McGill

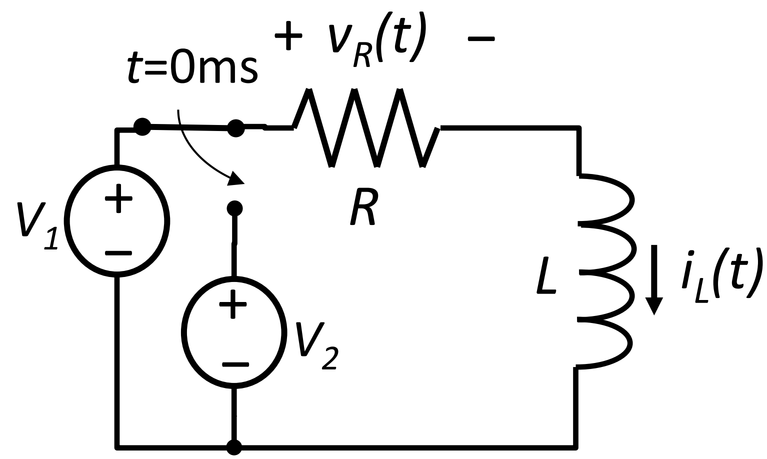
Today’s Outline

7. First Order Circuits

- response to a constant input
  - RL circuits
  - general procedure

response to a constant input

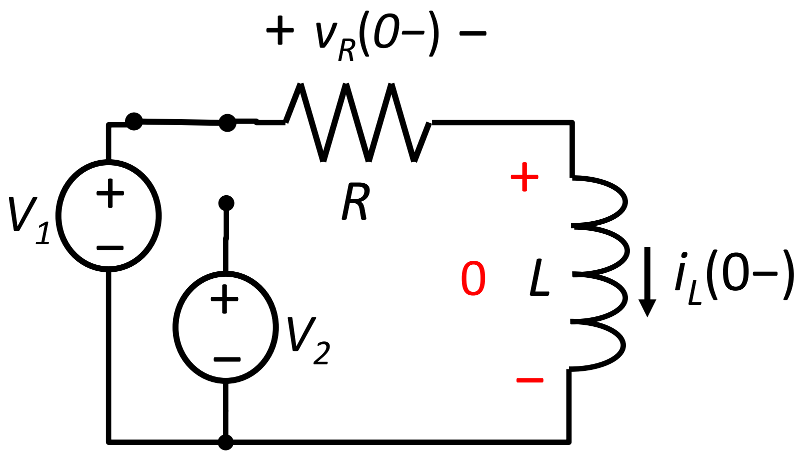
Consider an *RL* circuit being switched at t=0 between two different open circuit voltages. Assume steady state has been reached for t<0. What is the inductor current?



constant input: inductor current

Consider first the steady state conditions for $t < 0$.

$t < 0$



KVL: $0 = -V_1 + v_R(0-) + v_L(0-)$

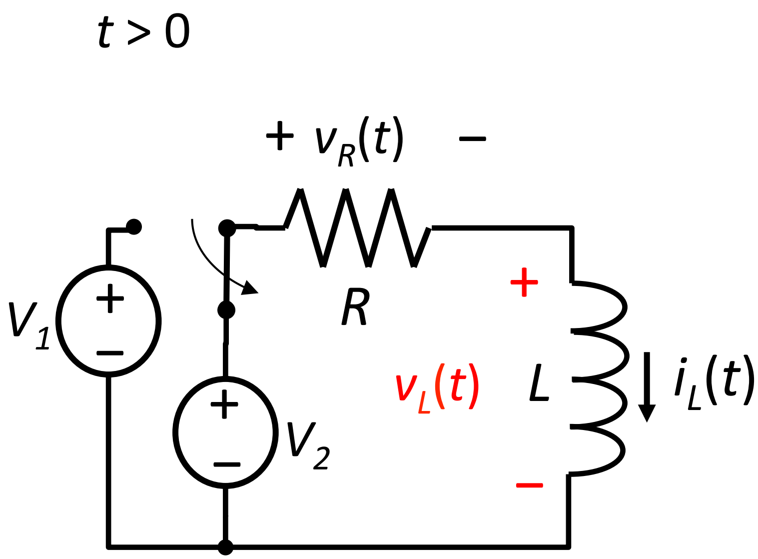
$= -V_1 + i_L(0-) \cdot R + 0$

$$i_L(0-) = \frac{V_1}{R}$$

Steady state: $v_L = L\dfrac{di_L}{dt} = 0$

constant input: inductor current

Consider the circuit equations for $t > 0$.



