Inference and Representation Homework 5

Alex Pine, 2015/09/29

I collaborated with Justin Mao-Jones, Peter Li, Maya Rotmensch, and Israel Malkin on these problems.

Question 1

We are given the following:

$$p(x) = \frac{1}{Z}\exp\left[\sum_{i<j}^{n} w_{ij}x_i x_j - \sum_{i}^{n} u_i x_i\right], x_i \in \{0, 1\}$$

Define $y \in (-1, 1)$, so $y = 2x - 1$, and $x = \frac{1}{2}(y + 1)$. Then let

$$S = \sum_{i<j}^{n} w_{ij}x_i x_j - \sum_{i}^{n} u_i x_i$$

.

$$S = x^T \mathbf{W} x - u^T x$$

Where $\mathbf{W}_{ij} = w_{ij}$ if $(i, j) \in E, \mathbf{W}_{ij} = 0$ otherwise.

$$S = \frac{1}{2}(y + \mathbf{1})^T \mathbf{W} \frac{1}{2}(y + \mathbf{1}) - u^T(\frac{1}{2}(y + \mathbf{1}))$$
$$S = \frac{1}{4}(y + \mathbf{1})^T \mathbf{W}(y + \mathbf{1}) - \frac{1}{2}u^T(y + \mathbf{1})$$
$$S = \frac{1}{4}(y^T + \mathbf{1}^T)(\mathbf{W}y + \mathbf{W}\mathbf{1}) - \frac{1}{2}u^T(y + \mathbf{1})$$
$$S = \frac{1}{4}(y^T \mathbf{W} y + y^T \mathbf{W}\mathbf{1} + \mathbf{1}^T \mathbf{W} y + \mathbf{1}^T \mathbf{W}\mathbf{1}) - \frac{1}{2}u^T(y + \mathbf{1})$$

Since $y^T \mathbf{W}\mathbf{1}$ is a scalar, $y^T \mathbf{W}\mathbf{1} = (y^T \mathbf{W}\mathbf{1})^T = \mathbf{1}^T \mathbf{W}^T y$. Substituting that into our expression for S, we get

$$S = \frac{1}{4}(y^T \mathbf{W} y + \mathbf{1}^T \mathbf{W}^T y + \mathbf{1}^T \mathbf{W} y + \mathbf{1}^T \mathbf{W}\mathbf{1}) - \frac{1}{2}u^T(y + \mathbf{1})$$
$$S = \frac{1}{4}(y^T \mathbf{W} y + \mathbf{1}^T(\mathbf{W}^T + \mathbf{W})y + \mathbf{1}^T \mathbf{W}\mathbf{1}) - \frac{1}{2}u^T(y + \mathbf{1})$$
$$S = \frac{1}{4}y^T \mathbf{W} y + \frac{1}{4}\mathbf{1}^T(\mathbf{W}^T + \mathbf{W})y + \frac{1}{4}\mathbf{1}^T \mathbf{W}\mathbf{1} - \frac{1}{2}u^T y - \frac{1}{2}u^T \mathbf{1}$$
$$S = \frac{1}{4}y^T \mathbf{W} y + \frac{1}{4}(\mathbf{1}^T(\mathbf{W}^T + \mathbf{W}) - 2u^T)y + \frac{1}{4}\mathbf{1}^T \mathbf{W}\mathbf{1} - \frac{1}{2}u^T \mathbf{1}$$

This implies

$$\mathbf{W}' = \frac{1}{4}\mathbf{W}$$
$$u' = \frac{1}{2}u - \frac{1}{4}((\mathbf{W} + \mathbf{W}^T)\mathbf{1})$$
$$Z' = \frac{Z}{\exp(\frac{1}{4}\mathbf{1}^T \mathbf{W}\mathbf{1} - \frac{1}{2}u^T \mathbf{1})}$$

Question 2 pairwise Markov random field

Let $\mathbf{X}$ represent a vector of n discrete random variables that can take on s states each. Let the distribution of $p(\mathbf{X}) \propto \psi(X_1, \ldots, X_n)$.

