Report on FEM Programming Homework 3

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1 Problem

$$\begin{cases} -(d(x)u^{'}(x))^{'} + c(x)u(x) = f(x), & x \in (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{1}$$

where $\beta < d(x), c(x) < \alpha < \infty$ for $x \in (0,1)$.

Test your code with $d(x) = \sin x + 2, c(x) = x^2 + 1, u(x) = x(x-1)$ for P^1 andP^2 finite element space and list the accuracy table.

2 Algorithm

2.1 Variations of PDE and FEM

The solution space is

$$V = H_0^1([0,1])$$

and the weak form is

$$\begin{cases} Find\ u \in V s.t. \\ a(u,v) = F(v), \forall v \in V \end{cases} \tag{2}$$

where $a(u,v) = \int_0^1 d(x)u^{'}(x)v^{'}(x) + c(x)u(x)v(x)dx, F(v) = \int_0^1 f(x)v(x)dx$

The finite element variation problem is

$$\begin{cases} Find\ u_h \in V_h(\subset V)\ s.t. \\ a(u,v) = F(v), \forall v \in V_h \end{cases} \tag{3}$$

where $V_h = P^1\ or\ P^2$

2.2 The basis of V_h

The basis of P^1 space is

$$\phi_j(x) = \begin{cases} \frac{x-x_{j-1}}{x_j-x_{j-1}}, & [x_{j-1}, x_j] \\ \frac{x_{j+1}-x}{x_{j+1}-x_j}, & [x_j, x_{j+1}] \\ 0, & elsewhere \end{cases} \quad j = 1, ..., N-1$$

. The basis of P^2 space is

$$\varphi_j(x) = \begin{cases} \frac{(2x-x_j-x_{j-1})(x-x_{j-1})}{h^2}, & x \in (x_{j-1}, x_j) \\ \frac{(2x-x_j-x_{j+1})(x-x_{j+1})}{h^2}, & x \in [x_j, x_{j+1}) \\ 0, & x \notin (x_{j-1}, x_{j+1}) \end{cases} \quad , \quad j = 1, \cdots, N-1.$$

$$\psi_{i+\frac{1}{2}}(x) = \begin{cases} \frac{4(x-x_i)(x_{i+1}-x)}{h^2}, & x \in (x_i, x_{i+1}) \\ 0, & x \notin (x_i, x_{i+1}) \end{cases} \quad , \quad i = 0, \cdots, N-1.$$

2.3 Stiffness Matrix

2.3.1 P^1 space

The stiffness matrix is $K = (k_{ij})_{(N-1)*(N-1)}$, where $k_{ij} = a(\phi_j^{'}, \phi_i^{'}) = \int_0^1 d(x)\phi_j^{'}(x)\phi_i^{'}(x) + c(x)\phi_j(x)\phi_i(x)dx$.
To calculate k_{ij}, we need to know derivatives of the basis function.

$$\phi_j^{'}(x) = \begin{cases} \frac{1}{x_j - x_{j-1}}, & [x_{j-1}, x_j] \\ \frac{-1}{x_{j+1} - x_j}, & [x_j, x_{j+1}] \\ 0, & elsewhere \end{cases} \quad j = 1, ..., N-1$$

.

2.3.2 P^2 space

The stiffness matrix is $K = (k_{ij})_{(2N-1)*(2N-1)}$, where $k_{ij} = a(\phi_j^{'}, \phi_i^{'}) = \int_0^1 d(x)\phi_j^{'}(x)\phi_i^{'}(x) + c(x)\phi_j(x)\phi_i(x)dx$.

$$\varphi_j^{'}(x) = \begin{cases} \frac{4x-3x_{j-1}-xj}{h^2}, & (x_{j-1}, x_j) \\ \frac{4x-3x_{j+1}-xj}{h^2}, & [x_j, x_{j+1}) \\ 0, & elsewhere \end{cases} \quad j = 1, ..., N-1$$

$$\psi_{i+\frac{1}{2}}^{'}(x) = \begin{cases} \frac{4(x_{i+1}-2x+x_i)}{h^2}, & (x_i, x_{i+1}) \\ 0, & elsewhere \end{cases} \quad i = 0, ..., N-1$$

.

2.4 Numerical Integration Method and Linear Equations Solution Method

We use Gauss integration method to compute all the integrations that we need.

$$\int_a^b f(x)dx = \int_{-1}^1 f(\frac{(b-a)t+a+b}{2})\frac{b-a}{2}dt = \frac{b-a}{2}\sum_1^n A_k f(\frac{(b-a)x_k+a+b}{2})$$

In this program, we take $n=3$, thus $x_1=-\sqrt{\frac{3}{5}}, x_2=0, x_3=\sqrt{\frac{3}{5}}$ and $A_1=A_3=\frac{5}{9}, A_2=\frac{8}{9}$ To solve linear equations, we just use $A\backslash b$.

2.5 Error Calculation of Different Norms

For each unit, we take its Gauss node as sampling points.
For $||u-u_h||_{L^1}$

$$\int_0^1 |u-u_h|dx = \sum_{j=1}^{N}\int_{-1}^{1}|u-u_h|(\frac{ht+jh+(j-1)h}{2})\frac{h}{2}dt = \frac{h}{2}\sum_{j=1}^{N}\sum_{1}^{n}A_k|u-u_h|(\frac{hx_k+jh+(j-1)h}{2})$$

where n, x_k, A_k are the same as above. For $||u-u_h||_{L^{\inf}}$, we take the maximum of the funciton value of all the sampling points.

