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## Chapter 2 Review of Elasticity

\* The essentials of linear elasticity will be reviewed

\* Linear elasticity: - Infinitesimal deformation (displacement-strain relation is linear) - Linear constitutive relation (stress-strain relation is linear) - Moreover, homogeneity and isotropy of material will be assumed

### 2.1 Displacements and strains

\* Index notation

1. Coordinates: \(x\_{i}\), \(i=1,2,3\)

2. Displacements: \(u\_{i}\), \(i=1,2,3\)

3. Derivatives: \(u\_{i,j}\), \(i,j=1,2,3\) where \(u\_{i,j}=\dfrac{\partial u\_{i}}{\partial x\_{j}}\)

4. Base vectors: \(\mathbf{e}\_{i}\), \(i=1,2,3\)

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Now consider a line element in a body which undergoes a deformation:

\[d\mathbf{r}^{\*}=(\mathbf{r}^{\*}+d\mathbf{r}^{\*})-\mathbf{r}^{\*}\] \[=(\mathbf{r}+d\mathbf{r}+\mathbf{u}+d\mathbf{u})-(\mathbf{r}+ \mathbf{u})\] \[=d\mathbf{r}+d\mathbf{u}\] or \[d\mathbf{u}=d\mathbf{r}^{\*}-d\mathbf{r}\]

\* Now, \[d\mathbf{u}=du\_{1}\mathbf{e}\_{1}+du\_{2}\mathbf{e}\_{2}+du\_{3}\mathbf{e}\_{3}\]

\* For a small deformation \[du\_{i}=\frac{\partial u\_{i}}{\partial x\_{1}}\,dx\_{1}+\frac{ \partial u\_{i}}{\partial x\_{2}}\,dx\_{2}+\frac{\partial u\_{i}}{\partial x\_{3}} \,dx\_{3}\] \[=\sum\_{j=1}^{3}u\_{i,j}dx\_{j}\] If we adopt Einstein's convention where repeated index implies summation \[du\_{i}=u\_{i,j}dx\_{j}\] Decompose \(u\_{i,j}\) into symmetric and antisymmetric parts:

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\[\begin{split} du\_{i}=&\Bigg{[}\frac{1}{2}\big{(}u\_{i,j}+u \_{j,i}\big{)}+\frac{1}{2}\big{(}u\_{i,j}-u\_{j,i}\big{)}\Bigg{]}dx\_{j}\\ =&\varepsilon\_{ij}dx\_{j}+w\_{ij}dx\_{j}\end{split}\] (1)

where

\[\begin{split}\varepsilon\_{ij}=&\frac{1}{2}\big{(}u\_{i, j}+u\_{j,i}\big{)}\\ w\_{ij}=&\frac{1}{2}\big{(}u\_{i,j}-u\_{j,i}\big{)} \end{split}\] (2)

\(\varepsilon\_{ij}\) is the tensorial component of strain and symmetric (\(\varepsilon\_{ij}=\varepsilon\_{ji}\))

\(w\_{ij}\) is the rotation and antisymmetric (\(w\_{ij}=-w\_{ji}\))

\(\bullet\)\(\varepsilon\_{ij}\) and \(w\_{ij}\):

- For \(i=j\), let's say \(i=j=1\).

\[\varepsilon\_{ij}=\varepsilon\_{11}=\frac{1}{2}\big{(}u\_{1,1}+u\_{1,1}\big{)}=u\_ {1,1}=\frac{\partial u\_{1}}{\partial x\_{1}}\]

\(\rightarrow\) Change of \(u\_{1}\) w.r.t the change of \(x\_{1}\).

\(\rightarrow\) normal strain in 1-direction

- For \(i\neq j\), let's say \(i=1\), \(j=2\).

\[\begin{split}\varepsilon\_{ij}=&\varepsilon\_{12}= \frac{1}{2}\big{(}u\_{1,2}+u\_{2,1}\big{)}=\varepsilon\_{21}\\ w\_{ij}=& w\_{12}=\frac{1}{2}\big{(}u\_{1,2}-u\_{2,1} \big{)}=-w\_{21}\end{split}\]

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\(\varepsilon\_{12}\): half of the angle change between the two line elements that are originally orthogonal to each other

\* Engineering shear strain \(\gamma\_{ij}=2\varepsilon\_{ij}\) \(\gamma\_{12}=2\varepsilon\_{12}\): total angle change between the two mutually orthogonal lines (line 1 and line 2)

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### Stresses

\* Stress vector, or traction (vector): \(\mathbf{t}\)

\[\mathbf{t}=\underset{\Delta A\to 0}{\lim}\frac{\Delta F}{\Delta A}\]

\[\mathbf{t}\]

is a force per unit area at point P in body B

\* Stress state at a point \(P\)

\[\mathbf{t}\_{3}=x\_{3}\]

\[\mathbf{t}\_{1}=t\_{11}\mathbf{e}\_{1}+t\_{12}\mathbf{e}\_{2}+t\_{13}\mathbf{e}\_{3}\]

\[\text{or}\ \ \mathbf{t}\_{i}=t\_{ij}\mathbf{e}\_{j}\ (i,j=1,2,3)\]

\(t\_{ij}\): the coordinate components of traction vector \(\mathbf{t}\_{i}\)

\(\rightarrow\) stress components

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We denote \(t\_{ij}\) as \(\sigma\_{ij}\), and call them tensorial components of stress.

