# An Impossibility Theorem of Quantifying Causal Effect

Yue Wang

Department of Applied Mathematics
University of Washington
yuewang@uw.edu

March 29, 2018

# Collaborator

- This work is collaborated with Dr. Linbo Wang at Department of Biostatistics, Harvard University.
- Full paper can be found at https://arxiv.org/abs/1711.04466

# Outline

- What is causal effect.
- Existing causal quantities and their problems.
- Criteria for a “good” causal quantity
- An impossibility theorem.

# What is causal effect

- Heating with fire causes water to boil. (Deterministic)
- HIV exposure causes AIDS. (Stochastic, strong effect)
- Smoking causes lung cancer. (Stochastic, weak effect)

# What is causal effect

- Skip 100 pages of philosophical discussions of causal effect...

# Purpose

- We have some random variables $X_1, X_2, \cdots, X_n, Y$.
- $X_1, \cdots, X_n$ (cause variables) are exactly all the direct causes of $Y$ (result variable). We assume there is no hidden cause of $Y$.
- Our purpose is to quantify the effect of a causal relationship $X_1 \to Y$, based on the joint probability distribution of $X_1, X_2, \cdots, X_n, Y$.

# Information theory

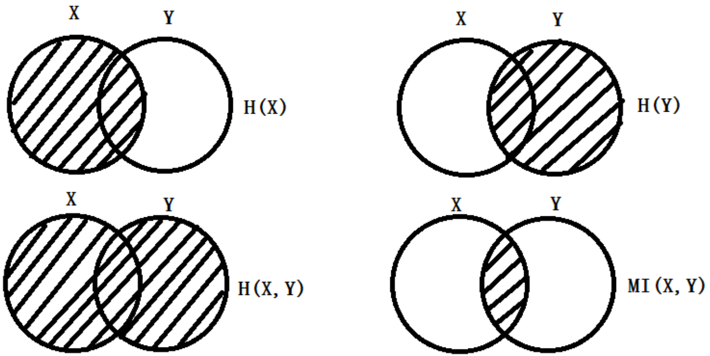
- Idea: if $X$ causes $Y$, then $X$ contains information of $Y$. Use information to quantify causal effect.
- Measure of information: entropy.

$$\mathrm{H}(X) = -\sum_i p_i \log p_i,$$

  where $p_i = \mathbb{P}(X = x_i)$.
- $\mathrm{H}(X) \geq 0$. Equality holds if and only if $X$ is deterministic.

# Mutual information (MI)

$$\mathrm{MI}(X, Y) = \mathrm{H}(X) + \mathrm{H}(Y) - \mathrm{H}(X, Y).$$



# Mutual information (MI)

- Intuition: the information shared between $X$ and $Y$. The information gain of $Y$ if we know $X$. The predict power of $X$ on $Y$.
- If $X$ causes $Y$, then $\mathrm{MI}(X, Y)$ can be used to describe the causal effect of $X \to Y$.
- $\mathrm{MI}(X, Y) \geq 0$. Equality holds if and only if $X$ and $Y$ are independent.

# Conditional Mutual information (CMI)

- Generalize MI for more variables.

$$\mathrm{CMI}(X_1, Y \mid X_2) = \mathrm{MI}(X_1 X_2, Y) - \mathrm{MI}(X_2, Y).$$

- Conditioned on the knowledge of $X_2$, how much extra information of $Y$ could $X_1$ provide.
- Can be used to describe the causal effect of $X_1 \to Y$ if $X_1$ and $X_2$ cause $Y$.
- $\mathrm{CMI}(X_1, Y \mid X_2) \geq 0$. Equality holds if and only if $X_1$ and $Y$ are independent conditioned on $X_2$. This means that with the knowledge of $X_2$, $X_1$ contains no new knowledge of $Y$.

