Today's Outline

8. Second Order Circuits

- the parallel RLC circuit
- underdamped, overdamped, and critically damped natural response

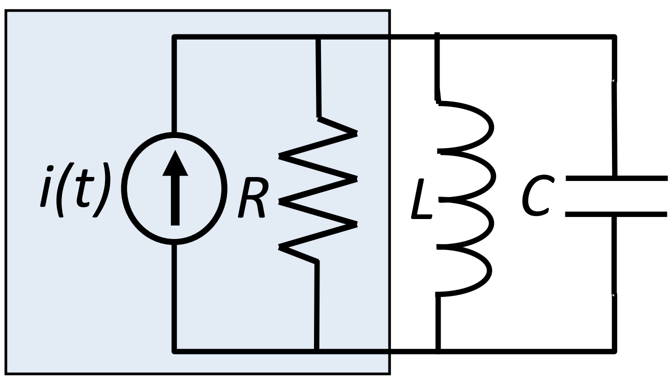
Overview

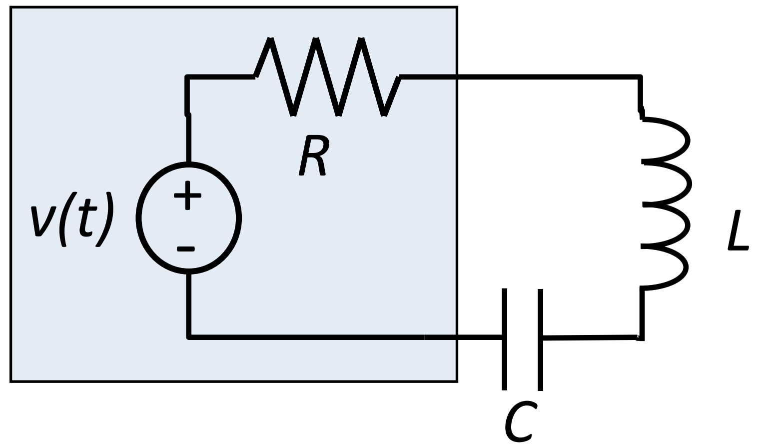
Second order circuit: a circuit composed of resistors, sources and two energy storage components, often one capacitor and one inductor. Some, not all, circuits with two capacitors or two inductors are second order circuits.

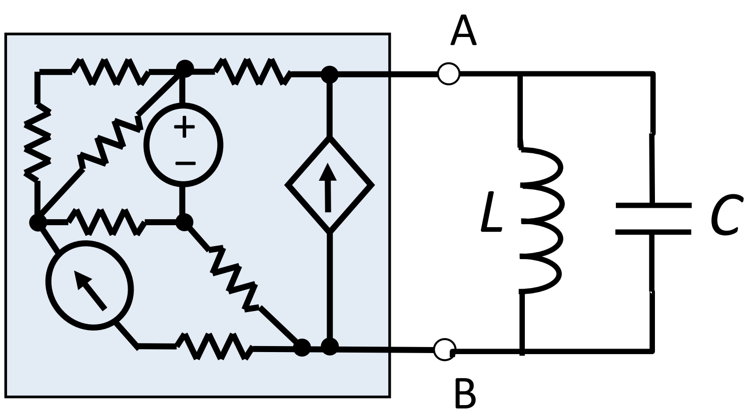
- **RLC circuits** include a resistor, an inductor and a capacitor
- "second order" refers to the second order differential equations that describe voltage and current variables, $v(t)$ and $i(t)$
- *RLC* circuits are useful because of the time dependent response of such circuits

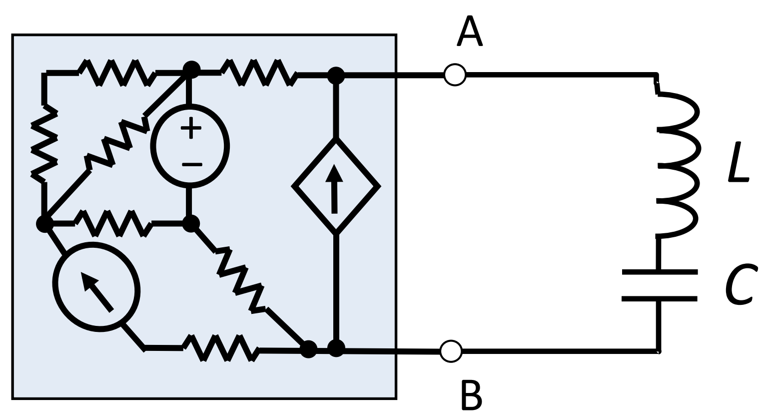
Overview

There are two classic forms of the *RLC* circuit: the ***parallel RLC*** and the ***series RLC***.

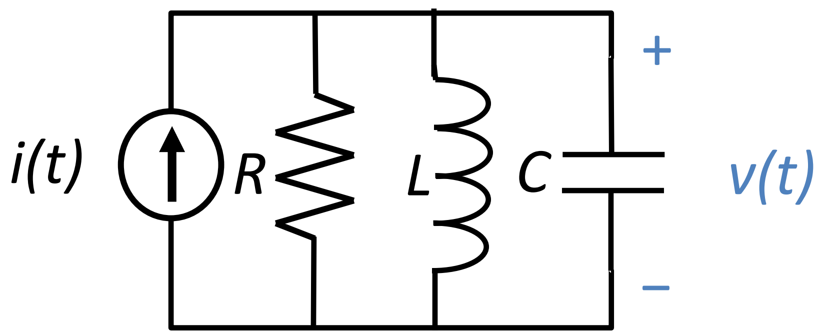








Parallel *RLC*



KCL: $$-i+\frac{v}{1/sC}+\frac{v}{R}+\frac{v}{sL}=0$$

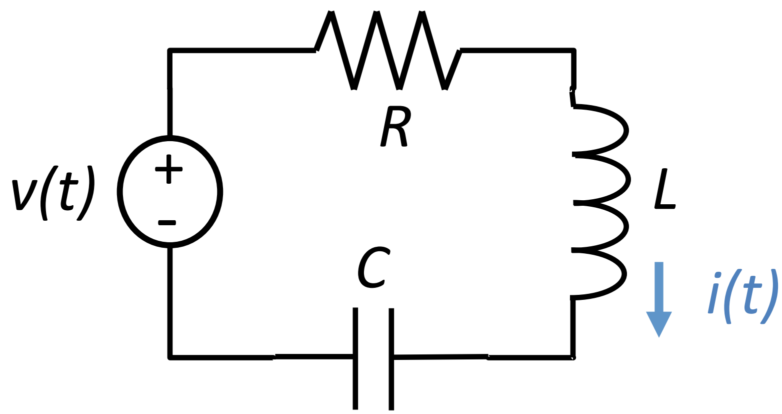
$$-i+C\frac{dv}{dt}+\frac{v}{R}+\frac{1}{L}\int_0^t v(t')\,dt'+i_L(0)=0$$

$$-si+s^2C\cdot v+\frac{s}{R}\cdot v+\frac{1}{L}\cdot v=0$$

Replacing *s* by *d*/*dt*: $$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{1}{LC}v=\frac{1}{C}\frac{di}{dt}$$

This is an **inhomogeneous linear second order differential equation**.

Series *RLC*



KVL: $$-v + R\cdot i + sL\cdot i + \frac{1}{sC}\cdot i = 0$$

$$-v + L\frac{di}{dt} + iR + \frac{1}{C}\int_0^t i(t')dt' + v_C(0) = 0$$

$$-sv + sR\cdot i + s^2L\cdot i + \frac{1}{C}\cdot i = 0$$

Replacing *s* by *d*/*dt*: $$\frac{d^2i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{1}{LC}i = \frac{1}{L}\frac{dv}{dt}$$

This is an **inhomogeneous linear second order differential equation**.

natural response

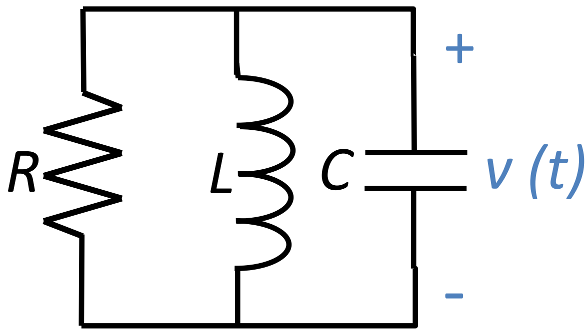
As in the case of first order circuits, it is useful to divide the response of a second order circuit into two parts.