ψ can take on s^n different states, one for each possible value of $\mathbf{X}$. We can impose an arbitrary order on these states, which we'll call χ. For example, if $n = 2$ and $s = 2$, then $\chi = [(0,0), (1,0), (0,1), (1,1)]$. $\chi_2 = (1,0)$, and $\chi_{2,1} = 1$.

Then let Y be a new random variable that can take on values $1, \ldots, s^n$. We can then make a new MRF that has Y and all $X_i \in \mathbf{X}$ as nodes, and has edges between each X_k and Y. This new pairwise MRF has the following potentials:

$$\phi(Y = i) = \psi(\chi_i)$$
$$\phi(Y = y, X_j = x) = 1, \text{if } y = \psi(\chi_i) \text{ and } x = \chi_{ij}, \text{ and 0 otherwise.}$$

This new MRF factorizes as

$$p'(Y, \mathbf{X}) = \phi(Y) \prod_{i=1}^{n} \phi(Y, X_i)$$

Marginalizing, we have

$$p'(\mathbf{X} = \chi_j) = \sum_{i=1}^{n} \left[\phi(Y = i) \prod_{k=1}^{n} \phi(Y = i, X_k = \chi_{jk}) \right]$$

In the sum, when $i = j$, then $\phi(Y = i) = \psi(\chi_i)$, and all the values of $\phi(Y = i, X_k = \chi_{jk}) = 1, k \in 1, \ldots, n$. However, when $i \neq j$ in the sum, $\phi(Y = i, X_k = \chi_{jk}) = 0$ for at least one value of $k \in 1, \ldots, n$, zeroing out that term in the sum. Therefore,

$$p'(\mathbf{X} = \chi_j) = \phi(Y = j) = \psi(\chi_j)$$

.

This is equivalent to the definition of $p(\mathbf{X})$, implying that this new pairwise MRF maps to the original probability distribution once Y is marginalized.

Question 3a.1

$$p(x \mid \mu, I) = (2\pi)^{-\frac{n}{2}} \exp\Big(-\frac{1}{2}(x-\mu)^T(x-\mu)\Big)$$
$$p(x \mid \mu, I) = (2\pi)^{-\frac{n}{2}} \exp\Big(-\frac{1}{2}x^T x + x^T \mu - \frac{1}{2}\mu^T \mu\Big)$$

From this, it's clear that

$$f(x) = (x^T, x^T x)$$
$$\eta = (\mu, -\frac{1}{2})$$

and also

$$-\log(Z(\eta)) = -\frac{1}{2}\mu^T \mu$$
$$Z(\eta) = \exp(\frac{1}{2}\mu^T \mu)$$
$$Z(\eta) = \exp(-\eta_2 \eta_1^T \eta_1)$$

All together, the solution is:

$$h(x) = (2\pi)^{-\frac{n}{2}}$$
$$f(x) = (x^T, x^T x)$$
$$\eta = (\mu, -\frac{1}{2})$$
$$Z(\eta) = \exp(\frac{1}{2}\eta_1^T \eta_1)$$

Question 3a.2

$$\mathrm{Dir}(x \mid \alpha) = \frac{1}{\beta(\alpha)} \prod_{i=1}^{k} x_i^{\alpha_i - 1}, \beta(\alpha) = \frac{\prod_{i=1}^{k} \Gamma(\alpha_i)}{\Gamma(\sum_{i=1}^{k} \alpha_i)}$$
$$\mathrm{Dir}(x \mid \alpha) = \exp(\log(\mathrm{Dir}(x \mid \alpha)))$$
$$\log(\mathrm{Dir}(x \mid \alpha)) = -\log(\beta(\alpha)) + \sum_{i=1}^{k} \log(x^{\alpha_i - 1}) = -\log(\beta(\alpha)) + \sum_{i=1}^{k} (\alpha_i - 1)\log(x)$$

