CCC 计算机竞赛预习材料—Basic Math Questions

1. Suppose that $n \geq 3$. A sequence $a_1, a_2, a_3, \ldots, a_n$ of n integers, the first m of which are equal to −1 and the remaining $p = n - m$ of which are equal to 1, is called an MP sequence. The sequence −1, −1, 1, 1, 1 is the MP sequence a_1, a_2, a_3, a_4, a_5 with $m = 2$ and $p = 3$. Consider all of the possible products $a_ia_ja_k$ (with $i < j < k$) that can be calculated using the terms from this sequence. Determine how many of these products are equal to 1.

   Solution: 4

   $$a_1a_2a_3 = (-1)\cdot(-1)\cdot 1 = 1$$
   $$a_1a_2a_4 = (-1)\cdot(-1)\cdot 1 = 1$$
   $$a_1a_2a_5 = (-1)\cdot(-1)\cdot 1 = 1$$
   $$a_1a_3a_4 = (-1)\cdot 1\cdot 1 = -1$$
   $$a_1a_3a_5 = (-1)\cdot 1\cdot 1 = -1$$
   $$a_1a_4a_5 = (-1)\cdot 1\cdot 1 = -1$$
   $$a_2a_3a_4 = (-1)\cdot 1\cdot 1 = -1$$
   $$a_2a_3a_5 = (-1)\cdot 1\cdot 1 = -1$$
   $$a_2a_4a_5 = (-1)\cdot 1\cdot 1 = -1$$
   $$a_3a_4a_5 = 1\cdot 1\cdot 1 = 1$$

2. If m and n are positive integers, an (m, n)-sequence is defined to be an infinite sequence $x_1, x_2, x_3, \ldots$ of A's and B's such that if $x_i = A$ for some positive integer i, then $x_{i+m} = B$ and if $x_i = B$ for some positive integer i, then $x_{i+n} = A$. For example, ABABAB . . . is a (1, 1)-sequence.
(a) Determine all (2, 2)-sequences.
(b) Show that there are no (1, 2)-sequences.
Solution:
(a)
A $(2, 2)$-sequence obeys the rules that if $x_i = A$, then $x_{i+2} = B$ and if $x_i = B$, then $x_{i+2} = A$.
Suppose that a $(2, 2)$-sequence has $x_1 = A$.
Then $x_{1+2} = x_3 = B$ and $x_{3+2} = x_5 = A$ and $x_7 = B$ and $x_9 = A$ and so on.
Following this pattern, every odd-numbered term in the sequence is determined by $x_1 = A$ and these terms alternate $A, B, A, B, \ldots$.
Similarly, suppose that a $(2, 2)$-sequence has $x_1 = B$.
Then $x_{1+2} = x_3 = A$ and $x_{3+2} = x_5 = B$ and $x_7 = A$ and $x_9 = B$ and so on.
Following this pattern, every odd-numbered term in the sequence is determined by $x_1 = B$

and these terms alternate $B, A, B, A, \ldots$.
Note that the value of x_1 does not affect any of the even-numbered terms.
Therefore, the value of x_1 determines all of the odd-numbered terms in the sequence.
If a $(2,2)$-sequence has $x_2 = A$, then we will have $x_4 = B$, $x_6 = A$, $x_8 = B$, and so on, and if a $(2,2)$-sequence has $x_2 = B$, then we will have $x_4 = A$, $x_6 = B$, $x_8 = A$, and so on.
Therefore, the value of x_2 determines all of the even-numbered terms in the sequence.
There are 2 possible values for x_1.
There are 2 possible values for x_2.
Thus, there are $2 \times 2 = 4$ possible $(2,2)$-sequences.
These are

$$AABBAABBAA\ldots \qquad ABBAABBAAB\ldots$$

$$BAABBAABBA\ldots \qquad BBAABBAABB\ldots$$

(b)

A $(1,2)$-sequence obeys the rules that if $x_i = A$, then $x_{i+1} = B$ and if $x_i = B$, then $x_{i+2} = A$.
There are only two possibilities: $x_1 = A$ or $x_1 = B$.
Suppose that a $(1,2)$-sequence exists with $x_1 = A$.
Then $x_{1+1} = x_2 = B$ and $x_{2+2} = x_4 = A$ and $x_{4+1} = x_5 = B$.
So x_1, x_2, x_3, x_4, x_5 is A, B, x_3, A, B.
Consider x_3. If $x_3 = B$, then we would have $x_5 = A$, which is not true. If $x_3 = A$, then we would have $x_4 = B$, which is not true.
Since there is no possible value for x_3, then a $(1,2)$-sequence cannot have $x_1 = A$.
Suppose that a $(1,2)$-sequence exists with $x_1 = B$.
Then $x_3 = A$ and $x_4 = B$.
So x_1, x_2, x_3, x_4 is B, x_2, A, B.
Consider x_2. If $x_2 = B$, then we would have $x_4 = A$, which is not true. If $x_2 = A$, then we would have $x_3 = B$, which is not true.
Since there is no possible value for x_2, then a $(1,2)$-sequence cannot have $x_1 = B$.
Therefore, a $(1,2)$-sequence cannot have $x_1 = A$ or $x_1 = B$, so no $(1,2)$-sequence exists.

