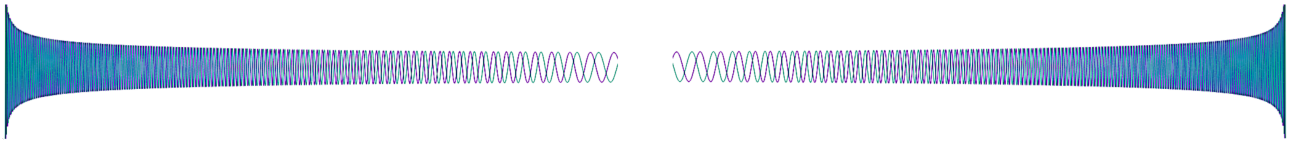
Tutorial on the essentials of gravitational wave detection (2019)

This tutorial focuses on an idealized detection of gravitational waves emitted during the inspiral and merger of compact binary objects. The used data are synthetic, and the templates use the approximations described by VIRGO and LIGO in Ann. Phys. 529 (2017).



Exercise 1: Gravitational wave signals ....

We focus on the inspiral and merger of two compact objects, e.g. two black holes or two neutron stars. From the masses m_1 and m_2, we define the total mass $M = m_1 + m_2$ and the reduced mass,

$$\eta = \frac{m_1 m_2}{M^2}. \tag{1}$$

Let then D be the (luminosity-)distance of the event, and let A be a geometric prefactor which depends on the relative orientation between the merging system and the interferometric detector. (Averaging over all angles, one has $A = 2/5$.)
The natural data domain of gravitational wave interferometers is the time domain: The interferometers detect a strain h, as a function of time t[1]. A gravitational wave signal from a binary merger can be represented in the time domain as

$$h(t) = \frac{4A\eta M}{D} \left[\pi M f(t)\right]^{\frac{2}{3}} \cos[\phi(t) + \phi_c]. \tag{2}$$

The function $f(t)$ describes how the wave frequency increases with time. Gravitational waves are quadrupolar, and the quadrupole moment of a binary system has spin-2. Consequently, the wave frequency is twice the orbital frequency of the inspiralling objects. The function

$$\phi(t) = 2\pi \int^t f(t')\mathrm{d}t', \tag{3}$$

tracks the progress of the objects on their orbits; hence the integral.

1) Scaling relations: Explain why the amplitude of the strain increases proportional to η, M and A, and why it decreases inversely with D.

2) Chirping: Employ Keplerian dynamics to explain how the 'chirping' of gravitational waves arises. Chirping describes the increase of frequency f as a function of time.

3) Computing the chirpline: We introduce the '*chirp mass*' as

$$\mathcal{M} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}. \tag{4}$$

[1] The t here used, is calendar time on Earth.

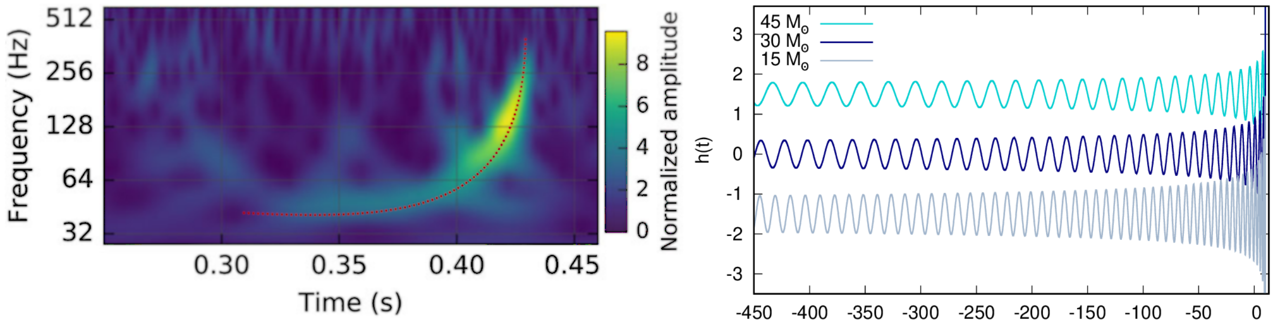


Fig. 1: ***Left:*** *Chirpline (red overlay), as computed from $f(t)$ given in Eq. (5). The background is the original plot from Abbott et al., Phys Rev. Lett. 116 (2016).*
Right: *Exemplary templates of gravitational wave forms resulting from our approximations. The ringdown is missing, but the acceleration of the phase during the inspiral is included. The legend indicates the chirp mass $\mathcal{M}$, and merger takes place at $t_c = 10s$.*

This leads to the 'chirpline' which relates chirp mass and timevolution of the frequency

$$f_{\mathrm{GW}}(t) = \frac{1}{40\pi}\left(\frac{G\mathcal{M}}{c^3}\right)^{-5/8}(t_c - t)^{-3/8}, \tag{5}$$

where t_c is the moment of merger. Expressing the chirp mass in multiples of solar masses, $\mathcal{M} = xM_\odot$, and plugging in G and c, the chirpline facilitates to

$$f_{\mathrm{GW}}(t) = B\,x^{-5/8}(t_c - t)^{-3/8}, \qquad B = 16.6\ \mathrm{s}^{-5/8}. \tag{6}$$

