Report on FEM Programming Homework 4

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1 Problem

$$\begin{cases} -\epsilon u''(x) + u'(x) = x, \quad x \in (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{1}$$

$$V_h = \{v \in H_0^1([0,1]), v|_{I_j} \in P^1(I_j)\}$$

Test your code with $\epsilon = 10^{-1}, \epsilon = 10^{-7}$ and the following two grids.

1. Grid1:uniform mesh(N cells in [0,1])
2. Grid2:Shinshkin mesh with $\tau = 1 - 2\epsilon \ln(N)$(N cells in $[0,\tau]$, and N cells in $[\tau,1]$)

2 Algorithm

2.1 Variations of PDE and FEM

The exact solution of the problem is

$$u(x) = -\frac{-e^{\frac{1}{\epsilon}}x^2 + e^{\frac{x}{\epsilon}} + 2\epsilon(-e^{\frac{1}{\epsilon}}x + e^{\frac{x}{\epsilon}} + x - 1) + x^2 - 1}{2(e^{\frac{1}{\epsilon}} - 1)}$$

We note that when ϵ is very small, $u(x)$ is not computable, so we change the solution to

$$u(x) = \frac{x^2}{2} + \epsilon x + \frac{(1+2\epsilon)(e^{x/\epsilon+b} - e^b)}{2(e^{1/\epsilon+b} - e^b)}$$

where $b = -\frac{1}{\epsilon}$.
The solution space is

$$V = H_0^1([0,1])$$

and the weak form is

$$\begin{cases} Find\ u \in V s.t. \\ a(u,v) = F(v), \forall v \in V \end{cases} \tag{2}$$

where $a(u,v) = \int_0^1 \epsilon u'(x)v'(x) + u'(x)v(x)dx, F(v) = \int_0^1 xv(x)dx$
The finite element variation problem is

$$\begin{cases} Find\ u_h \in V_h(\subset V)\ s.t. \\ a(u,v) = F(v), \forall v \in V_h \end{cases} \tag{3}$$

2.2 The basis of V_h

The basis of P^1 space is

$$\phi_j(x)=\begin{cases}\frac{x-x_{j-1}}{x_j-x_{j-1}}, & [x_{j-1},x_j]\\ \frac{x_{j+1}-x}{x_{j+1}-x_j}, & [x_j,x_{j+1}]\\ 0, & elsewhere\end{cases} \quad j=1,...,N-1$$

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2.3 Stiffness Matrix

The stiffness matrix is $K=(k_{ij})_{(N-1)*(N-1)}$, where $k_{ij}=a(\phi_j^{'},\phi_i^{'})=\int_0^1 \epsilon\phi_j^{'}(x)\phi_i^{'}(x)+\phi_j^{'}(x)\phi_i(x)dx$. To calculate k_{ij}, we need to know derivatives of the basis function.

$$\phi_j^{'}(x)=\begin{cases}\frac{1}{x_j-x_{j-1}}, & [x_{j-1},x_j]\\ \frac{-1}{x_{j+1}-x_j}, & [x_j,x_{j+1}]\\ 0, & elsewhere\end{cases} \quad j=1,...,N-1$$

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2.4 Numerical Integration Method and Linear Equations Solution Method

We use Gauss integration method to compute all the integrations that we need.

$$\int_a^b f(x)dx=\int_{-1}^1 f(\frac{(b-a)t+a+b}{2})\frac{b-a}{2}dt=\frac{b-a}{2}\sum_1^n A_k f(\frac{(b-a)x_k+a+b}{2})$$

In this program, we take $n=3$, thus $x_1=-\sqrt{\frac{3}{5}},x_2=0,x_3=\sqrt{\frac{3}{5}}$ and $A_1=A_3=\frac{5}{9},A_2=\frac{8}{9}$ To solve linear equations, we just use $A\backslash b$.