# Problem of CMI

- CMI measures unique information. When $X_1 \approx X_2$, they contain nearly the same information of $Y$. Both $\mathrm{CMI}(X_1, Y \mid X_2)$ and $\mathrm{CMI}(X_2, Y \mid X_1)$ are very small.
- Cannot distinguish:

$$\begin{tikzcd}
X_1 \arrow[dd] \arrow[dr, dashed] & \\
 & Y \\
X_2 \arrow[ur] &
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X_1 \arrow[dd] \arrow[dr] & \\
 & Y \\
X_2 \arrow[ur, dashed] &
\end{tikzcd}$$

$X_2 = X_1 + \epsilon$, $Y = X_2 + \delta$ $\qquad$ $X_2 = X_1 + \epsilon$, $Y = X_1 + \delta$

$X_1 \perp\!\!\!\perp Y \mid X_2$ $\qquad$ $X_2 \perp\!\!\!\perp Y \mid X_1$

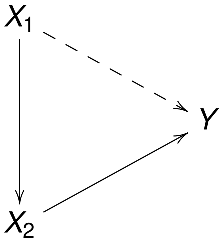
$\epsilon$ and $\delta$ are independent small noises.

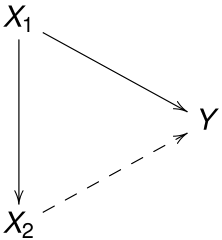
- $\mathrm{CMI}(X_1, Y \mid X_2)$ is 0 in the first case, and very small in the second case.

# New methods

- Utilize the slight difference between $X_1$ and $X_2$.
- Causal strength (CS) and part mutual information (PMI).
- $$\mathrm{CS}(X_1, Y) = \sum_{x_1, x_2, y} \mathbb{P}(x_1, x_2, y) \log \frac{\mathbb{P}(y \mid x_1, x_2)}{\sum_{x_1'} \mathbb{P}(y \mid x_1', x_2)\mathbb{P}(x_1')}.$$
- $$\mathrm{PMI}(X_1, Y \mid X_2) = \sum_{x_1, x_2, y} \mathbb{P}(x_1, x_2, y) \log \frac{\mathbb{P}(x_1, y \mid x_2)}{\sum_{x_1'} \mathbb{P}(y \mid x_1', x_2)\mathbb{P}(x_1') \sum_{y'} \mathbb{P}(x_1 \mid x_2, y')\mathbb{P}(y')}.$$

# New methods




$X_2 = X_1 + \epsilon$, $Y = X_2 + \delta$
$X_1 \perp\!\!\!\perp Y \mid X_2$

$X_2 = X_1 + \epsilon$, $Y = X_1 + \delta$
$X_2 \perp\!\!\!\perp Y \mid X_1$

$\epsilon$ and $\delta$ are independent small noises.
CS$(X_1, Y)$ and PMI$(X_1, Y \mid X_2)$ are 0 in the first case, and relatively large in the second case.

# Problem of new methods

- $$\begin{tikzcd} X_1 \arrow[dd] \arrow[dr, dashed] & \\ & Y \\ X_2 \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} X_1 \arrow[dd] \arrow[dr] & \\ & Y \\ X_2 \arrow[ur, dashed] & \end{tikzcd}$$
  $X_2 = X_1 + \epsilon,\ Y = X_2 + \delta$ $\qquad$ $X_2 = X_1 + \epsilon,\ Y = X_1 + \delta$
  $X_1 \perp\!\!\!\perp Y \mid X_2$ $\qquad$ $X_2 \perp\!\!\!\perp Y \mid X_1$
  $\epsilon$ and $\delta$ are independent small noises.
- These two joint distributions are almost the same, but the resulting causal effects are very different.
- CS and PMI may not be continuous with joint distribution (under total variation distance) when both $\{X_1\}$ and $\{X_2\}$ have all the information of $Y$ contained in $\{X_1, X_2\}$.

# Markov boundary (MB)

Assume we have cause variables $\mathcal{S} = \{X_1, \cdots, X_n\}$ and the result variable $Y$. A Markov boundary of $Y$, $\mathcal{S}_1$, is a subset of $\mathcal{S}$, which is minimal, and keeps its information of $Y$.

$$\mathrm{MI}(\mathcal{S}_1, Y) = \mathrm{MI}(\mathcal{S}, Y),$$

$$\forall \mathcal{S}_2 \subsetneqq \mathcal{S}_1, \quad \mathrm{MI}(\mathcal{S}_2, Y) < \mathrm{MI}(\mathcal{S}, Y).$$

This means $Y \perp\!\!\!\perp \mathcal{S} \backslash \mathcal{S}_1 \mid \mathcal{S}_1$.