\(\sigma\_{ij}\Rightarrow\)9 components for \(i,j\)=1,2,3

Moment equilibrium of the element requires \(\sigma\_{ij}=\sigma\_{ji}\)\((i\neq j)\)

\(\rightarrow\) 6 independent components

In matrix form, the stress tensor is

\(\sigma=\)\(\left[\sigma\_{ij}\right]=\)\(\left[\begin{array}[]{ccc}\sigma\_{11}&\sigma\_{12}&\sigma\_{13}\\ \sigma\_{12}&\sigma\_{22}&\sigma\_{23}\\ \sigma\_{13}&\sigma\_{23}&\sigma\_{33}\end{array}\right]\), \_Symmetric\_

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### Stress transformation

\*\*i. Transformation of a vector\*\*

\[Coordinate\ Transformation:\left(\mathbf{e}\_{1},\mathbf{e}\_{2},\mathbf{e}\_{3}\right) \rightarrow \left(\mathbf{e}\_{1}^{\prime},\mathbf{e}\_{2}^{\prime},\mathbf{e}\_{3}^{ \prime}\right)\]

\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\(x\_{3}\)\(x\_{1}\)\(x\_{2}\)\

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\(\mathbf{Q}\) is orthogonal

\[\mathbf{Q}^{\mathrm{T}}=\mathbf{Q}^{-1}\ or\ \ \mathbf{Q}\mathbf{Q}^{T}=I\] (4)

\*\*ii. Transformation of a stress tensor\*\*

Consider an inclined plane with direction normal \(\mathbf{n}\) on a tetrahedron OABC cut from a body in equilibrium:

\(\mathbf{n}\): unit normal to the plane

\(\mathbf{t}\): stress vector

\(A\_{1}=A\_{n}\mathbf{n}\cdot\mathbf{e}\_{1}=A\_{n}n\_{1}\)

\(A\_{2}=A\_{n}\mathbf{n}\cdot\mathbf{e}\_{2}=A\_{n}n\_{2}\)

\(A\_{3}=A\_{n}\mathbf{n}\cdot\mathbf{e}\_{3}=A\_{n}n\_{3}\)

\((A\_{n}=\) area of the inclined plane , \(A\_{i}\)=area of the plane normal to i-axis)

The force equilibrium in 1-direction:

\[t\_{1}A\_{n}-\sigma\_{11}A\_{1}-\sigma\_{21}A\_{2}-\sigma\_{31}A\_{3}=0\] \[t\_{1}=\sigma\_{11}n\_{1}+\sigma\_{12}n\_{2}+\sigma\_{13}n\_{3}\]

Force equilibrium of the element produces, therefore

\[t\_{i}=\sigma\_{ij}n\_{j}\] (5)

\[\begin{Bmatrix}t\_{1}\\ t\_{2}\\ t\_{3}\end{Bmatrix}=\begin{bmatrix}\sigma\_{11}&\sigma\_{21}&\sigma\_{31}\\ \sigma\_{12}&\sigma\_{22}&\sigma\_{23}\\ \sigma\_{13}&\sigma\_{23}&\sigma\_{33}\end{bmatrix}\begin{bmatrix}n\_{1}\\ n\_{2}\\ n\_{3}\end{bmatrix}\]

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Now for

\[t=\sigma n\] (a)

under coordinate transformation, we have

\[t^{\prime}=\sigma^{\prime}n^{\prime}\] (b)

However, \(t\) and \(n\) are vectors and follow vector transformation rule:

\[t^{\prime}=Qt\text{ and }n^{\prime}=Qn\] (c)

\(\bullet\) From (b) and (c)

\[Qt=\sigma^{\prime}Qn\] \[Q^{\text{T}}Qt=Q^{\text{T}}\sigma^{\prime}Qn\] \[t=Q^{\text{T}}\sigma^{\prime}Qn\text{ \ (}t=\sigma n\text{)}\]

\(\bullet\) Therefore, since \(t=\sigma n\),

\[\begin{split}\sigma=Q^{\text{T}}\sigma^{\prime}Q\\ or\ \sigma^{\prime}=Q\sigma Q^{\text{T}}\end{split}\] (6)

\(\bullet\) Strain tensor transforms under the same rule

\[\varepsilon^{\prime}=Q\varepsilon Q^{\text{T}}\]

\(\bullet\) For 2-D:

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\[\sigma^{{}^{\prime}}=\begin{bmatrix}\sigma^{{}^{\prime}}\_{11}&\sigma^{{}^{\prime}}\_{12}\\ \sigma^{{}^{\prime}}\_{21}&\sigma^{{}^{\prime}}\_{22}\end{bmatrix}=\begin{bmatrix} \cos\theta&\sin\theta\\ -\sin\theta&\cos\theta\end{bmatrix}\sigma\_{11}\quad\sigma\_{12}\quad\begin{bmatrix} \cos\theta&-\sin\theta\\ \sin\theta&\cos\theta\end{bmatrix}\]

\[\sigma^{{}^{\prime}}\_{11}=\sigma\_{11}\,cos^{2}\,\theta+\sigma\_{22}\, sin^{2}\,\theta+2\sigma\_{12}\,cos\,\theta\cdot sin\,\theta\]

\[=\frac{\sigma\_{11}+\sigma\_{22}}{2}+\frac{\sigma\_{11}-\sigma\_{22}}{2}\,cos\,2\, \theta+\sigma\_{12}\,sin\,2\,\theta\]

\[\sigma^{{}^{\prime}}\_{12}=-\frac{\sigma\_{11}-\sigma\_{22}}{2}\,sin\,2\,\theta+ \sigma\_{12}\,cos\,2\,\theta\]