2.6 The Structure of Our Code

pg3_1 and pg3_2 are the main program of our code. Gauss3 is a function to compute Gauss integration on a specific interval. ComputeCoef and ComputeCoef2 are the functions to solve the coefficient of u_h and Compute_Uh_1 and Compute_Uh_2 are functions for computing the function value of finite element solution u_h.

2.7 Improvement and Suggestions

The example we are assigned to is $u(x)=x(x-1)$, in this case, the accuracy of P^2 space is too small to see the order. Therefore, we change $u(x)=\sin(x)(x-1)$ to check our code. However, we show both results in the following section.

3 Results

Table 1: Accuracy test for function error in P^1 space,$u(x)=\sin(x)(x-1)$

N	$\|\|u-u_h\|\|_{L^1}$ error	order	$\|\|u-u_h\|\|_{L^\infty}$ error	order
N=10	1.511480273678e-03	–	2.616460770479e-03	–
N=20	3.776958189426e-04	2.000665432002	6.541798917812e-04	1.999857308706
N=40	9.441304370945e-05	2.000166718213	1.635615575677e-04	1.999853707888
N=80	2.360257867187e-05	2.000041701901	4.088974998515e-05,	2.000022559731
N=160	5.900602017578e-06	2.000010427967	1.022261041025e-05	1.999975596817
N=320	1.475147859726e-06	2.000002586485	2.555658334291e-06	1.999996764378