From this it's clear that

$$\eta = \alpha - \mathbf{1}$$
$$f(x) = \log(x)$$
$$h(x) = 1$$
$$Z(\eta) = \beta(\eta + \mathbf{1})$$

Question 3a.3

Since $\mu = 0$, the pdf for the lognormal distribution is given by

$$p(x \mid 0, \sigma) = (x\sigma\sqrt{2\pi})^{-1} \exp\Big(-\frac{(\log(x))^2}{2\sigma^2}\Big)$$

Note that

$$\frac{1}{\sigma} = \exp(\log(\sigma^{-1})) = \exp(-\log(\sigma))$$

which allows us to bring it inside the exponent, giving us

$$p(x \mid 0, \sigma) = (x\sqrt{2\pi})^{-1} \exp\Big(-\frac{(\log(x))^2}{2\sigma^2} - \log(\sigma)\Big)$$

Implying

$$\eta = -\frac{1}{2\sigma^2}$$
$$f(x) = (\log(x))^2$$
$$h(x) = \frac{1}{x\sqrt{2\pi}}$$
$$Z(\eta) = \sigma$$

Using that $\eta = -\frac{1}{2\sigma^2}$, we find

$$Z(\eta) = \frac{1}{\sqrt{-2\eta}}$$

Question 3a.4

The Boltzmann distribution is given by:

$$p(x \mid u, E, W) \propto \sum_{i,j \in E} W_{i,j} x_i x_j - \sum_{i=1}^{n} u_i x_i$$

In this distribution x is a vector with n elements, and each $x_i \in 0, 1$. u is a vector with n eliements of weights. E is a vector of integer pairs representing the edges. W is a upper triangular square matrix with dimensions (n, n). $W_{i,j} = w_{i,j}$ if $(i, j) \in E$, and zero otherwise.

We will define a vector w with n^2 entries, where each w_i is either a weight corresponding to an edge, or zero. To create the vector w, flatten out the matrix W.

Now define ξ to be a vector with the same dimensions as w. Each $\xi_i = x_j x_j$, where j is determined by the flattening process; it is determined by a function of i and the dimensions of W.

Then we can see that the Boltzmann distribution can be written:

$$p(x \mid u, E, W) \propto (w, -u) \cdot (\xi, x)$$

This implies that the Boltzmann distribution is in the exponential family with the following parameters:

$$h(x) = 1$$
$$\eta = (w, -u)$$

$$f(x) = (\xi, x)$$
$$Z(\eta) = 1$$

Question 3b

Define a s.t. $a = p(Y = 1 \mid x; \alpha) = (1 + \exp(-\alpha_0 - \sum_{i=1}^{n} \alpha_i x_i))^{-1}$

Then $p(Y = 0 \mid x; \alpha) = 1 - a$, and $p(Y \mid x; \alpha)$ can be expressed as follows:

$$p(Y \mid x; \alpha) = \exp(y \log(a) + (1 - y) \log(1 - a))$$

Let b be the arguments to the exponential function, so that $b = y \log(a) + (1 - y) \log(1 - a)$. Rearranging this expression, we get

$$b = y \log(\frac{a}{1 - a}) + \log(1 - a)$$

If we define c s.t. $c = \exp(-\alpha_0 - \sum_{i=1}^{n} \alpha_i x_i))$, then

$$\frac{a}{1 - a} = \frac{(1 + c)^{-1}}{1 - (1 + c)^{-1}} = \frac{1}{(1 + c) - 1} = c^{-1} = \exp(\alpha_0 + \sum_{i=1}^{n} \alpha_i x_i)$$

Plugging this into our expression for b, we get

$$b = y \log(\exp(\alpha_0 + \sum_{i=1}^{n} \alpha_i x_i)) + \log(1 - a) = y(\alpha_0 + \sum_{i=1}^{n} \alpha_i x_i) + \log(1 - a)$$