2.5 Error Calculation of Different Norms

For each unit of uniform mesh, we take its Gauss node as sampling points.
For $||u-u_h||_{L^1}$

$$\int_0^1 |u-u_h|dx=\sum_{j=1}^N\int_{-1}^1 |u-u_h|(\frac{ht+jh+(j-1)h}{2})\frac{h}{2}dt=\frac{h}{2}\sum_{j=1}^N\sum_1^n A_k|u-u_h|(\frac{hx_k+jh+(j-1)h}{2})$$

where n,x_k,A_k are the same as above. For $||u-u_h||_{L_\infty}$, we take the maximum of the funciton value of all the sampling points. As for Shinshkin mesh, we just separate the integration into two parts with different h and sum them up after intergration calculation when computing L^1 norm.

2.6 The Structure of Our Code

pg4_1 is a program of uniform mesh and pg4_2 is a program of Shinshkin mesh. Gauss3 is a function for computation of Gauss integration. ComputeUh_1 is a function for computation of the function value of FEM solution on uniform mesh and ComputeUh_Shinshkin the same functioin on Shinshkin mesh.

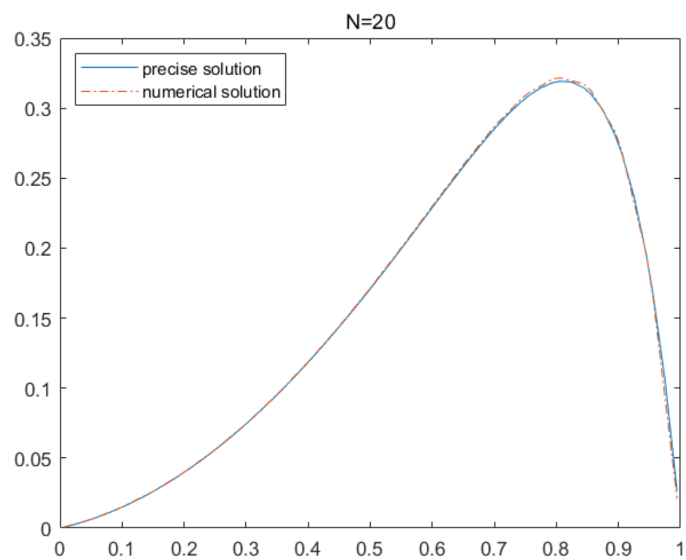
2.7 Improvement and Suggestions

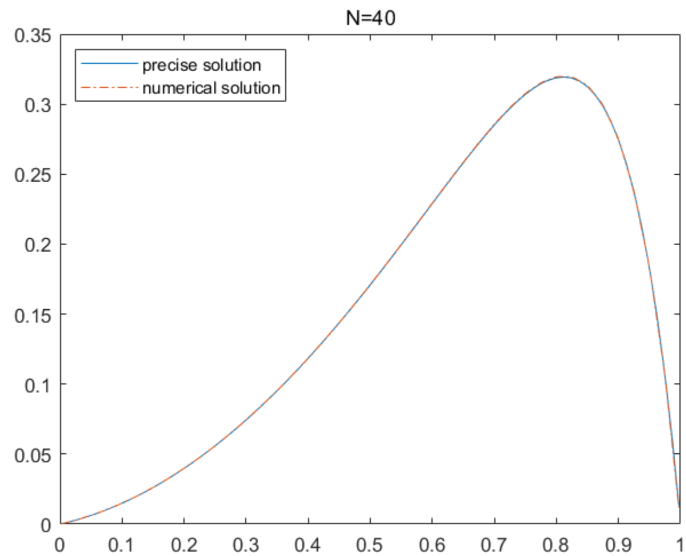
3 Results

3.1 Uniform mesh $\epsilon = 10^{-1}$

Table 1: Accuracy test for function error in P^1 space,uniform mesh,$\epsilon = 10^{-1}$