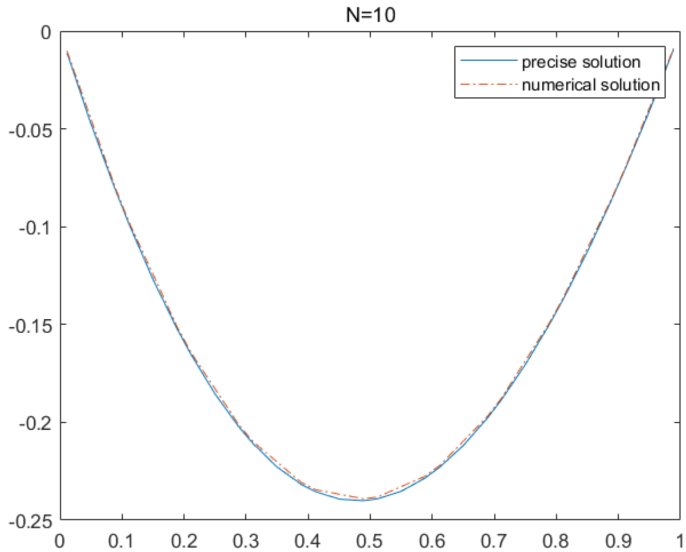

Figure 1: $u(x) = \sin(x)(x-1)$,N=10,P^1space

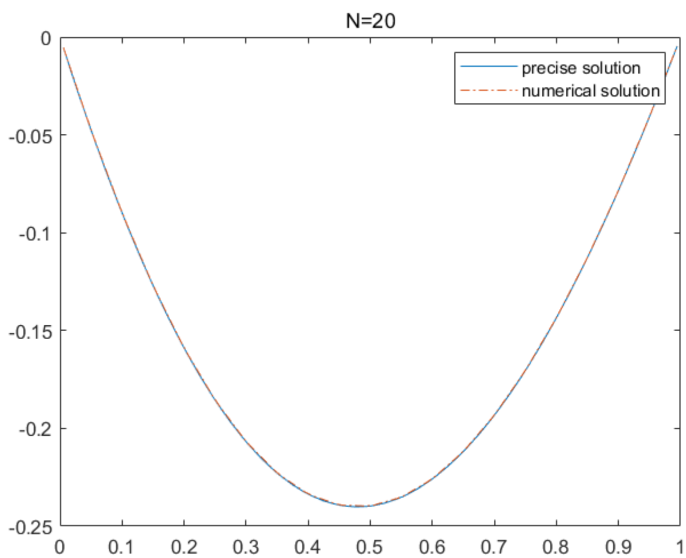

Figure 2: $u(x) = \sin(x)(x-1)$,N=20,P^1space

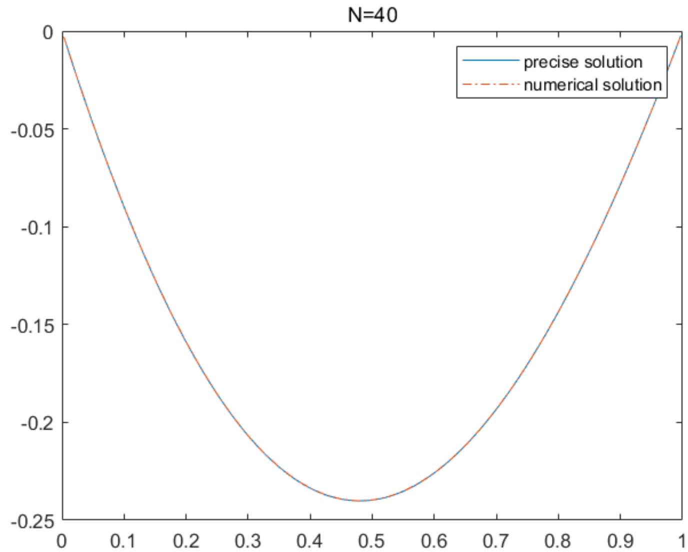

Figure 3: $u(x) = \sin(x)(x-1)$,N=40,P^1space

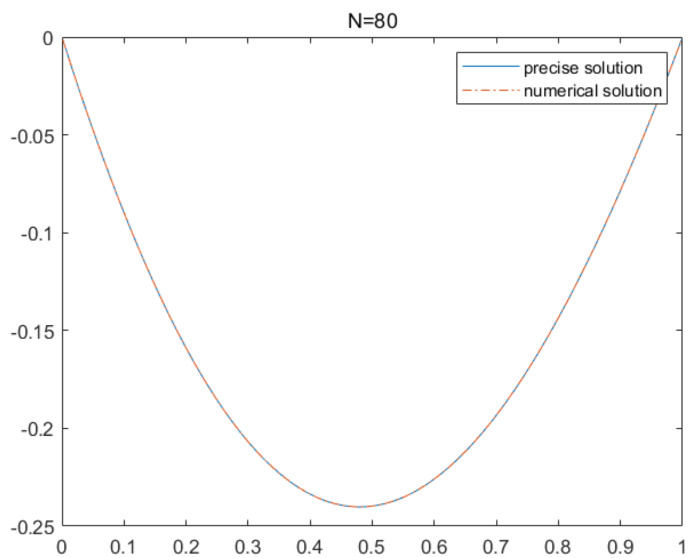

Figure 4: $u(x) = \sin(x)(x-1)$,N=80,P^1space

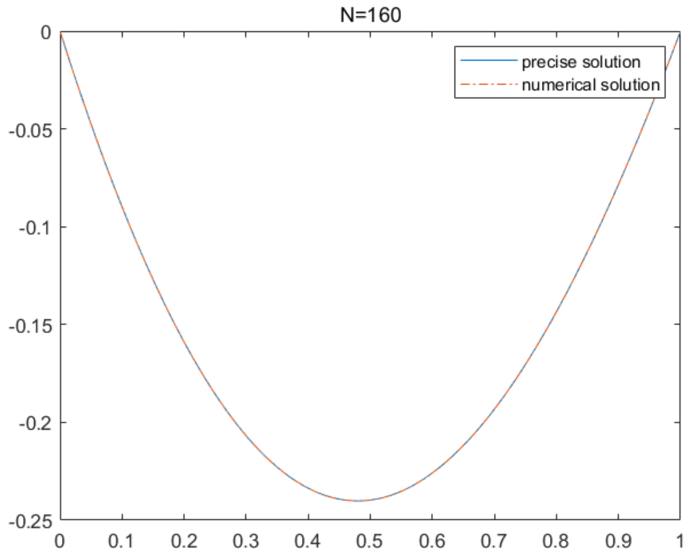

Figure 5: $u(x) = \sin(x)(x-1)$,N=160,P^1space

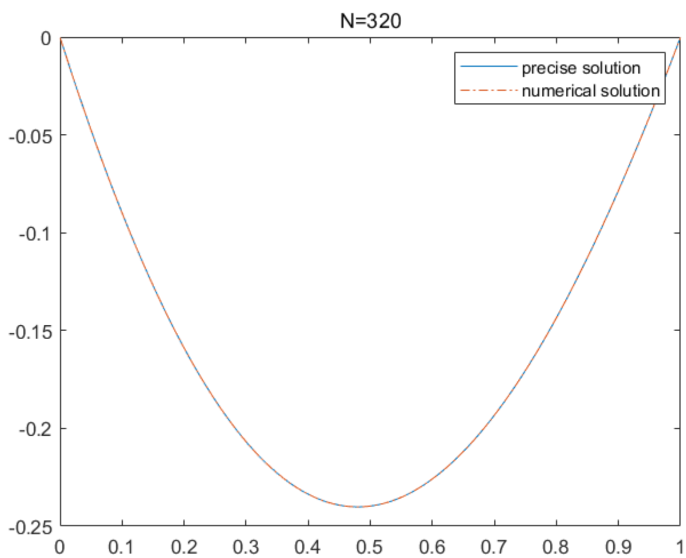

Figure 6: $u(x) = \sin(x)(x-1)$,N=320,P^1space

Table 2: Accuracy test for function error in P^2 space $u(x) = \sin(x)(x-1)$

N	$\|\|u - u_h\|\|_{L^1}$ error	order	$\|\|u - u_h\|\|_{L^\infty}$ error	order
N=10	4.262638092727e-06	–	1.557679228002e-05	–
N=20	5.293113749475e-07	3.009558009484	1.992934894660e-06	2.966431684621
N=40	6.575363560644e-08	3.008974087687	2.518934848522e-07	2.984008868079
N=80	8.193541403846e-09	3.004511613240	3.165739876948e-08	2.992199185843
N=160	1.022676029521e-09	3.002137956382	3.967754605666e-09	2.996148009721
N=320	1.277443462742e-10	3.001017843708	4.966276186407e-10	2.998086401346