The units of B are seconds to the $-5/8$, such that the units of f are correctly in Hertz. Typical values for x are 20-40 (solar masses). For the phase of the cosine, we require the integral of Eq. (6) which is

$$\phi(t) = 2\pi B x^{-5/8}\left(-\frac{3}{8}\right)(t_c - t)^{5/8}. \tag{7}$$

- Plot the chirplines Eq. (6) for different masses x and merger times t_c. You will recognize the typical upsweep in frequency prior to merger, as also visible in the original LIGO data, see Fig. 1.
- Plug the chirpline (Eq. 6), and the phase (Eq. 7) into the strain equation (Eq. 2). Ignore the prefactor. Plot strains $h(t)$ for different values of x. You should get waves similar to the right panel of Fig. 1.

OUTCOME OF THE EXERCISE: YOU HAVE UNDERSTOOD THE ESSENTIALS OF GRAVITATIONAL WAVE SCALING RELATIONS. YOU ARE ABLE TO COMPUTE CHIRPLINES $f(t)$ AND DETECTOR STRAINS $h(t)$ FOR DIFFERENT CHIRP MASSES.

Exercise 2: Creating a template bank for gravitational waves ....

Gravitational wave detection employs matched filtering, for which a template bank needs to be created first. This bank hosts the variety of potential signal shapes and in this exercise we will learn the essentials of how to create one.
In *Exercise 1* we saw that the signal shape is given by

$$h(t) = \frac{4A\eta M[\pi M]^{\frac{2}{3}}}{D}\left(16.6\,x^{-\frac{5}{8}}\,(t_c - t)^{-\frac{3}{8}}\right)^{\frac{2}{3}}\cos\left[-39.11\,x^{-\frac{5}{8}}(t_c - t)^{\frac{5}{8}} + \phi_c\right]. \tag{8}$$

1) Parameter study: Keeping all other parameters fixed, which effect does ϕ_c have? Keeping all other parameters fixed, which effect does t_c have, and which does x have?

2) Prior ranges: Which natural prior range does ϕ_c have? Given the LIGO and VIRGO sensitivities, which sensible prior range can you introduce for x?

3) Scale and shift symmetries: Parameters which simply scale an amplitude, and parameters which introduce mere shifts of the signal, are not regarded as real signal parameters in matched filtering. This is because in matched filtering, the amplitude does not determine *which* signal is detected, but only *how significantly* it is detected, in relation to the background noise. Shift parameters simply determine *where* in the data stream a signal is found, but not *which* signal is found. For the gravitational waves, which parameters are shift and scale parameters?

4) Dimensionality reduction: Efficient template banks sweep through as few parameters as possible. Currently, Eq. (8) employs six parameters: $m_1, m_2, A, D, t_c, \phi_c$. These are too many. Investigate Eq. (8), and find out which parameters are degenerate. Show that by replacing degenerate parameters with *effective parameters*, you can reduce the dimensionality of your template bank from 6 to 2[2]. If you succeed to reduce the dimensionality to 4 or 3 only, reconsider step (3).

5) Template spacing: Template spacing describes at which grid points between the prior boundaries templates are (pre-)computed. If the spacing is too large, the templates vary too rapidly and the detection significance will decrease due to *template mismatch*. If the spacing is too small, the template bank will be slow. Find sensible template spacings for your two parameters, by plotting how rapidly the templates change as you vary the parameters.

OUTCOME OF THE EXERCISE: YOU HAVE CREATED A TEMPLATE BANK FOR GRAVITATIONAL WAVES. IN THE NEXT EXERCISE, YOU CAN APPLY IT TO SYNTHETIC DATA.

Exercise 3: Filtering data streams for gravitational waves ....

The files *'GW1.dat, GW2.dat, GW3.dat'* may or may not contain synthetic gravitational wave signals. These files sample the time τ regularly (first column, in seconds). The second column stores the strain $h(\tau)$. The time τ is human detector time. The collapse time t_c from Exercise 1 and 2 may take place at any τ, so you will need to shift your signals as a function of t_c.

1.
2. Plot the filter output as a function of time. A large spike in the filter output indicates a correlation with the signal.
3. Signal-to-noise determination: The height of the spike can be measured in units of Signal-to-noise.

If you wish to restore the original amplitude of the signal, then the noise here added was uncorrelated Gaussian noise of variance $\sigma_2 =$.

Exercise R: Replacement exercise ....

The files *'Sine1.dat, Sine2.dat, Sine3.dat'* may or may not contain signals of the type

$$s(t) = \sin(\omega t), \tag{9}$$

hidden under additive Gaussian noise. Here, ω is a free parameter to be determined. Apply matched filtering to find the potential signals and their parameters. The first column samples t, the second is the data stream.

[2] After detection, the in-depth analysis then uses the original physical parameters.