\(\rightarrow\) Mohr's circle

### Principal stresses

\* We have the relation: \[t=on\] (a) For a surface with a particular orientation \(n\), the direction of

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becomes the same as that of \(\mathbf{n}\)

\* Thus (a) can be written: \[\mathbf{t}=\lambda\_{1}\mathbf{n}=\mathbf{\sigma}\mathbf{n}\] or \[\mathbf{\sigma}\mathbf{n}=\lambda\mathbf{n}\] (b) which means, on that particular plane, the traction is normal to the plane. The shear stress therefore is zero, and the magnitude of the normal stress is \(\lambda\)

\* The normal stress is called principal stress, and the plane is called principal plane

\* From (b), \[\mathbf{\sigma}\mathbf{n}-\lambda\mathbf{n}=0\] or \[(\mathbf{\sigma}-\lambda\mathbf{I})\mathbf{n}=0\] (c)

\* For (c) to have a solution \[\left|\mathbf{\sigma}-\lambda\mathbf{I}\right|=0\] (d)

\* For 3-D cases three \(\lambda\)'s can be determined from (d) and with each \(\lambda\), corresponding \(\mathbf{n}\) can be obtained from (c) \[\left|\mathbf{\sigma}-\lambda\mathbf{I}\right|=\left|\begin{array}[]{ccc} \sigma\_{11}-\lambda&\sigma\_{12}&\sigma\_{13}\\ \sigma\_{12}&\sigma\_{22}-\lambda&\sigma\_{23}\\ \sigma\_{13}&\sigma\_{23}&\sigma\_{33}-\lambda\end{array}\right|=0\] \[\rightarrow -\lambda^{3}+\mathrm{I}\_{1}\lambda^{2}-\mathrm{I}\_{2}\lambda^{2} +\mathrm{I}\_{3}=0\]

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where \(I\_{1}=\sigma\_{ii}=\sigma\_{11}+\sigma\_{22}+\sigma\_{33}\)

\[I\_{2} =\sigma\_{ii}\sigma\_{jj}-\sigma\_{ij}\sigma\_{ji}\] \[=\begin{vmatrix}\sigma\_{22}&\sigma\_{23}\\ \sigma\_{23}&\sigma\_{33}\end{vmatrix}+\begin{vmatrix}\sigma\_{11}&\sigma\_{13}\\ \sigma\_{13}&\sigma\_{33}\end{vmatrix}+\begin{vmatrix}\sigma\_{11}&\sigma\_{12}\\ \sigma\_{12}&\sigma\_{22}\end{vmatrix}\] \[I\_{3} =\begin{vmatrix}\sigma\_{11}&\sigma\_{12}&\sigma\_{13}\\ \sigma\_{12}&\sigma\_{22}&\sigma\_{23}\\ \sigma\_{13}&\sigma\_{23}&\sigma\_{33}\end{vmatrix}\]

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Notes: i) Since principal stresses are merely representing physical state at a point, \(\,\,\,I\_{1}\,,\,\,\,I\_{2}\,,\,\,\) and \(\,\,\,I\_{3}\,\,\) are independent of any coordinates - invariants w.r.t coordinate transformation ii) \(\lambda\_{1}\,,\,\,\,\lambda\_{2}\,,\,\) and \(\,\,\,\lambda\_{3}\,\,\) are all real. The corresponding unit normals \(\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\, \,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\, \,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\, \,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\, \,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\, \,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\, \,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\, \,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\, \,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\, \,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\, \,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\, \,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\, \,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\,\

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### Constitutive equations

\* Generalized Hooke's law for isotropic materials. \[\begin{split}&\varepsilon\_{11}=\frac{1}{E}\Big{(}\sigma\_{11}-v\big{(} \sigma\_{22}+\sigma\_{33}\big{)}\Big{)}\\ &\varepsilon\_{22}=\frac{1}{E}\Big{(}\sigma\_{22}-v\big{(}\sigma\_{33 }+\sigma\_{11}\big{)}\Big{)}\\ &\varepsilon\_{33}=\frac{1}{E}\Big{(}\sigma\_{33}-v\big{(}\sigma\_{1 1}+\sigma\_{22}\big{)}\Big{)}\\ &\varepsilon\_{12}=\frac{1}{2G}\sigma\_{12}\\ &\varepsilon\_{23}=\frac{1}{2G}\sigma\_{23}\\ &\varepsilon\_{31}=\frac{1}{2G}\sigma\_{31}\end{split}\] (8)

### Compatibility conditions

\* If we want to develop equations in terms of stresses, a problem arises. The stresses obtained can be used in Hooke's law to find strains. However, we have 6 strain components to determine 3 displacement components. Now, the 6 strain components are not totally independent. Removing displacement components from (2), we obtain (the strain components must be related through displacement components) \[\varepsilon\_{ij,kl}+\varepsilon\_{kj,kl}-\varepsilon\_{ik,jl}-\varepsilon\_{jl, ik}=0\] (9)

\* These are called "compatibility conditions".