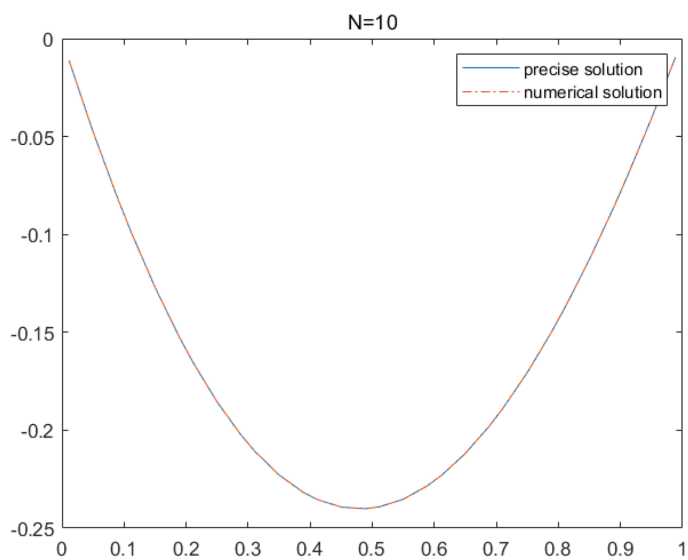

Figure 7: $u(x) = \sin(x)(x-1)$,N=10,P^2space

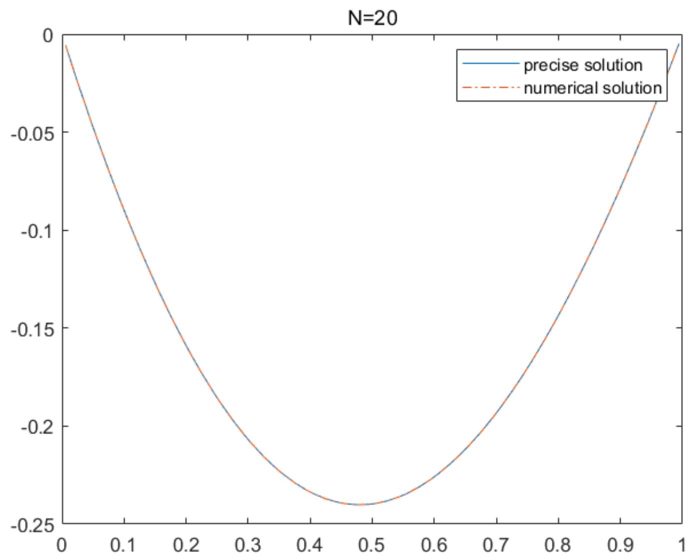

Figure 8: $u(x) = \sin(x)(x-1)$,N=20,P^2space

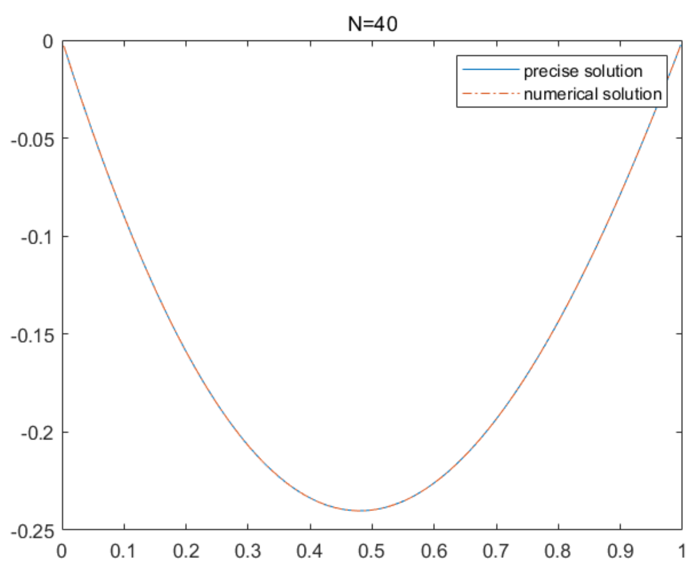

Figure 9: $u(x) = \sin(x)(x-1)$,N=40,P^2space

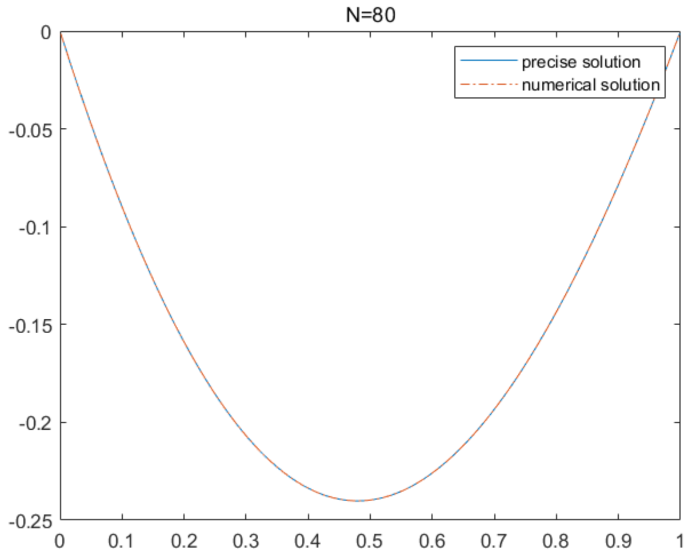

Figure 10: $u(x) = \sin(x)(x-1)$,N=80,P^2space

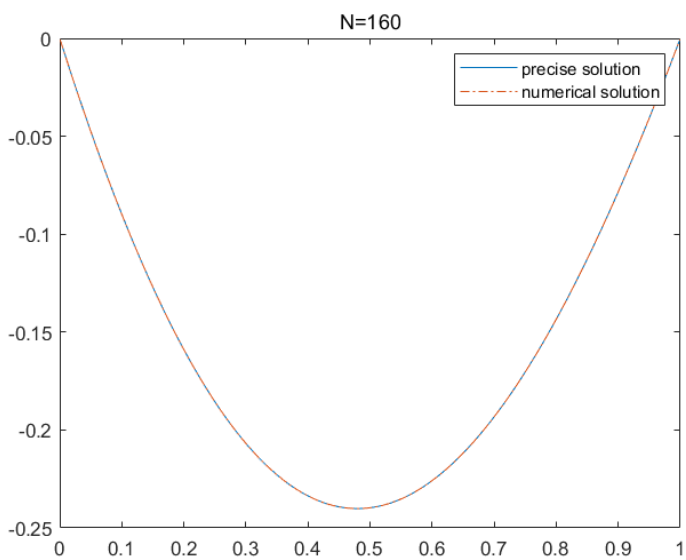

Figure 11: $u(x) = \sin(x)(x-1)$,N=160,P^2space

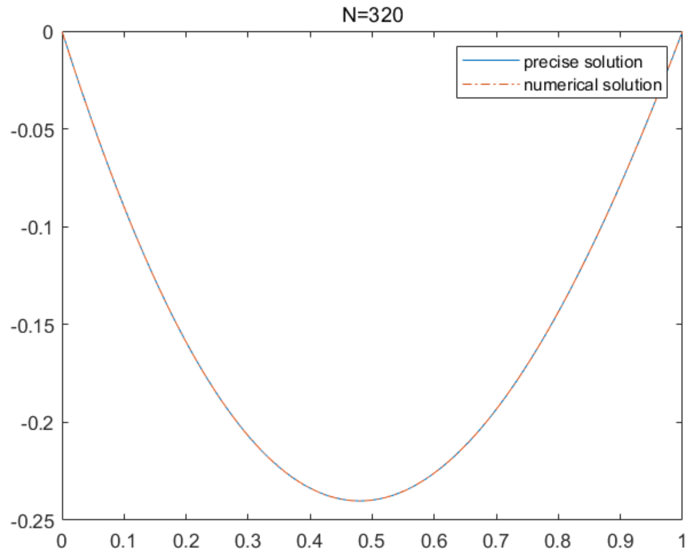

Figure 12: $u(x) = \sin(x)(x-1)$,N=320,P^2space

Table 3: Accuracy test for function error in P^1 space $u(x) = x(x-1)$

N	$\|\|u - u_h\|\|_{L^1}$ error	order	$\|\|u - u_h\|\|_{L^\infty}$ error	order
N=10	1.621872708486e-03	–	2.488194876649e-03	–
N=20	4.053796286239e-04	2.000315098537	6.234329428199e-04	1.996793186942
N=40	1.013393692403e-04	2.000078837131	1.560482006584e-04	1.998242668325
N=80	2.533449613630e-05	2.000019713034	3.903689840101e-05	1.999081385994
N=160	6.333602392655e-06	2.000004929567	9.762401035217e-06	1.999530507643
N=320	1.583399269416e-06	2.000001210672	2.441001762365e-06	1.999762681516