# Markov boundary (MB)

- MB may not be unique. (Set $X_1 = X_2$ in the above example.)
- Assume MB is unique. A cause variable inside MB has positive irreplaceable predict power of $Y$, and a cause variable outside MB has zero irreplaceable predict power of $Y$. Therefore the unique MB should be exactly all the cause variables with positive causal effect.

# Problem of new methods

- When there are multiple MB, both CS and PMI are not directly defined. (Contains $0/0$ in the expression.)
- Try to use continuation: choose a sequence of distributions (for which CS and PMI are defined) converging to the original distribution, and check whether the corresponding CS and PMI converge.

# Problem of new methods

**Theorem**

*In any arbitrarily small neighborhood of a distribution with multiple MB, CS (also PMI) can take any value in an interval with positive length.*

- When there are multiple MB, both CS and PMI cannot be well-defined.
- Similar to the behavior of a complex function near an essential singularity.
- Calculating CS and PMI in such case is not numerically feasible.

# Sketch of the proof

- Construct two sequences of distributions, both of which converge to the original distribution.
- CS (or PMI) of two sequences always exist, but converge to different values.
- Distribution of one sequence can continuously transform into distribution of the other sequence, during which CS (or PMI) is always defined.

# Purpose

- We have cause variables $\mathcal{S} = \{X_1, X_2, \cdots, X_n\}$ and result variable $Y$.
- Our purpose is to quantify the effect of a causal relationship $X_1 \rightarrow Y$.
- We propose several criteria for a “good” causal quantity.
- We focus on the case where MB is unique. In such case, the unique MB should be exactly all the variables with positive causal effect.

# Criteria for quantifying causal effect

- C0. The effect of $X_1 \to Y$ is identifiable from the joint distribution of cause variables and result variable.
- C1. If there is unique MB $\mathcal{M}$, and $X_1 \notin \mathcal{M}$, then the effect of $X_1 \to Y$ is 0.
- C2. If there is unique MB $\mathcal{M}$, and $X_1 \in \mathcal{M}$, then the effect of $X_1 \to Y$ is at least CMI$(X_1, Y \mid \mathcal{M} \backslash \{X_1\})$.
- C3. The effect of $X_1 \to Y$ is a continuous function of the joint distribution.
- CMI fails in C2. CS and PMI fail in C3.

# An impossibility theorem

**Theorem**

*Assume in a distribution, $Y$ has multiple MB. $X_1$ belongs to at least one MB, but not all MB. Then in any neighborhood of this distribution, the effect of $X_1 \to Y$ cannot be defined while satisfying criteria C0–C3.*

# Sketch of proof

For variable $X_1$, we define $X_1$ with $\epsilon$-noise to be $X_1^\epsilon$, which equals $X_1$ with probability $1 - \epsilon$, and equals an independent noise with probability $\epsilon$. Denote all cause variables by $\mathcal{S}$.

**Lemma (Strict Data Processing Inequality)**

*$\mathcal{S}_1$ is a group of variables without $X_1, Y$. If we add $\epsilon$-noise on $X_1$ to get $X_1^\epsilon$, then $CMI(X_1^\epsilon, Y \mid \mathcal{S}_1) \leq CMI(X_1, Y \mid \mathcal{S}_1)$, and the equality holds if and only if $CMI(X_1, Y \mid \mathcal{S}_1) = 0$.*

**Lemma**

*Assume $Y$ has multiple MB. For one MB $\mathcal{M}_0$, if we add $\epsilon$ noise on all variables of $\mathcal{S} \backslash \mathcal{M}_0$, then in the new distribution, $\mathcal{M}_0$ is the unique MB.*

