Consistent Interpretations of 'probability'

- „measures" are to a large degree arbitrary. If your colleagues pick one, and you „extend" their method but misinterpret their measure, then you will make a mistake

- Bayesian vs frequentist - discussions are about the meaning attributed to probability measures

Measure theory for probability

Measure theory	Kolmogorov axioms
May S be a set,	May S be a set, called sample space
$\mathcal{B}$ a σ-algebra on it,	E a σ-Algebra on it, now called „event space",
may there be a $\mu: \mathcal{B} \to \mathbb{R}^+$,	may there be a measure $\mathbb{P}: E \to [0,1]$
then μ is a measure if	then $\mathbb{P}$ is a probability measure if
(i) for all $A_i \in \mathcal{B}$ have $\mu(A_i) \geq 0$	(i) $\mathbb{P}(E_i) \in [0,1] \quad \forall E_i \in E$
(ii) $\mu(\emptyset) = 0$	(ii) $\mathbb{P}(E) + \mathbb{P}(\bar{E}) = 1$
(iii) $\sum_{i=1}^{\infty} \mu(A_i) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right)$	(iii) $\mathbb{P}(A \overset{\text{„or"}}{\cap} B) = \mathbb{P}(A) + \mathbb{P}(B)$ if A and B mutually exclusive

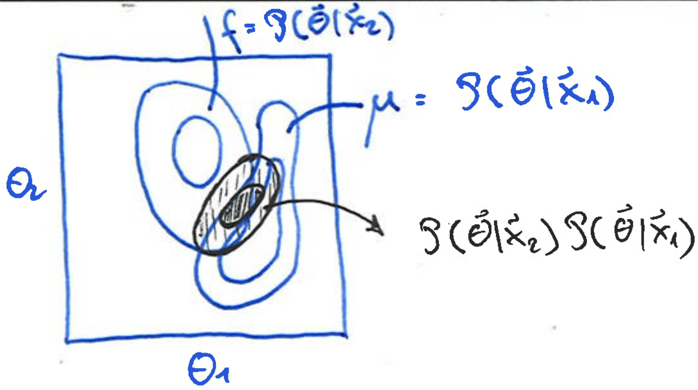
⇒ so it is the same, the only enhancement is:

Def.: May $(S, \mathcal{B}, \mu)$ be a measure space, then μ is a probability measure $\boxed{\text{if } \mu(S) = 1}$

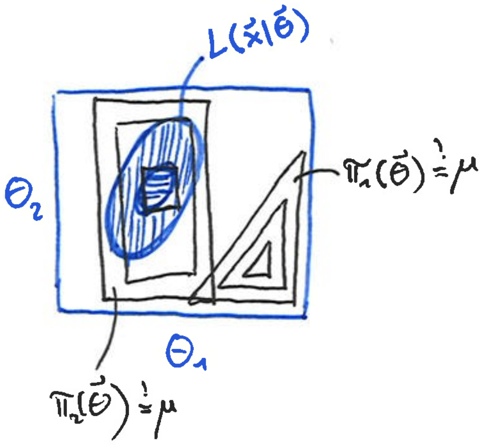
„Something needs to happen"

- Example: Bayes theorem:

$$\mathcal{P}(\vec{\theta}|\vec{x}) = \frac{L(\vec{x}|\vec{\theta})\,\pi(\vec{\theta})}{\int L(\vec{x}|\vec{\theta})\,\underbrace{\pi(\vec{\theta})\,d\vec{\theta}}_{\text{priors are measures}}} \quad \Big\} \text{ Lebesgue Integration!}$$



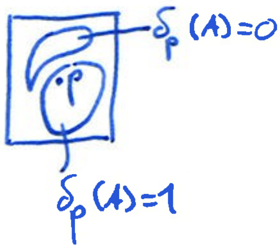
„ σ- additivity"



$\rightsquigarrow$ as measured by prior π_1 the outcome of the experiment is non-sensical (disfavoured)

$\rightsquigarrow$ π_2 assigns a large measure to it, so here things are fine.

- Probability measures satisfy $\mu(S)=1$.
- The Dirac δ-„function" is in reality a probability measure.
  - $\delta_p(A) = \begin{cases} 1 & \text{if } p \in A \\ 0 & \text{else} \end{cases}$



  - if $A = S$, then $\delta_p(S) \equiv 1$, since the point is definitely in S
- Lebesgue integration with Dirac-measure:

$$\int_X f(x)\, d\mu(x) \quad \text{, now with } \mu \overset{!}{=} \delta_p$$

$$\rightsquigarrow \int_X f(x)\, d\delta_p(x) = f(p)$$

$$„=" \int_X f(x) \underbrace{\delta_p(x)}_{\text{„Dirac } \delta\text{-function"}} \underbrace{dx}_{\text{Lebesgue measure}}$$

• Consequences :

$$\int_{-\infty}^{+\infty} \delta_D(ax)\, dx = \int_{-\infty}^{+\infty} \delta_D(x) \frac{dx}{|a|} = \frac{1}{|a|} \int_{-\infty}^{+\infty} \delta_D(x)\, dx = \frac{1}{|a|} \cdot 1$$

measures are positive!

etc, etc (the other properties of the δ-function)