If we define $x' = 1, x_1, \ldots, x_n$, then it's clear that $f(x, y) = yx'$, $\eta = \alpha$, and $\log(Z(\eta, x)) = -\log(1 - a)$. If we manipulate this last expression, we can find $Z(\eta, x)$.

$$\log(Z(\eta, x)) = -\log(1 - a) = \log((1 - a)^{-1})$$

$$Z(\eta, x) = (1 - a)^{-1} = (1 - (1 + c)^{-1})^{-1} = \frac{1}{1 - (1 - c)^{-1}} = \frac{1 - c}{(1 - c) - 1} = \frac{c - 1}{c} = 1 - c^{-1} = 1 -$$

Noting that $\alpha_0 + \sum_{i=1}^{n} \alpha_i x_i) = \eta^T x'$, we find

$$Z(\eta, x) = 1 - \exp(\eta^T x')$$

.

In conclusion,

$$h(x, y) = 1$$

$$f(x, y) = yx'$$
$$\eta = \alpha$$
$$Z(\eta, x) = 1 - \exp(\eta^T x')$$

Where $x' = 1, x_1, \ldots, x_n$.

Question 4a

$\theta \sim \mathrm{Dir}(\alpha), X \sim \mathrm{Cat}(\theta)$. Find $p(\theta \mid x, \alpha)$.

$$p(\theta \mid x, \alpha) = \frac{p(X \mid \theta, \alpha) p(\theta \mid \alpha)}{p(X \mid \alpha)}$$

Since the X's are observed, $p(X \mid \alpha)$ is a constant, so we will omit it going forward.

$$p(\theta \mid x, \alpha) \propto p(X \mid \theta, \alpha) p(\theta \mid \alpha)$$

First we'll find an expression for $p(X \mid \theta, \alpha)$. Since each x_i is categorial and i.i.d, we get

$$p(X \mid \theta, \alpha) = \prod_{i=1}^{n} p(x_i \mid \theta, \alpha)$$

$$p(X \mid \theta, \alpha) = \prod_{i=1}^{n} \theta_{x_i}$$

Then if you group the θ_{x_i} s that have the same category together, you get (assuming there are k categories):

$$p(X \mid \theta, \alpha) = \prod_{i=1}^{n} \prod_{j=1}^{k} \theta_j^{1(j=x_i)}$$

$$p(X \mid \theta, \alpha) = \prod_{j=1}^{k} \prod_{i=1}^{n} \theta_j^{1(j=x_i)}$$

$$p(X \mid \theta, \alpha) = \prod_{j=1}^{k} \theta_j^{\sum_{i=1}^{n} 1(j=x_i)}$$

Now we will find an expression for $p(\theta \mid \alpha)$. Since $\theta \sim \mathrm{Dir}(\alpha)$, then

$$p(\theta \mid \alpha) = \frac{1}{\beta(\alpha)} \prod_{i=1}^{k} \theta_i^{\alpha_i - 1}$$

Since $\frac{1}{\beta(\alpha)}$ is just a normalizing constant, I will omit it going forward and write

$$p(\theta \mid \alpha) \propto \prod_{i=1}^{k} \theta_i^{\alpha_i - 1}$$

Putting these two expressions together, we get

$$p(\theta \mid x, \alpha) \propto p(X \mid \theta, \alpha) p(\theta \mid \alpha)$$

$$p(\theta \mid x, \alpha) \propto \prod_{j=1}^{k} \theta_j^{\sum_{i=1}^{n} 1(j=x_i)} \prod_{i=1}^{k} \theta_i^{\alpha_i - 1}$$

$$p(\theta \mid x, \alpha) \propto \prod_{j=1}^{k} \theta_j^{\alpha_i - 1 + \sum_{i=1}^{n} 1(j=x_i)}$$

If we define $\alpha_i' = \alpha_i + \sum_{i=1}^{n} 1(j = x_i)$, then our expression becomes

$$p(\theta \mid x, \alpha) \propto \prod_{j=1}^{k} \theta_j^{\alpha_i' - 1}$$

This proves the posterier distribution for $\theta \sim \mathrm{Dir}(\alpha')$.

