Example exam

Notation: Vectors and matrices must be visually distinguishable. Transposed vectors and matrices are to be marked as such, also in scalar products and outer products.

Part I: Understanding . . . . total: 11 CP

I a) Artificial Neural Networks **2 CP**
Give, and explain the importance of, the *Universal Approximation Theorem* for artificial neural networks.

I b) Gaussians **4 CP**
Sketch the distribution of two random variables x and y, if they follow a joint Gaussian distribution of positive covariance (1 CP). Then...

- ...indicate $\langle x \rangle$ and $\langle y \rangle$ in your sketch (1 CP).
- ...add the marginal distribution of x to your plot (1 CP).
- ...and the conditional distribution for y, if $x = x_0 \neq \langle x \rangle$ (1 CP).

I b) Define and explain the following quantities **2 CP**

1. Posterior (0.5 CP)
2. Likelihood (0.5 CP)
3. Prior (0.5 CP)
4. Burnin (0.5 CP)

I b) Chi-squared test **3 CP**
Describe the χ^2/degF test (1 CP). Name one caveat against it, involving the value of χ^2 (1 CP). Name one caveat against it, involving the degrees of freedom degF (1 CP).

I b) Normalization **1 CP**
If $\mathcal{P}(\mathbf{x})$ is said to be normalized, with respect to which variable is it then normalized? Which equation describes 'normalization'? Finally, if $\mathcal{P}(\mathbf{x})$ is normalized, is $\mathcal{P}(\theta|\mathbf{x})$ normalized?

Part II: Analytics . . . . total: 9 CP

II a) Simple Malmquist bias **5 CP**
Imagine stars have an apparent brightness b which follows a Gaussian distribution $\mathcal{G}$ of mean b_0 and variance σ^2. Your telescope is only able to detect stars with $b > b_t$.

(i) Which sampling distribution do the apparent brightnesses of the stars observable through your telescope have? I.e.: Derive the sampling distribution of observable b (2 CP).

(ii) Imagine you measured 10 brightnesses, $\mathbf{b}^\top = (b_1, ..., b_10)$. Adopt a prior and give the posterior for the mean b_0 (2 CP).

(iii) If $b_t > b_0$, can you still infer b_0? Give a reason (1 CP).

II b) Sums **4 CP**
For $u \sim \mathcal{G}(0, 1)$ and $v \sim \mathcal{G}(0, 1)$, with u, v independent, derive the distribution of their sum $y = u + v$.

Part III: Numerics . . . . total: 10 CP

Important: Submit all source files and multiple screenshots of your code and your (labelled!) plots. Your code should contain your name in a introductory comment line. All submitted files should contain your name. Submit your files to: sellentin@strw.leidenuniv.nl, tutor1@leiden.nl, tutor2@leiden.nl. Before leaving the room, please come to the front and confirm receipt of your files.

III a) Markov Chain Manipulations 8 CP

Open the file 'MarkovChain.txt'. It contains 3 columns, the first being samples of a stellar mass M, the second being samples of stellar distances r in kpc (kpc: 'kiloparsec', an astronomical measure of distance). The third column is the log-likelihood. The first row is a header, the number of samples is 6000.

1. Read in the chain (0 CP).
2. Remove the burnin of the chain: Write up your argumentation line of what you are doing, and support it with a plot (2 CP).
3. Store the new, burnin-free chain in a new vector (0.5 CP) and plot the new chain (0.5 CP).
4. Are the mass M and the stellar distances r correlated (1 CP)? Support your answer by a plot or calculation (1 CP).
5. Plot the distributions $\mathcal{P}(M)$ (0.5 CP) and $\mathcal{P}(r)$ (0.5 CP).
6. Plot the distribution $\mathcal{P}(M|r = 5\,\mathrm{kpc})$ (1 CP).
7. Plot the distribution $\mathcal{P}(M|r = 10\,\mathrm{kpc})$ (1 CP).

III b) Sampling 2 CP

- Generate 5000 random samples from a uniform distribution with limits $[0, 1]$ (0.5 CP).
- For the remainder of this exercise, do not use a random number generator anymore. Instead work with the samples you already have.
  - Transform your samples to now sample from a Uniform distribution with limits $[-1, +1]$ (0.5 CP).
  - Use your samples to generate samples from a triangular distribution (1 CP).

. . . . HINTS . . . .

Integrals related to Gaussians are

$$\int_{-\infty}^{+\infty} e^{-a(x+b)^2}\mathrm{d}x = \sqrt{\frac{\pi}{a}} \tag{0.1}$$

$$\int e^{-ax^2}\mathrm{d}x = \sqrt{\frac{\pi}{4a}}\mathrm{erf}(\sqrt{a}x) \tag{0.2}$$