\* In 3-D, there are 81 equations in total. Only 6 are essential. The rests are due to repetitions and are identities.

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\* For 2-D case, there is only one equation (only 3 strain components exist) \[(a)\varepsilon\_{11,22}=u\_{1,122} \left(\frac{\partial^{2}\varepsilon\_{11}}{\partial x\_{2}^{2}}= \frac{\partial^{3}u\_{1}}{\partial x\_{1}\partial x\_{2}^{2}}\right)\] \[(b)\varepsilon\_{22,22}=u\_{2,112} \left(\frac{\partial^{2}\varepsilon\_{11}}{\partial x\_{1}^{2}}= \frac{\partial^{3}u\_{2}}{\partial x\_{1}^{2}\partial x\_{2}}\right)\] \[(c)\varepsilon\_{12,12}=\frac{1}{2}\big{[}u\_{1,112}+u\_{1,122}\big{]} \left(\frac{\partial^{2}\varepsilon\_{12}}{\partial x\_{1}\partial x\_{2}}=\frac{1 }{2}\bigg{[}\frac{\partial^{3}u\_{2}}{\partial x\_{1}^{2}\partial x\_{2}}+\frac{ \partial^{3}u\_{1}}{\partial x\_{1}\partial x\_{2}^{2}}\bigg{]}\right)\] Plugging (a) and (b) into (c): \[\varepsilon\_{11,22}+\varepsilon\_{22,11}-2\varepsilon\_{12,12}=0\] (10)

### Plane stress state

Stress components in a particular direction vanish:

\[\sigma\_{33}=\sigma\_{13}=\sigma\_{23}=0\]

The Hooke's law in this case

\[\begin{split}\varepsilon\_{11}&=\frac{1}{E}\big{(} \sigma\_{11}-\nu\sigma\_{22}\big{)}\\ \varepsilon\_{22}&=\frac{1}{E}\big{(}\sigma\_{22}-\nu \sigma\_{11}\big{)}\\ \varepsilon\_{33}&=-\frac{\nu}{E}\big{(}\sigma\_{11}+ \sigma\_{22}\big{)}\\ \varepsilon\_{12}&=\frac{1}{2G}\sigma\_{12}\end{split}\] (11)

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\*\*ii. Plane strain state\*\*

Strain components in a particular direction vanish

\[\varepsilon\_{33}=\varepsilon\_{13}=\varepsilon\_{23}=0\]

In this case

\[\varepsilon\_{33}=\frac{1}{E}\Big{[}\sigma\_{33}-v\big{(}\sigma\_{11}+\sigma\_{22} \big{)}\Big{]}=0\]

Therefore,

\[\sigma\_{33}=v\big{(}\sigma\_{11}+\sigma\_{22}\big{)}\]

Then Hooke's law becomes

\[\varepsilon\_{11}=\frac{1-v^{2}}{E}\Big{[}\sigma\_{11}-\frac{v}{1-v }\sigma\_{22}\Big{]}\] \[\varepsilon\_{22}=\frac{1-v^{2}}{E}\Big{[}\sigma\_{22}-\frac{v}{1-v }\sigma\_{11}\Big{]}\] \[\varepsilon\_{12}=\frac{1-v^{2}}{E}\Big{[}1+\frac{v}{1-v}\Big{]} \sigma\_{12}\] (2.12)

Or we can write the above as

\[\varepsilon\_{11}=\frac{1}{E^{{}^{\prime}}}\big{[}\sigma\_{11}-v^{{} ^{\prime}}\sigma\_{22}\big{]}\] \[\varepsilon\_{22}=\frac{1}{E^{{}^{\prime}}}\big{[}\sigma\_{22}-v^{{ }^{\prime}}\sigma\_{11}\big{]}\] \[\varepsilon\_{12}=\frac{1+v^{{}^{\prime}}}{E^{{}^{\prime}}}\sigma\_ {12}\] \[\text{where }E^{{}^{\prime}}=\frac{E}{1-v^{2}}\text{ and }v^{{}^{ \prime}}=\frac{v}{1-v}\] (2.13)

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### Total potential energy

\*\*i. Work potential\*\*

Consider a work done by \*\*a fixed force F\*\* in bringing the system from configuration A to configuration B. Then the work is

\[W=\int\_{A}^{B}\mathbf{F}\cdot d\mathbf{s}=\int\_{A}^{B}F\_{s}ds\] (a)

\(F\_{s}\) is the tangential components of \(\mathbf{F}\) to the path "s"

If there is no dissipative force involved, the work done moving the system around any closed path is zero, which implies that the work \(W\) is path independent.

\[W=\int\_{A}^{B}F\_{s}\cdot ds=\int\_{A}^{B}d\Pi=\Pi\_{B}-\Pi\_{A}=-\Delta\Pi\] (b)

i.e., \(F\_{s}ds\) is an exact differential of some function \(\Pi\) (Work potential)

The (-) sign implies that the system has lost potential in doing the work.