KVL: $0 = -V_2 + v_R + v_L$

$$= -V_2 + R \cdot i_L + L\frac{di_L}{dt}$$

$$\boxed{\frac{di_L}{dt} + \frac{R}{L} i_L = \frac{V_2}{L} \qquad t > 0}$$

continuity of inductor current:

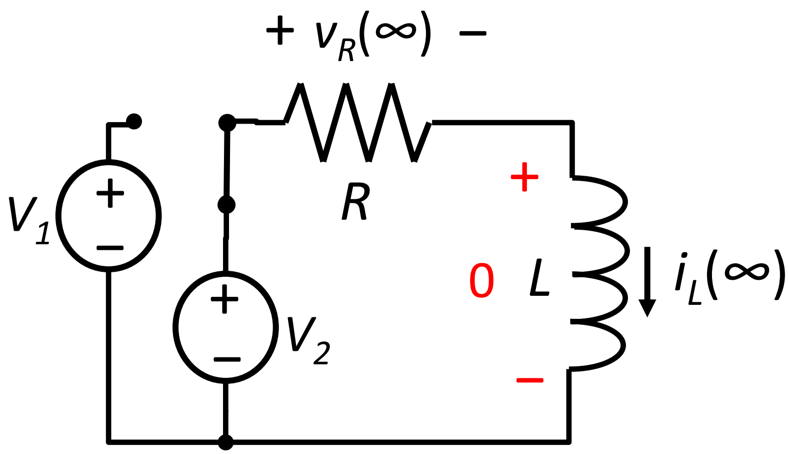
$$\boxed{i_L(0+) = i_L(0-) = \frac{V_1}{R}}$$

We have a first-order linear differential equation with initial conditions.

constant input: inductor current

Consider steady state as $t \to \infty$.

$t \to \infty$



Steady state: $v_L = L\dfrac{di_L}{dt} = 0$

KVL:

$$0 = -V_2 + v_R(\infty) + v_L(\infty)$$
$$= -V_2 + R \cdot i_L(\infty) + 0$$

$$\boxed{i_L(\infty) = \frac{V_2}{R}}$$

This can also be concluded from the circuit equation for t > 0 :

$$\left.\frac{di_L}{dt}\right|_{t\to\infty} + \frac{R}{L} i_L(\infty) = \frac{V_2}{L}$$

$$0 \quad + \frac{R}{L} i_L(\infty) = \frac{V_2}{L}$$

$$i_L(\infty) = \frac{V_2}{R}$$

constant input: inductor current

Solve the differential equation.

$$\frac{di_L}{dt} + \frac{R}{L} i_L = \frac{V_2}{L} \qquad t > 0$$

$$i_L(0+) = \frac{V_1}{R}$$

Recall:

$$\frac{dx}{dt} + kx = G$$

$$x(t) = c_1 + c_2 \exp(-kt)$$

The form of the solution is: $i_l(t) = c_1 + c_2 \exp\left(-\frac{t}{L/R}\right)$

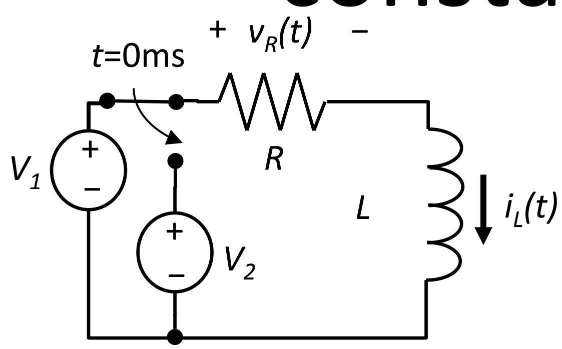
Use our initial and final conditions: $i_L(\infty) = \lim_{t\to\infty}\left[c_1 + c_2 \exp\left(-\frac{t}{RC}\right)\right] = c_1$