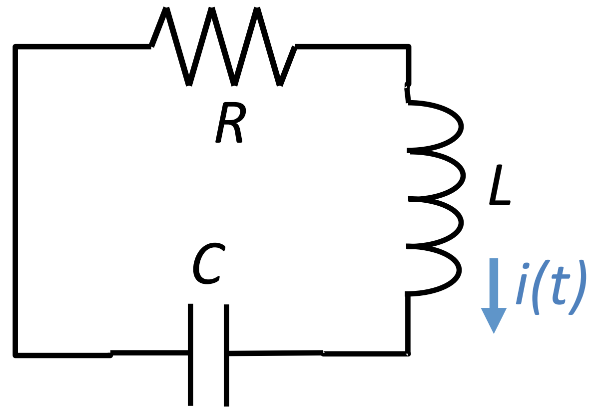
total response = transient (natural) response
+ steady state (forced) response

Solution to the homogeneous equation gives the "short-lived" transient or natural response.

Particular solution of the inhomogeneous equation gives the "long-lived" steady state response.

natural response





The differential equations that are to be solved are thus:

$$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{1}{LC}v=0 \quad t>0$$

$$i_L(0+)=I_0$$

$$v(0+)=V_0$$

$$\frac{d^2i}{dt^2}+\frac{1}{L/R}\frac{di}{dt}+\frac{1}{LC}i=0 \quad t>0$$

$$v_C(0+)=V_0$$

$$i(0+)=I_0$$

mathematical review

Consider the ***homogeneous linear second order differential equation,*** which corresponds to ***unforced (natural)*** behaviour:

$$a\frac{d^2x}{dt^2} + b\frac{dx}{dt} + cx = 0$$

Assume a solution of the form: $x(t) = K\exp(+st)$

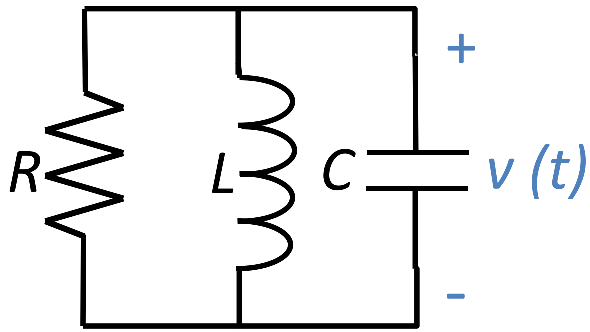
Substitution returns an operator equation:

$$as^2 \cdot x + bs \cdot x + c \cdot x = \left(as^2 + bs + c\right) \cdot x = 0$$

The ***characteristic equation*** for the value of *s* is: $as^2 + bs + c = 0$

The roots of the characteristic equation are: $s_{1,2} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

parallel RLC



The differential equation to be solved is:

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

The characteristic equation to be solved is:

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

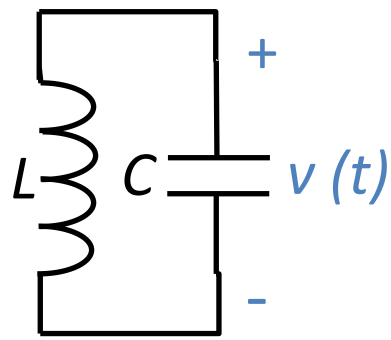
We identify the two coefficients as meaningful parameters:

$$s^2 + 2\alpha \cdot s + \omega_0^2 = 0$$

$$\alpha = \frac{1}{2RC} = \text{damping rate}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \text{resonant frequency}$$

undamped parallel RLC



$$\alpha = \frac{1}{2RC} \rightarrow 0$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

Consider the special case where $R \rightarrow \infty$, the damping rate $\alpha \rightarrow 0$ and the characteristic equation:

$$s^2 + \omega_0^2 = 0$$

$$s = \pm j\omega_0 \qquad j = \sqrt{-1}$$

The solution to $v(t)$ is:

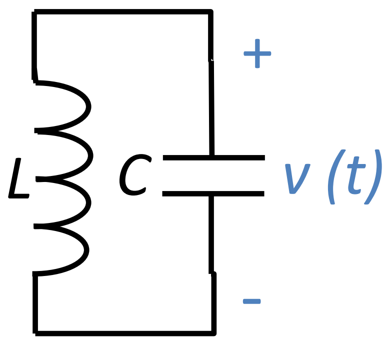
$$v(t) = A_1 \exp(+j\omega_0 t) + A_2 \exp(-j\omega_0 t)$$

By Euler's theorem:

$$v(t) = B_1 \sin(\omega_0 t) + B_2 \cos(\omega_0 t)$$

The constants are determined by the initial conditions at $t = 0+$.

undamped parallel RLC

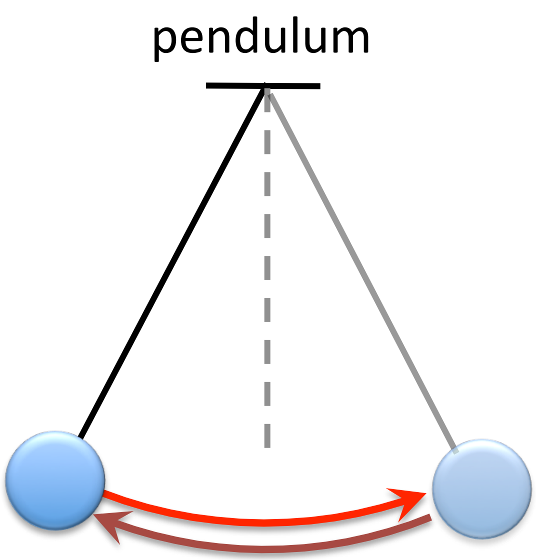


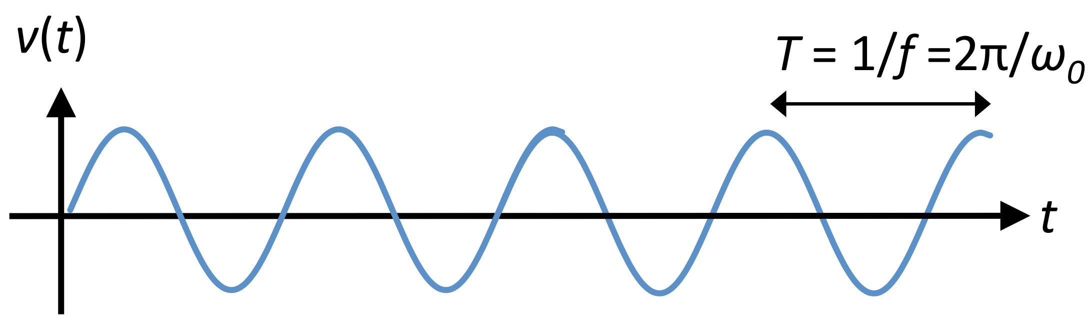
The solution is an undamped oscillation, like that of an undamped mechanical pendulum.

$$v(t) = B_1 \sin(\omega_0 t) + B_2 \cos(\omega_0 t)$$

$$\omega_0 = \frac{1}{\sqrt{LC}}$$

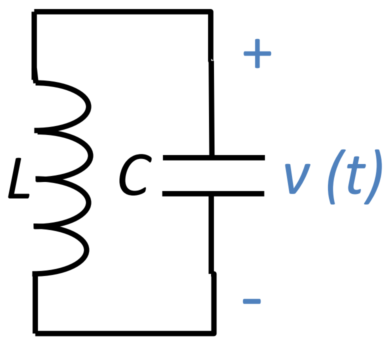
The angular frequency ω_0 is called the ***undamped resonant frequency*** of the *LC* circuit.