N	$\|\|u - u_h\|\|_{L^1}$ error	order	$\|\|u - u_h\|\|_{L^\infty}$ error	order
N=20	1.181971619503e-03	–	1.240866833753e-02	–
N=40	2.919203095731e-04	2.017548904080	3.757395943819e-03	1.723543241658
N=80	7.286918309486e-05	2.002193863509	1.038582231160e-03	1.855117703091
N=160	1.820367452490e-05	2.001079120566	2.733736370442e-04	1.925669420470
N=320	4.550484002071e-06	2.000137789163	7.015144412327e-05	1.962329414175
N=640	1.137564817871e-06	2.000071250818	1.776994094014e-05	1.981033916717

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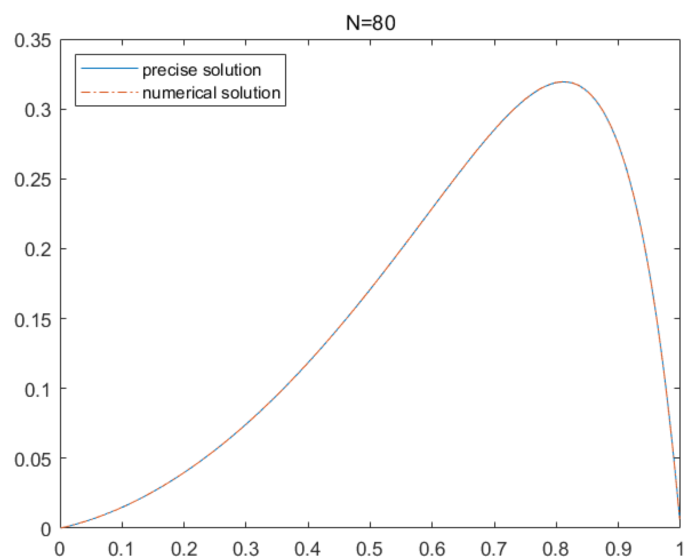
precise solution
numerical solution
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1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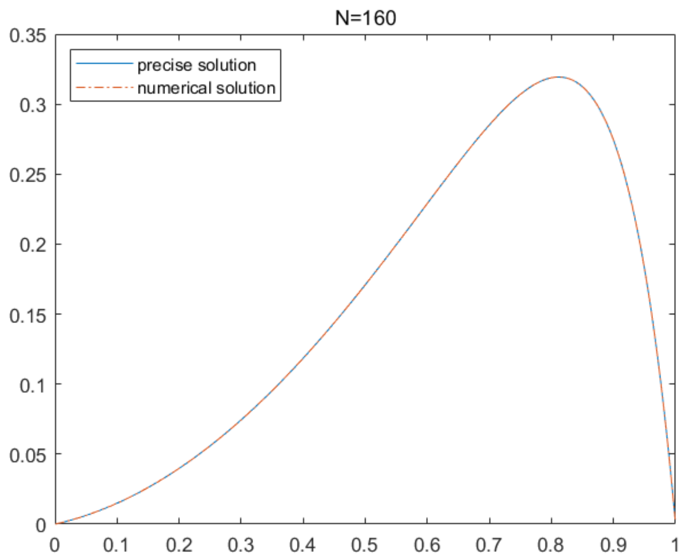
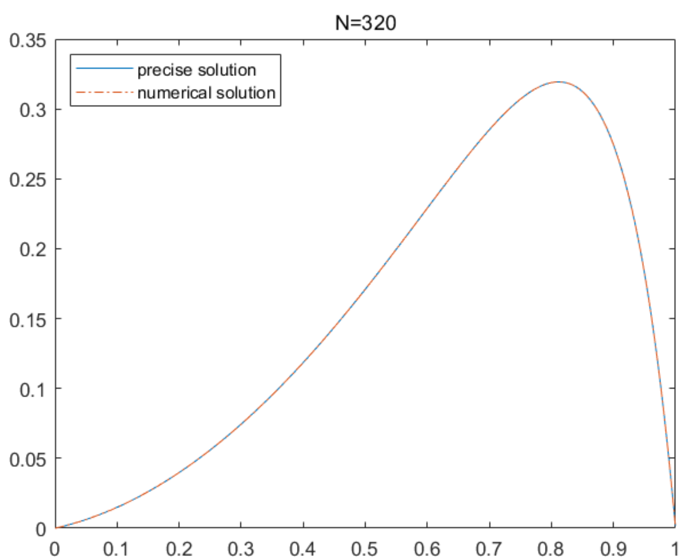
precise solution
numerical solution
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1


