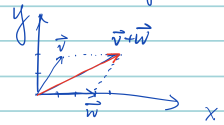
Main Structure of Linear Algebra : Vector Space.

$\mathbb{R}^2 = \{ \binom{x}{y} \mid x, y \in \mathbb{R} \}$



$\vec{v} = \binom{1}{2}$ $\quad \vec{v}+\vec{w} = \binom{1}{2} + \binom{3}{0} = \binom{1+3}{2+0} = \binom{4}{2}$

$\vec{w} = \binom{3}{0}$

§ 1.1 Vector spaces

Defn. A vector space V is a collection of objects called vectors along w/ 2 operations : $\begin{cases} \text{Vector addition} \\ \text{Scalar multiplication} \end{cases}$

$2\binom{1}{2} = \binom{2\cdot 1}{2\cdot 2} = \binom{2}{4}$

if scalars are real #s, then real V.S.

. . . . complex #s, . . . Complex V.S.

Axioms of V.S.

0). Closedness $\begin{cases} \text{Addition} & \text{if } \vec{u}, \vec{v} \in V \text{, then } \vec{u}+\vec{v} \in V \\ \text{Scalar Mult.} & \text{if } \vec{v} \in V \text{ then } \alpha\vec{v} \in V, \ \forall \alpha \in \mathbb{R} \text{ or } \mathbb{C} \end{cases}$

1) Addition for $\vec{u}, \vec{v}, \vec{w} \in V$,

a) Commutativity : $\vec{v}+\vec{u} = \vec{u}+\vec{v}$ $\quad \binom{1}{2} + \binom{3}{4} = \binom{4}{6} = \binom{3}{4} + \binom{1}{2}$

b) Associativity : $(\vec{v}+\vec{u}) + \vec{w} = \vec{v} + (\vec{u}+\vec{w})$

c) Zero vector : $\exists$ a vector $\vec{0}$ s.t. $\vec{v}+\vec{0} = \vec{0}+\vec{v} = \vec{v}$

d) Additive Inverse : $\forall \vec{v} \in V$, $\exists \vec{u} \in V$, s.t. $\vec{u}+\vec{v} = \vec{0}$

$\vec{u}$ denote by $-\vec{v}$

2) Multiplication

a) Identity $1\vec{v} = \vec{v}$, $\forall \vec{v} \in V$

b) Associativity $\alpha(\beta\vec{v}) = (\alpha\beta)\vec{v}$

c) Distributivity $\alpha(\vec{v}+\vec{u}) = \alpha\vec{v} + \alpha\vec{u}$

$\vec{v}(\alpha+\beta) = \alpha\vec{v} + \beta\vec{v}$

Ex. $\mathbb{R}^2$ is a vector space ---> $\mathbb{R}^n$ is a V.S.

real #

$$V = \mathbb{R}^n = \left\{ \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \middle| v_i \in \mathbb{R} \right\}$$

$$\vec{v} + \vec{u} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} + \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} v_1+u_1 \\ v_2+u_2 \\ \vdots \\ v_n+u_n \end{pmatrix}$$ component-wise

$$\lambda \vec{v} = \begin{pmatrix} \lambda v_1 \\ \lambda v_2 \\ \vdots \\ \lambda v_n \end{pmatrix}$$

Side note: All scalars i.e. $\mathbb{R}$ satisfy the following:

defn of a ~~field~~ ring ⋮

1. add. & mult. are closed
2. " " are commutative, associative & distrib.
3. 0 is the additive identity
4. 1 is the mult. identity.

def:

Ex. $V = \left\{ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \middle| v_1, v_2 \in \mathbb{R} \right\}$, $\alpha \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \alpha v_1 \\ \alpha v_2 \end{pmatrix}$, $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} + \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} v_1+w_1 \\ v_2-w_2 \end{pmatrix}$

Is this a V.S.?

$\vec{w} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$\vec{v} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\vec{w} + \vec{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

$\neq$

$\vec{v} + \vec{w} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

It does not satisfy add. commutativity.

$\Rightarrow$ V is not a V.S.

Prop. In a V.S. V, the additive identity $\vec{0}$ is unique.

To proof uniqueness: Suppose 2 things satisfy the desired property & prove they have to be the same.

Pf. Suppose $\vec{0}$ & $\vec{0}'$ are both additive identity in V.S

$$\Rightarrow \begin{cases} \vec{0}+\vec{v} = \vec{v} & \xrightarrow{\vec{v} := \vec{0}'} \quad \underline{\vec{0}+\vec{0}'} = \underline{\vec{0}'} \\ \vec{0}'+\vec{v} = \vec{v} & \xrightarrow{v := \vec{0}\text{; commut.}} \quad \underline{\vec{0}+\vec{0}'} = \underline{\vec{0}} \end{cases}$$

$$\Rightarrow \vec{0}' = \vec{0}$$

Prop. Additive inverse is unique.

Prop. $-\vec{v} = (-1)\vec{v}$

Pf. We know that $-\vec{v}$ is the unique vec. s.t. $\vec{v}+(-\vec{v}) = \vec{0}$

$$\vec{v}+(-1)\vec{v} = 1\vec{v}+(-1)\vec{v} \qquad \text{by mult. id.}$$

$$= (1+(-1))\vec{v} \qquad \text{by distributivity}$$

$$= 0\vec{v} = \vec{0} \qquad \text{by defn of zero vec.}$$

$$\Rightarrow -\vec{v} = (-1)\vec{v} \qquad \text{by uniqueness of add. inv.}$$

$$\vec{v} + \bigcirc = \vec{0}$$

add. inv.

§ 1.2

Defn. Linear Combination of $\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_n \in V$ is a sum of the form:

$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \alpha_3\vec{v}_3 + \cdots + \alpha_n\vec{v}_n$$

$$= \sum_{i=1}^{n} \alpha_i \vec{v}_i \quad , \quad (\alpha_i\text{'s are Scalars})$$

Ex. $\mathbb{R}^2$, $\vec{v}_1 = \binom{1}{2}$, $\vec{v}_2 = \binom{-1}{4}$

one linear combination is

$$6\binom{1}{2} + (-2)\binom{-1}{4} = \binom{6+2}{12-8} = \binom{8}{4}$$

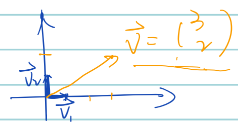
a linear combination of $\vec{v}_1$ & $\vec{v}_2$

Defn. A collection of vectors $\vec{v}_1, \vec{v}_2, \cdots \vec{v}_n \in V$ is called a basis for V if any $\vec{v} \in V$ admits a unique representation as a linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$

i.e. $\vec{v} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_n \vec{v}_n$ for some unique choice of $\alpha_1, \alpha_2, \dots, \alpha_n$.

The $\alpha_1, \alpha_2, \dots, \alpha_n$ is called the coordinates of $\vec{v}$ w.r.t. the basis $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$

Ex. $\mathbb{R}^2$, one basis $\{ \underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\vec{v}_1}, \underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{\vec{v}_2} \}$ standard basis.



$$\vec{v} = 3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$= 3 \vec{v}_1 + 2 \vec{v}_2$$

Other basis ? $\vec{w} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\vec{u} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$ ✓

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \in \mathbb{R}^2$

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \alpha_1 + 2\alpha_2 \\ 2\alpha_1 \end{pmatrix}$$

$$\begin{cases} v_1 = \alpha_1 + 2\alpha_2 \\ v_2 = 2\alpha_1 \end{cases} \Rightarrow \begin{cases} \alpha_1 = \frac{v_2}{2} \\ \alpha_2 = \frac{v_1}{2} - \frac{v_2}{4} \end{cases}$$

$\downarrow$

unique solution ✓

$\Rightarrow$ combination is unique

$\Rightarrow \{\vec{w}, \vec{u}\}$ is a basis of $\mathbb{R}^2$

Side Note.

$A, B \in M_{m \times n}(\mathbb{R})$

$(A+B)_{i,j} = A_{i,j} + B_{i,j}$ ✓

$(\alpha A)_{i,j} = \alpha A_{i,j}$ ✓

For matrix A, the transpose of A, written as A^T is defined by $(A^T)_{i,j} = A_{j,i}$

Ex: $A = \begin{pmatrix} 1 & 4 & 3 \\ 2 & 6 & 7 \end{pmatrix}$ $\quad A^T = \begin{pmatrix} 1 & 2 \\ 4 & 6 \\ 3 & 7 \end{pmatrix}$

6/13.

Ex. T/F: $\underbrace{\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}}_{}$ is a basis in $\mathbb{R}^2$? spanning set of

$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$\Rightarrow \begin{cases} v_1 = \alpha_1 - \alpha_2 + 2\alpha_3 \\ v_2 = 2\alpha_1 + \alpha_2 - \alpha_3 \end{cases} \quad \Rightarrow 5\alpha_1 + \alpha_2 = v_1 + 2v_2$

$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \Rightarrow 5\alpha_1 + \alpha_2 = 6 \quad \Rightarrow \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ or

$\frac{20}{3} + \alpha_2 = \frac{18}{3}$ $\quad \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 4/3 \\ -2/3 \\ 0 \end{pmatrix}$

$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 1 \underset{\vec{v}_1}{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} + 1 \underset{\vec{v}_2}{\begin{pmatrix} -1 \\ 1 \end{pmatrix}} + 1 \underset{\vec{v}_3}{\begin{pmatrix} 2 \\ -1 \end{pmatrix}}$

$\Rightarrow$ Not unique. $\Rightarrow$ not a basis.

Defn. Given $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$, the span is all vectors $\vec{v}$ that can be written as a linear combination of $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$

Ex. Given $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ in $\mathbb{R}^2$

$\mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} \alpha \\ 0 \end{pmatrix} \mid \alpha \in \mathbb{R} \right\}$

If $\mathrm{Span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\} = V$, then we say $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a spanning set / generating set / complete set.

Given $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$: spanning set of $\mathbb{R}^2$

$\mathrm{Span}(\dots) = \mathbb{R}^2$ $\quad \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 15 \end{pmatrix}, \dots$

$2\begin{pmatrix}1\\0\end{pmatrix}+2\begin{pmatrix}0\\1\end{pmatrix}$

V itself is a spanning set of V.

$\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ is a spanning set of $\mathbb{R}^2$

$\left\{\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}-1\\1\end{pmatrix}, \begin{pmatrix}2\\-1\end{pmatrix}\right\}$ is . . . . $\mathbb{R}^2$

Basis: 1. Span.
2. uniqueness / linear independence.

Defn. Vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly independent if

$$\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \dots + \alpha_n\vec{v}_n = \vec{0} \text{ implies } \begin{pmatrix}\alpha_1\\\alpha_2\\\vdots\\\alpha_n\end{pmatrix} = \begin{pmatrix}0\\0\\\vdots\\0\end{pmatrix}$$

AKA. there is no nontrivial linear combinations of zero vector.

Prop. $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ are linearly dependent

$\iff$ Some $\vec{v}_k$ is a linear combination of others
iff.

Pf.

$\Rightarrow$ Suppose $\vec{v}_1, \vec{v}_2, \dots \vec{v}_n$ are linearly dep.

$\Rightarrow \exists$ coefficients $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \dots + \alpha_k\vec{v}_k + \dots + \alpha_n\vec{v}_n = 0$ s.t. $\alpha_k \neq 0$ for some k

$$\alpha_k\vec{v}_k = \sum_{\substack{i=1\\i\neq k}}^{n} -\alpha_i\vec{v}_i \Rightarrow \vec{v}_k = \sum_{\substack{i=1\\i\neq k}}^{n} \frac{-\alpha_i}{\alpha_k}\vec{v}_i = -\frac{\alpha_1}{\alpha_k}\vec{v}_1 + -\frac{\alpha_2}{\alpha_k}\vec{v}_2 + \dots + -\frac{\alpha_n}{\alpha_k}\vec{v}_n$$

$\Leftarrow$ Suppose $\exists\vec{v}_k$ can be written as a L.C. of the others

i.e. $\vec{v}_k = \sum_{\substack{i=1\\i\neq k}}^{n} \beta_i\vec{v}_i$ $\leftarrow$ $\vec{v}_k = \beta_1\vec{v}_1 + \beta_2\vec{v}_2 + \dots + \beta_n\vec{v}_n$

$-\sum$ $\quad -\sum$

$$1\cdot\vec{v}_k + \sum_{\substack{i=1\\i\neq k}}^{n} -\beta_i\vec{v}_i = \vec{0}$$

$\downarrow$

$\alpha_k = 1 \Rightarrow$ L.C. is nontrivial $\Rightarrow \vec{v}_1, \dots \vec{v}_n$ linearly dep.

$\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ $\quad \alpha_1\begin{pmatrix}1\\0\end{pmatrix} + \alpha_2\begin{pmatrix}0\\1\end{pmatrix} = \vec{0}$

$\begin{pmatrix}\alpha_1\\ \end{pmatrix} = \begin{pmatrix}0\\ \end{pmatrix}$

$\circledast \quad \begin{pmatrix} \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \end{pmatrix}$

$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\}$ → linearly dep.

$\alpha_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \alpha_3 \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \vec{0}$

$\Rightarrow$ ① $\alpha_1, \alpha_2, \alpha_3 = 0$

② $1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + (-5) \begin{pmatrix} -1 \\ 1 \end{pmatrix} + (-3) \begin{pmatrix} 2 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 1 + 5 - 6 \\ 2 - 5 + 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

when $\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -5 \\ -3 \end{pmatrix}$

$\ast$ Non-trivial Solution

$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$

a basis of $\mathbb{R}^2$

Prop. $\vec{v}_1, \cdots \vec{v}_n$ are a basis of V $\Leftrightarrow$ 1. they span V

2. they are linearly indep.

$\ast$ Used as defn. of basis.

Prop. Every spanning sets $\{\vec{v}_1, \dots, \vec{v}_n\}$ contains a basis of V.