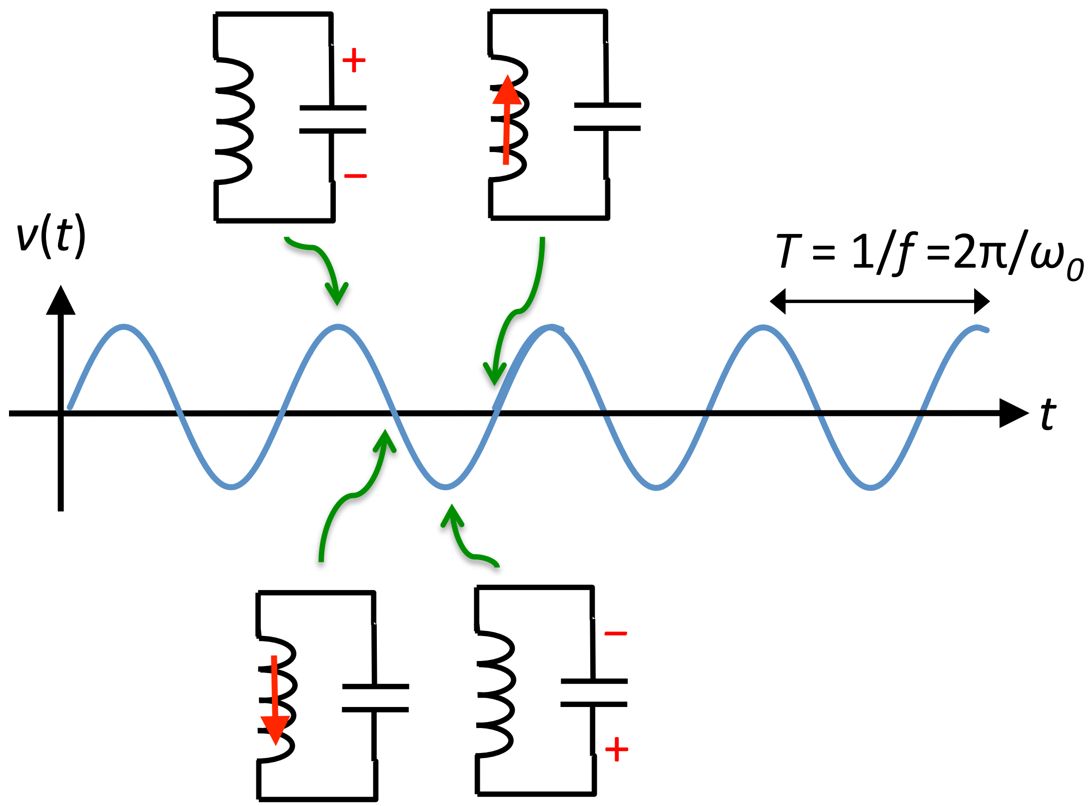




undamped parallel RLC

Energy oscillates between electric energy stored in *C* and magnetic energy stored in *L*. Without *R*, there is no energy lost to heat.



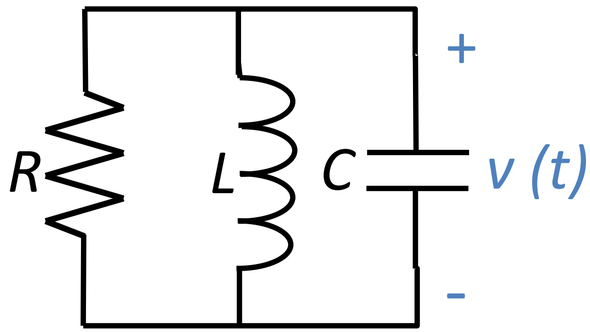


$$U_C = \frac{1}{2}Cv^2$$

$$U_L = \frac{1}{2}Li^2 = \frac{1}{2}L\left(C\frac{dv}{dt}\right)^2$$

$$U_{tot} = U_L + U_C = \text{constant}$$

parallel RLC



$$\alpha = \frac{1}{2RC} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

The case of non-zero damping gives:

$$s^2 + 2\alpha \cdot s + \omega_0^2 = 0$$

The general solution to the roots are:

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

These roots can are in general complex numbers, with three classes of solution:

$$\alpha < \omega_0 \qquad s = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}$$

$$\alpha = \omega_0 \qquad s = -\alpha$$

$$\alpha > \omega_0 \qquad s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

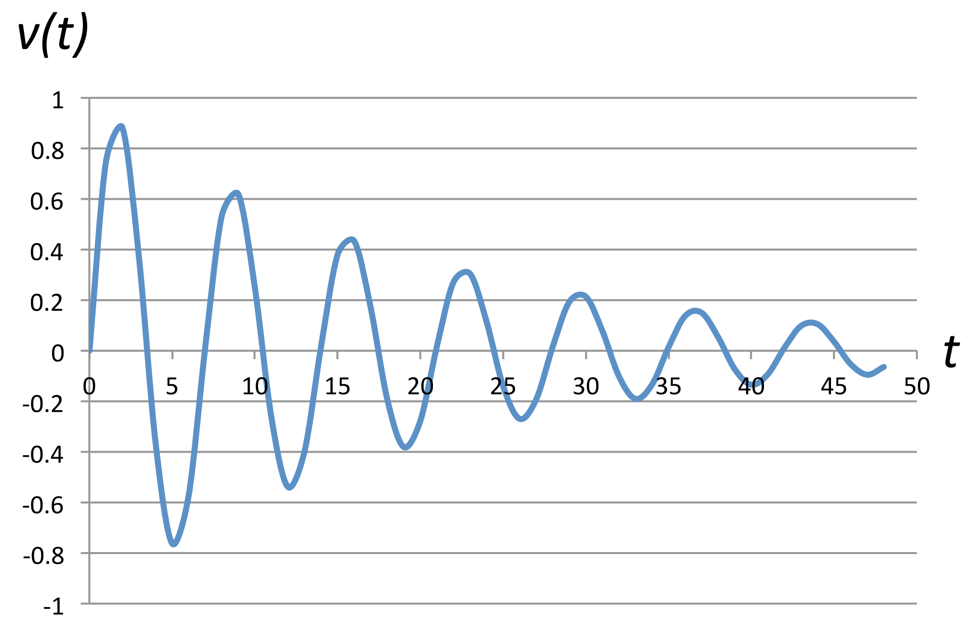
underdamped natural response

underdamped response: $\alpha < \omega_0$

The roots are complex:

$$s = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}$$
$$= -\alpha \pm j\omega_d$$

The natural response has an oscillation at the damped frequency ω_d, decaying with time constant α.



$$v(t) = \exp(-\alpha t)\left[A_1 \exp(+j\omega_d t) + A_2 \exp(-j\omega_d t)\right]$$
$$= \exp(-\alpha t)\left[B_1 \sin(\omega_d t) + B_2 \cos(\omega_d t)\right]$$

The coefficients can be found:

$$v(0+) = B_2$$

$$\frac{dv}{dt}(0+) = -\alpha B_2 - \omega_d B_1$$

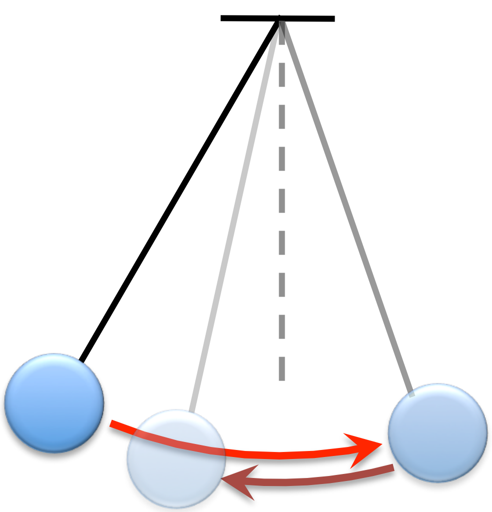
underdamped natural response

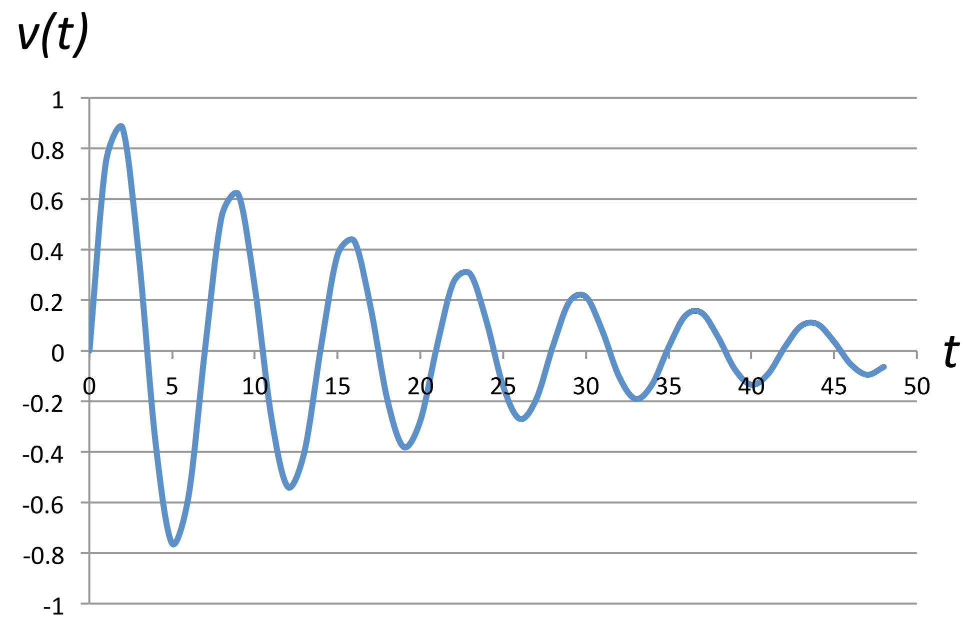
underdamped response: $\alpha < \omega_0$

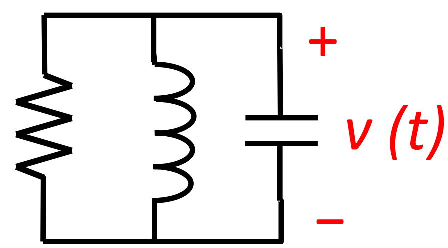
$$s = -\alpha \pm j\omega_d$$

By analogy, an underdamped pendulum oscillates, losing a fraction of the initially stored energy during each oscillation (eg. by friction).

pendulum







The resistor dissipates energy at each moment that $v(t) \neq 0$.

critically damped natural response

critically damped response: $\alpha = \omega_0$

The roots are identical:

$$s = -\alpha = -\omega_0$$

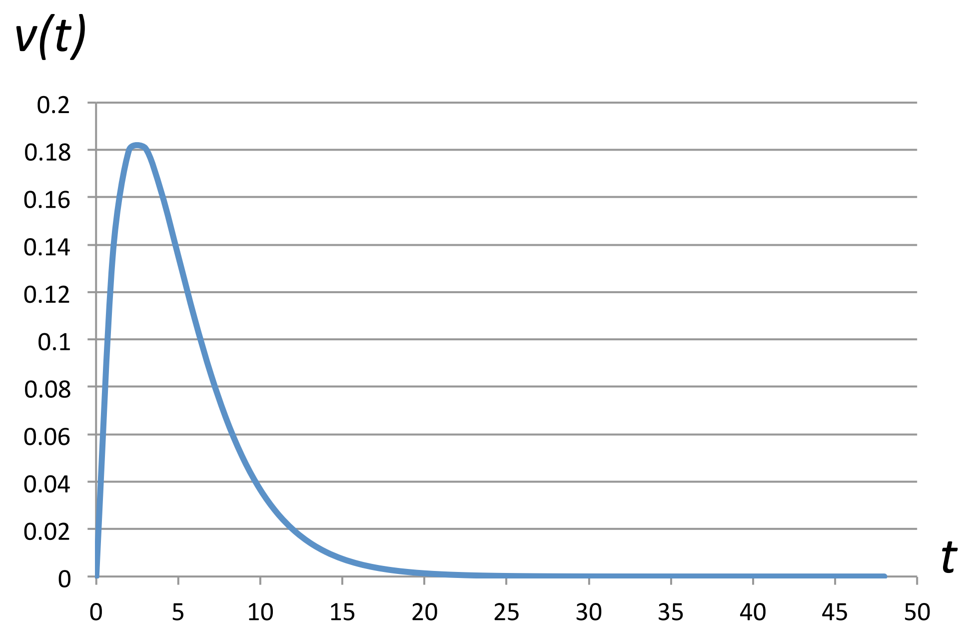
The natural response has the form*:

$$v(t) = \exp(-\alpha t)\left[A_1 t + A_2\right]$$

The coefficients can be found by using the initial conditions:

$$v(0+) = A_2$$

$$\frac{dv}{dt}(0+) = A_1 + sA_2$$



**refer to theory of ordinary differential equations for explanation*

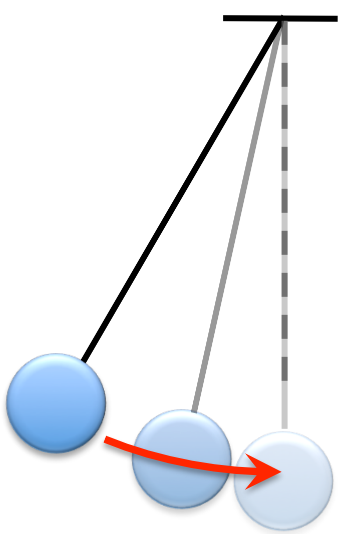
critically damped natural response

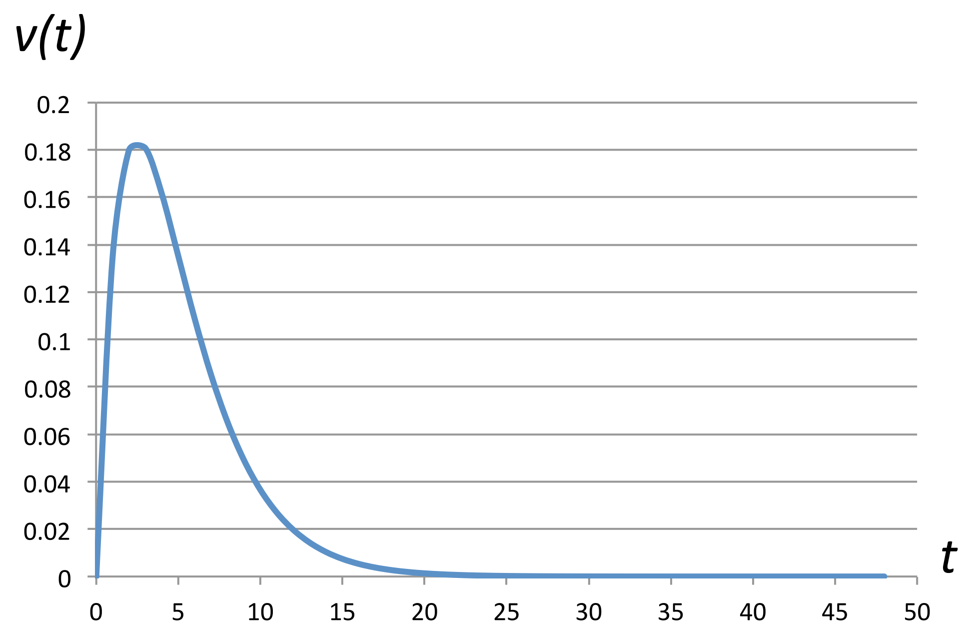
critically damped response: $\alpha = \omega_0$

$$s = -\alpha = -\omega_0$$

By analogy, a critically damped pendulum does not oscillate, but approaches rest.

pendulum





overdamped natural response

overdamped response: $\alpha > \omega_0$

Two real roots are found.