N=160
precise solution
numerical solution
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


N=320
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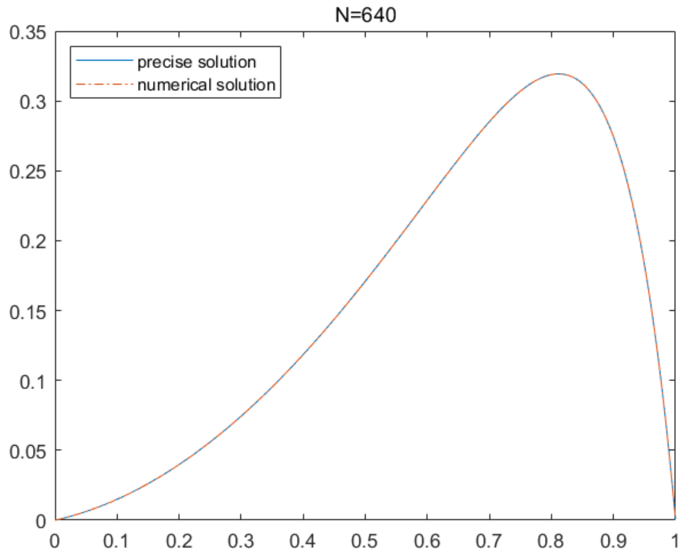
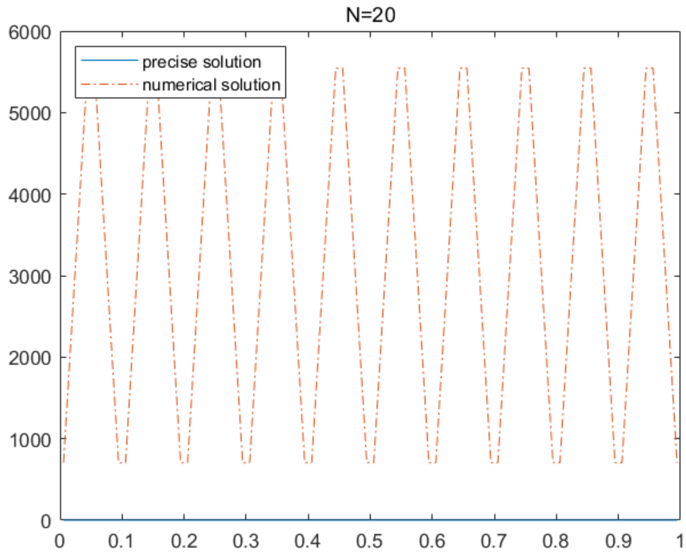
numerical solution
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



3.2 Uniform mesh $\epsilon = 10^{-7}$

Table 2: Accuracy test for function error in P^1 space,uniform mesh,$\epsilon = 10^{-7}$