Question 4b

Find $p(x_{n'} \mid x_n, \alpha)$.

We can find this by integrating the joint distribution with θ, that is

$$p(x_{n'} \mid x_n, \alpha) = \int p(x_{n'}, \theta \mid x_n, \alpha) d\theta$$

Using Bayes' rule, this equals

$$\int p(x_{n'} \mid \theta, x_n, \alpha) p(\theta \mid x_n, \alpha) d\theta$$

Since the x's are independent, this equals

$$\int p(x_{n'} \mid \theta, \alpha) p(\theta \mid x_n, \alpha) d\theta$$

Since $X \sim \text{Cat}(\theta)$, this equals

$$\int \theta_{n'} p(\theta \mid x_n, \alpha) d\theta$$

And from the previous problem we know that $p(\theta \mid x_n, \alpha) = \frac{1}{\beta(\alpha')} \prod_k^c \theta_k^{\alpha'_k - 1}$. Substituting this in our previous expression, we get

$$\int \theta_{n'} \frac{1}{\beta(\alpha')} \prod_k^c \theta_k^{\alpha'_k - 1} d\theta$$

which equals

$$\frac{1}{\beta(\alpha')} \int \prod_k^c \theta_k^{1(k=n') + \alpha'_k - 1} d\theta$$

If we define $\alpha'' = \alpha'_k + 1(k = n')$, then

$$\text{Dir}(\alpha'') = \frac{1}{\beta(\alpha'')} \prod_k^c \theta_k^{\alpha''_k - 1}$$

,

Since this is a pdf, then

$$\int \frac{1}{\beta(\alpha'')} \prod_k^c \theta_k^{\alpha''_k - 1} = 1$$
$$\frac{1}{\beta(\alpha'')} \int \prod_k^c \theta_k^{\alpha''_k - 1} = 1$$
$$\int \prod_k^c \theta_k^{\alpha''_k - 1} = \beta(\alpha'')$$

If we substitute this into our expression for $p(x_{n'} \mid x_n, \alpha)$, we get

$$p(x_{n'} \mid x_n, \alpha) = \frac{1}{\beta(\alpha')} \int \prod_k^c \theta_k^{1(k=n')+\alpha_k'-1} d\theta = \frac{1}{\beta(\alpha')} \int \prod_k^c \theta_k^{\alpha_k''-1} d\theta = \frac{\beta(\alpha'')}{\beta(\alpha')}$$

Substituting the definition of the Beta distribution, we get

$$p(x_{n'} \mid x_n, \alpha) = \frac{\prod_{i=1}^c \Gamma(\alpha_i'')}{\Gamma(\sum_{i=1}^c \alpha_i'')} \frac{\Gamma(\sum_{i=1}^c \alpha_i')}{\prod_{i=1}^c \Gamma(\alpha_i')}$$

Question 5: Text message analysis with pymc3

Code, data summary, and graphical summary given below:

