236861 Numerical Geometry of Images

Tutorial 10

Laplace-Beltrami operator
Diffusion geometry

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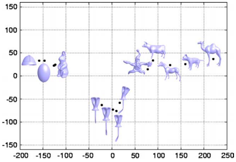
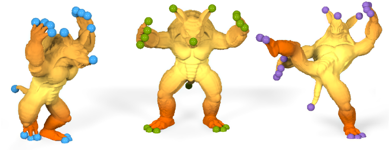
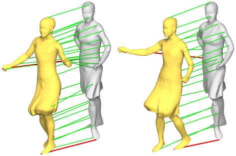
Applications for shape analysis



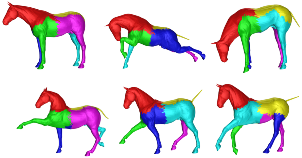
Shape descriptor, Reuter'06



Point descriptors, Sun'09



Shape matching, Ovsjanikov'10



Shape segmentation, Skraba'10

Heat diffusion in Euclidean domain $\mathbb{R}^m$

Heat diffusion on $\mathbb{R}^m$ is governed by the *heat equation*

$$\left(\Delta + \frac{\partial}{\partial t}\right) u(x;t) = 0; \quad u(x;0) = u_0(x), \;\; u(\partial\Omega) = ...$$

- $u : \Omega \in \mathbb{R}^m \times \mathbb{R}^+ \to \mathbb{R}$ - heat distribution at $x \in \Omega$ at time $t > 0$.
- $u_0(x)$ - initial heat distribution.
- Δ - the Laplacian. It is defined as

$$\Delta u(x) = \nabla^2 u(x) = \mathrm{div}\nabla u(x) = \sum_{i=1}^{m} \frac{\partial^2 u}{\partial x_i^2}.$$

Heat kernel in Euclidean domain

For an initial distribution $u_0(x) = \delta(x - x_0)$ the solution is called the *heat kernel*

$$h_t(x - x_0) = \frac{1}{(4\pi t)^{m/2}} e^{-\|x - x_0\|^2/4t}$$

(we saw it in HW1).

The solution $u(x;t)$ for a general initial distribution $u_0(x)$ is given by

$$u(x;t) = \int_{\mathbb{R}^m} h_t(y - x)\, u_0(y) dy.$$

Spectral decomposition

The Laplacian eigenvalue problem (the Helmholtz equation)

$$\Delta\phi = -\lambda\phi.$$

λ is an *eigenvalue* of the Laplacian, and ϕ is its corresponding *eigenfunction*.
In 1D Euclidean domain $\mathbb{R}$, the eigenfunctions of Δ are the *Fourier basis functions*

$$\phi_k(x) = e^{if_k x}.$$

Spectral decomposition in 1D: example

In 1D, the Laplacian can be discretized as follows (finite differences scheme, with periodic boundary condition)

$$L = \begin{pmatrix} 2 & -1 & 0 & 0 & \dots & 0 & -1 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ 0 & -1 & 2 & -1 & \dots & 0 & 0 \\ & & & \vdots & & & \\ -1 & 0 & 0 & 0 & \dots & -1 & 2 \end{pmatrix}$$

Question: what are the eigenvectors of L?

Next: what happens on surfaces?

Heat diffusion

Function defined on a surface X: with each $x \in X$ we associate a function value $u(x)$

$$u : X \to \mathbb{R}.$$

Heat diffusion on X is governed by the *heat equation*

$$\left(\Delta_X + \frac{\partial}{\partial t}\right) u(x;t) = 0; \quad u(x;0) = u_0(x).$$

- $u(x;t)$ - heat distribution at $x \in X$ at time $t > 0$.
- $u_0(x)$ - initial heat distribution.
- Δ_X - the Laplace-Beltrami operator defined on X.

The Laplace-Beltrami operator Δ_X

Generalization of the Laplacian for Riemannian manifolds - surfaces in our case.

$$\Delta_X f = \text{div}_X \left(\nabla_X f\right) = \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} g^{ij} \partial_i\right),$$

where the gradient and the divergence are calculated on the manifold X.