N	$\|\|u - u_h\|\|_{L^1}$ error	order	$\|\|u - u_h\|\|_{L^\infty}$ error	order
N=20	3.124750839902e+03	–	5.545542937834e+03	–
N=40	7.810002348926e+02	2.000346269954	1.386339376548e+03	2.000048241586
N=80	1.950626586392e+02	2.001385356820	3.465406854229e+02	2.000183826898
N=160	4.857856457119e+01	2.005545859360	8.659257457502e+01	2.000709518734
N=320	1.195874100790e+01	2.022254347468	2.160818401402e+01	2.002665486377
N=640	2.808582125459e+00	2.090151619168	5.370838171199e+00	2.008358673521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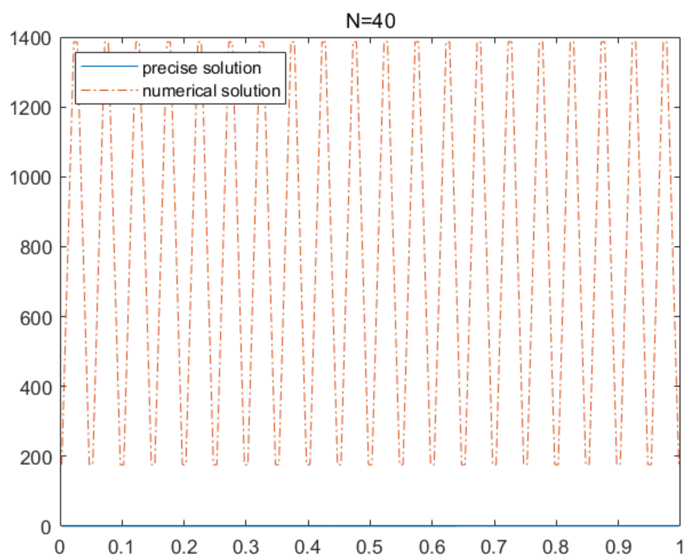
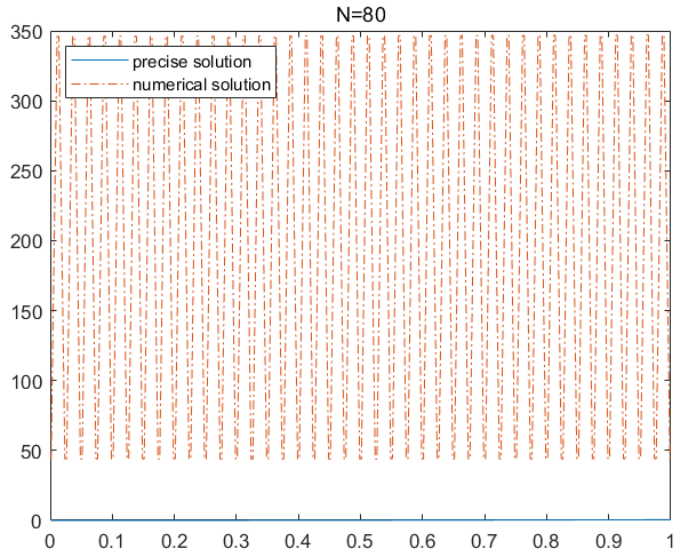
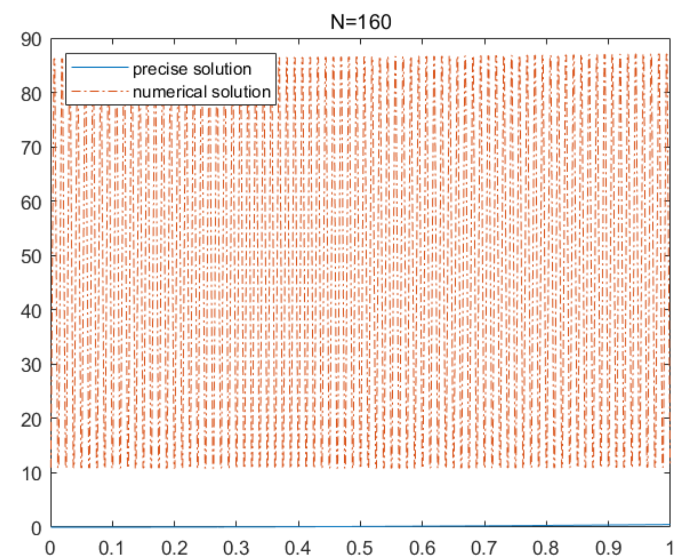
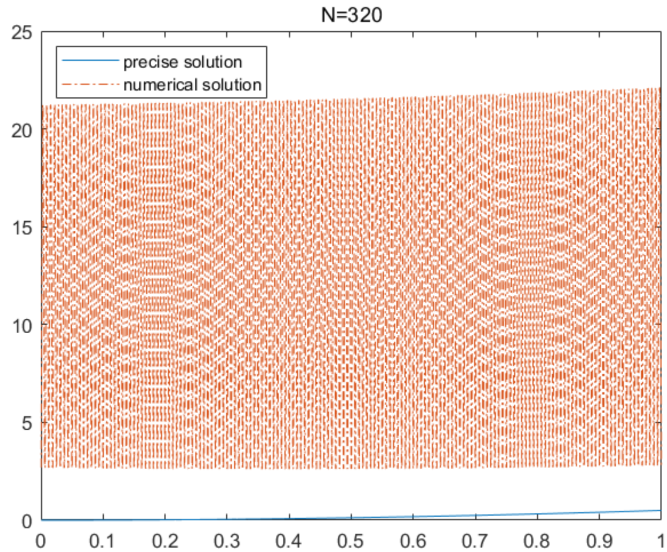
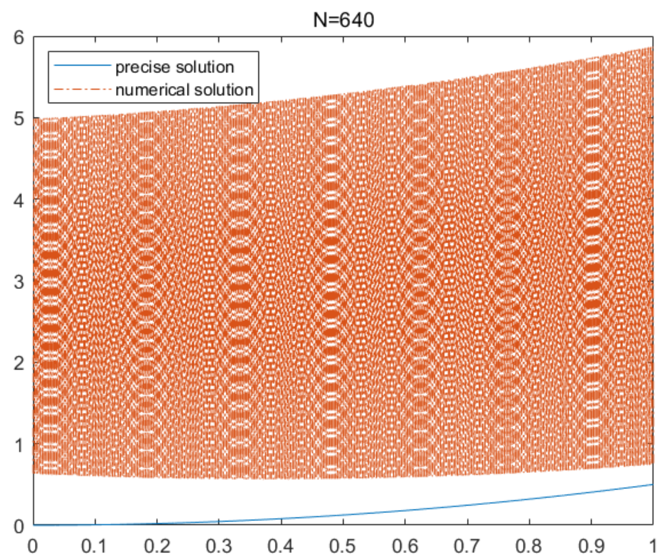
precise solution
numerical solution
6000
5000
4000
3000
2000
1000
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