① Span $\mathbb{R}^2$? $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$ is a basis of $\mathbb{R}^2$ $\Leftarrow$

$\begin{cases} v_1 = \alpha_1 - \alpha_2 & ① \\ v_2 = 2\alpha_1 + \alpha_2 & ② \end{cases} \Rightarrow v_1 + v_2 = 3\alpha_1$

$\Rightarrow \begin{cases} \alpha_1 = \dfrac{v_1 + v_2}{3} \\ \alpha_2 = \dfrac{v_1 + v_2}{3} - v_1 = \dfrac{v_2 - 2v_1}{3}. \end{cases}$

spans ✓.

② $\alpha_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \vec{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{cases} \alpha_1 = \dfrac{0+0}{3} = 0 \\ \alpha_2 = \dfrac{0-2 \cdot 0}{3} = 0 \end{cases} \Rightarrow \begin{pmatrix} 1 \\ 2 \end{pmatrix} \& \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ lin. indep.

§1.3 Linear Transformation.

Defn. Let V & W be vector spaces. A function (transformation) $T: V \to W$ is linear if $\forall \vec{v}, \vec{u} \in V$, scalars α

① $T(\vec{v}+\vec{u}) = T(\vec{v}) + T(\vec{u})$

② $T(\alpha\vec{u}) = \alpha T(\vec{u})$

Ex. $T: \mathbb{R}^2 \to \mathbb{R}^2$ by $T\left(\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}\right) = \begin{pmatrix} 2u_1+u_2 \\ v_1 \end{pmatrix}$ Is this linear T?

$\Rightarrow \vec{v}, \vec{u} \in \mathbb{R}^2, \vec{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \vec{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, α is a scalar

① LHS: $T(\vec{v}+\vec{u}) = T\left(\begin{pmatrix} v_1+u_1 \\ v_2+u_2 \end{pmatrix}\right) = \begin{pmatrix} 2(v_1+u_1)+(v_2+u_2) \\ v_1+u_1 \end{pmatrix}$

RHS: $T(\vec{v}) + T(\vec{u}) = \begin{pmatrix} 2v_1+v_2 \\ v_1 \end{pmatrix} + \begin{pmatrix} 2u_1+u_2 \\ u_1 \end{pmatrix}$

$= \begin{pmatrix} 2v_1+v_2+2u_1+u_2 \\ v_1+u_1 \end{pmatrix} = \begin{pmatrix} 2(v_1+u_1)+(v_2+u_2) \\ v_1+u_1 \end{pmatrix}$

LHS = RHS

② LHS $= T(\alpha\vec{u}) = T\left(\begin{pmatrix} \alpha u_1 \\ \alpha u_2 \end{pmatrix}\right) = \begin{pmatrix} 2\alpha u_1+\alpha u_2 \\ \alpha u_1 \end{pmatrix}$

RHS $= \alpha T(\vec{u}) = \alpha \begin{pmatrix} 2u_1+u_2 \\ u_1 \end{pmatrix} = \begin{pmatrix} 2\alpha u_1+\alpha u_2 \\ \alpha u_1 \end{pmatrix}$

LHS = RHS

$\Rightarrow$ T is a linear transformation.

Ex. $T: \mathbb{R}^2 \to \mathbb{R}^2$, $T\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+1 \\ y \end{pmatrix}$

$\vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

LHS: $T(\vec{u}+\vec{v}) = T\left(\begin{pmatrix} 3 \\ 3 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ LHS $\neq$ RHS $\Rightarrow$ Not linear.

RHS: $T(\vec{u})+T(\vec{v}) = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

Ex. $T: \mathbb{R}^2 \to \mathbb{R}^2$ rotate θ radians ccw about $(0,0)$

$$T\left(\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\right) = \begin{pmatrix} v_1\cos\theta - v_2\sin\theta \\ v_1\sin\theta + v_2\cos\theta \end{pmatrix}$$

Prop: $T: V \to W$ is linear $\iff \forall \vec{v}, \vec{u} \in V$ & scalars α, β

$$T(\alpha\vec{u} + \beta\vec{v}) = \alpha T(\vec{u}) + \beta T(\vec{v})$$

Prop. If $T: V \to W$ is linear, then $T(\vec{0}) = \vec{0}$. $\not\Leftarrow$

$$\vec{0} + T(\vec{0}) = T(\vec{0}+\vec{0}) = T(\vec{0}) + T(\vec{0})$$

$$\underset{W}{\vec{0}} = T(\underset{V}{\vec{0}})$$

Prop If $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear and we know $T(\vec{e}_1), T(\vec{e}_2), \dots, T(\vec{e}_n)$
Then we can complete $T(\vec{v})$, $\forall \vec{v} \in \mathbb{R}^n$.

$*$ $\forall$ Linear transformation is determined by what it does on a basis.

$$\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \in \mathbb{R}^n \qquad \vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + \dots + v_n\vec{e}_n$$

$$\vec{e}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \vec{e}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

$$T(\vec{v}) = T(v_1\vec{e}_1 + \dots + v_n\vec{e}_n) = v_1T(\vec{e}_1) + v_2T(\vec{e}_2) + \dots + v_nT(\vec{e}_n)$$

by linearity

Ex. $T: \mathbb{R}^2 \to \mathbb{R}^2$
$$\begin{cases} T(\vec{e}_1) = T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 2 \\ 3 \end{pmatrix} \\ T(\vec{e}_2) = T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \end{cases}$$

$$T\left(\begin{pmatrix} 3 \\ -1 \end{pmatrix}\right) = T\left[3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + (-1)\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]$$
$$= 3T\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right) - 1\,T\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right) = 3\begin{pmatrix} 2 \\ 3 \end{pmatrix} + (-1)\begin{pmatrix} -1 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

$$T(\vec{v}) = A\vec{v} \qquad \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 6+1 \\ 9-4 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix}$$

Thm: $T: \mathbb{R}^n \to \mathbb{R}^m$ need to know

$$T(\vec{e}_1) = \vec{a}_1 \in \mathbb{R}^m = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$$
$$T(\vec{e}_2) = \vec{a}_2$$
$$\vdots$$
$$T(\vec{e}_n) = \vec{a}_n$$

$$\underset{m\times n}{A} = \overset{n}{\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix}}\ m$$

$$T(\vec{v}) = A\vec{v}$$

Defn. $A\vec{v} = \sum_{k=1}^{n} v_k \vec{a}_k = v_1\vec{a}_1 + v_2\vec{a}_2 + \dots + v_n\vec{a}_n$

k^{th} column of A

k^{th} entry of $\vec{v}$

Ex. $\underbrace{\begin{pmatrix} 1 & 0 & 3 \\ -1 & 5 & 2 \end{pmatrix}}_{\substack{A \\ 2\times 3}} \underbrace{\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}}_{\substack{\vec{v} \\ 3\times 1}} = 2\begin{pmatrix} 1 \\ -1 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 5 \end{pmatrix} + 1\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -10 \end{pmatrix}$

2×1

Defn. $\vec{r} = (r_1\ r_2\ \dots\ r_n)$ row vector, $\vec{c} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$ be column vector,

~~then~~ dot product is $\vec{r}\cdot\vec{c} = r_1c_1 + r_2c_2 + \dots + r_nc_n = \_ \in \mathbb{R}$

So for $A\vec{v}$, k^{th} entry is the k^{th} row of A dot product w/ $\vec{v}$.

Notation $T(\vec{v}) = A\vec{v}$, the matrix associate w/ T, often written as $[T]$. Sometimes just T.

Defn. If V, W are v.s. Let $\mathcal{L}(V,W) = \{T: V \to W \mid T$ is a linear trasformation$\}$

If $S, T \in \mathcal{L}(V,W)$, we can define $S+T$ by $(S+T)(\vec{v}) = S(\vec{v}) + T(\vec{v})$

If $T \in \mathcal{L}(V,W)$, α is a scalar, $(\alpha T)(\vec{v}) = \alpha(T(\vec{v}))$

Thm. $\mathcal{L}(V,W)$ is a Vector Space.

Check 8 axioms. $S+T$ is closed: is linear.

$(S+T)(\vec{v}) = \overset{\in W}{S(\vec{v})} + \overset{\in W}{T(\vec{v})} \in W$, Let $\vec{v}, \vec{u} \in V$.

$(S+T)(\alpha\vec{u}+\beta\vec{v}) = S(\alpha\vec{u}+\beta\vec{v}) + T(\alpha\vec{u}+\beta\vec{v})$

$= \alpha S(\vec{u}) + \beta S(\vec{v}) + \alpha T(\vec{u}) + \beta T(\vec{v})$

$= \alpha(S(\vec{u}) + T(\vec{u})) + \beta(S(\vec{v}) + T(\vec{v}))$

$= \alpha(S+T)(\vec{u}) + \beta(S+T)(\vec{v})$

$\Rightarrow S+T \in \mathcal{L}(V,W)$

Add. Commutativity ... ...

§ Composition of Linear Transformations

Suppose $T_1 : \mathbb{R}^n \to \mathbb{R}^m$, $T_2 : \mathbb{R}^m \to \mathbb{R}^k$ both linear.

$$(T_2 \circ T_1)\vec{v} = T_2(T_1(\vec{v}))$$

Prop. If T_1, T_2 are linear, $T_2 \circ T_1$ is linear.

Skipped Pf.

Q. If $T_2 \circ T_1 = T \Rightarrow [T_1], [T_2], [T]$

Matrix multiplication?

Defn. For matrices A & B, the entry in j^{th} row & k^{th} column of AB is $(AB)_{j,k} = (\text{row } j \text{ of } A \cdot \text{col } k \text{ of } B)$

$$\Rightarrow (AB)_{j,k} = \sum_{\ell=1}^{n} A_{j,\ell} B_{\ell,k}$$

Ex.

$$\begin{pmatrix} 1 & 0 & 3 \\ 2 & 4 & 0 \end{pmatrix} \begin{pmatrix} 5 & 0 \\ -1 & 2 \\ 0 & -3 \end{pmatrix} = \begin{pmatrix} 5 & -9 \\ 6 & 8 \end{pmatrix}$$

$$A_{m\times n} \quad B_{n\times k} \Rightarrow M_{m\times k}$$

Properties. A, B, C where product exists. * Not Commutative.

① Associativity $(AB)C = A(BC)$

② Distributivity $A(B+C) = AB + AC$; $(A+B)C = \underline{AC + BC}$

③ Scalar Multiplication $A(\alpha B) = \alpha(AB)$

Thm. If $T_1 : \mathbb{R}^n \to \mathbb{R}^m$ is linear & associated w/ matrix $A_{m\times n}$

$T_2 : \mathbb{R}^m \to \mathbb{R}^k$ 〃 〃 〃 $B_{k\times m}$

Then $T_2 \circ T_1 : \mathbb{R}^n \to \mathbb{R}^k$ associated with $B \times A_{k\times n}$

Thm. If $A \in M_{m\times n}$, ~~then~~ the map $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\vec{v}) = A\vec{v}$ is a linear transformation.

Pf: $T(\alpha\vec{v} + \beta\vec{u}) = A(\alpha\vec{v} + \beta\vec{u}) = \alpha A\vec{v} + \beta A\vec{u} = \alpha T(\vec{v}) + \beta T(\vec{u})$

Prop. $(AB)^T = B^T A^T$

Pf. $LHS = (AB)^T_{j,k} = (AB)_{k,j} = \sum_{l=1}^{n} A_{k,l} B_{l,j}$

$RHS = (B^T A^T)_{j,k} = \sum_{l=1}^{n} B^T_{j,l} A^T_{l,k} = \sum_{l=1}^{n} B_{l,j} A_{k,l}$

$\Big\}$ = b/c scalar mult. is commut.

$LHS = RHS$

§ 1.6 Invertable transformations

Defn. $\forall$ vector space V, $\exists$ an identity transformation $I_V : V \to V$, $I_V(\vec{v}) = \vec{v}$

Claim: I_V is linear. Check: $I_V(\alpha\vec{v} + \beta\vec{u}) = \alpha\vec{v} + \beta\vec{u} = \alpha I_V(\vec{v}) + \beta I_V(\vec{u})$

Prop. If $T: V \to W$ then $T \circ I_V = T$; $I_W \circ T = T$.

Defn. Matrix for $I: \mathbb{R}^n \to \mathbb{R}^n$, $[I] = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 1 \end{pmatrix}_{n\times n}$ $n\times n$ identity matrix.

Defn. Matrix A has a left inverse if $\exists$ matrix B s.t. $BA = I$
Matrix A has a right inverse if $\exists$ matrix C s.t. $AC = I$

Matrix A has a 2-sided inverse (A is invertible) if it has a left & right inverse.

Ex. $A = \underset{1\times 2}{(1 \ \ 1)}$ has right inverse $\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$

$(1 \ \ 1)\begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix} = (1)$ 1×1 id.

$(1 \ \ 1)\begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \end{pmatrix} = (1)$

Left inv? $\begin{pmatrix} a \\ b \end{pmatrix}(1 \ \ 1) = \begin{pmatrix} a & a \\ b & b \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{cases} a = 1 \\ a = 0 \end{cases} \longrightarrow$ no left inv.

Thm. If A has a left inv. B & a right inv. C. then B & C are equal & unique
i.e. $A^{-1} = B = C$.

Pf. We know $BA = I$, $AC = I$

$BAC = (BA)C = IC = C$
$BAC = B(AC) = BI = B$ $\Big| \Rightarrow B = C = B'$ uniqueness
$B'AC = \cdots\cdots C$

Ex. I invertible. $I^{-1} = I$ b/c $II = I = II^{-1}$

Ex. $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$

Left $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2-1 & 2-2 \\ -1+1 & -1+2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Right $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2-1 & -1+1 \\ 2-2 & -1+2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Defn. Linear transf. $T: V \to W$ has a left inv. if $\exists$ a L.T. $S: W \to V$ s.t. $S \circ T = I_V$.