Reminder: in local coordinates (v, w) the first fundamental form is given by

$$G = \{g_{ij}\} = \begin{pmatrix} \langle X_v, X_v \rangle & \langle X_v, X_w \rangle \\ \langle X_w, X_v \rangle & \langle X_w, X_w \rangle \end{pmatrix},$$

and $g \triangleq \det(g_{ij})$, and $g^{ij} = (g^{-1})_{ij}$.

Derivation of Δ_X (Aflalo'12)

For $X : \mathbb{R}^n \to \mathbb{R}^m$ and some $dv, dw \in \mathbb{R}^n$, the scalar product induced by the metric G on the tangent plane of X is

$$dv^T G dw.$$

This scalar product implies a new definition of the gradient of $f : X \to \mathbb{R}$ induced by G, $\nabla_G f$,

$$\begin{aligned} f(v + dv) &= f(u) + \langle \nabla_G f, dv \rangle_G + o(\|dv\|) \\ &= f(u) + \nabla_G f^T G dv + o(\|dv\|) \\ &= f(v) + \nabla f^T dv + o(\|dv\|). \end{aligned}$$

Thus we obtain

$$\nabla_G f = G^{-1} \nabla f.$$

Derivation of Δ_X (Aflalo'12) II

In Euclidean domain $\Omega \subset \mathbb{R}^m$, the Laplacian has the following property

$$\int\limits_{x\in\Omega} \Delta f g da = -\int\limits_{x\in\Omega} \langle \nabla f, \nabla g \rangle da, \quad \forall g | g_{|\partial\Omega} = 0,$$

where da is the infinitesimal area element.
Hence, a natural extension of the Laplacian for a given scalar product and a given definition of the gradient would consist of finding $\Delta_G f$ such that

$$\int\limits_{v\in\Omega} \Delta_G f h da = -\int\limits_{v\in\Omega} \langle \nabla_G f, \nabla_G h \rangle_G \sqrt{g} dv_1 \dots dv_n, \quad \forall h | h_{|\partial\Omega} = 0$$

where $g = \det(G)$ and $da = \sqrt{g} dv_1 \dots dv_n$ is the local infinitesimal area element according to the metric G.

Derivation of Δ_X (Aflalo'12) III

Since $\langle \nabla_G f, \nabla_G h \rangle_G = \nabla f^T G^{-1} \nabla h$, we have

$$\begin{aligned}
\int\limits_{v\in\Omega} \Delta_G fhda &= \int\limits_{u\in\Omega} \nabla f^T G^{-1} \nabla h \sqrt{g} du_1 \dots du_n \\
&= \int\limits_{u\in\Omega} \left(\sqrt{g} G^{-1} \nabla f\right)^T \nabla h du_1 \dots du_n \\
&= -\int\limits_{u\in\Omega} \left(\sum_{i=1}^{n} \partial_i \left(\sqrt{g} G^{-1} \nabla f\right)_i h \right) du_1 \dots du_n \\
&= \int\limits_{u\in\Omega} \Delta_G fh \sqrt{g} du_1 \dots du_n.
\end{aligned}$$

Using Einstein summation convention $\Delta_G f = \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} g^{ij} \partial_j f\right)$.

Spectral decomposition of Δ_X

The Laplace-Beltrami operator Δ_X has a discrete set of eigenvectors and eigenvalues

$$\Delta_X \phi = \lambda \phi, \quad \phi : X \to \mathbb{R}^+,$$

where

$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \ldots$$

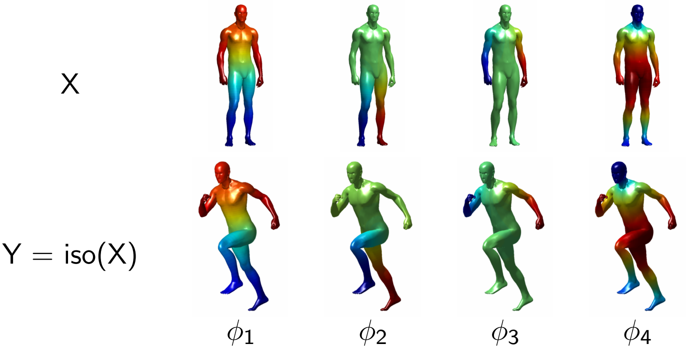
There exist $\lambda_0 = 0$ when X has a boundary, with $\phi_0 = \mathit{const}$.