\*\*ii. Strain energy\*\*

Now, we investigate the work done by internal forces:

Consider

--- Page 18 ---

The work done by the force \(f\_{x}\) for the deformation \(\delta\) due to \(d\varepsilon\_{xx}\):

\[f\_{x}\cdot\delta=\big{(}\sigma\_{xx}\Delta d\big{)}\big{(}\Delta xd\varepsilon\_{ xx}\big{)}=\sigma\_{xx}d\varepsilon\_{xx}\Delta V\]

The total incremental work done by the internal forces:

\[W=\int\left[\begin{array}[]{c}\int\_{0}^{\varepsilon\_{11}}\sigma\_{11}d \varepsilon\_{11}+\int\_{0}^{\varepsilon\_{22}}\sigma\_{22}d\varepsilon\_{22}+\int\_ {0}^{\varepsilon\_{33}}\sigma\_{33}d\varepsilon\_{33}\\ +\int\_{0}^{\varepsilon\_{12}}\sigma\_{12}d\varepsilon\_{12}+\int\_{0}^{\varepsilon\_ {23}}\sigma\_{23}d\varepsilon\_{23}+\int\_{0}^{\varepsilon\_{13}}\sigma\_{13}d \varepsilon\_{13}\end{array}\right]dV\]

The integrals are path independent, therefore \(dW\) is the exact differential of a potential, which is denoted as \(U\_{0}\):

Then,

\[dU\_{0}=\sigma\_{11}d\varepsilon\_{11}+\sigma\_{22}d\varepsilon\_{22}+...+\sigma\_{1 3}d\varepsilon\_{13}\] (a)

Since \(dU\_{0}\) is an exact differential, we can say:

\[dU\_{0}=\frac{\partial U\_{0}}{\partial\varepsilon\_{11}}d\varepsilon\_{11}+\frac {\partial U\_{0}}{\partial\varepsilon\_{22}}d\varepsilon\_{22}+...\] (b)

From (a) and (b):

\[\sigma\_{ij}=\frac{\partial U\_{0}}{\partial\varepsilon\_{ij}}\] (14)

(For \(i=1\), \(j=2\)

\[\sigma\_{12}=\frac{\partial U\_{0}}{\partial\varepsilon\_{12}}\text{: in this case, the usual relation }\varepsilon\_{12}=\frac{1}{2G}\sigma\_{12}\text{ \ holds}\text{)}\]

\(\bullet\) (2.14) is valid both for linear and non-linear elastic materials.

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\(U\_{0}=\int\_{0}^{\varepsilon\_{11}^{\*}}\sigma\_{11}d\varepsilon\_{11}=\frac{1}{2} \sigma\_{11}^{\*}\varepsilon\_{11}^{\*}=A\)

\* \(U\_{0}\) is called Strain Energy Density, which is the strain energy stored per unit volume of the body. This energy is provided by the external forces and stored as an elastic potential.

\* Total strain energy of a body is \(U=\int U\_{0}dV\)

\* \*\*Total potential energy\*\* By definition, total potential energy is the sum of those two potentials: \[\Pi=U-W\] (15) where \(U\) is the total strain energy, and -\(W\) is the work potential of the externally applied force F.

## Chapter 2 Review of Elasticity

- ⚫ The essentials of linear elasticity will be reviewed

- ⚫ Linear elasticity:

- - Infinitesimal deformation (displacement-strain relation is linear)

- - Linear constitutive relation (stress-strain relation is linear)

- - Moreover, homogeneity and isotropy of material will be assumed

## 2.1 Displacements and strains

- ⚫ Index notation

<!-- image -->

- 1. Coordinates: , 1,2,3 i x i =

- 2. Displacements:

, 1,2,3 i u i =

- 3. Derivatives:

$$, , , , 1,2,3 where i i j i j j u u i j u x  = = $$

4. Base vectors: , 1,2,3 i i = e

- ⚫ Now consider a line element in a body which undergoes a deformation:

<!-- image -->

<!-- formula-not-decoded -->

- ⚫ Now,

<!-- formula-not-decoded -->

- ⚫ For a small deformation

<!-- formula-not-decoded -->

If we adopt Einstein's convention where repeated index implies summation

<!-- formula-not-decoded -->

Decompose , i j u into symmetric and antisymmetric parts:

where

<!-- formula-not-decoded -->

- ij  is the tensorial component of strain and symmetric ( ij ji  = )

- ij w is the rotation and antisymmetric ( ij ji w w = -)

- ⚫ ij  and ij w :

- - For i j = , let's say 1 i j = = .

<!-- formula-not-decoded -->

- → Change of 1 u w.r.t the change of 1 x .

- → normal strain in 1-direction

- - For i j  , let's say 1, 2 i j = = .

<!-- formula-not-decoded -->

<!-- formula-not-decoded -->

<!-- formula-not-decoded -->

<!-- formula-not-decoded -->

<!-- image -->

12  : half of the angle change between the two line elements that are originally orthogonal to each other

- ⚫ Engineering shear strain 2 ij ij   =

12 12 2   = : total angle change between the two mutually orthogonal lines (line 1 and line 2)

## 2.2 Stresses

- ⚫ Stress vector, or traction (vector): t

<!-- image -->

<!-- formula-not-decoded -->

A 

is a force per unit area at point P in body B t

- ⚫ Stress state at a point P

$

<!-- image -->

<!-- formula-not-decoded -->

ij t : the coordinate components of traction vector i t

- → stress components

We denote ij t as ij  , and call them tensorial components of stress.