N=40
precise solution
numerical solution
1400
1200
1000
800
600
400
200
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


N=80
precise solution
numerical solution
350
300
250
200
150
100
50
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


N=160
precise solution
numerical solution
90
80
70
60
50
40
30
20
10
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

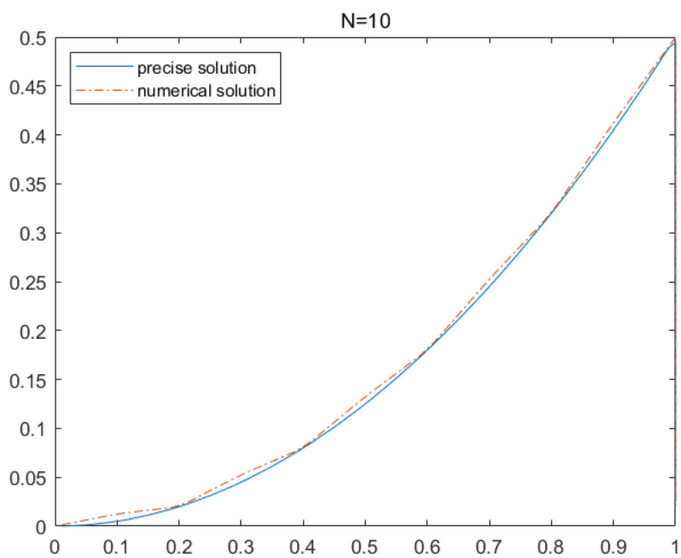

N=320
precise solution
numerical solution
25
20
15
10
5
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