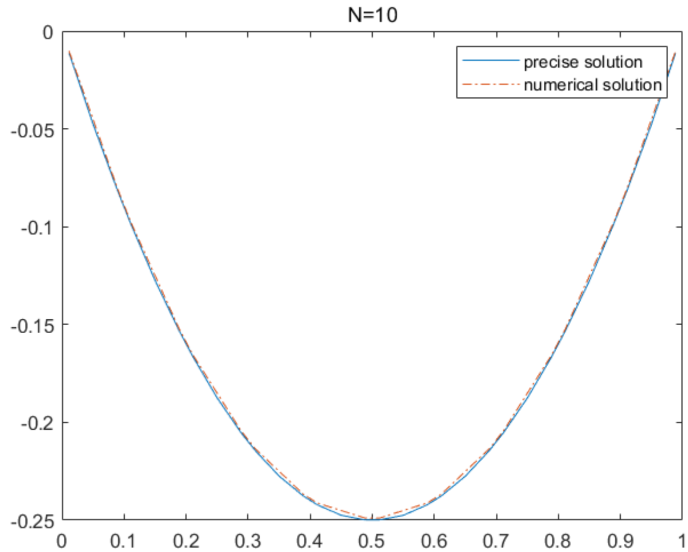

Figure 13: $u(x) = x(x-1)$,N=10,P^1space

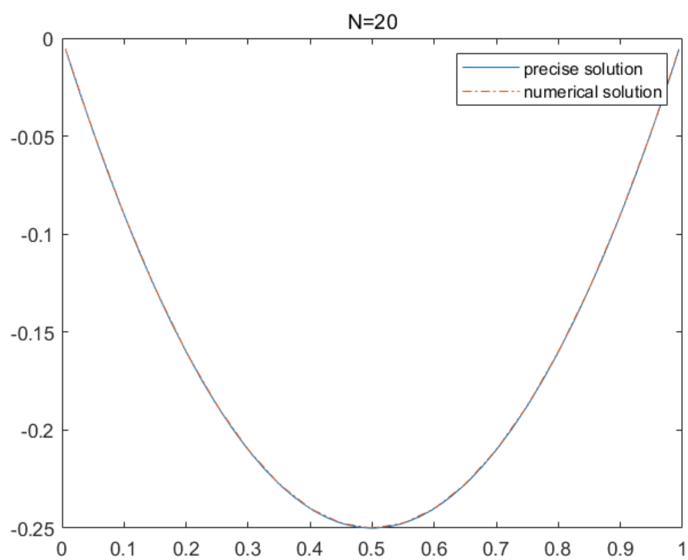

Figure 14: $u(x) = x(x-1)$,N=20,P^1space

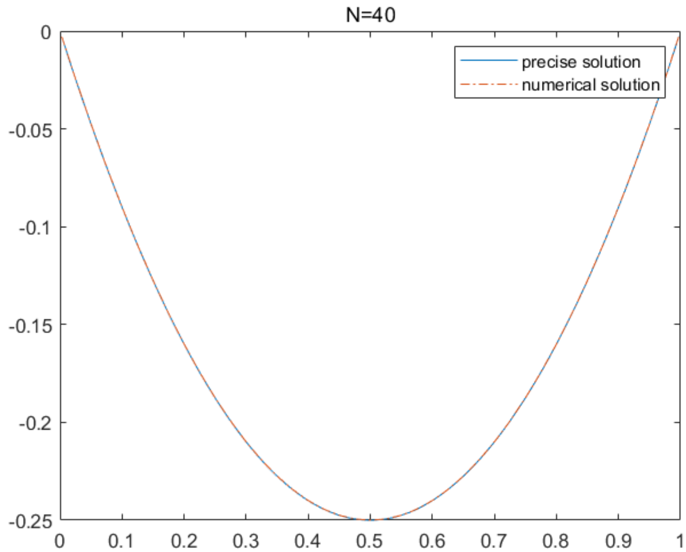

Figure 15: $u(x) = x(x-1)$,N=40,P^1space

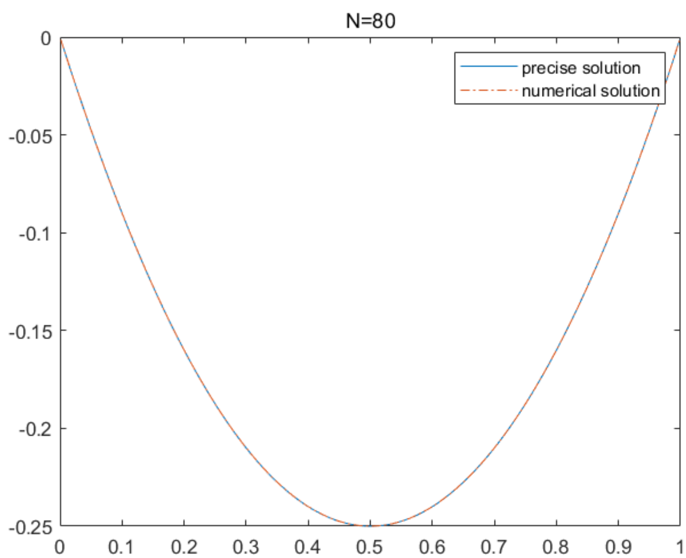

Figure 16: $u(x) = x(x-1)$,N=80,P^1space

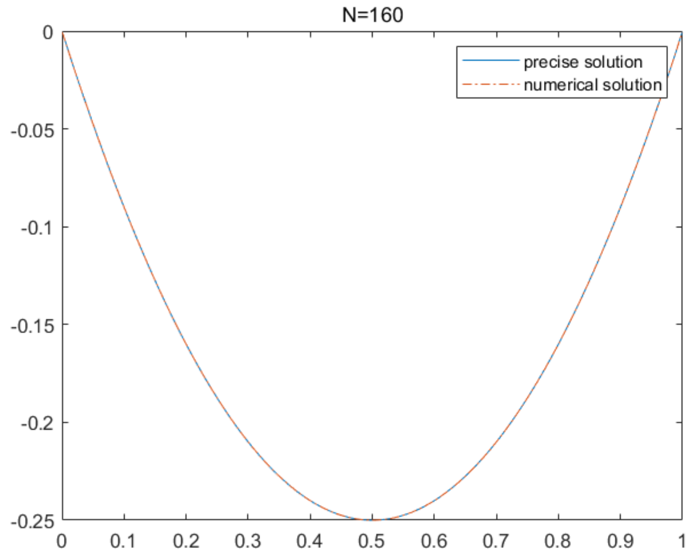

Figure 17: $u(x) = x(x-1)$,N=160,P^1space

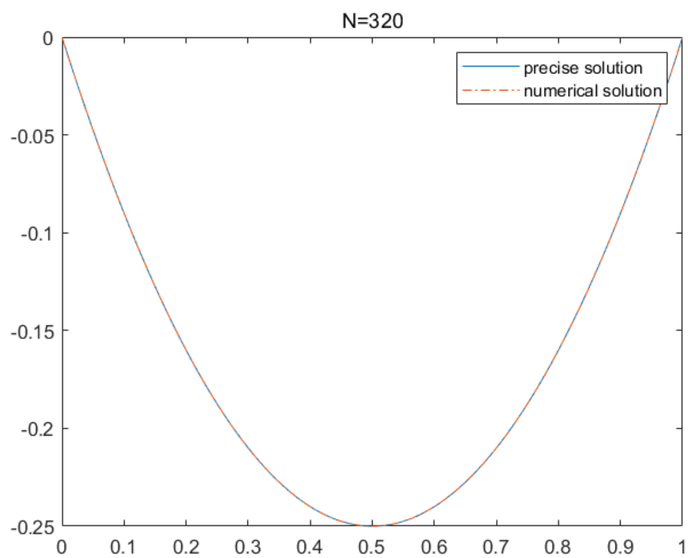

Figure 18: $u(x) = x(x-1)$,N=320,P^1space

Table 4: Accuracy test for function error in P^2 space $u(x) = x(x-1)$

N	$\|\|u-u_h\|\|_{L^1}$ error	order	$\|\|u-u_h\|\|_{L^\infty}$ error	order
N=10	1.636728649988e-11	–	4.748209464500e-11	–
N=20	2.580731057097e-13	5.986891546584	7.562943327155e-13	5.972291961675
N=40	4.063586730803e-15	5.988882306416	1.501576640806e-14	5.654397790774
N=80	2.472164665302e-14	-2.604949249363	6.422640197457e-14	-2.096688365281
N=160	1.411522209036e-13	-2.513405082267	2.227107387398e-13	-1.793932741505
N=320	6.540287148364e-14	1.109825945376	2.343403249228e-13	-0.073434108160