# Sketch of proof

- Assume $X_1 \in \mathcal{M}_0$, $X_1 \notin \mathcal{M}_1$ for MB $\mathcal{M}_0, \mathcal{M}_1$.
- We can add $\epsilon$-noise on $\mathcal{S} \backslash \mathcal{M}_1$, such that $\mathcal{M}_1$ is the unique MB. Criterion C1 shows that the effect of $X_1^\epsilon \to Y$ is 0.
- We can add $\epsilon$-noise on $\mathcal{S} \backslash \mathcal{M}_0$, such that $\mathcal{M}_0$ is the unique MB. Criterion C2 shows that the effect of $X_1 \to Y$ is at least $\mathrm{CMI}(X_1, Y \mid \mathcal{M}_0 \backslash \{X_1\}) > 0$.
- Let $\epsilon \to 0$. Criterion C3 shows that the effect of $X_1 \to Y$ should be at least $\mathrm{CMI}(X_1, Y \mid \mathcal{M}_0 \backslash X)$, and should be 0.

# Summary

- Quantifying causal effect with multiple MB is an essentially ill-posed problem.
- When a distribution with unique MB is close to another distribution with multiple MB, a reasonable causal quantity is either very small (CMI) or fluctuate violently (CS, PMI). Therefore in such case, quantitative method is not feasible.
- A practical problem: detecting whether MB is unique from data.

Algorithm 1: An assumption-free algorithm for determining the uniqueness of MB

(1) **Input**

Observations of $\mathcal{S} = \{X_1, \ldots, X_k\}$ and $Y$

(2) **Set** $\mathcal{E} = \emptyset$

(3) **For** $i = 1, \ldots, k$,

Test whether $X_i \perp\!\!\!\perp Y \mid \mathcal{S} \setminus \{X_i\}$

**If** $X_i \not\!\perp\!\!\!\perp Y \mid \mathcal{S} \setminus \{X_i\}$

$\mathcal{E} = \mathcal{E} \cup \{X_i\}$

(4) **If** $Y \perp\!\!\!\perp \mathcal{S} \mid \mathcal{E}$

**output:** $Y$ has a unique MB

**Else**

**output:** $Y$ has multiple MB

Algorithm 2: An assumption-free algorithm for producing one MB

(1) **Input**

Observations of $\mathcal{S} = \{X_1, \ldots, X_k\}$ and $Y$

(2) **Set** $\mathcal{M}_0 = \mathcal{S}$

(3) **Repeat**

**Set** $X_0 = \arg\min_{X \in \mathcal{M}_0} \Delta(X, Y \mid \mathcal{M}_0 \setminus \{X_i\})$

**If** $X_0 \perp\!\!\!\perp Y \mid \mathcal{M}_0 \setminus \{X_0\}$

**Set** $\mathcal{M}_0 = \mathcal{M}_0 \setminus \{X_0\}$

**Until** $X_0 \not\perp\!\!\!\perp Y \mid \mathcal{M}_0 \setminus \{X_0\}$

(4) **Output** $\mathcal{M}_0$ is a MB

Algorithm 3: A general algorithm for determining the uniqueness of MB

(1) **Input**

  Observations of $\mathcal{S} = \{X_1, \ldots, X_k\}$ and $Y$

  Algorithm $\Omega$ which can correctly produce one MB

(2) **Set** $\mathcal{M}_0 = \{X_1, \ldots, X_m\}$ to be the result of Algorithm $\Omega$ on $\mathcal{S}$

(3) **For** $i = 1, \ldots, m$,

  **Set** $\mathcal{M}_i$ to be the result of Algorithm $\Omega$ on $\mathcal{S} \mid \{X_i\}$

  **If** $Y \perp\!\!\!\perp \mathcal{M}_0 \mid \mathcal{M}_i$

    **Output** $Y$ has multiple MB

    **Terminate**

(4) **Output** $Y$ has a unique MB

Algorithm 4: An assumption-free algorithm for determining the uniqueness of MB

(1) **Input**
  Observations of $\mathcal{S} = \{X_1, \ldots, X_k\}$ and $Y$
(2) **Set** $\mathcal{M}_0 = \{X_1, \ldots, X_m\}$ to be the result of Algorithm 2 on $\mathcal{S}$
(3) **For** $i = 1, \ldots, m$,
  **If** $Y \perp\!\!\!\perp X_i \mid \mathcal{S} \setminus \{X_i\}$
  **If** $X_i \not\perp\!\!\!\perp Y \mid \mathcal{S} \setminus \{X_i\}$
    **Output** $Y$ has multiple MB
    **Terminate**
(4) **Output** $Y$ has a unique MB