<!-- formula-not-decoded -->

<!-- image -->

Moment equilibrium of the element requires ( ) ij ji i j  = 

→ 6 independent components

In matrix form, the stress tensor is

<!-- formula-not-decoded -->

## 2.3 Stress transformation

## i. Transformation of a vector

( ) ( ) 1 2 3 1 2 3 : , , , , Coordinate Transformation    → e e e e e e

<!-- image -->

## ⚫ For any vector

<!-- formula-not-decoded -->

<!-- formula-not-decoded -->

Dot product with j e  on both sides

<!-- formula-not-decoded -->

In matrix notation or

or

<!-- formula-not-decoded -->

<!-- formula-not-decoded -->

<!-- formula-not-decoded -->

## ※ Q is orthogonal

<!-- formula-not-decoded -->

## ii. Transformation of a stress tensor

- ⚫ Consider an inclined plane with direction normal n on a tetrahedron OABC cut from a body in equilibrium:

<!-- image -->

( 𝐴𝑛 = area of the inclined plane , 𝐴𝑖 =area of the plane normal to i-axis)

The force equilibrium in 1-direction:

<!-- formula-not-decoded -->

Force equilibrium of the element produces, therefore

<!-- formula-not-decoded -->

- ⚫ Now for

<!-- formula-not-decoded -->

under coordinate transformation, we have

<!-- formula-not-decoded -->

However, t and n are vectors and follow vector transformation rule:

<!-- formula-not-decoded -->

- ⚫ From (b) and (c)

<!-- formula-not-decoded -->

- ⚫ Therefore, since 𝒕 = 𝝈𝒏 ,

<!-- formula-not-decoded -->

- ⚫ Strain tensor transforms under the same rule

<!-- formula-not-decoded -->

- ⚫ For 2-D:

<!-- image -->

<!-- formula-not-decoded -->

𝜎11 ′ = 𝜎11 𝑐𝑜𝑠 2 𝜃 + 𝜎 22 𝑠𝑖𝑛 2 𝜃 + 2𝜎 12 𝑐𝑜𝑠 𝜃 ⋅ 𝑠𝑖𝑛 𝜃

<!-- formula-not-decoded -->

<!-- formula-not-decoded -->

## → Mohr's circle

<!-- image -->

## 2.4 Principal stresses

- ⚫ We have the relation:

<!-- formula-not-decoded -->

For a surface with a particular orientation n , the direction of t

## becomes the same as that of n

- ⚫ Thus (a) can be written:

<!-- formula-not-decoded -->

or

<!-- formula-not-decoded -->

which means, on that particular plane, the traction is normal to the plane. The shear stress therefore is zero, and the magnitude of the normal stress is 

- ⚫ The normal stress is called principal stress, and the plane is called principal plane

- ⚫ From (b),

<!-- formula-not-decoded -->

<!-- formula-not-decoded -->

or

- ⚫ For (c) to have a solution

<!-- formula-not-decoded -->

- ⚫ For 3-D cases three  's can be determined from (d) and with each  , corresponding n can be obtained from (c)

<!-- formula-not-decoded -->

→ 3 2 2 1 2 3 I I I 0    -+ -+ =

<!-- formula-not-decoded -->

## Notes:

- i) Since principal stresses are merely representing physical state at a point, 1 I , 2 I , and 3 I are independent of any coordinates - invariants w.r.t coordinate transformation

- ii) 1  , 2  , and 3  are all real. The corresponding unit normals 1 n , 2 n , and 3 n are mutually orthogonal.

## 2.5 Equilibrium equations

- ⚫ Consider an infinitesimal element in a body

- ⚫ Force equilibrium of the infinitesimal element in 1, 2, and 3 directions yields

<!-- image -->

<!-- formula-not-decoded -->

where F is a body force.

Without a body force:

<!-- formula-not-decoded -->

- ⚫ For 1 i =

<!-- formula-not-decoded -->

## 2.6 Constitutive equations

- ⚫ Generalized Hooke's law for isotropic materials.

<!-- formula-not-decoded -->

## 2.7 Compatibility conditions

- ⚫ If we want to develop equations in terms of stresses, a problem arises. The stresses obtained can be used in Hooke's law to find strains. However, we have 6 strain components to determine 3 displacement components. Now, the 6 strain components are not totally independent. Removing displacement components from (2.2), we obtain (the strain components must be related through displacement components)

<!-- formula-not-decoded -->

- ⚫ These are called 'compatibility conditions'.

- ⚫ In 3-D, there are 81 equations in total. Only 6 are essential. The rests are due to repetitions and are identities.

- ⚫ For 2-D case, there is only one equation (only 3 strain components exist)

<!-- formula-not-decoded -->

<!-- formula-not-decoded -->

<!-- formula-not-decoded -->

## 2.8 Plane stress state and plane strain state

## i. Plane stress state

Stress components in a particular direction vanish:

<!-- formula-not-decoded -->

The Hooke's law in this case

<!-- formula-not-decoded -->

## ii. Plane strain state

Strain components in a particular direction vanish

<!-- formula-not-decoded -->

In this case

Therefore,

<!-- formula-not-decoded -->

Then Hooke's law becomes

<!-- formula-not-decoded -->

Or we can write the above as

<!-- formula-not-decoded -->

<!-- formula-not-decoded -->

<!-- formula-not-decoded -->

<!-- formula-not-decoded -->

<!-- formula-not-decoded -->

<!-- formula-not-decoded -->

## 2.9 Total potential energy

## i. Work potential

Consider a work done by a fixed force F in bringing the system from configuration A to configuration B. Then the work is

<!-- formula-not-decoded -->

s F is the tangential components of F to the path ' s '

If there is no dissipative force involved, the work done moving the system around any closed path is zero, which implies that the work W is path independent.