The corresponding set of eigenvalues forms an orthogonal basis for functions defined on X

$$\{\phi_i\}_{i \geq 1}.$$

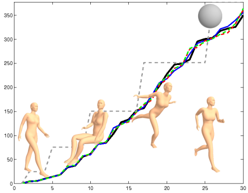
Spectral decomposition of Δ_X: example

As the Laplace-Beltrami operator Δ_X depends only on g_{ij}, it is invariant to isometric transformations of X, and so are its eigenvalues and eigenfunctions (up to a sign).



Applications

Shape DNA, Reuter'06: use $\{\lambda_i\}_{i\geq 1}$ as an isometry invariant descriptor of X.



Reuter'07

Global Point Signature (GPS), Rustamov'07: $\left[\frac{1}{\sqrt{\lambda_1}}\phi_1, \frac{1}{\sqrt{\lambda_2}}\phi_2, \ldots\right]$
Canonical representation, defined up to isometry.

Diffusion geometry on surfaces: heat kernel

The *heat kernel* is given by

$$h_t(x, x') = \sum_{i>0} e^{-\lambda_i t} \phi_i(x) \phi_i(x')$$

It corresponds to the initial heat distribution $u_0(x) = \delta(x - x_0)$.

The value $h_t(x, x')$ can be interpreted as the probability density of a random walk of length t from the point x to the point x'.

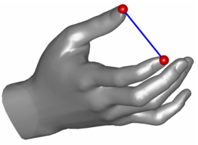
Diffusion geometry on surfaces: diffusion distance

The *diffusion distance* is defined as

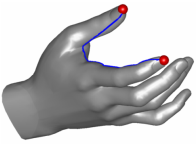
$$d_t^2(x, x') = \int_X \left(h_t(x, z) - h_t(x', y)\right)^2 da$$

$$= \sum_{i>0} e^{-2\lambda_i t} \left(\phi_i(x) - \phi_i(x')\right)^2.$$

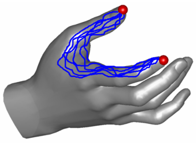
In practice, we sum over N smallest eigenvalues and their corresponding eigenvectors.



Euclidean distance

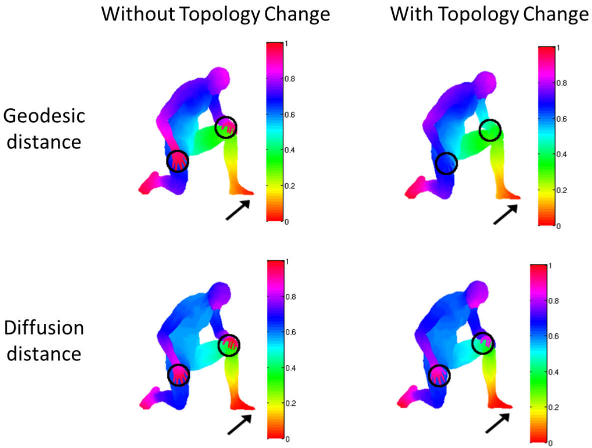


Geodesic distance



Diffusion distance

Comparison between geodesic and diffusion distances



Heat kernel signature and heat kernel map

Heat kernel signature (HKS) and heat kernel map (HKM) are intrinsic point signatures introduced in Sun'09 and Ovsjanikov'10.

$$HKS(x) = (h_{t_1}(x,x), h_{t_2}(x,x), ..., h_{t_K}(x,x))$$

$$HKM(x) = (h_{t_1}(x_0,x), h_{t_2}(x_0,x), ..., h_{t_K}(x_0,x))$$

To defined latter, we should choose a point $x_0 \in X$.

Both can be used for

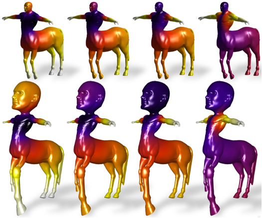
- Shape matching.
- Symmetry detection.
- Interest point detection.