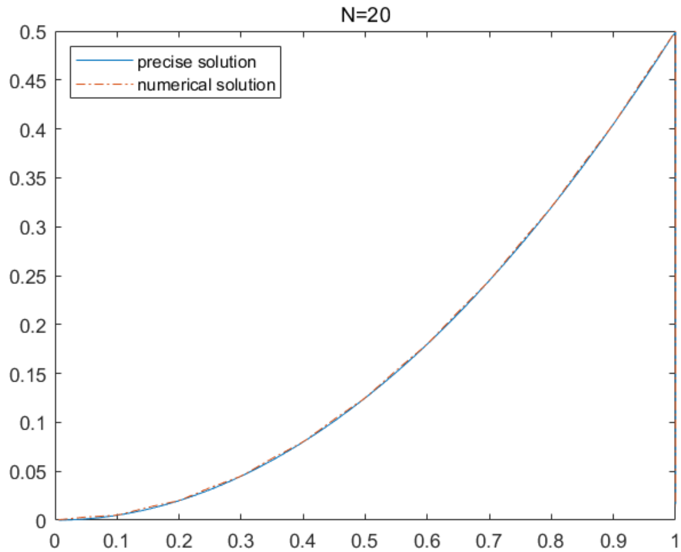

N=640
precise solution
numerical solution
6
5
4
3
2
1
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

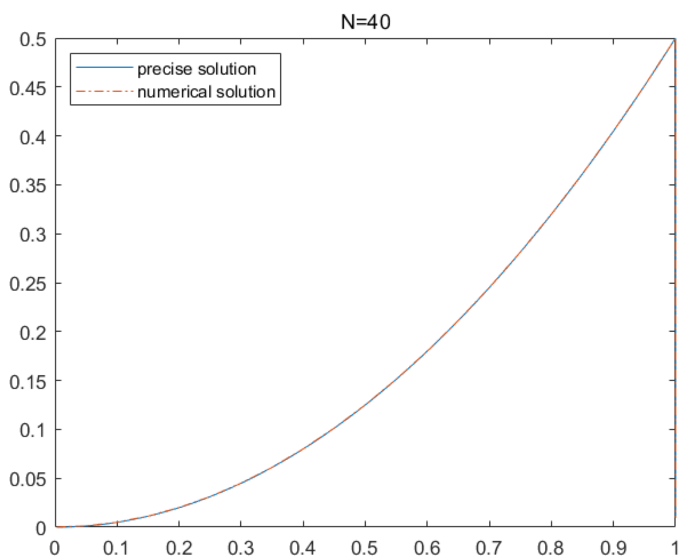
3.3 Shinshkin mesh $\epsilon = 10^{-7}$

Table 3: Accuracy test for function error in P^1 space, Shinshkin mesh,$\epsilon = 10^{-7}$

N	$\|\|u-u_h\|\|_{L^1}$ error	order	$\|\|u-u_h\|\|_{L^\infty}$ error	order
N=10	4.639296088719e-03	–	8.386228834696e-03	–
N=20	1.197195767135e-03	1.954246841580	4.261507328685e-03	0.976658384173
N=40	3.060237043336e-04	1.967943770814	1.805210401082e-03	1.239196817248
N=80	7.728927487654e-05	1.985303271034	6.816756726444e-04	1.405009593285
N=160	1.930045173885e-05	2.001633615974	2.381147755147e-04	1.517428352066
N=320	4.713707366070e-06	2.033700513209	7.875919711751e-05	1.596136836212
N=640	1.066522338885e-06	2.143948012705	2.504767516724e-05	1.652771700270
N=1280	1.856037670493e-07	2.522616286932	7.737266597965e-06	1.694780814183
N=2560	2.156629925750e-08	3.105375453290	2.337274631050e-06	1.726996723171
N=5120	3.541126177174e-09	2.606498477801	6.931914377659e-07	1.753501521848
N=10240	8.136989525150e-10	2.121641212047	2.025022952354e-07	1.775315574452




N=20
precise solution
numerical solution
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