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

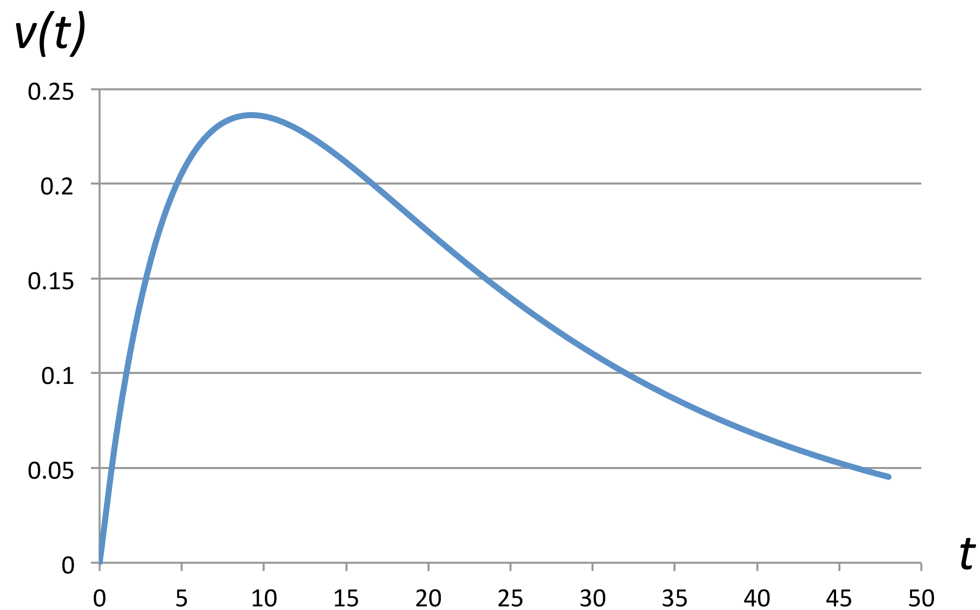
The natural response is:

$$v(t) = A_1 \exp(s_1 t) + A_2 \exp(s_2 t)$$

The coefficients can be found by using the initial conditions:

$$v(0+) = A_1 + A_2$$

$$\frac{dv}{dt}(0+) = s_1 A_1 + s_2 A_2$$



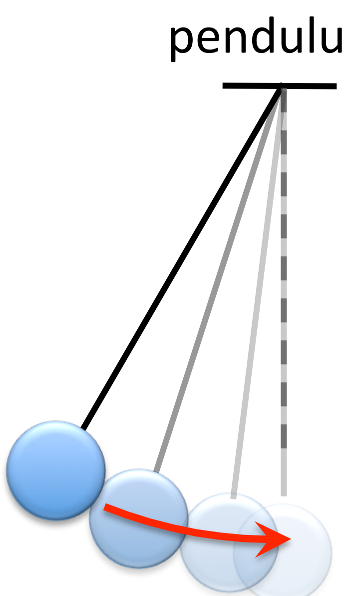
overdamped natural response

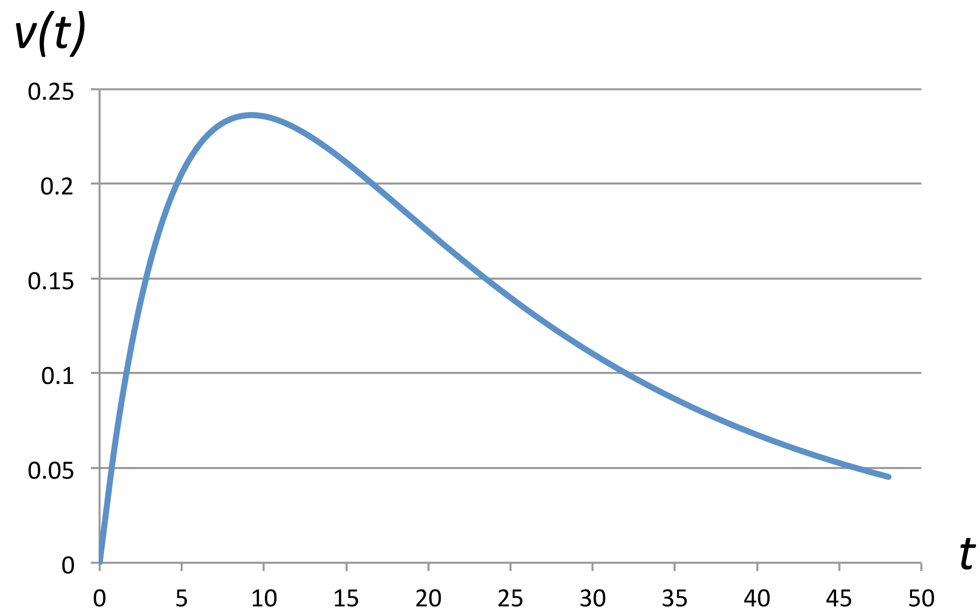
overdamped response: $\alpha > \omega_0$

Two real roots are found.

$$s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

By analogy, an overdamped pendulum approaches rest at a rate dominated by damping (eg. as if placed in a highly viscous fluid)