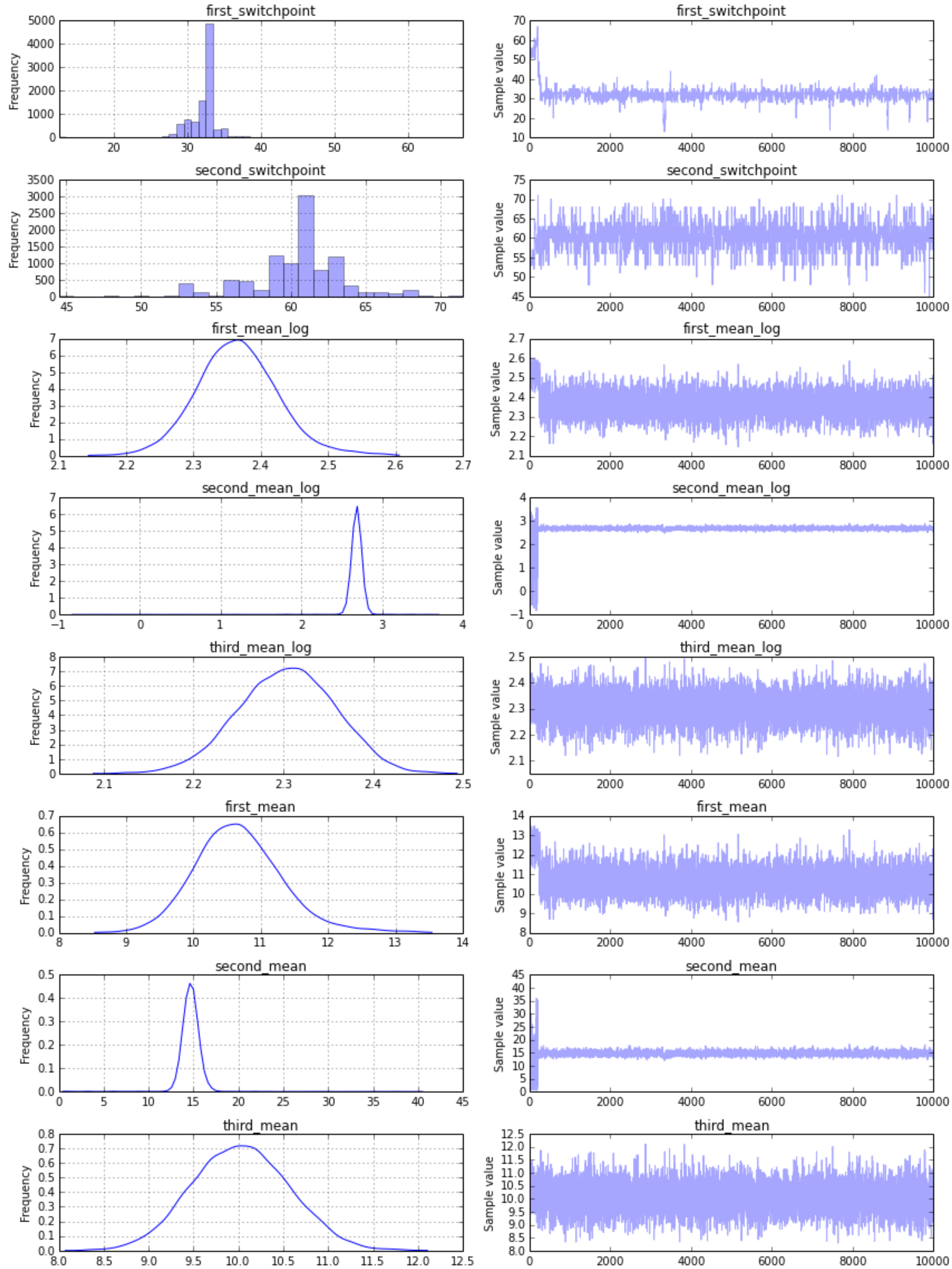
In [6]:
```
%matplotlib inline

import matplotlib.pyplot as plt
import numpy as np
import pymc3 as pm
import theano.tensor as t

@pm.theano.compile.ops.as_op(itypes=[t.lscalar, t.lscalar, t.dscala
r, t.dscalar,
                                     t.dscalar],
                             otypes=[t.dvector])
def rateFunc(first_switchpoint, second_switchpoint, first_mean, sec
ond_mean,
             third_mean):
    out = np.empty(n_count_data)
    out[:first_switchpoint] = first_mean
    out[first_switchpoint:second_switchpoint] = second_mean
    out[second_switchpoint:] = third_mean
    return out

count_data = np.loadtxt('/Users/pinesol/inference/hw2/text_data.cs
v')
n_count_data = len(count_data)

with pm.Model() as text_model:
    first_switchpoint = pm.DiscreteUniform('first_switchpoint', low
er=0,
                                           upper=n_count_data)
    second_switchpoint = pm.DiscreteUniform('second_switchpoint',
                                            lower=0,
                                            upper=n_count_data)
    alpha = 1.0 / count_data.mean()
    first_mean = pm.Exponential('first_mean', lam=alpha)
    second_mean = pm.Exponential('second_mean', lam=alpha)
    third_mean = pm.Exponential('third_mean', lam=alpha)

    rate = rateFunc(first_switchpoint, second_switchpoint, first_me
an,

                    second_mean, third_mean)

    text_count = pm.Poisson('text_count', rate, observed=count_dat
a)

    step1 = pm.Slice([first_mean, second_mean, third_mean])
    step2 = pm.Metropolis([first_switchpoint, second_switchpoint])
    trace = pm.sample(10000, step=[step1, step2])
    pm.summary(trace)
    pm.traceplot(trace)
    plt.show()
```

```
 [-----------------100%-----------------] 10000 of 10000 complete
in 66.5 sec
first_switchpoint:
```

```
  Mean             SD               MC Error        95% HPD inter
val
  ------------------------------------------------------------------
---

  32.670           4.087            0.344           [27.000, 37.0
00]

  Posterior quantiles:
  2.5            25             50             75             97.5
  |--------------|==============|==============|--------------|

  28.000         32.000         33.000         33.000         42.0
00


second_switchpoint:

  Mean             SD               MC Error        95% HPD inter
val
  ------------------------------------------------------------------
---

  60.409           3.162            0.111           [53.000, 66.0
00]

  Posterior quantiles:
  2.5            25             50             75             97.5
  |--------------|==============|==============|--------------|

  53.000         59.000         61.000         62.000         67.0
00


first_mean_log:

  Mean             SD               MC Error        95% HPD inter
val
  ------------------------------------------------------------------
---

  2.366            0.060            0.002           [2.243, 2.48
