of the rod immediately after the collision.

First, we point out that the horizontal linearmomentum of the rod and the block is *not conserved*, because there is a horizontal force *exerted by the pivot* on the rod during the collision (can you see why?). The correct way to do the problem is using the conservation of the *angular* momentum. Due to the force from the pivot, we should choose as the coordinate origin to make the torque of the pivot to be zero. Then, since the gravity does not contribute to the torque neither, we obtain

Here the velocity of the block before collision is found from energy conservation:

And the moments of inertia with respect to the pivot are

(the parallel axis theorem is used when finding )

The mechanical energy is conserved when the rod is swinging:

Here is the height change of the CM of rod + block:

Finally, we obtain

1. I invented this name by myself. [↑](#footnote-ref-2)