$$\sum_{i=0}^{N}\left[\left(\vec{F_i}-m_i\ddot{\vec{r_i}}\right)\delta\vec{r_i}\right]=0$$ (1) 达朗贝尔原理

$$\vec{r_i}=(q_1,q_2,\cdots,q_n,t)$$

全微分：$$d\vec{r_i}=\sum_{s=1}^{n}\frac{\partial\vec{r_i}}{\partial q_s}\cdot dq_s+\frac{\partial\vec{r_i}}{\partial t}\cdot dt$$

d 和 δ 一样

$$\delta\vec{r_i}=\sum_{s=1}^{n}\frac{\partial\vec{r_i}}{\partial q_s}\cdot\delta q_s+\underbrace{\frac{\partial\vec{r_i}}{\partial t}\cdot\delta t}_{\text{分析此刻，}\delta t=0}$$

$$\delta\vec{r_i}=\sum_{s=1}^{n}\left(\frac{\partial\vec{r_i}}{\partial q_s}\cdot\delta q_s\right)$$ (2)

(2) -> (1)：

$$\sum_{i=0}^{N}\left[\left(\vec{F_i}-m_i\ddot{\vec{r_i}}\right)\cdot\sum_{s=1}^{n}\left(\frac{\partial\vec{r_i}}{\partial q_s}\cdot\delta q_s\right)\right]=0$$

整理：

$$\sum_{s=1}^{n}\left[\boxed{\sum_{i=0}^{N}\left(\vec{F_i}\frac{\partial\vec{r_i}}{\partial q_s}-m_i\ddot{\vec{r_i}}\frac{\partial\vec{r_i}}{\partial q_s}\right)}\cdot\underline{\delta q_s}\right]=0$$

q_s 广义坐标，显式表示

相互无约束

$$\sum_{i=0}^{N}\left(\vec{F}_i \frac{\partial \vec{r}_i}{\partial q_s} - m_i \ddot{\vec{r}}_i \frac{\partial \vec{r}_i}{\partial q_s}\right) = 0 \quad (3)$$

$$Q_s = \sum_{i=0}^{N}\left(\vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_s}\right) \quad (5)$$

?

$$\sum_{i=0}^{N}\left(m_i \cdot \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_s}\right) \quad (4)$$

$$\frac{d(f \cdot g)}{dt} = \dot{f} \cdot g + f \cdot \dot{g}$$

$$\dot{f} \cdot g = \frac{d(f \cdot g)}{dt} - f \cdot \dot{g}$$

$$\sum_{i=0}^{N}\left(m_i \cdot \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_s}\right) = \sum_{i=0}^{N}\left(\frac{d\overbrace{\boxed{(m_i \dot{\vec{r}}_i)}}^{f}}{dt} \cdot \overbrace{\boxed{\frac{\partial \vec{r}_i}{\partial q_s}}}^{g}\right)$$

$$= \sum_{i=0}^{N}\left(\frac{d\left[\boxed{m_i \cdot \dot{\vec{r}}_i} \cdot \boxed{\frac{\partial \vec{r}_i}{\partial q_s}}\right]}{dt} - \boxed{m_i \dot{\vec{r}}_i} \cdot \frac{d\left(\frac{\partial \vec{r}_i}{\partial q_s}\right)}{dt}\right)$$

$$= \sum_{i=0}^{N}\left(\frac{d\left[\boxed{m_i \cdot \dot{\vec{r}}_i} \cdot \boxed{\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_s}}\right]}{dt} - \boxed{m_i \dot{\vec{r}}_i} \cdot \frac{\partial\left(\frac{d\vec{r}_i}{dt}\right)}{\partial q_s}\right)$$

拉格朗日关系：

① $$\frac{\partial \vec{r}_i}{\partial q_s} = \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_s}$$

② $$\frac{d\left(\frac{\partial \vec{r}_i}{\partial q_s}\right)}{dt} = \frac{\partial\left(\frac{d\vec{r}_i}{dt}\right)}{\partial q_s}$$

$$\boxed{\begin{aligned} T &= \sum_{i=0}^{N} \frac{1}{2} m_i \, \dot{\vec{r}}_i(q_1, q_2, \ldots, q_n)^2 \\ \frac{\partial T}{\partial \dot{q}_s} &= \sum_{i=0}^{N} m_i \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_s} \end{aligned}}$$

$$= \sum_{i=0}^{N} \left(\frac{d\left(\frac{\partial \left(\frac{1}{2} m_i \dot{\vec{r}}_i^{\,2}\right)}{\partial \dot{q}_s}\right)}{dt} - \frac{m_i \dot{\vec{r}}_i \cdot \partial\left(\frac{d\vec{r}_i}{dt}\right)}{\partial q_s} \right)$$

$$= \frac{d\left(\frac{\partial T}{\partial \dot{q}_s}\right)}{dt} - \frac{\partial T}{\partial q_s}$$

P :

$$\sum_{i=0}^{N} \left(m_i \cdot \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_s} \right) = \frac{d\left(\frac{\partial T}{\partial \dot{q}_s}\right)}{dt} - \frac{\partial T}{\partial q_s} \qquad (6)$$

(5) (6) $\rightarrow$ (3)

$$Q_s = \frac{d\overbrace{\left(\frac{\partial T}{\partial \dot{q}_s}\right)}^{P_s}}{dt} - \frac{\partial T}{\partial q_s}$$

$$Q_s = \frac{d P_s}{d t} - \frac{\partial T}{\partial q_s}$$