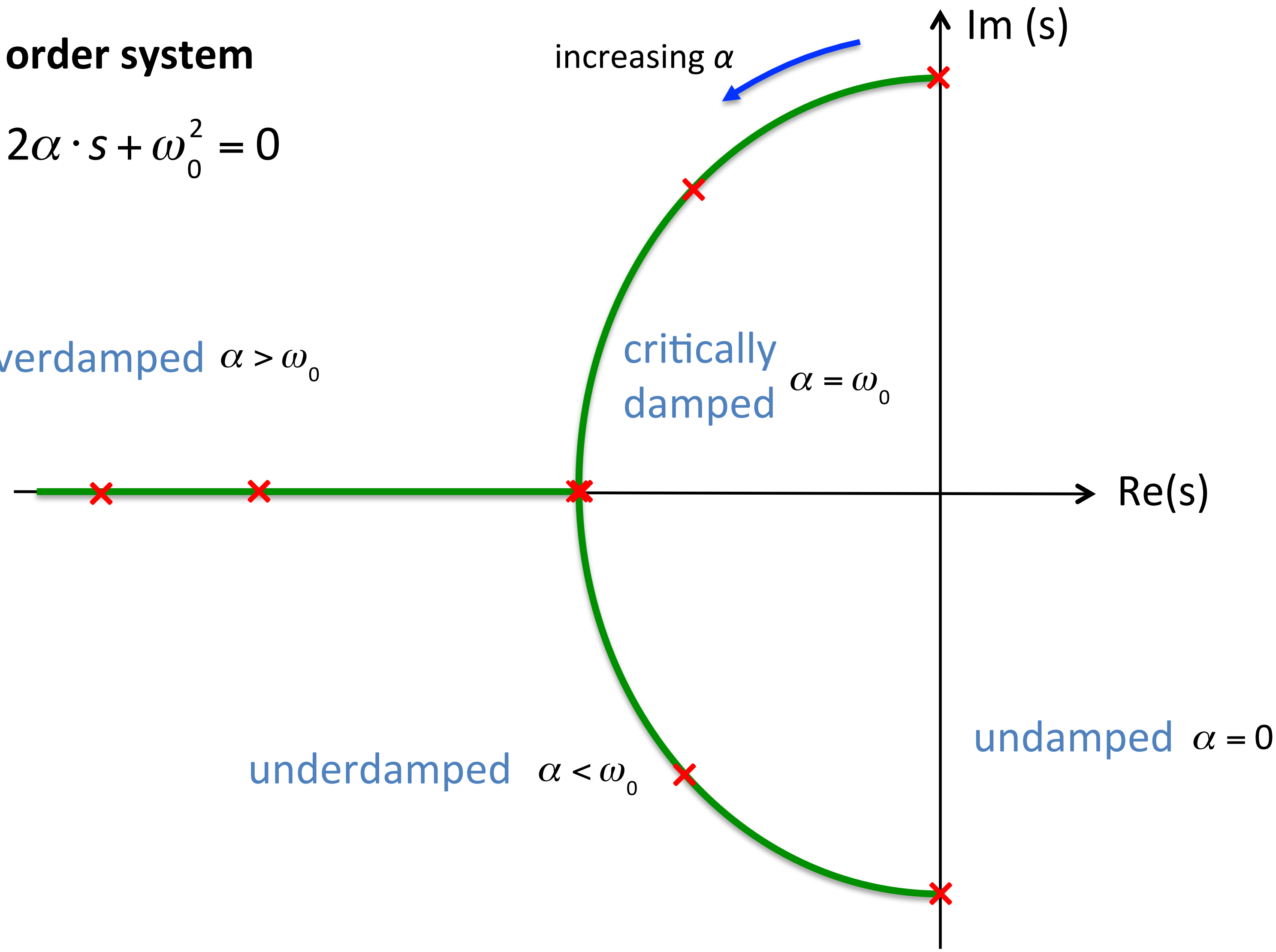




the complex *s*-plane

2nd order system

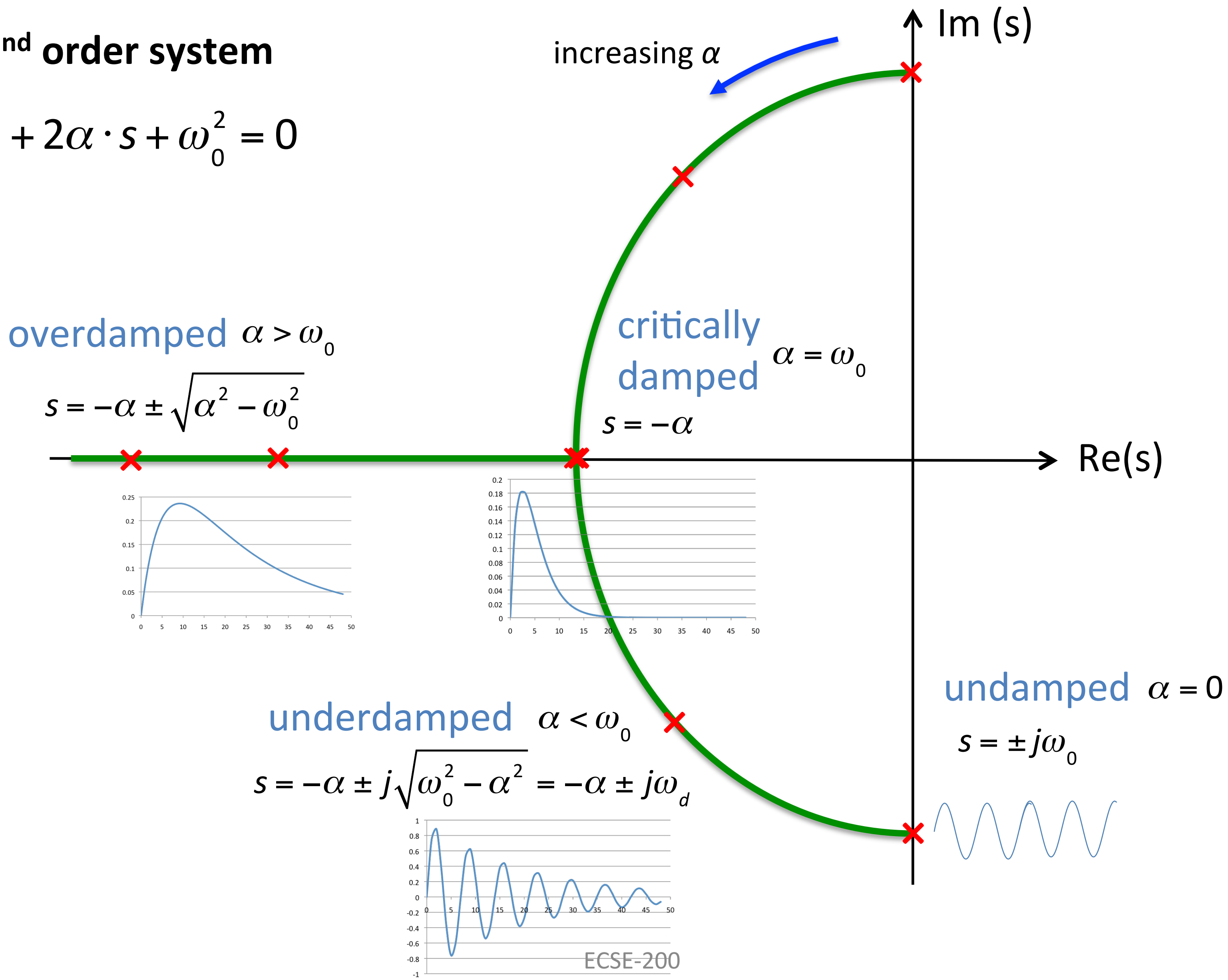
$$s^2 + 2\alpha \cdot s + \omega_0^2 = 0$$



the complex *s*-plane

2nd order system

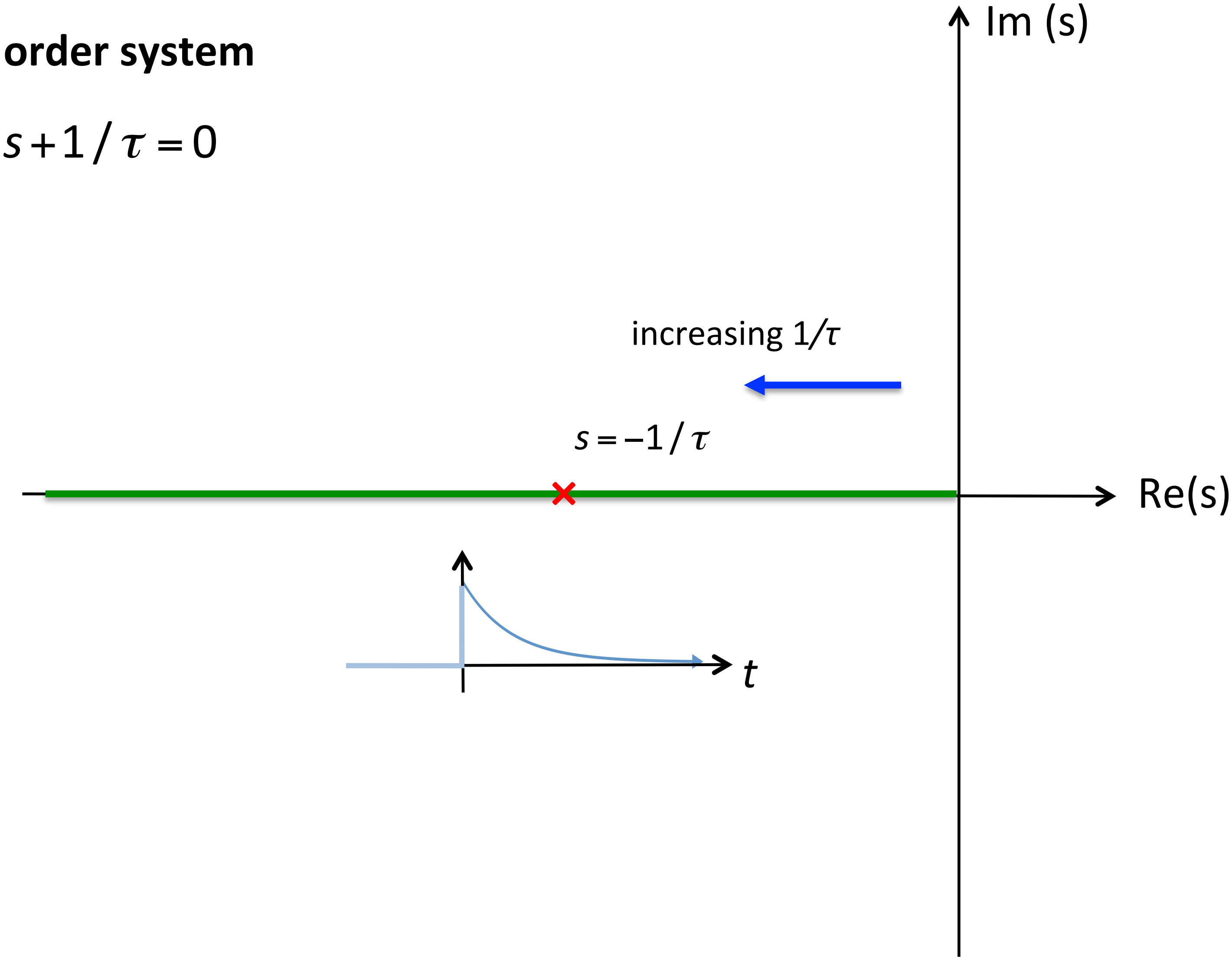
$$s^2 + 2\alpha \cdot s + \omega_0^2 = 0$$



the complex s-plane

1st order system

$$s+1/\tau=0$$



RLC natural response: general procedure

step #1: Find the initial value and initial deritvative of the circuit variable of interest, $x(0+)$ and $dx/dt(0+)$, using circuit analysis and continuity of capacitor voltage and inductor current.

step #2: Using the operator method, or by directly writing down the equation, determine the characteristic equation coefficients α and ω_0 in terms of R, L, C.