T has a right inv. if $\exists$ L.T. $R: W \to V$ s.t. $T \circ R = I_W$

If T has a 2 sided inv., we say T is invertible & has both S & R

(As before S & R are unique & agree)

If $T(x) = 2\vec{x}$ then $T^{-1}(x) = \frac{1}{2}\vec{x}$

e^x ... $\ln(x)$

If $T: \mathbb{R}^n \to \mathbb{R}^m$ is linear, with matrix A. T is invertible $\Leftrightarrow$ A invertible.

T_θ is rotation, ccw, by angle θ. $T_\theta^{-1} = T_{(-\theta)}$

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \qquad A_\theta^{-1} = A_{-\theta} = \begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix}$$

$$= \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin(\theta) & \cos\theta \end{pmatrix}$$

$$A_\theta A_{-\theta} = \begin{pmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \cos\theta\sin\theta \\ \sin\theta\cos\theta - \sin\theta\cos\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thm. A, B are invertible, AB is defined $\Rightarrow$ (AB) is invertible.

and $(AB)^{-1} = B^{-1}A^{-1}$ cannot be reversed.

$$(AB)(AB)^{-1} = (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AIA^{-1} = AA^{-1} = I = (B^{-1}A)(AB)$$

Ex. $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \end{pmatrix}$ not invertible $\quad AB = \begin{pmatrix} -1 & 13 \\ -1 & 2 \end{pmatrix} \quad AB^{-1} = \begin{pmatrix} 8/5 & \frac{-13}{5} \\ \frac{1}{5} & \frac{-1}{5} \end{pmatrix}$

$B = \begin{pmatrix} 1 & 2 \\ -1 & 4 \\ 0 & 1 \end{pmatrix}$ not invertible

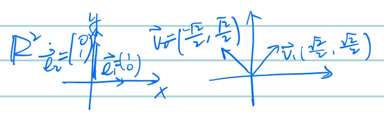
Thm. If A is invertible, A^T is invertible. $(A^T)^{-1} = (A^{-1})^T$ $\quad + (AB)^T = B^T A^T$

right $A^T(A^T)^{-1} = A^T(A^{-1})^T = (A^{-1}A)^T = I^T = I$

left $(A^T)^{-1}A^T = (A^{-1})^T A^T = (AA^{-1})^T = I$

Thm. If A invertible, then A^{-1} is invertible. $(A^{-1})^{-1} = A$.

Defn. A linear transformation $T: V \to W$ is called an isomorphism if T is invertible.
and we call V & W are isomorphic. $V \cong W$.



by T_{45°

Thm. $T: V \to W$ an isomorphism. Then

$\{\vec{v}_1, \vec{v}_2, \dots \vec{v}_n\}$ is a basis of $V \iff \{T(\vec{v}_1), T(\vec{v}_2), \dots, T(\vec{v}_n)\}$ is a basis of W.

$P_n(\mathbb{R})$ $\quad 1+\alpha_1 t + \alpha_2 t^2 + \dots + \alpha_n t^n$

$\{1, t, t^2, \dots t^n\}$

$1 + \beta_1 t + \beta_2 t^2 + \dots + \beta_n t^n$

$T: P_n(\mathbb{R}) \to P_{n-1}(\mathbb{R})$ $\qquad (x^3 + 5x^2 + 1)' = 3x^2 + 10x$

$T(p) = p'$

Pf. Let $p, q \in P_n(\mathbb{R})$ $\quad T(\alpha p + \beta q) = (\alpha p + \beta q)'$

$= (\alpha p)' + (\beta q)'$ by property of derivatives

$= \alpha p' + \beta q'$

$= \alpha T(p) + \beta T(q)$

[Derivative operation is linear.]

Jacobian 1^{st}

Hessian 2^{nd}

* Isomorphism sends a basis to a basis.

Any linear map sending a basis to a basis is isomorphism.

Thm: Let $T: V \to W$ be linear transformation & let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis for V & $\{\vec{w}_1, \dots, \vec{w}_n\}$ be a basis for W.

If $T(\vec{v}_k) = \vec{w}_k$, $\forall k$. Then T is isomorphism.

Pf. Set $T^{-1}(\vec{w}_1) = \vec{v}_1$, $T^{-1}(\vec{w}_2) = \vec{v}_2$, $\dots$, $T^{-1}(\vec{w}_n) = \vec{v}_n$.

$\underline{T^{-1} \circ T(\vec{v})} = T^{-1}(T(\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \dots + \alpha_n \vec{v}_n))$

$= T^{-1}(\alpha_1 T(\vec{v}_1) + \alpha_2 T(\vec{v}_2) + \dots + \alpha_n T(\vec{v}_n))$

$= T^{-1}(\alpha_1 \vec{w}_1 + \alpha_2 \vec{w}_2 + \dots + \alpha_n \vec{w}_n)$

$= \alpha_1 T^{-1}(\vec{w}_1) + \alpha_2 T^{-1}(\vec{w}_2) + \dots + \alpha_n T^{-1}(\vec{w}_n)$

$= \alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n = \underline{\vec{v}} = I_V(\vec{v})$

$T \circ T^{-1}(\vec{w}) = \vec{w} = I_W(\vec{w})$

$\Rightarrow T$ is invertible

Corollary: An $m \times n$ matrix is invertible $\iff$ it's columns form a basis for $\mathbb{R}^m$.

span $\mathbb{R}^m$ + linear indep.

$A\vec{e}_1 = (\vec{a}_1 \vdots \vec{a}_2 \vdots \dots \vdots \vec{a}_n)\,\vec{e}_1 = \vec{a}_1$

Ex. $P_n(\mathbb{R}) \cong \mathbb{R}^{n+1}$

$\{1, t, t^2, \dots, t^n\} \to \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n, \vec{e}_{n+1}\}$

$T(1) = \vec{e}_1,\ T(t) = \vec{e}_2, \dots\ T(t^n) = \vec{e}_{n+1}$

$P_2(\mathbb{R}) \cong \mathbb{R}^3$

$3t^2 + bt - 4 \to \begin{pmatrix} -4 \\ b \\ 3 \end{pmatrix}$

§ 1.7 Subspaces

Defn. A subspace of v.s. V is a non-empty subset $X \subseteq V$ which is closed under addition & scalar mult.

$\vec{0}$ inside.

if $\vec{v}, \vec{w} \in X$, then $\vec{v} + \vec{w} \in X$

if $\vec{v} \in X$, then $\alpha \vec{v} \in X$.

A subspace X with operations inherited from V is a vector space.

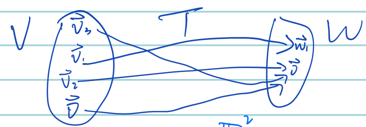
Ex. $\{\vec{0}\} \subset V$ is a subspace

1. $\vec{0} \in \{\vec{0}\}$
2. $\vec{0} + \vec{0} = \vec{0} \in \{\vec{0}\}$
3. $\alpha \vec{0} = \vec{0} \in \{\vec{0}\}$

Ex. $V \subseteq V$ is a subspace

Ex. $\mathbb{Z} \subseteq \mathbb{R}$

Defn. Let $T: V \to W$ be linear. The kernel (or nullspace) of T in V is $\operatorname{Ker} T = \{\vec{v} \in V \mid T(\vec{v}) = \vec{0}\}$



$\vec{v}_2, \vec{v}_3 \in \operatorname{Ker} T$.

$\mathbb{R}^2 \to \mathbb{R}$.

Prop. $\operatorname{Ker} T$ is a subspace of V.

① non empty: $\vec{0} \in \operatorname{Ker} T$

② $\vec{v}_1, \vec{v}_2 \in \operatorname{Ker} T$ $\vec{0} = T(\vec{v}_1) + T(\vec{v}_2) = \underline{T(\vec{v}_1 + \vec{v}_2)}$

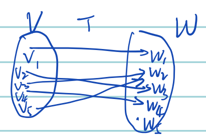
$\vec{v}_1 + \vec{v}_2 \in \operatorname{Ker} T$.

$\mathbb{R}^2$ $\vec{v}_1 = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$T: 2v_1 - v_2 = 0$

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Defn. For l.t. $T: V \to W$, The image of T (Range) is $\operatorname{Im} T \{\vec{w} \in W \mid \vec{w} = T(\vec{v}), \forall \vec{v} \in V\}$



$\operatorname{Ran}(W) = \{w_1, w_2, w_3, w_4\}$

Prop. $\operatorname{Im}(T)$ is a subspace of W.

Chapter 2.

Defn. An equation is linear if exponents on x_i's ≤ 1.

A system of equations is a set of equations to be solved simultaneously.

$$\begin{cases} 2x_1 + x_2 = 8 \\ x_1 - x_2 = 1 \end{cases} \Rightarrow \begin{pmatrix} 2 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$$

$3x_1 = 9$

$$\begin{cases} x_1 = 3 \\ x_2 = 2 \end{cases}$$

Augmented matrix: $\left(\begin{array}{cc|c} 2 & 1 & 8 \\ 1 & -1 & 1 \end{array}\right)$

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + \dots\dots a_{mn}x_n = b_m \end{cases} \Rightarrow \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{m1} & \cdots & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

$$A\vec{x} = \vec{b}$$

Defn. Elementary row operations

1. Row Exchange : $R_i \leftrightarrow R_j$
2. Scaling : αR_i
3. Row Replacement : Replace R_i with $R_i + \alpha R_j$, $\alpha \neq 0$

Ex.

$$\begin{cases} 2x_1 + x_2 = 8 \\ x_1 - x_2 = 1 \end{cases} \qquad \begin{array}{c} R_1 \\ R_2 \end{array} \left(\begin{array}{cc|c} 2 & 1 & 8 \\ 1 & -1 & 1 \end{array}\right)$$

$$\begin{cases} x_1 - x_2 = 1 \\ 2x_1 + x_2 = 8 \end{cases} \qquad \underline{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 2 & 1 & 8 \end{array}\right) \quad \text{not necessary}$$

$$\begin{cases} x_1 - x_2 = 1 \\ 3x_2 = 6 \end{cases} \qquad \underline{R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 3 & 6 \end{array}\right)$$

$$\begin{cases} x_1 - x_2 = 1 \\ x_2 = 2 \end{cases} \qquad \underline{\tfrac{1}{3}R_2} \left(\begin{array}{cc|c} 1 & -1 & 1 \\ 0 & 1 & 2 \end{array}\right)$$

$$\begin{cases} x_1 = 3 \\ x_2 = 2 \end{cases} \qquad \underline{R_1 + R_2} \left(\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 2 \end{array}\right)$$

Defn. A matrix is in echelon form if:

- All rows w/ all zero entries are below all non-zero rows.
- For any non-zero row, the leading entry is strictly to the right of the leading entry of the prev. row.

$$\left(\begin{array}{ccc|c} 2 & 0 & 1 & \vdots \\ & 1 & 2 & \vdots \\ & & 3 & \vdots \end{array}\right)$$

Reduced Row Echelon Form.

- It's in echelon form
- All pivot entries $= 1$
- All entries above pivots are 0.

$$\left(\begin{array}{ccc|c} 1 & 0 & 3 & \vdots \\ & 1 & 2 & \vdots \\ 0 & 0 & 0 & \vdots \\ 0 & 0 & 0 & \vdots \\ 0 & 0 & 0 & \vdots \end{array}\right)$$

$$\begin{cases} y - 2z = 3 \\ x + y - 3z = 7 \\ x + 3y - 7z = 13 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 0 & 1 & -2 & 3 \\ 1 & 1 & -3 & 7 \\ 1 & 3 & -7 & 13 \end{array}\right)$$

$$R_1 \leftrightarrow R_2 \left(\begin{array}{ccc|c} 1 & 1 & -3 & 7 \\ 0 & 1 & -2 & 3 \\ 1 & 3 & -7 & 13 \end{array}\right)$$

$$R_3 - R_1 \left(\begin{array}{ccc|c} 1 & 1 & -3 & 7 \\ 0 & 1 & -2 & 3 \\ 0 & 2 & -4 & 6 \end{array}\right)$$

$$R_3 - 2R_2 \left(\begin{array}{ccc|c} 1 & 1 & -3 & 7 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \text{Echelon Form}$$

pivot variables

x y $z \to$ a free variable

$$R_1 - R_2 \left(\begin{array}{ccc|c} 1 & 0 & -1 & 4 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x - z = 4$$
$$y - 2z = 3$$

$$\begin{cases} x = 4 + z \\ y = 3 + 2z \\ z = z \end{cases}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} + z \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

$\begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix}$ → a particular solution to $A\vec{x} = \vec{b}$

$z \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ → The full soln to homogeneous system $A\vec{x} = \vec{0}$

Defn.