<!-- formula-not-decoded -->

i.e., s Fds is an exact differential of some function  (Work potential)

The (-) sign implies that the system has lost potential in doing the work.

## ii. Strain energy

Now, we investigate the work done by internal forces:

Consider

<!-- image -->

The work done by the force x f for the deformation  due to xx d  :

<!-- formula-not-decoded -->

The total incremental work done by the internal forces:

<!-- formula-not-decoded -->

The integrals are path independent, therefore dW is the exact differential of a potential, which is denoted as 0 U :

Then,

<!-- formula-not-decoded -->

Since 0 dU is an exact differential, we can say:

<!-- formula-not-decoded -->

From (a) and (b):

<!-- formula-not-decoded -->

<!-- formula-not-decoded -->

<!-- formula-not-decoded -->

- ⚫ (2.14) is valid both for linear and non-linear elastic materials.

<!-- image -->

- ⚫ 0 U is called Strain Energy Density, which is the strain energy stored per unit volume of the body. This energy is provided by the external forces and stored as an elastic potential.

- ⚫ Total strain energy of a body is 0 U U dV = 

## iii. Total potential energy

By definition, total potential energy is the sum of those two potentials:

<!-- formula-not-decoded -->

where U is the total strain energy, and W is the work potential of the externally applied force F.

Image: figure-1-1.jpg

The image shows two 3D coordinate systems with labeled axes and an arrow pointing from the first to the second, indicating a transformation or change of basis.

- \*\*Left Coordinate System:\*\*

- Axes labeled as \(x, u\), \(y, v\), and \(z, w\).

- Unit vectors labeled as \(i\), \(j\), and \(k\).

- \*\*Right Coordinate System:\*\*

- Axes labeled as \(x\_1, u\_1\), \(x\_2, u\_2\), and \(x\_3, u\_3\).

- Unit vectors labeled as \(e\_1\), \(e\_2\), and \(e\_3\).

The arrow between the two systems suggests a transformation from the first set of axes and unit vectors to the second set.

Image: figure-10-10.jpg

The image is a diagram of Mohr's Circle, a graphical representation used in engineering to determine stress states. Here's a summary of the text content:

- The circle is centered at \(\sigma\_{\text{avg}}\), which is calculated as \(\frac{\sigma\_{11} + \sigma\_{22}}{2}\).

- The radius \(R\) is given by the formula:

\[

R^2 = \left(\frac{\sigma\_{11} - \sigma\_{22}}{2}\right)^2 + \sigma\_{12}^2

\]

- The equation of the circle is:

\[

(\sigma'\_{11} - \sigma\_{\text{avg}})^2 + (\sigma'\_{12})^2 = R^2

\]

- Points labeled on the circle include \((\sigma\_{11}, \sigma\_{12})\), \((\sigma\_{22}, \sigma\_{12})\), and \((\sigma'\_{11}, \sigma'\_{12})\) as "any point."

- The angle \(2\theta\) is shown, indicating the relationship between the original and transformed stress components.

The diagram visually represents the relationship between normal and shear stresses on different planes.

Image: figure-10-9.jpg

The image is a diagram showing two sets of coordinate axes. The original axes are labeled \(x\_1\) and \(x\_2\), with unit vectors \(\mathbf{e\_1}\) and \(\mathbf{e\_2}\) respectively. There is a rotated set of axes labeled \(x\_1'\) and \(x\_2'\), with unit vectors \(\mathbf{e\_1'}\) and \(\mathbf{e\_2'}\). The angle of rotation between the original and rotated axes is labeled \(\theta\). The diagram illustrates a typical 2D coordinate transformation involving a rotation.

Image: figure-13-11.jpg

The image shows a 3D rectangular box with three axes labeled as \(dx\_1\), \(dx\_2\), and \(dx\_3\). Each axis has arrows indicating direction. The box represents a differential volume element in a coordinate system, often used in physics or engineering to analyze small changes in volume.

Image: figure-17-12.jpg

The image shows a 3D rectangular block with dimensions labeled as Δx, Δy, and Δz. There is an equation pointing to one face of the block:

\[

\sigma\_{xx} \Delta A = \sigma\_{xx} (\Delta y \Delta z) = f\_x

\]

This represents a stress component (\(\sigma\_{xx}\)) acting on the area (\(\Delta A\)) of the face, which is equal to the product of the dimensions \(\Delta y\) and \(\Delta z\), resulting in a force component (\(f\_x\)). The image is likely illustrating a concept from mechanics or material science related to stress and force on a material element.

Image: figure-19-13.jpg

The image contains two graphs and related equations, illustrating concepts in mechanics or materials science.

1. \*\*Linear Graph:\*\*

- The graph is labeled "Linear."

- It shows a straight line from the origin to a point \((\varepsilon\_{11}^\*, \sigma\_{11}^\*)\).

- The area under the line is labeled \(A\).

- The equation is:

\[

U\_0 = \int\_0^{\varepsilon\_{11}^\*} \sigma\_{11} d\varepsilon\_{11} = \frac{1}{2} \sigma\_{11}^\* \varepsilon\_{11}^\* = A

\]

2. \*\*Nonlinear Graph:\*\*

- The graph is labeled "Nonlinear."