$$s^2 + 2\alpha s + \omega_0^2 = 0$$

step #3: Find the roots of the characteristic equation:

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

step #4: Construct the solution, using the over-/critical/under-damped nature of the response:

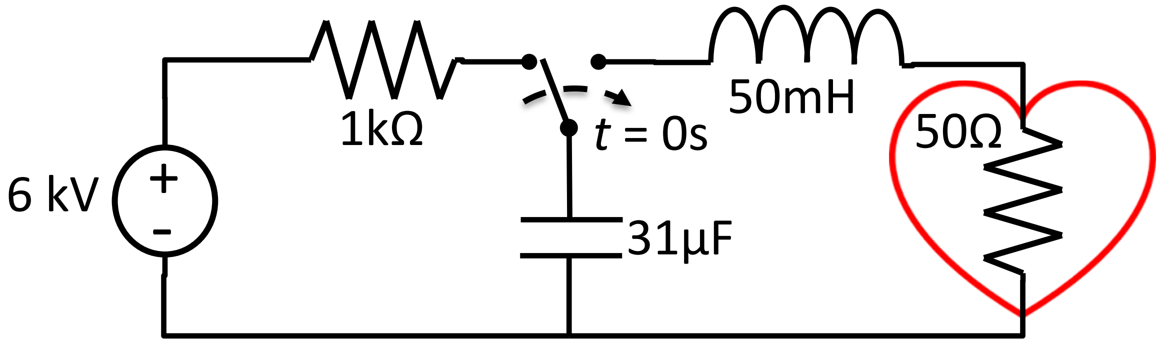
$$x(t) = A_1 \exp(s_1 t) + A_2 \exp(s_2 t)$$

$$x(t) = \exp(-\alpha t)\left[A_1 t + A_2\right]$$

$$x(t) = \exp(-\alpha t)\left[B_1 \sin(\omega_d t) + B_2 \cos(\omega_d t)\right]$$

example: defibrillator

Dr. Bernard Lown



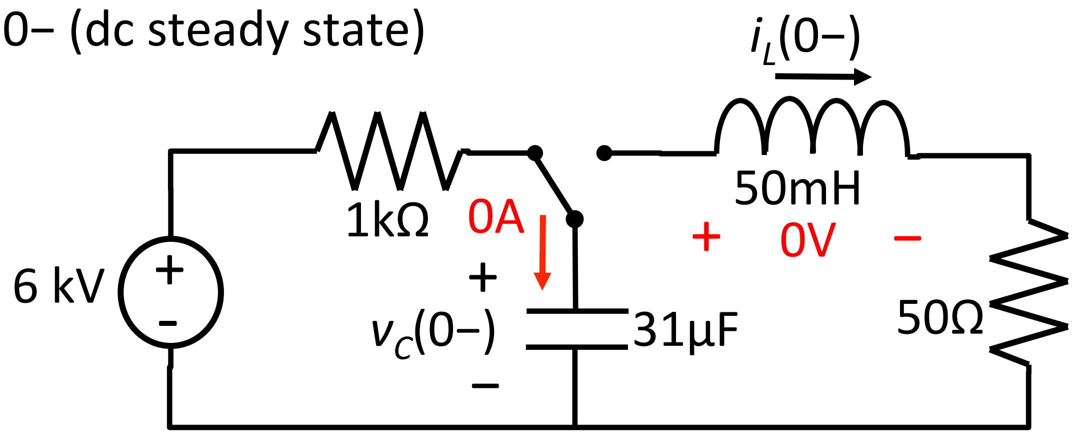
Find the voltage versus time on the heart, crudely approximated as a 50Ω resistor. The circuit is in dc steady state for $t < 0$, and the switch moves instantaneously at t = 0s.

The circuit has been designed to deliver a voltage pulse to the human heart, with shape and amplitude that have the desired physiological effect.

example: defibrillator

step #1 : initial conditions

t = 0− (dc steady state)



KVL:

$v_C(0-) = 6\text{kV}$

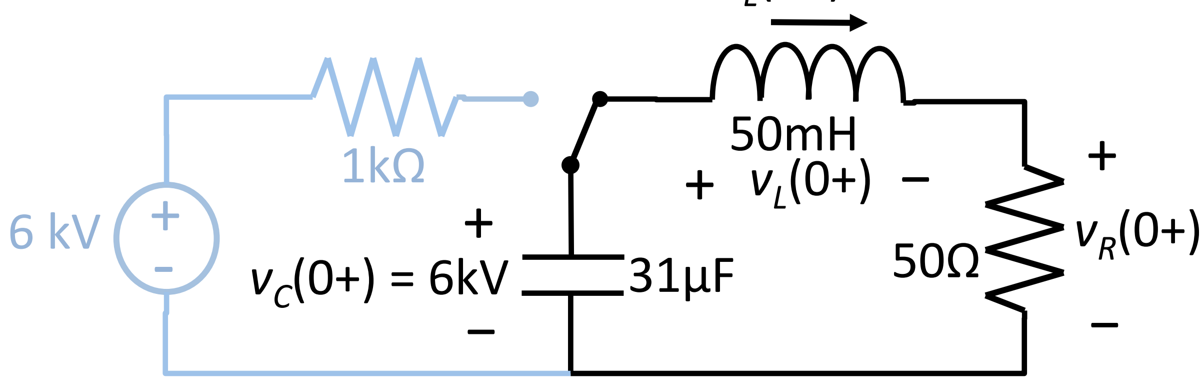
KCL:

$i_L(0-) = 0\text{A}$

example: defibrillator

step #1 : initial conditions

$t = 0+$



continuity:

$v_C(0+) = v_C(0-) = 6kV$

$i_L(0+) = i_L(0-) = 0A$

Ohm: $v_R(0+) = i_L(0+)\ 50\Omega = 0V$

KVL: $0 = -v_C(0+) + v_L(0+) + v_R(0+)$

$v_L(0+) = v_C(0+) - v_R(0+) = 6kV$

Ohm, inductor: $$\left.\frac{dv_R}{dt}\right|_{t=0+} = 50\Omega \cdot \left.\frac{di_L}{dt}\right|_{t=0+} = 50\Omega \cdot \frac{v_L(0+)}{50mH} = 6\ MV/s$$

initial conditions:

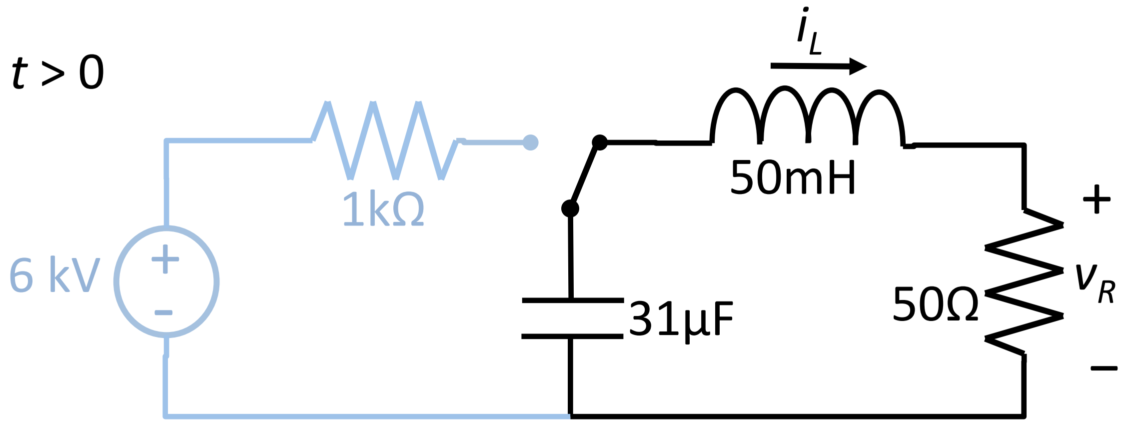
$$v_R(0+) = 0\ V$$

$$\left.\frac{dv_R}{dt}\right|_{t=0+} = 6\ MV/s$$

example: defibrillator

step #2 : find the characteristic equation

$t > 0$



Note that the natural response of v_R and i_L are determined by the same characteristic equation. Why?