A system $A\vec{x}=\vec{0}$ is called homogeneous. A system $A\vec{x}=\vec{b}$ has an associated homogeneous system $A\vec{x}=\vec{0}$

Associated:

$$\begin{cases} y-2z=0 \\ x+y-3z=0 \\ x+3y-7z=0 \end{cases} \Rightarrow \left(\begin{array}{ccc|c} 0 & 1 & -2 & 0 \\ 1 & 1 & -3 & 0 \\ 1 & 3 & -7 & 0 \end{array}\right) \Rightarrow \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \quad \begin{cases} x=z \\ y=2z \\ z=z \end{cases}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = z\underbrace{\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}}_{\text{basis of } \operatorname{Ker} A:\ k_1}$$

The general solution to the system can be written as $\vec{x} = \vec{x}_p + \vec{x}_h$

$(A \mid \vec{b})$ ← l.i.T

$$\overset{A}{\begin{pmatrix} 0 & 1 & -2 \\ 1 & 1 & -3 \\ 1 & 3 & -7 \end{pmatrix}} \overset{k_1}{\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}} = 1\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} + \begin{pmatrix} -2 \\ -3 \\ -7 \end{pmatrix} = \begin{pmatrix} 0+2-2 \\ 1+2-3 \\ 1+6-7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Span Ran A

Ex. $A = \begin{pmatrix} 1 & 1 & 2 & 2 & 1 \\ 2 & 2 & 5 & 1 & 1 \\ 3 & 3 & 7 & 3 & 2 \\ 1 & 1 & 3 & -1 & 0 \end{pmatrix}$ Solve $A\vec{x}=\vec{0}$ ⟶ $(A' \mid 0) =$

pivot column

$$\begin{array}{ccccc|c} x_1 & x_2 & x_3 & x_4 & x_5 & \\ \hline 1 & 1 & 0 & 8 & 3 & 0 \\ 0 & 0 & 1 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}$$

free columns

$$\begin{cases} x_1+x_2+8x_4+3x_5=0 \\ x_3-3x_4-x_5=0 \end{cases}$$

Ker?
$$\begin{cases} x_1 = -x_2 - 8x_4 - 3x_5 \\ x_2 = x_2 \\ x_3 = 3x_4 + x_5 \\ x_4 = x_4 \\ x_5 = x_5 \end{cases} \qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = x_2\underbrace{\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}_{k_1} + x_4\underbrace{\begin{pmatrix} -8 \\ 0 \\ 3 \\ 1 \\ 0 \end{pmatrix}}_{k_2} + x_5\underbrace{\begin{pmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}}_{k_3}$$

$\{k_1, k_2, k_3\}$ is a basis of kernel of A.

f = # of free columns in A' RREF of A. $\dim(\operatorname{Ker} A) = 3$

p = # of pivot columns in A'

$\dim \operatorname{Ker}(A) = f$

$\dim \operatorname{Im}(A) = p$ a basis of $\operatorname{Im}(A) = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 7 \\ 3 \end{pmatrix} \right\}$: rank $A = 2$

§2.3 Analyzing the pivots.

Defn $A\vec{x}=\vec{b}$ is consistent if it has a solution
inconsistent otherwise.

Ex $\begin{cases} x+y=0 \\ x+y=-1 \end{cases}$: inconsistent $\quad \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 1 & -1 \end{array}\right) \xrightarrow{R_1 - R_2} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)$

The augmented matrix row reduced to having a pivot in the augmented col.

$\Leftrightarrow$ $A\vec{x}=\vec{b}$ is inconsistent

Notation: $A' = $ RREF of A

prop. ① $A\vec{x}=\vec{b}$ has at most 1 solution $\Leftrightarrow$ A' has a pivot in every column.
② $A\vec{x}=\vec{b}$ is consistent ($\exists$ soln) $\Leftrightarrow$ A' has a pivot in every row.
③ $A\vec{x}=\vec{b}$ has a unique solution $\Leftrightarrow$ A' has a pivot in every row & col.

Pf. ① $A\vec{x}=\vec{b}$ has ≤ 1 soln $\Leftrightarrow$ $\exists$ no free variables $\Leftrightarrow$ A' has a pivot per col.
② ($\Leftarrow$) A' pivot in every row $\Rightarrow$ $(A|\vec{b})$ has no pivot in augmented col. $\Rightarrow$ Consist.
($\Rightarrow$) By contradiction.
Suppose $A'\vec{x}=\vec{b}'$ is inconsistent $\Rightarrow$ $(A'|\vec{b}') = \left(\begin{array}{c|c} \vdots & \\ 00000 & 1 \end{array}\right)$

$R_2 - 8R_4 - 2R_3$ $\begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \longrightarrow 0\ 1\ 0\ 0 \longrightarrow \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Defn. Elementary Matrices for Elementary Row operations are:

1. Row Exchange $\quad E_{R_i \Leftrightarrow R_j} = \begin{array}{c} \\ i \\ j \end{array}\begin{pmatrix} 1 & & & & \\ & 0 & \cdots & 1 & \\ & \vdots & & \vdots & \\ & 1 & \cdots & 0 & \\ & & & & 1 \end{pmatrix}$ $\leftarrow$ $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = EA$

2. Scaling $\quad E_{\alpha R_i} = i\begin{pmatrix} 1 & & & \\ & \ddots & & 0 \\ 0 & & \alpha & \\ & & & \ddots \ 1 \end{pmatrix}$ $\qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

3. Row Replacement $\qquad AE = \begin{pmatrix} b & a \\ d & c \end{pmatrix}$

R_j $\quad E_{R_j + \alpha R_i} \quad \begin{array}{c} i \\ j \end{array}\begin{pmatrix} 1 & & & \\ & \ddots & & \\ & \alpha & \ddots & \\ & & & 1 \end{pmatrix}$

§ 2.4 Finding A^{-1}

Any invertible matrix can be row reduced to the identity

To find $A^{-1} = X$, need to find $A \cdot X = I = (\vec{e}_1\ \vec{e}_2 \cdots \vec{e}_n)$

$X = (\vec{x}_1\ \vec{x}_2 \cdots \vec{x}_n)$ then $A(\vec{x}_1\ \vec{x}_2 \cdots \vec{x}_n) = (\vec{e}_1\ \vec{e}_2 \cdots \vec{e}_n)$: $\begin{cases} A\vec{x}_1 = \vec{e}_1 \\ \vdots \\ A\vec{x}_n = \vec{e}_n \end{cases}$

Steps to find A^{-1}: ① row reduce $(A \mid I)$

② if RREF $(I \mid X)$, then A is invertible & $X = A^{-1}$

③ Otherwise, A is not invertible

Defn. The dimension of V.S. V is the number of vectors in a basis for V.

* if $V = \{\vec{0}\}$ then $\dim V = 0$

$\mathbb{R}^2 = \{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\}$: $\dim \mathbb{R}^2 = 2$

§ 2.7

Rank $A = \dim(\text{Ran } A) = \dim(\text{Im } A)$

Thm. A $\text{rank}(A) = \text{rank}(A^T)$

Thm. Rank-Nullity Thm. $A_{m \times n}$

① $\dim(\ker A) + \underbrace{\dim(\text{Ran } A)}_{\text{rank}(A)} = n$

② $\dim(\ker A^T) + \text{rank}(A^T) = m$

Defn. We say A is similar to B if $\exists$ an invertible matrix Q s.t.

$$A = Q^{-1} B Q \Rightarrow QA = BQ$$

Note: Q & Q^{-1} is invertible $\Rightarrow$ Q is $n \times n$ square mat.

$\Rightarrow$ A & B are $n \times n$

$\Rightarrow$ Similar matrice are square & has same size

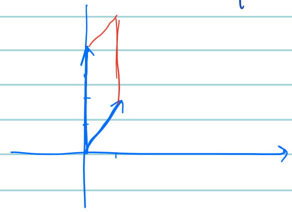
Chapter 3 Determinant.

The determinant of $A_{n \times n}$ is scalar

Det(A) is the "volume" of the parallel piped determined by the column of A.

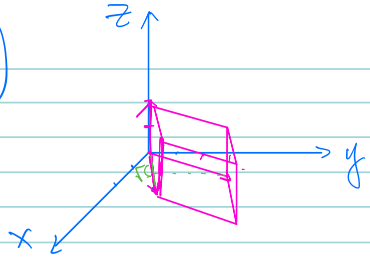
Ex.

$A = \begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix} \in \mathbb{R}^2$



$\Rightarrow \text{Det}(A) = 4$

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$



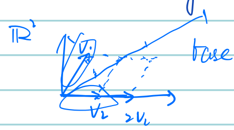
$\text{Det}(A) = 10$

The determinant of $A = (\vec{v}_1\ \vec{v}_2 \cdots \vec{v}_n) = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$

$$\text{Det}(A) = \text{Det}(\vec{v}_1 \cdots \vec{v}_n) = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{vmatrix}$$

Properties:

1. Linearity in each argument

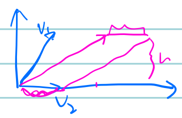


true $\quad \text{Det}(\vec{v}_1, 2\vec{v}_2) = 2\,\text{Det}(\vec{v}_1, \vec{v}_2)$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

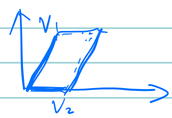
$\begin{pmatrix} 3 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} 6 \\ 2 \end{pmatrix}$

2. Col Replacement



$\text{Det}(\vec{v}_1 + 2\vec{v}_2, \vec{v}_2) = \text{Det}(\vec{v}_1, \vec{v}_2)$

3. Antisymmetry

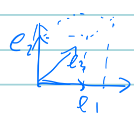


$\text{Det}(\vec{v}_2, \vec{v}_1) = -\text{Det}(\vec{v}_1, \vec{v}_2)$

$\text{Det}(\vec{v}_1, \vec{v}_2, \cdots \vec{v}_i, \cdots \vec{v}_j, \cdots \vec{v}_n)$

$= -\text{Det}(\vec{v}_1, \vec{v}_2, \cdots, \vec{v}_j, \cdots \vec{v}_i, \cdots \vec{v}_n)$

4. Normalization



$\text{Det}(\vec{e}_1, \vec{e}_2) = 1$

$\text{Det}(\vec{e}_1, \vec{e}_2, \vec{e}_3) = 1$

$\therefore \text{Det}(I) = 1$

Prop.

1. A has a col of all 0's then $\text{Det}(A) = 0$
2. A has 2 equal cols. then $\text{Det}(A)=0$
3. One coln is multiple of the other $\Rightarrow$ $\text{Det}(A)=0$
4. A has cols that are linearly dependent $\Rightarrow$ $\text{Det}(A)=0$

$\Downarrow$ $\Uparrow$
A is not invertible

Defn: A square matrix is called Diagonal if all entries off the main diagonal is 0.

$$A = \begin{pmatrix} a_{11} & 0 & & & \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & & \\ \vdots & & & \ddots & \\ 0 & & & & a_{nn} \end{pmatrix}$$
$$\vec{a}_1 \; \vec{a}_2 \; \cdots \; \vec{a}_n$$

$\text{Det}(I) = 1 \Rightarrow \text{Det}(A) = \text{Det}(\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_n)$

$= \text{Det}(a_{11}\vec{e}_1\ \ a_{22}\vec{e}_2 \cdots a_{nn}\vec{e}_n)$

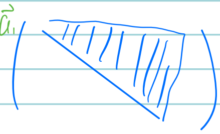
$= a_{11}\,\text{Det}(\vec{e}_1\ a_{22}\vec{e}_2 \cdots)$

$= a_{11} a_{22} \cdots a_{nn}\,\text{Det}(I)$

$= a_{11} a_{22} \cdots a_{nn}$ (product of diagonal entries)

A is upper triangular if all entries below the main diagonal are 0's.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{pmatrix} \quad \vec{a}_2 \leftarrow 2\vec{a}_2 - 2\vec{a}_1$$



A is lower triangular ...... above ...... are 0's

Claim.
Determinant of a triangular matrix = product of diagonal entries

Thm $\det A = 0 \Leftrightarrow A$ is not invertible : A is invertible $\Leftrightarrow \det A \neq 0$

$\Leftarrow$

$\Rightarrow$ Pf by contradiction. A is invertible $\Rightarrow A^T$ is invertible.

$\Rightarrow$ row operations to A^T to I $\Rightarrow$ col operations get to diagonal
$\Rightarrow \text{Det}(A) = \underset{\neq 0}{\alpha}(\text{Det } I) = \alpha \Rightarrow \det(A) \neq 0$ contradiction.

Thm

For square A & B, $\det(AB) = \det(A) \cdot \det(B)$

Ex. $A = \begin{pmatrix} 1 & 0 \\ 2 & 4 \end{pmatrix}$ $\quad B = \begin{pmatrix} 5 & 0 \\ 0 & 2 \end{pmatrix}$ $\quad AB = \begin{pmatrix} 5 & 0 \\ 10 & 8 \end{pmatrix}$

$\operatorname{Det} A = 4$ $\quad \operatorname{Det} B = 10$ $\quad \operatorname{Det}(AB) = 40$

$= 4 \cdot 10$

E_i	Effect	$\operatorname{Det}(E)$	$\operatorname{Det}(E^T)$	$\operatorname{Det}(AE)$
	Coln swap	-1	-1	$-\operatorname{Det}(A)$
	Coln. Scaling	α	α	$\alpha \operatorname{Det}(A)$
Triangular	Coln. Replacement	1	1	$\operatorname{Det}(A)$

Thm $\det(A) = \det(A^T)$

if $\det A = 0 \Rightarrow \det A^T = 0$

A is not invertible $\Rightarrow A^T$ is not invertible

Define Determinant.

$$A_{n\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ a_{n1} & \cdots & & a_{nn} \end{pmatrix} = \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_n \end{pmatrix}$$

$$\vec{v}_k = \begin{pmatrix} a_{1k} \\ \vdots \\ a_{nk} \end{pmatrix} = a_{1k}\vec{e}_1 + a_{2k}\vec{e}_2 + \cdots + a_{nk}\vec{e}_n$$

$$\mathrm{Det}(A) = \left| \vec{v}_1 \cdots \vec{v}_n \right| = \left| \sum_{j=1}^{n} a_{j,1} \vec{e}_{j,1}, \vec{v}_2, \vec{v}_3, \ldots \vec{v}_n \right|$$

$$= \sum_{j_1=1}^{n} a_{j_1,1} \left| \vec{e}_{j_1}, \vec{v}_2, \ldots, \vec{v}_n \right|$$

$$= \sum_{j_1=1}^{n} \sum_{j_2=1}^{n} \cdots \sum_{j_n=1}^{n} a_{j_1,1} a_{j_2,2} \cdots a_{j_n,n} \left| \vec{e}_{j_1}, \vec{e}_{j_2}, \ldots, \vec{e}_{j_n} \right|$$

* Only keep track of terms when all j's are distinct.