3]

  Posterior quantiles:
  2.5            25             50             75             97.5
  |--------------|==============|==============|--------------|

  2.252          2.326          2.364          2.403          2.49
5


second_mean_log:
```

```
second_mean_log:

  Mean             SD               MC Error        95% HPD inter
val
  ------------------------------------------------------------------
---

  2.678            0.157            0.006           [2.573, 2.80
8]

  Posterior quantiles:
  2.5             25              50              75             97.5
  |---------------|===============|===============|---------------|

  2.564           2.648           2.687           2.724          2.80
2


third_mean_log:

  Mean             SD               MC Error        95% HPD inter
val
  ------------------------------------------------------------------
---

  2.304            0.054            0.001           [2.198, 2.40
6]

  Posterior quantiles:
  2.5             25              50              75             97.5
  |---------------|===============|===============|---------------|

  2.197           2.268           2.305           2.341          2.40
6


first_mean:

  Mean             SD               MC Error        95% HPD inter
val
  ------------------------------------------------------------------
---

  10.670           0.649            0.027           [9.359, 11.90
3]

  Posterior quantiles:
  2.5             25              50              75             97.5
  |---------------|===============|===============|---------------|

  9.505           10.235          10.635          11.054         12.1
17


second_mean:
```

```
second_mean:

  Mean             SD               MC Error        95% HPD inter
val
  ------------------------------------------------------------------
---

  14.674           1.440            0.032           [12.913, 16.3
78]

  Posterior quantiles:
  2.5            25             50             75             97.5
  |--------------|==============|==============|--------------|

  12.989         14.119         14.690         15.238         16.4
73


third_mean:

  Mean             SD               MC Error        95% HPD inter
val
  ------------------------------------------------------------------
---

  10.031           0.540            0.010           [9.000, 11.08
5]

  Posterior quantiles:
  2.5            25             50             75             97.5
  |--------------|==============|==============|--------------|

  9.000          9.661          10.024         10.395         11.0
88
```



The summary statistics and their associated charts indicate that the first switch point probably occured somewhere around day 33, and the second one around day 60. The mean rate of text messages per day was around 10.67 before the first switch point, 14.67 messages per day afterwards, and then 10.03 after the second switchpoint.