mesh equation: $0 = i_L \cdot sL + i_L \cdot R + i_L \cdot \frac{1}{sC}$

$$0 = \left(sL + R + \frac{1}{sC} \right) \cdot i_L$$

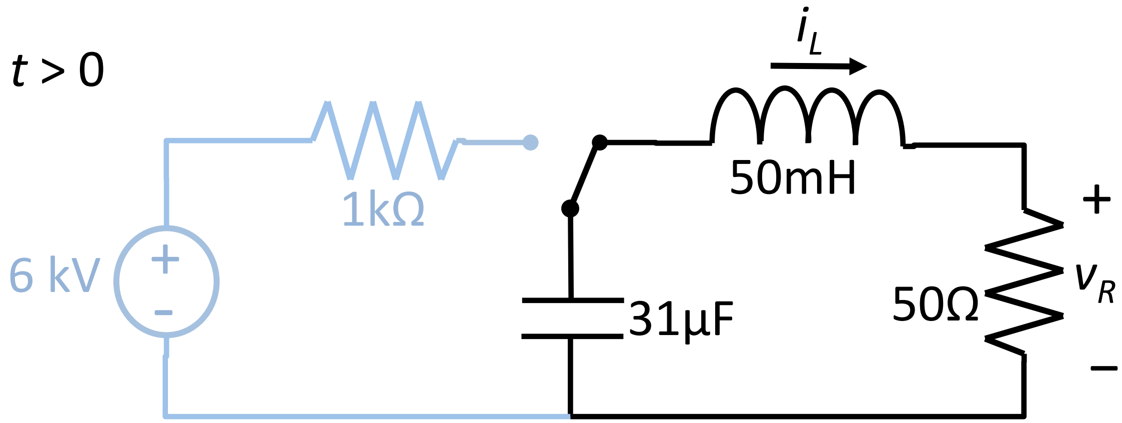
$$i_L \neq 0 \qquad \therefore 0 = sL + R + \frac{1}{sC} \quad \rightarrow \quad 0 = s^2 + s \cdot \frac{R}{L} + \frac{1}{LC}$$

$$0 = s^2 + s \cdot \frac{50\Omega}{50\text{mH}} + \frac{1}{50\text{mH} \cdot 31\mu\text{F}}$$

$$0 = s^2 + s \cdot 1000 + 645161$$

example: defibrillator

step #3 : find the roots of the characteristic equation



$$0 = a \cdot s^2 + b \cdot s + c$$

$$= s^2 + s \cdot 1000 + 645161$$

$$s_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-1000 \pm \sqrt{(1000)^2 - 4 \cdot 1 \cdot 645161}}{2 \cdot 1} = -500 \pm i \cdot 629 \quad [s^{-1}] \text{ units}$$

example: defibrillator

step #4 : construct the solution

$$v_R(0+) = 0\ \text{V} \qquad \left.\frac{dv_R}{dt}\right|_{t=0+} = 6\ \text{MV/s} \qquad s_{1,2} = -500 \pm i \cdot 629 \quad [\text{s}^{-1}]$$

Two complex roots correspond to ***underdamped*** natural response.

$$\begin{aligned} v_R(t) &= A_1 \exp(s_1 t) + A_2 \exp(s_2 t) \\ &= \exp(-500\text{s}^{-1} \cdot t)\left[B_1 \cos(629\text{s}^{-1} \cdot t) + B_2 \sin(629\text{s}^{-1} \cdot t)\right] \end{aligned}$$

Apply first of two initial conditions:

$$\begin{aligned} v_R(0+) &= 1 \cdot \left[B_1 \cdot 1 + B_2 \cdot 0\right] = B_1 \\ 0\text{V} &= B_1 \end{aligned}$$

example: defibrillator

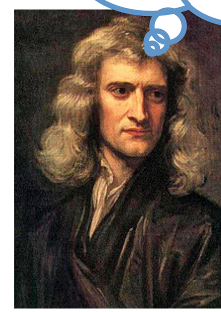
step #4 : construct the solution

$$v_R(t) = B_2 \cdot \exp(-500\text{s}^{-1} \cdot t) \cdot \sin(629\text{s}^{-1} \cdot t)$$

Apply second of two initial conditions:

$$\frac{dv_R}{dt} = B_2 \cdot \left[-500\text{s}^{-1} \exp(-500\text{s}^{-1} \cdot t)\right] \cdot \sin(629\text{s}^{-1} \cdot t)$$
$$+B_2 \cdot \exp(-500\text{s}^{-1} \cdot t) \cdot \left[629\text{s}^{-1} \cos(629\text{s}^{-1} \cdot t)\right]$$

$$(fg)' = f'g + fg'$$

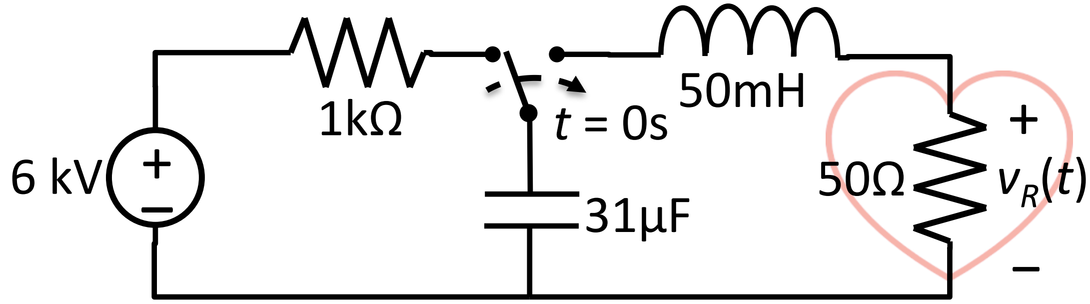


$$\left.\frac{dv_R}{dt}\right|_{t=0+} = B_2 \cdot \left[-500\text{s}^{-1} \cdot 1\right] \cdot 0 + B_2 \cdot 1 \cdot \left[629\text{s}^{-1} \cdot 1\right]$$

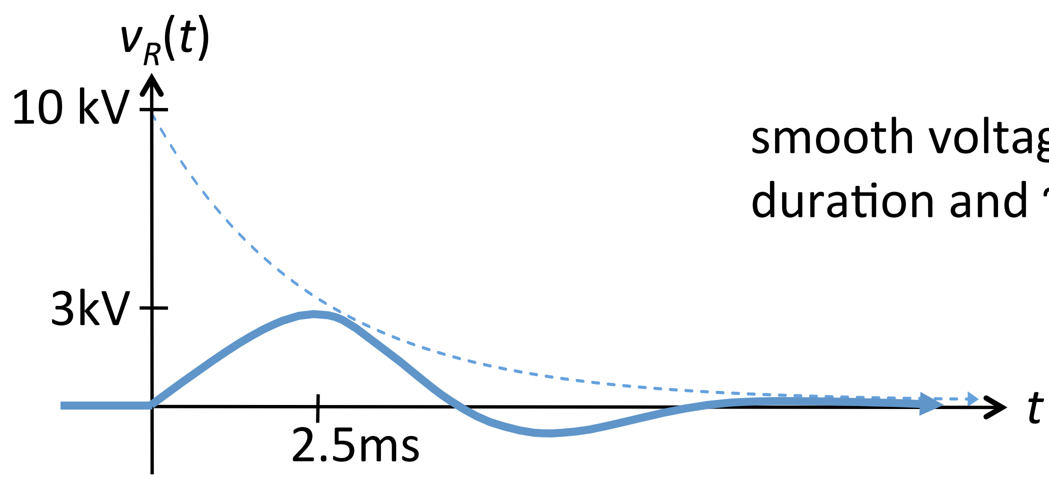
$$6\text{MV/s} = B_2 \cdot 629\text{s}^{-1}$$

$$B_2 = 9.54\ \text{kV}$$

example: defibrillator



solution: $$v_R(t) = 9.54\text{kV} \cdot \exp(-500\text{s}^{-1} \cdot t) \cdot \sin(629\text{s}^{-1} \cdot t) \qquad t > 0$$



smooth voltage pulse of several ms duration and ~3kV amplitude

The energy delivered to the heart is: $$U = \frac{1}{2} C\, v_C^2(0) = \frac{1}{2} \cdot 31\mu\text{F} \cdot \left(6\text{kV}\right)^2 = 558\text{ J}$$