# Algorithms performances

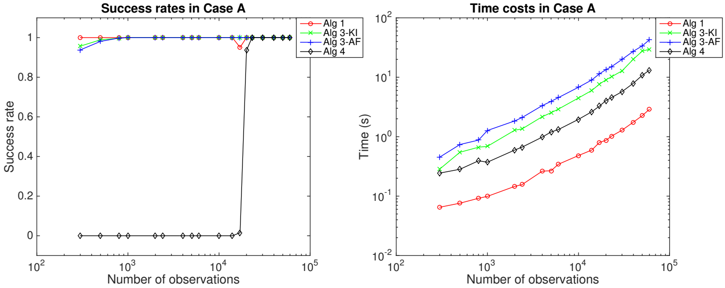



Figure: Success rates and average time costs per execution (in seconds) of Algorithms 1 (red circle), 3-KI (green ‘x’), 3-AF (blue ‘+’), 4 (black diamond) with different numbers of observations in Case A. Number of observations and time costs are in logarithm.

# Algorithms performances

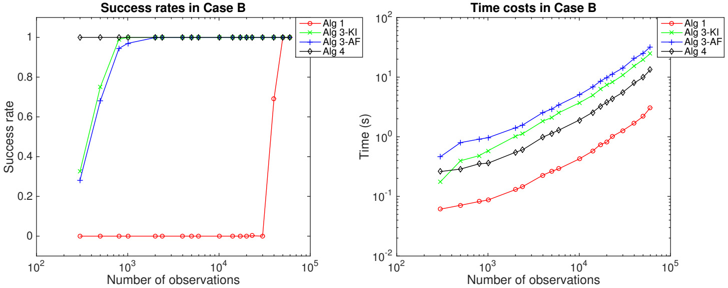



Figure: Success rates and average time costs per execution (in seconds) of Algorithms 1 (red circle), 3-KI (green ‘x’), 3-AF (blue ‘+’), 4 (black diamond) with different numbers of observations in Case B. Number of observations and time costs are in logarithm.

# Algorithms performances

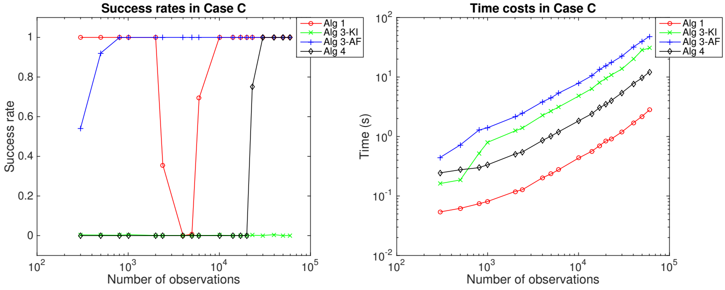



Figure: Success rates and average time costs per execution (in seconds) of Algorithms 1 (red circle), 3-KI (green 'x'), 3-AF (blue '+'), 4 (black diamond) with different numbers of observations in Case C. Number of observations and time costs are in logarithm.

# References


- CMI: DOBRUSHIN, R. L. (1963). General formulation of Shannon's main theorem in information theory. Amer. Math. Soc. Trans. 33, 323–438.
- CS: JANZING, D., BALDUZZI, D., GROSSE-WENTRUP, M. & SCHÖLKOPF, B. (2013). Quantifying causal influences. Ann. Stat. 41, 2324–2358.
- PMI: ZHAO, J., ZHOU, Y., ZHANG, X. & CHEN, L. (2016). Part mutual information for quantifying direct associations in networks. Proc. Natl. Acad. Sci. 113, 5130–5135.
- MB: PEARL, J. (1988). Probabilistic Reasoning in Intelligent Systems: Networks of Plausible Inference.

Thank you!