N=40
precise solution
numerical solution
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
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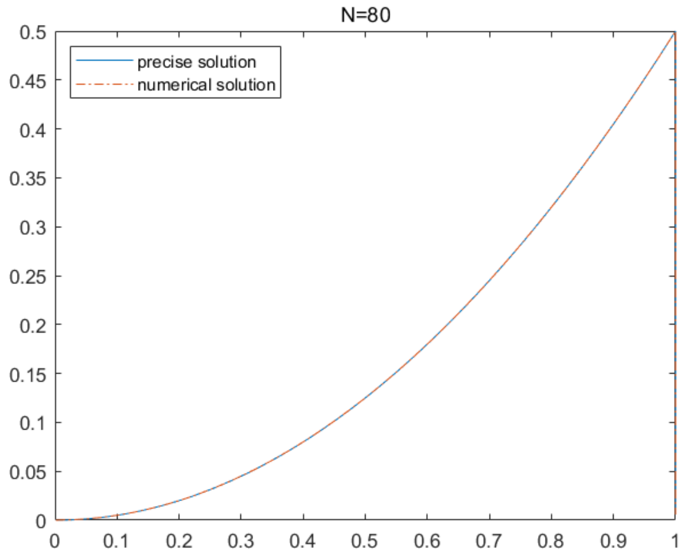
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1


N=80
precise solution
numerical solution
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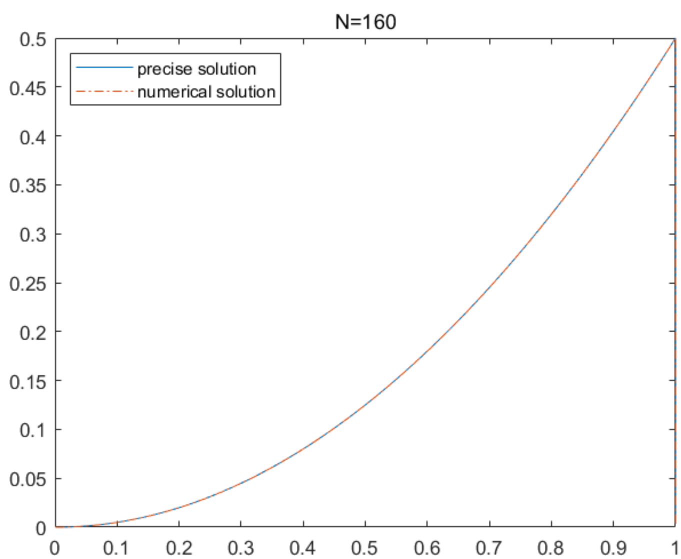
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1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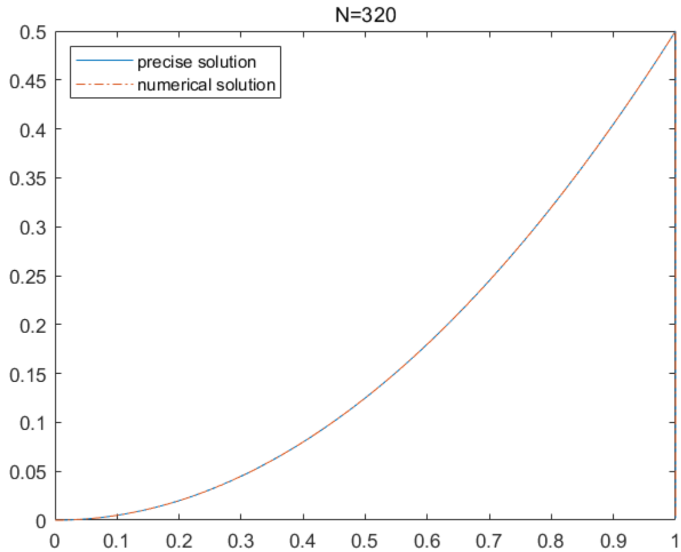
0.5
0.45
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1



4 Conclusions

The orders of uniform mesh are both 2 w.r.t different ϵ, but we can see from the pictures and the errors that when $\epsilon = 10^{-7}$ the FEM solution does not appoximate precise solution.
However, the FEM solution is a good approximation of the orginal function on Shinshkin mesh when $\epsilon = 10^{-7}$, and the order is around 2.
The reason for the better behavior of Shinshkin mesh is that Shinshkin mesh can be seen as a adaptive mesh in this case, which inspires us to adjust the length of units according to the properties of the original function.