- It shows a curve starting from the origin and reaching \((\varepsilon\_{11}^\*, \sigma\_{11}^\*)\).

- The area under the curve is labeled \(A\).

- The equation is:

\[

U\_0 = \int\_0^{\varepsilon\_{11}^\*} \sigma\_{11} d\varepsilon\_{11} = A

\]

Both graphs depict stress-strain relationships, with the linear graph showing a constant slope and the nonlinear graph showing a curve. The area under each curve represents the energy, denoted as \(U\_0\).

Image: figure-2-2.jpg

The image is a diagram illustrating vectors and their differentials. Here's a summary of the text content and elements:

- Vectors labeled as \( r \), \( r + dr \), \( r^\* \), and \( r^\* + dr^\* \).

- Differential vectors \( dr \) and \( dr^\* \) are shown.

- A vector \( u \) and its differential \( u + du \) are highlighted in red.

- The diagram includes arrows and ellipses to represent the vectors and their differentials in a spatial context.

The diagram seems to depict a transformation or mapping between two sets of vectors, possibly in a mathematical or physical context.

Image: figure-4-3.jpg

The image is a diagram illustrating a transformation process involving geometric shapes and mathematical expressions. Here's a summary of the text content and the visual elements:

1. \*\*Initial Shape\*\*: A square with axes labeled \(x\_1\) and \(x\_2\).

2. \*\*Transformation\*\*:

- The square is transformed into a parallelogram.

- Angles and derivatives are labeled:

- \(\alpha\_1 = \frac{\partial u\_2}{\partial x\_1} = u\_{2,1}\)

- \(\alpha\_2 = \frac{\partial u\_1}{\partial x\_2}\)

- \(\frac{\pi}{2} - (\alpha\_1 + \alpha\_2)\)

3. \*\*Decomposition\*\*:

- The transformation is decomposed into two parts:

- \*\*Shear\*\*:

- \(\alpha\_{\text{avg}} = \frac{1}{2}(\alpha\_1 + \alpha\_2)\)

- \(\frac{\pi}{2} - 2\alpha\_{\text{avg}} = \frac{\pi}{2} - (\alpha\_1 + \alpha\_2)\)

- \(\frac{1}{2}\gamma = \varepsilon\_{12}\)

- \*\*Rigid-body rotation\*\*:

- \(\alpha\_1 - \alpha\_{\text{avg}} = \frac{1}{2}(\alpha\_1 - \alpha\_2)\)

- \(\frac{1}{2}(u\_{2,1} - u\_{1,2}) = \omega\_{21}\)

4. \*\*Visual Elements\*\*:

- Arrows indicate the direction of transformation.

- Points and angles are marked on the shapes.

- The transformation is shown as a combination of shear and rotation.

The diagram visually represents the mathematical decomposition of a transformation into shear and rotation components.

Image: figure-5-4.jpg

The image is a diagram featuring two overlapping ellipses. The inner ellipse is labeled "A" and contains a smaller area labeled "ΔA" with a point at its center. The outer ellipse is labeled "B." There is an arrow labeled "P" pointing from the center of "ΔA" towards the edge of ellipse "A." Another arrow labeled "ΔF" extends from the center of "ΔA" outward, beyond the ellipses.

Summary: The diagram illustrates two overlapping ellipses with labels indicating specific areas and vectors, possibly representing forces or changes in a physical system.

Image: figure-5-5.jpg

The image shows a 3D diagram of a cube with labeled axes and vectors. The axes are labeled as \(x\_1\), \(x\_2\), and \(x\_3\). There is a point labeled \(P\) on the cube. Three vectors are emanating from point \(P\), labeled as \(\mathbf{t}\_1\), \(\mathbf{t}\_2\), and \(\mathbf{t}\_3\). The vectors appear to represent directions or forces in a three-dimensional space.

Image: figure-6-6.jpg

The image shows a 3D diagram of a rectangular block with arrows indicating stress components. The text labels on the diagram are:

- σ11

- σ12

- σ13

These labels represent different stress components acting on the block, typically used in the context of mechanics or materials science to describe the state of stress at a point in a material. The arrows indicate the direction of each stress component.

Image: figure-7-7.jpg

The image shows a 3D coordinate system with two sets of basis vectors. The axes are labeled \(x\_1\), \(x\_2\), and \(x\_3\).

- The black vectors are labeled as \(\mathbf{e}\_1\), \(\mathbf{e}\_2\), and \(\mathbf{e}\_3\).

- The red vectors are labeled as \(\mathbf{e}'\_1\), \(\mathbf{e}'\_2\), and \(\mathbf{e}'\_3\).

The red vectors appear to represent a rotated or transformed basis in the same 3D space.

Image: figure-8-8.jpg

The image is a diagram showing a three-dimensional coordinate system with axes labeled \(x\_1\), \(x\_2\), and \(x\_3\). There is a triangular plane with vertices labeled \(A\), \(B\), and \(C\), and a point \(O\) at the origin. Vectors are shown on the diagram:

- \(\mathbf{t}\) is a vector pointing outward from the plane.

- \(\mathbf{n}\) is a vector normal to the plane.

- \(\sigma\_{11}\) and \(\sigma\_{22}\) are stress components acting along the \(x\_1\) and \(x\_2\) axes, respectively.

The diagram likely represents a stress analysis scenario in a material or structure.