$\Rightarrow j_1 \cdots j_n$ include all #'s from 1 to n

① j_i's $= 1$ $\quad \vec{e}_2 \; \vec{e}_2$

② j_i's $= 1$ except $j_n = 2$ $\qquad n^n$

③ j_i's $= 3$ except $j_2 = 4$

a permutation of $\{1, 2, \ldots, n\}$ ☆

$$\mathrm{Det}(A) = \sum_{\sigma \in \mathrm{Perm}(n)} a_{\sigma(1),1}\, a_{\sigma(2),2} \cdots a_{\sigma(n),n} \underbrace{\left| \vec{e}_{\sigma(1)}, \ldots, \vec{e}_{\sigma(n)} \right|}_{= \pm 1}$$

is related to I by some col. swaps.

Defn.

The set of permutations of $\{1, 2, \dots, n\}$ is

$$\mathrm{Perm}(n) = \{\{1,2,3,\dots,n\}, \{2,1,3,\dots,n\}, \dots\}$$

$$|\mathrm{Perm}(n)| = n!$$

↖ Cardinality.

An inversion of σ is $i<j$ w/ $\sigma(i) > \sigma(j)$

Ex:

$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & 5 \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \sigma = [& 3 & 1 & 4 & 5 & 2] \end{array}$$

$$\begin{cases} 3>1 \\ 3>2 \\ 4>2 \\ 5>2 \end{cases}$$

# of inversions of σ = 4

$\sigma[1] = 3$

$\sigma[2] = 1$ 1 2 3 4

```
For i in (1, 5):
    for j in (2, 6):    2 3 4 5
        if i<j:
            if σ[i] > σ[j]:
                inversion += 1
```

The sign of σ is $(-1)^{\#\text{ of inversions of }\sigma}$

$$\mathrm{sgn}(\sigma) = (-1)^4 = 1$$

$$|\vec{e}_{\sigma(1)} \cdots \vec{e}_{\sigma n}| = \mathrm{sgn}(\sigma)$$

$$\Rightarrow \mathrm{Det}(A) = \sum_{\sigma = \mathrm{Perm}(n)} a_{\sigma(1),1} \cdots a_{\sigma(n),n} \, \mathrm{sgn}(\sigma)$$

Ex. $A = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 0 & 4 \\ 1 & 2 & 1 \end{pmatrix} \to$ col operations $\to$ Det of triangular.

$$\det A = \sum_{\sigma \in Perm(3)} a_{\sigma(1),1}\, a_{\sigma(2),2}\, a_{\sigma(3),3}\ sgn(\sigma)$$

6 perms:

1. $1\ 2\ 3 \to a_{1,1} a_{2,2} a_{3,3} \cdot (-1)^0 = 1 \cdot 0 \cdot 1 \cdot 1 = 0$
2. $1\ 3\ 2 \to a_{1,1} a_{3,2} a_{2,3} (-1)^1 = 1 \cdot 2 \cdot 4 \cdot (-1) = -8$
3. $2\ 1\ 3 \to\ = 2$
4. $2\ 3\ 1 \to\ = -6$
5. $3\ 1\ 2 \to\ = 8$
6. $3\ 2\ 1 \to\ = 0$

$\det A = \sum (i) = -4$

$$\Rightarrow Det(A) = \left| \begin{matrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{matrix} \right| \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{matrix}$$

$-\quad -\quad -\qquad +\quad +\quad +$

Ex. $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ $\quad \det A = \sum_{\sigma \in Perm(2)} a_{\sigma(1),1}\, a_{\sigma(2),2}\ sgn(\sigma)$

$1\ 2 \quad a_{11}\ a_{22}\ (+)$

$2\ 1 \quad a_{21}\ a_{12}\ (-)$

$$\Rightarrow \left| \begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix} \right|$$

$-\qquad +$

$Det(A) = a_{11} a_{22} - a_{12} a_{21}$

§ 3.5 Cofactor Expansion

Defn. If A is $n \times n$. $A_{j,k}$ denotes $(n-1)\times(n-1)$ matrix where j^{th} row & k^{th} col is crossed.

Ex. $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ 0 & 4 & -2 \end{pmatrix}$ $\quad A_{3,2} = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$

The corresponding cofactor $C_{j,k} = (-1)^{j+k} \operatorname{Det}(A_{j,k})$

Ex. $C_{3,2} = (-1)^5 \det\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} = (-1)(-5) = 5$

Thm. $A_{n\times n}$

① Expansion along row j:

$$\det A = \sum_{i=1}^{n} a_{j,i}\, C_{j,i}$$

② Expansion along col k:

$$\det(A) = \sum_{m=1}^{n} a_{m,k}\, C_{m,k}$$

Ex. $\begin{vmatrix} 1 & 0 & 2 \\ 3 & -1 & 1 \\ 0 & 4 & -2 \end{vmatrix} \overset{\text{along } 1^{st} \text{ row}}{=} 1\cdot(-1)^2 \begin{vmatrix} -1 & 1 \\ 4 & -2 \end{vmatrix} + 0\cdot(-1)^{1+2}\begin{vmatrix} 3 & 1 \\ 0 & -2 \end{vmatrix}$

$+ 2\cdot(-1)^{1+3}\begin{vmatrix} 3 & -1 \\ 0 & 4 \end{vmatrix}$

$= -2 + 2\cdot 12 = 22$

along 2^{nd} coln exercise.

4x4.

$$\begin{array}{c|cccc|}
 & & 2 & & \\
1 & 1 & 3 & 1 & 2 \\
 & 2 & 0 & -1 & 1 \\
3 & -1 & 6 & 5 & 0 \\
 & 4 & 0 & 3 & 4
\end{array}$$

* Choose 2nd col w/ more 0's

$$3\cdot(-1)^{1+2}\begin{vmatrix} 2 & -1 & 1 \\ -1 & 5 & 0 \\ 4 & 3 & 4 \end{vmatrix} + 0 + 6(-1)^{3+2}\begin{vmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \\ 4 & 3 & 4 \end{vmatrix} + 0$$

$$= -3\left((-1)(-1)^{2+1}\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} + 5(-1)^{2+2}\begin{vmatrix} 2 & 1 \\ 4 & 4 \end{vmatrix}\right)$$

$$-6\left((-1)^{1+1}\begin{vmatrix} -1 & 1 \\ 3 & 4 \end{vmatrix} - 1\begin{vmatrix} 2 & 1 \\ 4 & 4 \end{vmatrix} + 2\begin{vmatrix} 2 & -1 \\ 4 & 3 \end{vmatrix}\right)$$

$$= -3(-7+5\cdot 4) - 6(-7-4+20)$$

$$= -39 - 54 = -93$$

Thm Let A be an invertible matrix, C is cofactor matrix

$$A^{-1} = \frac{1}{\det A} C^T$$

Ex. $A = \begin{pmatrix} 1 & 3 \\ 2 & 8 \end{pmatrix}$, $\det A = 2 \Rightarrow A$ is invertible

$$C_{1,1} = (-1)^2 \cdot 8$$
$$C_{1,2} = -1 \cdot 2 = -2$$
$$C_{2,1} = -1 \cdot 3 = -3$$
$$C_{2,2} = 1$$

$$\Rightarrow C^T = \begin{pmatrix} 8 & -3 \\ -2 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{2}\begin{pmatrix} 8 & -3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 4 & -\frac{3}{2} \\ -1 & \frac{1}{2} \end{pmatrix}$$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Chapter 4

Easiest transf.

$T: V \to V$ Ex: $T(\vec{v}) = \vec{v}$

$T(\vec{v}) = \lambda\vec{v}$, λ is a scalar ($\mathbb{R}$, $\mathbb{C}$)

Defn $A_{n\times n}$. If there's a scalar λ & a vector $\vec{v}$ ($\vec{v} \neq \vec{0}$) w/ the property $A\vec{v} = \lambda\vec{v}$ then λ is an eigenvalue for A & $\vec{v}$ is an eigenvector of A for the e.value λ.

Ex.
$$\begin{pmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 7 \\ -9 \\ 18 \end{pmatrix} + \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} + \begin{pmatrix} -9 \\ 9 \\ -24 \end{pmatrix} = \begin{pmatrix} -2 \\ -2 \\ -6 \end{pmatrix}$$
$$= -2\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

So $\lambda = -2$ is an eigenvalue for A

$\vec{v} = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ ∴ eigenvector of A for -2.

$$\begin{pmatrix} 7 & 0 & -3 \\ -9 & -2 & 3 \\ 18 & 0 & -8 \end{pmatrix}\underbrace{\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}}_{\vec{v}} = \begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix} - \begin{pmatrix} -3 \\ 3 \\ -8 \end{pmatrix} = \underbrace{\begin{pmatrix} 3 \\ -7 \\ 8 \end{pmatrix}}_{\text{soln}}$$

→ How to find?

Defn. The spectrum of A is $\sigma(A) = \{\lambda \mid \lambda \text{ is an eigenvalue of } A\}$

$$A\vec{v} = \lambda\vec{v}$$

$$A\vec{v} = \lambda I \vec{v}$$

$$A\vec{v} - \lambda I \vec{v} = \vec{0}$$

$$(A - \lambda I)\vec{v} = \vec{0} \quad \Rightarrow \vec{v} \in \ker(A - \lambda I)$$

So if λ is an eigen value for A, then the set of eigenvectors for $\lambda \cup \{\vec{0}\}$ is a subspace for $\mathbb{R}^n$. It's the kernel of a matrix, & we call it the eigenspace of A.

To find: $A\vec{v} = \lambda\vec{v}$ for $\vec{v} \neq 0$

$\Leftrightarrow$ $(A - \lambda I)\vec{v} = \vec{0}$ for $\vec{v} \neq 0$

$\Leftrightarrow$ has a nontrivial solution

$\Leftrightarrow$ $(A - \lambda I)$ is not invertible

$\Leftrightarrow$ $\det(A - \lambda I) = 0$

Ex. $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$ find eigenvalues & eigenvectors

$$A - \lambda I = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 2 \\ -1 & 4-\lambda \end{pmatrix}$$

$$\det(A - \lambda I) = (1-\lambda)(4-\lambda) + 2 = \lambda^2 - 5\lambda + 6$$

$$= 4 + \lambda^2 - 5\lambda + 2 = (\lambda - 2)(\lambda - 3) = 0$$

characteristic polynomial

$$\Rightarrow \lambda = 2 \text{ or } \lambda = 3$$

① $\lambda = 2$. $\vec{v} \in \ker(A - \lambda I)$

$$\ker(A - 2I) = \ker\begin{pmatrix} 1-2 & 2 \\ -1 & 4-2 \end{pmatrix} = \ker\begin{pmatrix} -1 & 2 \\ -1 & 2 \end{pmatrix}$$

To find ker: $\left(\begin{array}{cc|c} -1 & 2 & 0 \\ -1 & 2 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$$\begin{cases} x_1 = 2x_2 \\ x_2 = x_2 \end{cases} \Rightarrow \vec{v} = x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

eigenspace for $\lambda = 2$ is $\text{Span}\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\}$.

② $\lambda = 3$

$$\ker\begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix} \rightsquigarrow \vec{v} = x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$ one basis $\text{Span}\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$

geometric mult. is 1

Prop. λ is an eigenvalue of A

$\Leftrightarrow$ λ is root of the characteristic polynomial

if A is $n \times n$, then Char. Poly. also has a degree of n

$\Rightarrow$ n eigenvalues but could be same & could be complex.

Ex. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} \Rightarrow \lambda^2 + 1 = 0 \Rightarrow \begin{cases} \lambda = i \\ \lambda = -i \end{cases}$$

① $\lambda = i \Rightarrow \ker\begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \xrightarrow[R_2 - iR_1]{} \left(\begin{array}{cc|c} -i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$

$\xrightarrow{iR_1} \left(\begin{array}{cc|c} 1 & -i & 0 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow$ eigenspace $\text{Span}\{\begin{pmatrix} i \\ 1 \end{pmatrix}\}$

Prop. Similar Matrices have the same eigenvalues.

Lemma: Similar Matrices have the same determinant

Pf: $A = QBQ^{-1} \Rightarrow AQ = QB$

$\det(AQ) = \det(QB)$

$\Rightarrow \det(A)\cancel{\det(Q)} = \cancel{\det(Q)}\det(B)$

$\Rightarrow \det(A) = \det(B)$

Pf prop: Let $A = QBQ^{-1}$

Consider $Q(B-\lambda I)Q^{-1}$.

$= Q(BQ^{-1} - \lambda I Q^{-1})$

$= QBQ^{-1} - Q\lambda I Q^{-1}$

$= A - \lambda I$.

So $(A-\lambda I)$ is similar to $(B-\lambda I)$

$\Rightarrow \det(A-\lambda I) = \det(B-\lambda I)$ by lemma

$\Rightarrow$ Same characteristic polynomials

$\Rightarrow$ Same eigenvalues

Multiplicity

Defn. If λ is an eigenvalue, then

- Algebraic multiplicity: # of times λ is a root.

  Ex. $A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$ $\det\begin{pmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{pmatrix} = (\lambda-3)^2 = 0$

  $\Rightarrow \lambda = 3$ with (algebraic) mult. of 2.

- Geometric mult. is $\dim(\ker(A-\lambda I))$
  dim. of eigenspace for λ.

* Geometric multiplicity $\leq$ Algebraic multiplicity.

Trace of nxn matrix.

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \ddots & & \\ & & \ddots & \\ a_{n,1} & & & a_{n,n} \end{pmatrix}$$

$$Tr(A) = \sum_{i=1}^{n} a_{i,i}$$

Lemma: $Tr(AB) = Tr(BA)$

Thm. $A_{n\times n}$ & $\lambda_1, \lambda_2 \cdots \lambda_n$ are its eigenvalues repeated w/ multiplicity

then $Tr(A) = \lambda_1 + \lambda_2 + \cdots + \lambda_n = \sum \lambda_i$

$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n = \prod \lambda_i$

Thm. Let A be triangular, then the eigenvalues of A, repeated w/ mult. are the diagonal entries.