Varying metric g

Reminder:

$$\Delta_X f = -\mathrm{div}_X \left(\nabla_X f \right) = \frac{1}{\sqrt{g}} \partial_i \left(\sqrt{g} g^{ij} \partial_i \right)$$

By changing the metric g we obtain scale-invariant (Aflalo'11) or affine-invariant (Raviv'11) Laplace-Beltrami operator.



Scale invariant HKS signature

Affine invariant matching

Discretization of Δ_X

Discrete version of the Laplace-Beltrami operator

$$(Lu)_i = \sum_{j \in \mathcal{N}_i} w_{ij} \left(u_i - u_j \right), \quad u_i = u(x_i).$$

Some properties of smooth Laplacians:

- $\Delta u = 0$ for $u = const$.
- Local support: for any $x \neq y \in X$, $\Delta(x)$ is independent of $u(y)$.
- Positive semi-definiteness: $\int_X u \Delta u da \geq 0$.

Wardetzky'07: "discrete Laplacians cannot satisfy all natural properties".

Discretization of Δ_X: existing schemes

There exist various discretization schemes for the Laplace-Beltrami operator

- Graph Laplacian (polygonal meshes, point clouds).
- Cotangent weights, mean-value coordinates (triangulated meshes).
- Finite elements (polygonal meshes).
- Belkin and Nyogi (triangulated meshes, point clouds).
- Many more.

Graph Laplacian

Interpret the surface as a graph (X, E), where X are the graph vertices, and E are the edges.

$$(Lu)_i = \sum_{j \in \mathcal{N}_i} w_{ij} \left(u_i - u_j \right), \quad \mathcal{N}_i = \{ j : (i,j) \in E \}$$

where $w_{ij} = 1$, or $w_{ij} \sim \| x_i - x_j \|^{-1}$, etc.

The matrix L is therefore

$$(L)_{ij} = \begin{cases} -w_{ij}, & i \neq j, (i,j) \in E, \\ \sum_{j' \in \mathcal{N}_i} w_{ij'}, & i = j, \\ 0, & \text{otherwise.} \end{cases}$$

Cotangent weight scheme (Meyer'02)

A different *geometric* discretization was suggested by Meyer et al., for surfaces given by triangulated meshes

$$w_{ij} = \frac{1}{2A_i}\left(\cot\alpha_{ij} + \cot\beta_{ij}\right), \quad i \neq j, j \in \mathcal{N}_i,$$

where $\mathcal{N}_i$ is the 1-ring neighborhood of x_i, and A_i is the Voronoi area of the vertex x_i

$$A_i = \frac{1}{8}\sum_{j\in\mathcal{N}_i}\left(\cot\alpha_{ij} + \cot\beta_{ij}\right)\|x_i - x_j\|^2.$$

(Note that for obtuse triangles adjacent to the vertex i we will use a mixed area defined in the next slide instead).

Cotangent weight scheme (Meyer'02)

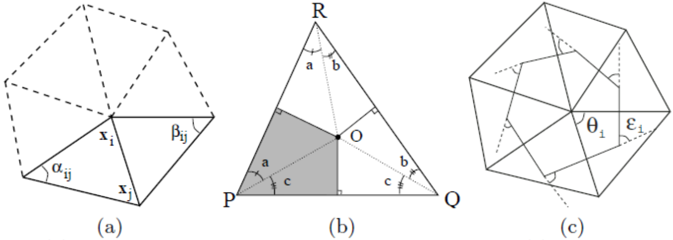


Fig. 3. *(a) 1-ring neighbors and angles opposite to an edge; (b) Voronoi region on a non-obtuse triangle; (c) External angles of a Voronoi region.*

$\mathcal{A}_{\text{Mixed}} = 0$
For each triangle T from the 1-ring neighborhood of $\mathbf{x}$
    If T is non-obtuse, `// Voronoi safe`
        $\mathcal{A}_{\text{Mixed}} += $ Voronoi region of $\mathbf{x}$ in T
    Else `// Voronoi inappropriate`
        If the angle of T at $\mathbf{x}$ is obtuse
            $\mathcal{A}_{\text{Mixed}} += \text{area}(T)/2$
        Else
            $\mathcal{A}_{\text{Mixed}} += \text{area}(T)/4$

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