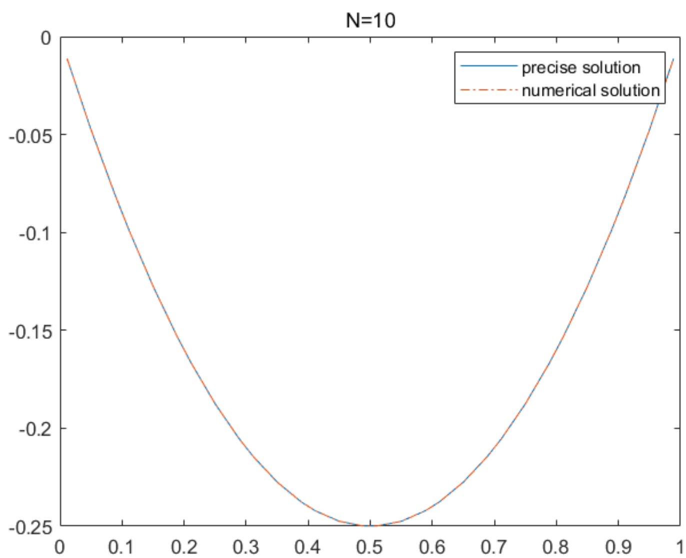

Figure 19: $u(x) = x(x-1)$,N=10,P^2space

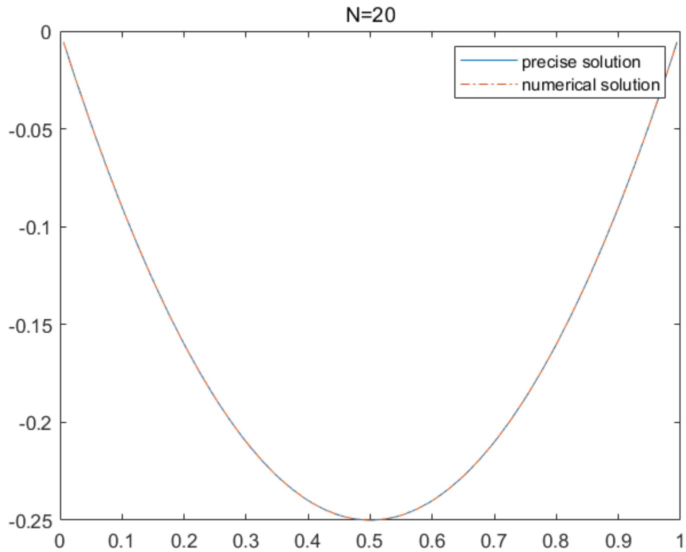

Figure 20: $u(x) = x(x-1)$,N=20,P^2space

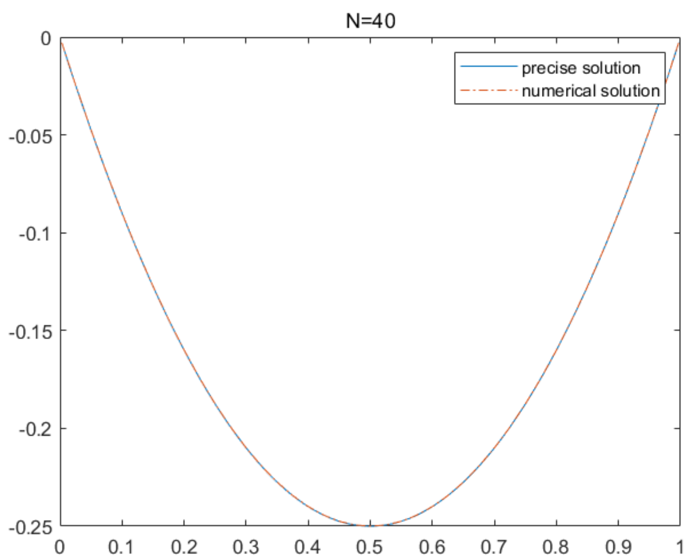

Figure 21: $u(x) = x(x-1)$,N=40,P^1space

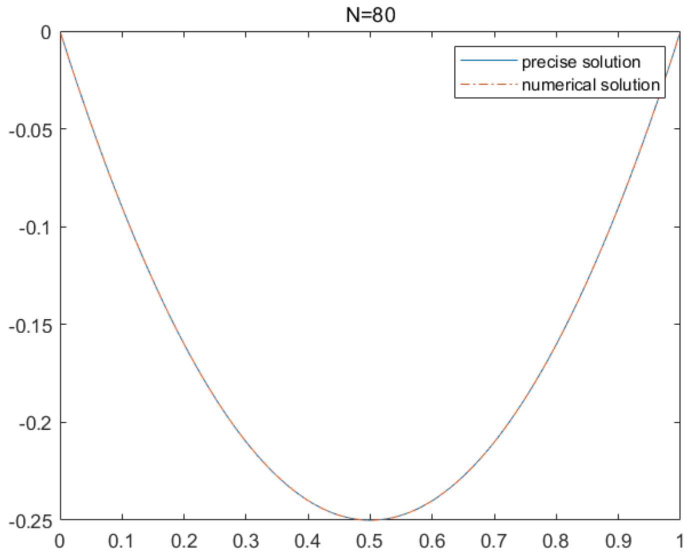

Figure 22: $u(x) = x(x-1)$,N=80,P^2space

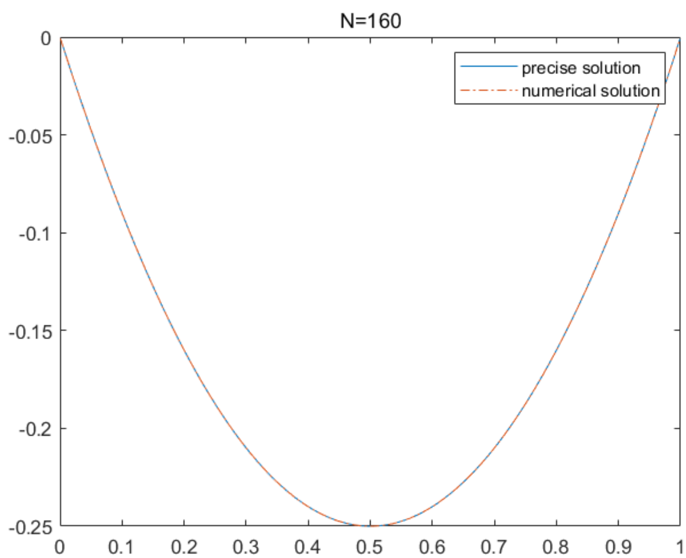

Figure 23: $u(x) = x(x-1)$,N=160,P^2space

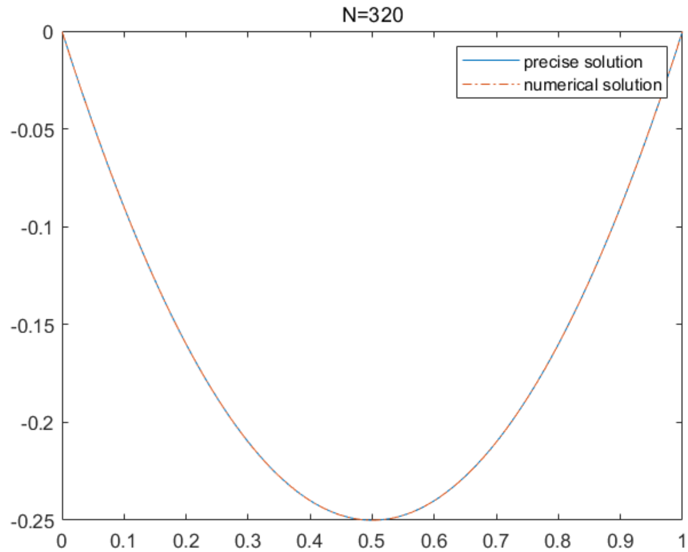

Figure 24: $u(x) = x(x-1)$,N=320,P^2space

4 Conclusions

From the test of function $u(x) = sin(x)(x-1)$,we can tell the order of P^1 space is 2 and the order of P^2 space is 3, no matter $L_1 or L_\infty$ norm, which are the same as theoretical values.However, the finite element solution of function $u(x) = x(x-1)$ only shows that the order of P^1 space is 2 and the order of P^2 space is not computable due to the very small accuracy.