Ex. $A = \begin{pmatrix} 2 & 4 \\ 0 & 5 \end{pmatrix}$ $\lambda_1 = 2$, $\lambda_2 = 5$

"Eigendecomposition"

$$A = Q \Lambda Q^{-1}$$

$$A\vec{v} = \lambda \vec{v}$$

$$A(\vec{v}_1 \vdots \vec{v}_2 \vdots \vec{v}_3 \vdots \cdots \vdots \vec{v}_n) = (\lambda_1 \vec{v}_1 \vdots \lambda_2 \vec{v}_2 \vdots \cdots \vdots \lambda_n \vec{v}_n)$$

$$= (\vec{v}_1 \vdots \vec{v}_2 \vdots \cdots \vdots \vec{v}_n) \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

$$A \quad Q \quad = \quad Q \Lambda$$

$$A = Q \Lambda Q^{-1}$$

Need: algebraic multiplicity for λ_i = geometric mult. for λ_i

§4.2 Diagonalization

$A = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}$ $\det A = a_1 \cdot a_2 \cdots a_n$

$B = \begin{pmatrix} b_1 & & \\ & \ddots & \\ & & b_n \end{pmatrix}$ eigenvalue $\lambda = a_1, a_2, \cdots, a_n$

$$AB = \begin{pmatrix} a_1 b_1 & & \\ & \ddots & \\ & & a_n b_n \end{pmatrix} = \begin{pmatrix} b_1 a_1 & & \\ & \ddots & \\ & & b_n a_n \end{pmatrix} = BA$$

$$A^N = \begin{pmatrix} a_1^N & & \\ & \ddots & \\ & & a_n^N \end{pmatrix}$$

Ex. $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\lambda_1 = 3$ $am(\lambda_1) = 1$

$\lambda_2 = 1$ $am(\lambda_2) = 2$

$\lambda_2 = 1$ $A - \lambda I = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\vec{v} = x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$gm(\lambda_2) = 2$

$\lambda_1 = 3$ $A - \lambda I = \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) = \vec{0}$ $\vec{v} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ $gm(\lambda_1) = 1$

$\Rightarrow$ Standard basis vectors = eigenvectors.

Defn. A linear transformation $T: V \to W$ is diagonalizable if $\exists$ an ordered basis $\mathcal{B}$ of V s.t. $[T]_{\mathcal{B}\mathcal{B}}$ is diagonal.

A matrix A is diagonalizable if $T(\vec{v}) = A\vec{v}$ is diag.

$\to$ Defn. For a vector space V with an ordered basis $\mathcal{B} = \{\vec{b}_1, \cdots, \vec{b}_n\}$,

the coordinate of $\vec{v} \in V$ in the basis $\mathcal{B}$ is

$$[\vec{v}]_{\mathcal{B}} = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} \quad \text{where } \vec{v} = \alpha_1 \vec{b}_1 + \cdots \alpha_n \vec{b}_n$$

Ex. $\mathbb{R}^2$ $S = \{ \underset{\vec{e}_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \underset{\vec{e}_2}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \}$ $\vec{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix} = 1\vec{e}_1 + 3\vec{e}_2$

$\mathcal{B} = \{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix} \}$ $\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} \alpha_1 = 2 \\ \alpha_2 = 1 \end{cases}$ $[\vec{v}]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Defn. Let $T: V \to W$ be linear & let $\mathcal{A} = \{\vec{a}_1, \cdots \vec{a}_n\}$ be an ordered basis for V & $\mathcal{B} = \{\vec{b}_1, \cdots, \vec{b}_n\}$ be an ordered basis of W.

Then the matrix of T w.r.t. bases $\mathcal{A}, \mathcal{B}$ is

$$[T]_{\mathcal{B}\mathcal{A}} = \left([T(\vec{a}_1)]_{\mathcal{B}} \; [T(\vec{a}_2)]_{\mathcal{B}} \cdots \; [T(\vec{a}_n)]_{\mathcal{B}} \right)$$

Ex. $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$

$T: \mathbb{R}^2 \to \mathbb{R}^2$, $T(\vec{v}) = A\vec{v}$

$\lambda_1 = 2$ Span $\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\}$

$\lambda_2 = 3$ Span $\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$

$\mathcal{B} = \{\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$

$$[T]_{\mathcal{B}\mathcal{B}} = \left([T(\vec{b}_1)]_{\mathcal{B}} \; [T(\vec{b}_2)]_{\mathcal{B}} \right) = \left(\overset{A}{\begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}} \overset{\vec{b}_1}{\begin{pmatrix} 2 \\ 1 \end{pmatrix}} \;\vdots\; \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right)$$

$$= \left(\left[\begin{pmatrix} 4 \\ 2 \end{pmatrix}\right]_{\mathcal{B}} \; \left[\begin{pmatrix} 3 \\ 3 \end{pmatrix}\right]_{\mathcal{B}} \right)$$

$$= \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \leftarrow \text{Diagonal}$$

$$A = Q \Lambda Q^{-1}$$

$$= \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \frac{1}{1} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 4 & 3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$$

Thm The following are equivalent:

① A is diagonalizable ✓

② $\exists$ is a collection of eigenvectors for A, ✓
$\mathcal{B} = \{\vec{E}_1, \ldots, \vec{E}_n\}$ that forms a basis for $\mathbb{R}^n$ or $\mathbb{C}^n$.

③ $\exists$ an invertible matrix Q w/
$Q^{-1}AQ = D$ where D is a diagonal ma. ✓

Ex. $A = \begin{pmatrix} 2 & -2 & 1 \\ -1 & 3 & -1 \\ 2 & -4 & 3 \end{pmatrix}$

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & -2 & 1 \\ -1 & 3-\lambda & -1 \\ 2 & -4 & 3-\lambda \end{vmatrix} \begin{matrix} 2-\lambda & -2 \\ -1 & 3-\lambda \\ 2 & -4 \end{matrix} = (2-\lambda)(9+\lambda^2-6\lambda) + 4 + 4 - 6 + 2\lambda - 4(2-\lambda) - 2(3-\lambda)$$

$= 18 - 9\lambda + 2\lambda^2 - \lambda^3 - 12\lambda + 6\lambda^2 + 8 - 20 + 8\lambda$

$= -\lambda^3 + 8\lambda - 13\lambda + 6 = 0 \Rightarrow (\lambda - 1)(-\lambda + 1)(\lambda - 6)$

$\Rightarrow \lambda_1 = 1$

$$\begin{array}{r l} & -\lambda^2 + 7\lambda - 6 \\ \lambda - 1 \,\big) & -\lambda^3 + 8\lambda^2 - 13\lambda + 6 \\ & -\lambda^3 + \lambda^2 \\ \hline & 7\lambda^2 - 13\lambda + 6 \\ & 7\lambda^2 - 7\lambda \\ \hline & -6\lambda + 6 \\ & -6\lambda + 6 \\ \hline & 0 \end{array}$$

$\Rightarrow \lambda_1 = 1$ mult. $= 2$

$\lambda_2 = 6$ mult. $= 1$

$$\operatorname{Ker}\begin{pmatrix} 1 & -2 & 1 \\ -1 & 2 & -1 \\ 2 & -4 & 2 \end{pmatrix} \rightsquigarrow \left(\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$= \operatorname{Span}\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\} \quad @ \lambda_1 = 1$$

$$\operatorname{Ker}\begin{pmatrix} -4 & -2 & 1 \\ -1 & -3 & -1 \\ 2 & -4 & -3 \end{pmatrix} \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x_1 + x_2 = 0 \qquad x_1 = -x_2$$
$$2x_2 + x_3 = 0 \qquad x_3 = -2x_2 \quad \rightsquigarrow \quad x_2\begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix}$$

$$\Rightarrow \Lambda = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix} \qquad Q = \begin{pmatrix} 2 & -1 & -1 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{pmatrix} \begin{matrix} 2 & -1 \\ 1 & 0 \\ 0 & 1 \end{matrix}$$

$$\operatorname{Det} Q = \quad -1 -2 -2 \quad = -5$$

$\Rightarrow$ Eigenvectors form a basis

Thm. $\lambda_1, \cdots \lambda_r$ are distinct eigenvalues, $\vec{v}_1, \cdots, \vec{v}_r$ are corresponding eigenvectors.

$\Rightarrow \vec{v}_1, \cdots, \vec{v}_r$ are linearly independent.

Corollary: if $A_{n\times n}$ has n distinct eigenvalues

$\Rightarrow A$ is diagonalizable

Pf. Base Step: $\vec{v}_1$ of λ_1. $\{\vec{v}_1\}$ L.I. b/c $\vec{v}_1 \neq \vec{0}$

Inductive Step. IH: $\{\vec{v}_1, \cdots, \vec{v}_{r-1}\}$ is linearly indep.

Let $\vec{v}_1, \cdots, \vec{v}_r$ be eigenvec for $\lambda_1, \cdots, \lambda_r$

Assume $\alpha_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r = \vec{0}$

$$(A - \lambda_r I)(\alpha_1\vec{v}_1 + \cdots + \alpha_r\vec{v}_r) = \vec{0}$$

$\alpha_1 (A - \lambda_r I)\vec{v}_1$

$\alpha_1 A \vec{v}_1 - \alpha_1 \lambda_r \vec{v}_1 + \cdots \alpha_r A \vec{v}_r - \alpha_r \lambda_r \vec{v}_r = \vec{0}$

$\alpha_1 \lambda_1 \vec{v}_1 - \alpha_1 \lambda_r \vec{v}_1 + \cdots + \underline{\alpha_r \lambda_r \vec{v}_r - \alpha_r \lambda_r \vec{v}_r} = \vec{0}$

$\vec{0}$

$\alpha_1 (\lambda_1 - \lambda_r)\vec{v}_1 + \cdots + \alpha_{r-1}(\lambda_{r-1} - \lambda_r)\vec{v}_{r-1} = \vec{0}$

λ_i's are distinct.

$\lambda_i - \lambda_r \neq 0$

$\Rightarrow \quad \alpha_i$'s $= 0$

$\Rightarrow \quad \vec{0} + \alpha_r \vec{v}_r = \vec{0} \quad , \quad \vec{v}_r \neq \vec{0}$

Δ

$\Rightarrow \alpha_r = 0$

$\Rightarrow \{\vec{v}_1, \ldots, \vec{v}_r\}$ is L.I. $\boxed{\square}$

Direct Sum of Subspace

Def. Let $V_1, V_2, \ldots, V_k$ be subspaces for V.
The subspaces are linearly indep. if $\vec{v}_1 + \vec{v}_2 + \cdots + \vec{v}_k = \vec{0}$ w/ $\vec{v}_i \in V_i$ implies that $\vec{v}_1 = \vec{v}_2 = \cdots = \vec{v}_k = \vec{0}$

Ex. $\mathbb{R}^3 \qquad V_1 = \left\{ \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \middle| \, x, y \in \mathbb{R} \right\} \qquad \vec{v}_1 \in V_1 \qquad \vec{v}_1 + \vec{v}_2 = \vec{0}$

$V_2 = \left\{ \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} \middle| \, z \in \mathbb{R} \right\} \qquad \vec{v}_2 \in V_2$

$$\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow x = y = z = 0$$

V_1 & V_2 are L.I.

Def. Subspaces $V_1, \ldots, V_k$ are spanning if $\forall \vec{v} \in V$, $\vec{v} = \vec{v}_1 + \cdots + \vec{v}_k$ for $\vec{v}_i \in V_i$

Ex. $\mathbb{R}^3 \qquad \begin{pmatrix} 4 \\ 3 \\ 7 \end{pmatrix} \rightarrow \quad \vec{v}_1 = \begin{pmatrix} 4 \\ 3 \\ 0 \end{pmatrix} \in V_1 \qquad V_1, V_2$ spans $\mathbb{R}^3$

$\vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 7 \end{pmatrix} \in V_2$

Def. V is a direct sum of $V_1, \dots, V_k$,
written as $V = V_1 \oplus V_2 \oplus \dots \oplus V_k$ ($V_1, \dots, V_k$ are a basis for V)
if $V_1, \dots, V_k$ are LI & Spanning.

$$\mathbb{R}^3 = V_1 \oplus V_2$$

$$\underbrace{\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\end{pmatrix}\right\}}_{\mathcal{B}_1} \qquad \underbrace{\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}}_{\mathcal{B}_2}$$

$\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$
basis for $\mathbb{R}^3$

Thm. if $\{V_1, \dots, V_k\}$ is a basis of subspaces for V & each subspace V_i has a basis of vectors $\mathcal{B}_i$
$\Rightarrow \mathcal{B} = \bigcup_{i=1}^{k} \mathcal{B}_i$ is a basis of vectors for V.

Thm. $T: V \to V$ where $V \in \mathbb{R}^n$ is diagonalizable $\Leftrightarrow$ $\forall \lambda$ of A.
$A\vec{v} = T(\vec{v})$. $gm(\lambda) = am(\lambda)$

Lemma. Let $A: V \to V$. be linear w/ eigenvalues $\lambda_1 \dots \lambda_k$
then the eigenspaces $E_i = \ker(A - \lambda_i I)$ are L.I.