3. Given a sequence $a_1, a_2, a_3, \ldots$ of positive integers, we define a new sequence $b_1, b_2, b_3, \ldots$ by $b_1 = a_1$ and, for every positive integer $n \geq 1$,

$$b_{n+1} = \begin{cases} b_n + a_{n+1} & \text{if } b_n \leq a_{n+1} \\ b_n - a_{n+1} & \text{if } b_n > a_{n+1} \end{cases}$$

For example, when $a_1, a_2, a_3, \cdots$ is the sequence $1, 2, 1, 2, 1, 2, \ldots$ we have

n	1	2	3	4	5	6	7	8	9	10	$\cdots$
a_n	1	2	1	2	1	2	1	2	1	2	$\cdots$
b_n	1	3	2	4	3	1	2	4	3	1	$\cdots$

(a) Suppose that $a_n = n^2$ for all $n \geq 1$. Determine the value of b_{10}.

(b) Suppose that $a_n = n$ for all $n \geq 1$. Determine all positive integers n with $n < 2015$ for which $b_n = 1$.

Solution:

(a)

When $a_n = n^2$, we obtain

n	1	2	3	4	5	6	7	8	9	10	$\cdots$
a_n	1	4	9	16	25	36	49	64	81	100	$\cdots$
b_n	1										$\cdots$

Since $b_1 \leq a_2$, then $b_2 = b_1 + a_2 = 1 + 4 = 5$.
Since $b_2 \leq a_3$, then $b_3 = b_2 + a_3 = 5 + 9 = 14$.
Since $b_3 \leq a_4$, then $b_4 = b_3 + a_4 = 14 + 16 = 30$.
Since $b_4 > a_5$, then $b_5 = b_4 - a_5 = 30 - 25 = 5$.
Since $b_5 \leq a_6$, then $b_6 = b_5 + a_6 = 5 + 36 = 41$.
Since $b_6 \leq a_7$, then $b_7 = b_6 + a_7 = 41 + 49 = 90$.
Since $b_7 > a_8$, then $b_8 = b_7 - a_8 = 90 - 64 = 26$.
Since $b_8 \leq a_9$, then $b_9 = b_8 + a_9 = 26 + 81 = 107$.
Since $b_9 > a_{10}$, then $b_{10} = b_9 - a_{10} = 107 - 100 = 7$.
In tabular form, we have

n	1	2	3	4	5	6	7	8	9	10	$\cdots$
a_n	1	4	9	16	25	36	49	64	81	100	$\cdots$
b_n	1	5	14	30	5	41	90	26	107	7	$\cdots$

(b)

As in (a), we start by calculating the first several values of b_n:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	$\cdots$
a_n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	$\cdots$
b_n	1	3	6	2	7	1	8	16	7	17	6	18	5	19	4	20	3	21	2	22	1	23	$\cdots$

We will show that if k is a positive integer with $b_k = 1$, then $b_{3k+3} = 1$ and $b_n \neq 1$ for each n with $k < n < 3k + 3$.
We note that this is consistent with the table shown above.
Using this fact without proof, we see that $b_1 = 1$, $b_6 = 1$, $b_{21} = 1$, $b_{66} = 1$, $b_{201} = 1$, $b_{606} = 1$, $b_{1821} = 1$, $b_{5466} = 1$, and no other b_n with $n < 5466$ is equal to 1.
(This list of values of b_n comes from the fact that $6 = 3(1)+3$, $21 = 3(6)+3$, $66 = 3(21)+3$, and so on.)
Thus, once we have proven this fact, the positive integers n with $n < 2015$ and $b_n = 1$ are $n = 1, 6, 21, 66, 201, 606, 1821$.

Suppose that $b_k = 1$.
Since $a_{k+1} = k + 1 > 1 = b_k$, then $b_{k+1} = b_k + a_{k+1} = k + 2$.
Since $a_{k+2} = k + 2 = b_{k+1}$, then $b_{k+2} = b_{k+1} + a_{k+2} = 2k + 4$.
Continuing in this way, we find that $b_{k+2m-1} = k+3-m$ for $m = 1, 2, \ldots, k+2$ and that $b_{k+2m} = 2k + m + 3$ for $m = 1, 2, \ldots, k + 1$.
To justify these statements, we note that each is true when $m = 1$ and that

- if $b_{k+2m} = 2k+m+3$ for some m with $1 \le m \le k+1$, then since $a_{k+2m+1} = k+2m+1$ and $b_{k+2m} - a_{k+2m+1} = k - m + 2 > 0$ when $m \le k + 1$, then

$$b_{k+2m+1} = b_{k+2m} - a_{k+2m+1} = k - m + 2$$

  which can be re-written as $b_{k+2(m+1)-1} = k+3-(m+1)$ and so is of the desired form; and
- if $b_{k+2m-1} = k + 3 - m$ for some with $1 \le m \le k + 1$, then since $a_{k+2m} = k + 2m$ and $b_{k+2m-1} - a_{k+2m} = 3 - 3m \le 0$ when $m \ge 1$, then

$$b_{k+2m} = b_{k+2m-1} + a_{k+2m} = (k + 3 - m) + (k + 2m) = 2k + m + 3$$

  and so is of the desired form.

This tells us that, for these values of m, the terms in the sequence b_n have the desired form.
Now $b_{k+2m} = 2k + m + 3 \neq 1$ for $m = 1, 2, \ldots, k + 1$ and $b_{k+2m-1} = k + 3 - m = 1$ only when $m = k + 2$.
Since $k + 2(k + 2) - 1 = 3k + 3$, then $b_{3k+3} = 1$ and no other n with $k < n < 3k + 3$ gives $b_n = 1$.

Thus, the positive integers n with $n < 2015$ and $b_n = 1$ are $n = 1, 6, 21, 66, 201, 606, 1821$.