Chapter 5

§ 5.1 Inner Product Spaces

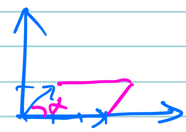
Defn. The standard inner product on $\mathbb{R}^n$ or dot product of

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \& \quad \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$ is the scalar

$$\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$$

$$\langle \vec{x}, \vec{y} \rangle = \vec{x}^T \vec{y} = \vec{y}^T \vec{x}$$

$\mathbb{R}^2 \quad x = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad y = \begin{pmatrix} 3 \\ 0 \end{pmatrix}$



$\langle x, y \rangle = 3 + 0 = 3$

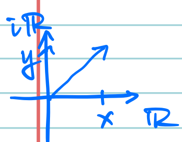
$=$ area of $\square$

$= |x||y| \cos \alpha$

Length of $\vec{x} = \|\vec{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ the norm of $\vec{x}$

$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$

$z \in \mathbb{C} \qquad z = x + iy \qquad x, y \in \mathbb{R}$



$\|z\| = \sqrt{x^2 + y^2}$

$\bar{z} = x - iy$
complex conjugate

$\vec{z} \in \mathbb{C}^n \qquad \vec{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} x_1 + iy_1 \\ \vdots \\ x_n + iy_n \end{pmatrix}$

$\langle \vec{z}, \vec{z} \rangle = z_1 \bar{z}_1 + z_2 \bar{z}_2 + \cdots$

$= (x_1 + iy_1)(x_1 - iy_1) + \cdots$

$= x_1^2 + y_1^2$

$\|\vec{z}\| = \sqrt{x_1^2 + y_1^2 + x_2^2 + y_2^2 \cdots}$

$= \sqrt{\langle \vec{z}, \vec{z} \rangle}$

$\langle \vec{z}, \vec{w} \rangle = z_1 \overline{w_1} + z_2 \overline{w_2} + \cdots + z_n \overline{w_n}$

$\vec{z} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} \qquad \vec{w} = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}$

Defn. Hermitian Adjoint of $A \in M_{m \times n}(\mathbb{C})$ is written A^+
where $A^+ = \bar{A}^T = \overline{A^T}$

Ex. $A = \begin{pmatrix} 1 & 1-i \\ 2 & i \end{pmatrix}$ $\quad A^{\dagger} = \begin{pmatrix} 1 & 2 \\ 1+i & -i \end{pmatrix}$

Hermitian Matrix: $A = A^{\dagger}$

$A = \begin{pmatrix} 1 & 1-i \\ 1+i & 2 \end{pmatrix}$ $\quad A^{\dagger} = \begin{pmatrix} 1 & 1-i \\ 1+i & 2 \end{pmatrix}$

$\langle \vec{z}, \vec{w} \rangle = \vec{z}^{T} \overline{\vec{w}} = \overline{\vec{w}}^{T} \vec{z} = \vec{w}^{\dagger} \vec{z}$

$\| \vec{z} \| = \sqrt{\langle \vec{z}, \vec{z} \rangle}$

Properties:

IP1: $\langle \vec{x}, \vec{y} \rangle = \vec{y}^{\dagger} \vec{x} = \overline{\vec{y}}^{T} \vec{x} = \vec{x}^{T} \overline{\vec{y}}$

$\langle \vec{y}, \vec{x} \rangle = \vec{x}^{\dagger} \vec{y} = \vec{y}^{T} \overline{\vec{x}} = \overline{\overline{\vec{y}}^{T} \vec{x}} = \overline{\vec{x}^{T} \overline{\vec{y}}}$

$\Rightarrow \langle \vec{x}, \vec{y} \rangle = \overline{\langle \vec{y}, \vec{x} \rangle}$

Conjugate Symmetry

IP2: $\langle \alpha \vec{x} + \beta \vec{y}, \vec{z} \rangle = \vec{z}^{\dagger} (\alpha \vec{x} + \beta \vec{y}) = \alpha \vec{z}^{\dagger} \vec{x} + \beta \vec{z}^{\dagger} \vec{y}$

$\alpha \langle \vec{x}, \vec{z} \rangle + \beta \langle \vec{y}, \vec{z} \rangle$

Linearity

IP3: $\vec{x} = \begin{pmatrix} a_1 + i b_1 \\ \vdots \\ a_n + i b_n \end{pmatrix}$ $\quad \langle \vec{x}, \vec{x} \rangle = \vec{x}^{\dagger} \vec{x} = a_1^2 + b_1^2 + \cdots + a_n^2 + b_n^2 \in \mathbb{R}$

≥ 0

Non-negativity

IP4: $\langle \vec{x}, \vec{x} \rangle = 0 \iff \vec{x} = \vec{0}$

Non-degeneracy.

Defn: Let V be V.S. w/ scalar in $\mathbb{R}$ or $\mathbb{C}$. An inner product is a function $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{F}$ where $\mathbb{F} = \mathbb{R} \cup \mathbb{C}$ s.t. $\forall \vec{x}, \vec{y}, \vec{z} \in V$ & $\alpha, \beta \in \mathbb{F}$

IP1 through IP4 hold. In this case, V is an inner product space.

Lemma. Let V be a v.s. $\vec{x} \in V$

$\langle \vec{x}, \vec{y} \rangle = \vec{0}$ for all $\vec{y} \in V \iff \vec{x} = \vec{0}$

Pf. assume $\langle \vec{x}, \vec{y} \rangle = \vec{0}$ let $\vec{y} = \vec{x}$

$\Rightarrow \langle \vec{x}, \vec{x} \rangle = \vec{0} \xRightarrow{\text{by IP4}} \vec{x} = 0$

Corollary $\vec{x}, \vec{y} \in V$ an IPS. $\vec{x} = \vec{y} \iff \langle \vec{x}, \vec{z} \rangle = \langle \vec{y}, \vec{z} \rangle$ $\forall \vec{z} \in V$.

Pf. $\vec{x} = \vec{y} \Rightarrow \vec{x} - \vec{y} = \vec{0} \xRightarrow{\text{IP4}} \langle \vec{x} - \vec{y}, \vec{z} \rangle = \vec{0}$

$\xRightarrow{\text{IP2}} \langle \vec{x}, \vec{z} \rangle - \langle \vec{y} - \vec{z} \rangle = \vec{0} \iff \langle \vec{x}, \vec{z} \rangle = \langle \vec{y}, \vec{z} \rangle$

Def. For IPS V, the norm on V is $\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$

"L2-norm"

Def Absolute Value

$\mathbb{R}: \quad |x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases} = \sqrt{x^2}$

$\mathbb{C}: \quad |z| = |x + iy| = \sqrt{x^2 + y^2}$

Thm. Cauchy-Schwartz Inequality

For $\vec{x}, \vec{y} \in V$, IPS. $\quad |\langle \vec{x}, \vec{y} \rangle| \leq \|\vec{x}\| \, \|\vec{y}\|$

Pf. WLOG ① $\vec{y} = \vec{0}$ the $\langle \vec{x}, \vec{0} \rangle = 0 = \|\vec{x}\| \, \|\vec{0}\|$ ✓

② $\vec{y} \neq \vec{0}$ Let t be a scalar $\in \mathbb{R}$ Consider $\|\vec{x} - t\vec{y}\|^2$

$\|\vec{x} - t\vec{y}\|^2 = \langle \vec{x} - t\vec{y}, \vec{x} - t\vec{y} \rangle \geq 0$

$= \langle \vec{x}, \vec{x} \rangle - t \langle \vec{y}, \vec{x} \rangle - t \langle \vec{x}, \vec{y} \rangle + t^2 \|\vec{y}\|^2$

Let $t = \dfrac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{y}\|^2}$ $\quad \vec{y} \neq \vec{0} \Rightarrow \|y\|^2 \neq 0$

$= \|x\|^2 - 2 \dfrac{\langle \vec{x}, \vec{y} \rangle}{\|\vec{y}\|^2} \langle \vec{x}, \vec{y} \rangle + \dfrac{\cancel{\|y\|^2}}{\cancel{\|y\|^2}} \cdot \dfrac{\langle \vec{x}, \vec{y} \rangle^2}{\|y\|^2}$

$$= \|x\|^2 - \frac{\langle \vec{x}, \vec{y} \rangle^2}{\|y\|^2} \geq 0$$

$$\|x\|^2 \geq \frac{\langle \vec{x}, \vec{y} \rangle^2}{\|y\|^2}$$

$$\Rightarrow \|x\| \, \|y\| \geq \langle \vec{x}, \vec{y} \rangle$$

Cor: $|\langle \vec{x}, \vec{y} \rangle| = \|\vec{x}\| \, \|\vec{y}\| \iff$ one vector is a multiple of the other

Thm Triangle Inequality

For $\vec{x}, \vec{y} \in V$, IPS $\quad \|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

Pf. $\underline{\|x+y\|^2} = \langle \vec{x}+\vec{y}, \vec{x}+\vec{y} \rangle \overset{\text{I2}}{=} \langle \vec{x}, \vec{x}+\vec{y} \rangle + \langle y, \vec{x}+\vec{y} \rangle$

$$\overset{\text{I2}}{=} \langle \vec{x}, \vec{x} \rangle + \langle \vec{x}, \vec{y} \rangle + \langle \vec{y}, \vec{x} \rangle + \langle \vec{y}, \vec{y} \rangle$$

$$\leq \|\vec{x}\|^2 + 2\,\underline{|\langle x, y \rangle|} + \|\vec{y}\|$$

Lemma:
$\vec{x}, \vec{y} \in V$, IPS.
$\|\vec{x}+\vec{y}\|^2 + \|\vec{x}-\vec{y}\|^2 = 2(\|\vec{x}\|^2 + \|\vec{y}\|^2)$

Lemma $z \in \mathbb{R} \cup \mathbb{C} \quad z + \bar{z} \leq 2|z|$

$z = x+iy \qquad x+iy+x-iy = 2x$

$|z| = \sqrt{x^2+y^2} \Rightarrow$ want $\cancel{2}x \leq \cancel{2}\sqrt{x^2+y^2}$

$\sqrt{x^2+y^2} \geq \sqrt{x^2}$ $\square$

$$\underline{\leq} \|\vec{x}\|^2 + 2\|\vec{x}\| \, \|\vec{y}\| + \|\vec{y}\|^2$$

$$= \underline{(\|x\| + \|y\|)^2}$$

$$\Rightarrow \|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

Defn. Normed Space

a v.s. V w/ scalars $\in \mathbb{F} = \mathbb{R} \cup \mathbb{C}$ & a function $\|\cdot\| : V \to \mathbb{R}$ which satisfies $\forall \vec{x}, \vec{y} \in V, \alpha \in \mathbb{F}$:

N1: Homogeneity: $\|\alpha \vec{x}\| = |\alpha| \cdot \|\vec{x}\|$

N2: Triangle Inequality: $\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$

N3: Non-negativity: $\|\vec{x}\| \geq 0$

N4: Non-degeneracy: $\|\vec{x}\| = 0 \Leftrightarrow \vec{x} = \vec{0}$

Prop: V is IPS $\Rightarrow$ V is a normed space.

$\|\vec{x}\| = \sqrt{\langle \vec{x}, \vec{x} \rangle}$

Ex. Norms but not IPS

Let $V = \mathbb{R}^n$ or $\mathbb{C}^n$ define p-norm on V, $p = 1, 2, 3, \dots$

$$\|\vec{x}\|_p = (|x_1|^p + |x_2|^p + \dots + |x_n|^p)^{\frac{1}{p}}$$

$\vec{x} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ $\quad \|\vec{x}\|_3 = (|-1|^3 + |2|^3)^{\frac{1}{3}} = 9^{\frac{1}{3}}$

N1: $$\|\alpha \vec{x}\|_p = (|\alpha x_1|^p + |\alpha x_2|^p + \dots + |\alpha x_n|^p)^{\frac{1}{p}}$$
$$= [|\alpha|^p (|x_1|^p + \dots + |x_n|^p)]^{\frac{1}{p}}$$
$$= |\alpha| (|x_1|^p + \dots + |x_n|^p)^{\frac{1}{p}}$$
$$= |\alpha| \, \|\vec{x}\|_p$$

Check parallelogram identity.

Let $\vec{x} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\vec{y} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\in \mathbb{C}^n$ $\quad$ Check $\|\cdot\|_1$

$$LHS = \|\vec{x} + \vec{y}\|_1^2 + \|\vec{x} - \vec{y}\|_1^2 = \left\| \begin{pmatrix} 1 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\|_1^2 + \left\| \begin{pmatrix} 1 \\ -1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\|_1^2 = 2^2 + 2^2 = 8$$

$$RHS = 2(\|\vec{x}\|_1^2 + \|\vec{y}\|_1^2) = 2(1^2 + 1^2) = 4$$

LHS $\neq$ RHS $\Rightarrow$ Parallelogram Identity doesn't hold.

$\Rightarrow$ $\|\vec{x}\|_1$ is not from inner product.

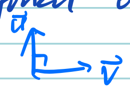
Thm. V is a normed space & the norm in V comes from an IPS

$\Leftrightarrow$ The norm satisfies the paral. Ident.

§ 5.2 Orthogality

* V is IPS.

Defn. $\vec{u}, \vec{v} \in V$ are orthogonal or $\vec{u} \perp \vec{v}$ if $\langle \vec{u}, \vec{v} \rangle = 0$.

Ex. 

Prop. Pythagorean Identity

if $\vec{u} \perp \vec{v}$ then $\|\vec{u}\|^2 + \|\vec{v}\|^2 = \|u+v\|^2$

Pf. $\|\vec{u}+\vec{v}\|^2 = \langle \vec{u}+\vec{v}, \vec{u}+\vec{v} \rangle$

$= \langle \vec{u}, \vec{u} \rangle + \langle \vec{v}, \vec{v} \rangle + \langle \vec{u}, \vec{v} \rangle + \langle \vec{v}, \vec{u} \rangle$

$= \|\vec{u}\|^2 + \|\vec{v}\|^2 + 0 + \cancel{\langle \vec{u}, \vec{v} \rangle} \to 0$

Defn. Let $\vec{v} \in V$ & W is is a subspace of V. Ex $\mathbb{R}^3$ $W = \text{Span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$

$\vec{v} \perp W$ if $\vec{v} \perp \vec{w}, \forall \vec{w} \in W$. $\vec{v} = \begin{pmatrix}0\\1\\0\end{pmatrix}$ $\vec{v} \perp W$

Defn. U & W are subspaces of V.

$U \perp W$ if $\forall \vec{u} \in U, \forall \vec{w} \in W, \vec{u} \perp \vec{w}$.

Ex. $\mathbb{R}^3$ $W = \text{Span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$ $U = \text{Span}\left\{\begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$

$\begin{pmatrix}2\\0\\0\end{pmatrix} \perp \begin{pmatrix}0\\-3\\0\end{pmatrix}$ $W \perp U$.

Prop let $W = \text{Span}\{\vec{w}_1, \dots, \vec{w}_k\}$
then $\vec{v} \perp W \iff \vec{v} \perp \vec{w}_i$ for all i.

Defn. $\vec{v}_1, \dots, \vec{v}_n$ form an orthogonal system if $\vec{v}_i \perp \vec{v}_j$, $i \neq j$.

pairwise orthogonal.

Ex. $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ $\langle \vec{e}_1, \vec{e}_2 \rangle = 0$. $\langle \vec{e}_2, \vec{e}_3 \rangle = 0$. $\langle \vec{e}_1, \vec{e}_3 \rangle = 0$

Defn. An orthogonal System is orthonormal if $\|\vec{v}_i\| = 1$, $\forall i$.

Prop. Any orthogonal system of non-zero vectors $\vec{v}_1, \dots, \vec{v}_k$ is linearly independent.

$\vec{v}_1 \neq 3\vec{v}_2 - \vec{v}_4 + 6\vec{v}_5$

Pf. $\alpha_1\vec{v}_1 + \cdots + \alpha_k\vec{v}_k = \vec{0}$

$$\langle \alpha_1\vec{v}_1 + \cdots + \alpha_k\vec{v}_k , \vec{v}_i \rangle = \langle \vec{0}, \vec{v}_i \rangle \overset{IP4}{=} 0.$$

$$\overset{IP2}{\Rightarrow} \alpha_1 \underbrace{\langle \vec{v}_1, \vec{v}_i \rangle}_{0} + \cdots + \underbrace{\alpha_i\langle \vec{v}_i, \vec{v}_i\rangle}_{} + \cdots + \alpha_k \underbrace{\langle \vec{v}_k, \vec{v}_i \rangle}_{0} = 0$$

$$\Rightarrow \alpha_i \langle \vec{v}_i, \vec{v}_i \rangle + 0 = 0$$

since $\vec{v}_i \neq \vec{0}$

$$\Rightarrow \alpha_i = 0 \quad \Rightarrow L.I.$$

Defn. An orthogonal basis of V is an orthogonal system which is also a basis

An orthonormal basis is an orthonormal system which is also a basis

Thm.

Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be an ordered basis for V, $\vec{x} \in V$, $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{pmatrix}$

- If $\mathcal{B}$ is an orthogonal basis $\Rightarrow \alpha_i = \dfrac{\langle \vec{x}, \vec{v}_i\rangle}{\|\vec{v}_i\|^2}$
- " " " orthonormal " $\Rightarrow \alpha_i = \langle \vec{x}, \vec{v}_i \rangle$

Defn. Let W is a subspace of V & orthogonal complement of W,

$$W^\perp = \{\vec{v} \in V \mid \vec{v} \perp W\}$$

Ex $\mathbb{R}^3$ If $W = \left\{\begin{pmatrix}x\\y\\0\end{pmatrix}\right\}$ x-y plane, $W^\perp = \left\{\begin{pmatrix}0\\0\\z\end{pmatrix}\right\}$ z-axis

Prop. W is a subspace of V

① $W^\perp$ is a subspace of V.

② $V = W \oplus W^\perp$ aka. $\{W, W^\perp\}$ is a basis of V.

§ 5.3 Orthogonal Projection

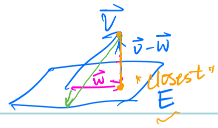
* V: IPS, E: subspaces of V.

Defn. If $\vec{v} \in V$, the orthogonal projection of $\vec{v}$ onto E is $\vec{w} \in V$ w/

OP1: $\vec{w} \in E$

OP2: $(\vec{v} - \vec{w}) \perp E$ we write $\vec{w} = P_E \vec{v}$

Ex.



$\langle \vec{v}_i, \vec{v}_j \rangle = 0 \quad , i \neq j$

Prop. If $\vec{v}_1, \dots, \vec{v}_r$ are orthogonal basis for E & $\vec{v} \in V$, then

$$\vec{w} = \alpha_1 \vec{v}_1 + \cdots + \alpha_r \vec{v}_r$$

where $\alpha_i = \dfrac{\langle \vec{v}, \vec{v}_i \rangle}{\|\vec{v}_i\|^2}$ satisfies OP1 & OP2. so $\vec{w} = P_E \vec{v}$.

So: orthogonal proj exists if $\exists$ orthogonal basis for E.

Pf. $P_E \vec{v} = \displaystyle\sum_{i=1}^{r} \frac{\langle v, \vec{v}_i \rangle}{\|\vec{v}_i\|^2} \vec{v}_i$ OP1: $\vec{w} \in E$ by closure.

OP2: to show $(\vec{v}-\vec{w}) \perp E$ It suffices to show $(\vec{v}-\vec{w}) \perp \vec{v}_i$, $i = 1$ to r.

$\langle \vec{v}-\vec{w}, \vec{v}_i \rangle = \langle \vec{v} - \alpha_1 \vec{v}_1 - \cdots - \alpha_r \vec{v}_r, \vec{v}_i \rangle = \langle \vec{v}, \vec{v}_i \rangle - \alpha_1 \cancel{\langle \vec{v}_1, \vec{v}_i \rangle} - \cdots - \alpha_r \langle \vec{v}_r, \vec{v}_i \rangle$

0 $\quad \alpha_i \langle \vec{v}_i, \vec{v}_i \rangle$

$= \langle \vec{v}, \vec{v}_i \rangle - \alpha_i \langle \vec{v}_i, \vec{v}_i \rangle$

$= \langle \vec{v}, \vec{v}_i \rangle - \dfrac{\langle v, \vec{v}_i \rangle}{\|\vec{v}_i\|^2} \langle \vec{v}_i, \vec{v}_i \rangle$

$= \langle \vec{v}, \vec{v}_i \rangle - \dfrac{\langle v, \vec{v}_i \rangle}{\cancel{\|\vec{v}_i\|^2}} \cancel{\|\vec{v}_i\|^2}$

$= 0 \quad \Rightarrow (\vec{v}-\vec{w}) \perp \vec{v}_i \Rightarrow (\vec{v}-\vec{w}) \perp E$. $\blacksquare$

Prop. Suppose $\vec{v}_1, \dots, \vec{v}_r$ is orthogonal basis for E.
the map $P_E : V \to V$ is linear.

Pf. $P_E(\alpha \vec{v} + \beta \vec{w}) = \displaystyle\sum_{i=1}^{r} \frac{\langle \alpha \vec{v} + \beta \vec{w}, \vec{v}_i \rangle}{\|\vec{v}_i\|^2} \vec{v}_i = \sum_{i=1}^{r} \frac{\langle \alpha \vec{v}, \vec{v}_i \rangle}{\|\vec{v}_i\|^2} \vec{v}_i + \sum_{i=1}^{r} \frac{\langle \beta \vec{w}, \vec{v}_i \rangle}{\|\vec{v}_i\|^2} \vec{v}_i$

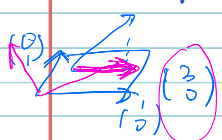
$= \alpha \displaystyle\sum_{i=1}^{r} \frac{\langle \vec{v}, \vec{v}_i \rangle}{\|\vec{v}_i\|^2} \vec{v}_i + \beta \sum_{i=1}^{r} \frac{\langle \vec{w}, \vec{v}_i \rangle}{\|\vec{v}_i\|^2} \vec{v}_i$

$= \alpha P_E \vec{v} + \beta P_E \vec{w}$ $\blacksquare$

Thm. $\vec{v} \in V$, $\vec{w} \in V$ satisfying OP1 & OP2.

① $\forall \vec{x} \in E \quad \|\vec{v}-\vec{w}\| \leq \|\vec{v}-\vec{x}\|$

② If for some $\vec{x}$ $\|\vec{v}-\vec{w}\| = \|\vec{v}-\vec{x}\|$ then $\vec{x} = \vec{w}$. (uniqueness).



coordinates
$\mathcal{B}$

Pf. ① $\vec{x} \in E$, let $\vec{y} = \vec{w} - \vec{x}$ $\quad \vec{v}-\vec{x} = \vec{v}-\vec{w}+\vec{w}-\vec{x} = \vec{v}-\vec{w}+\vec{y}$

$(\vec{v}-\vec{w}) \perp E$ by OP2, $\vec{w}$ & $\vec{x} \in E$ $\quad (\vec{v}-\vec{w}) \perp (\vec{w}-\vec{x}) = \vec{y}$

By Pathagorean Identity $\|\vec{v}-\vec{x}\|^2 = \|\vec{v}-\vec{w}+\vec{y}\|^2 = \|\vec{v}-\vec{w}\|^2 + \|\vec{y}\|^2$

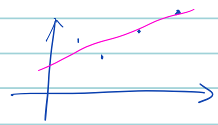
$$\Rightarrow \|\vec{v}-\vec{x}\|^2 \geq \|\vec{v}-\vec{w}\|^2$$

$$\Rightarrow \|\vec{v}-\vec{x}\| \geq \|\vec{v}-\vec{w}\|$$

② If $\|\vec{v}-\vec{w}\| = \|\vec{v}-\vec{x}\| \Rightarrow \|\vec{y}\|^2 = 0$

$\Rightarrow \vec{y} = \vec{0}$ by non-degeneracy N4 $\Rightarrow \vec{w} = \vec{x}$.

x	y
1	6
2	5
3	7
4	10



§5.4 Least Squares.

$y = \alpha + \beta x$

$\alpha + \beta(1) = 6$
$\alpha + \beta(2) = 5$
$\alpha + \beta(3) = 7$
$\alpha + \beta(4) = 10$

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 7 \\ 10 \end{pmatrix}$$

Goal: Find best $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ that's "closest" to being a sol.

MAE MSE

$\Rightarrow A\vec{x} - \vec{b} = \vec{0}$

>> Have a system $A\vec{x} = \vec{b}$ w/ no sol. to find $\vec{x}$ closest to being a sol.,

take $\|A\vec{x} - \vec{b}\|$ & find $\vec{x}$ that minimize it.

want: $\min \|A\vec{x} - \vec{b}\|^2$

$A_{m\times n}$ then $\|A\vec{x} - \vec{b}\|^2 = \sum_{k=1}^{m} \left((A\vec{x})_k - \vec{b}_k \right)^2$ ← Least square.

How to find $\vec{x}$?

If we check all $\vec{x}$ & look at $A\vec{x}$, get all vectors in Ran(A)

$\Rightarrow$ minimizing $\|A\vec{x} - \vec{b}\|^2$ = minimizing the distance from $\vec{b}$ to Ran(A)

* To find a vector in Ran A closest to $\vec{b}$: Orthog. Proj.

$$A\vec{x} = P_{\text{Ran}A}\,\vec{b} \in \text{Ran}A.$$

Need: ① find a orthogonal basis for Ran A
② Use formula to compute $P_{\text{Ran}A}\,\vec{b}$
③ Row reduction to find $\vec{x}$.

Recall: if $\vec{w} = P_W \vec{v}$ then $(\vec{v}-\vec{w}) \perp W$ OP2

So if $A\vec{x} = P_{\text{Ran}A}\vec{b}$, $(\vec{b} - A\vec{x}) \perp \text{Ran}A$

The columns of A $\vec{a}_1, \dots, \vec{a}_n$ span $\text{Ran}A$.

$(\vec{b} - A\vec{x}) \perp \text{Ran}A \iff (\vec{b} - A\vec{x}) \perp \vec{a}_i$, $\forall i \in [1,n]$.

$\iff \langle \vec{b} - A\vec{x}, \vec{a}_i \rangle = 0$ for all i.

$\iff \underset{\text{for all } i}{\vec{a}_i^{\,T}(\vec{b} - A\vec{x}) = 0} \iff A^T(\vec{b} - A\vec{x}) = \vec{0}$

$\iff A^T\vec{b} - A^TA\vec{x} = \vec{0} \iff A^T\vec{b} = A^TA\vec{x}$.

Find $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$:

$$\begin{pmatrix}1&1\\1&2\\1&3\\1&4\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \begin{pmatrix}6\\5\\7\\10\end{pmatrix}$$

$$\text{LHS} = \underbrace{\begin{pmatrix}1&1&1&1\\1&2&3&4\end{pmatrix}}_{A^T}\underbrace{\begin{pmatrix}6\\5\\7\\10\end{pmatrix}}_{\vec{b}} = \underbrace{\begin{pmatrix}1&1&1&1\\1&2&3&4\end{pmatrix}}_{A^T}\underbrace{\begin{pmatrix}1&1\\1&2\\1&3\\1&4\end{pmatrix}}_{A}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$\begin{pmatrix}6+5+7+10\\6+10+21+40\end{pmatrix} = \begin{pmatrix}4&10\\10&30\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$\begin{pmatrix}28\\77\end{pmatrix} = \begin{pmatrix}4&10\\10&30\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$\left[\begin{array}{cc|c}4&10&28\\10&30&77\end{array}\right] \underset{R_2 + R_1/7}{\overset{R_1/2}{\rightsquigarrow}} \left[\begin{array}{cc|c}2&5&14\\2&\frac{40}{7}&15\end{array}\right] \underset{R_2 - R_1}{\rightsquigarrow} \left[\begin{array}{cc|c}2&5&14\\0&\frac{5}{7}&1\end{array}\right]$$

$$\begin{cases}\frac{5}{7}\beta = 1\\ 2\alpha + 5\cdot\frac{7}{5} = 14\end{cases} \Rightarrow \begin{cases}\beta = \frac{7}{5}\\ \alpha = \frac{7}{2}\end{cases}$$