$$\therefore c_1 = i_L(\infty) = \frac{V_2}{R}$$

$$i_L(0+) = c_1 + c_2 \exp\left(0\right) = c_1 + c_2$$

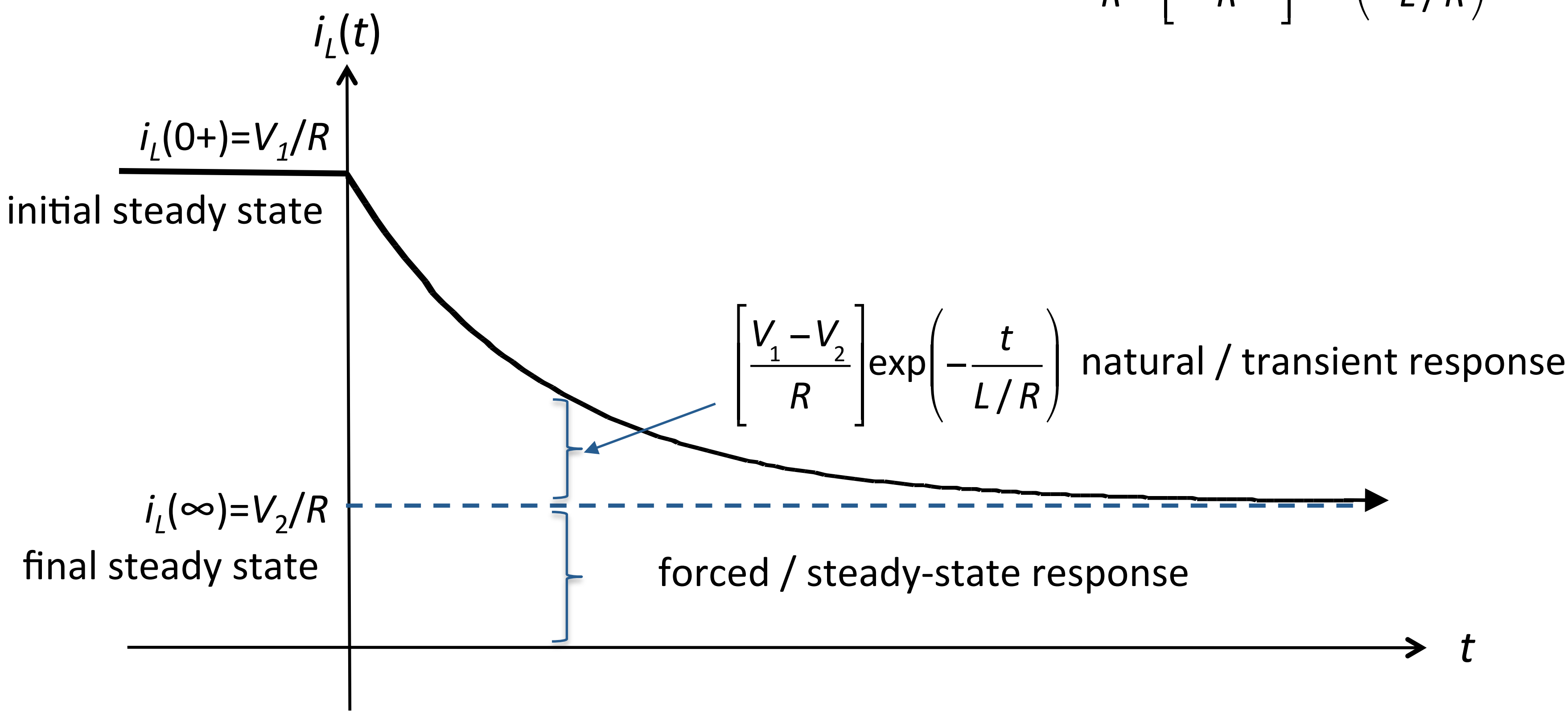
$$\therefore c_2 = i_L(0+) - c_1 = \frac{V_1 - V_2}{R}$$

constant input: inductor current



The response is again a sum of a natural / transient response and a forced / steady-state response.

solution for $i_L(t)$ for $t>0$: $i_L(t) = \frac{V_2}{R} + \left[\frac{V_1 - V_2}{R}\right]\exp\left(-\frac{t}{L/R}\right)$



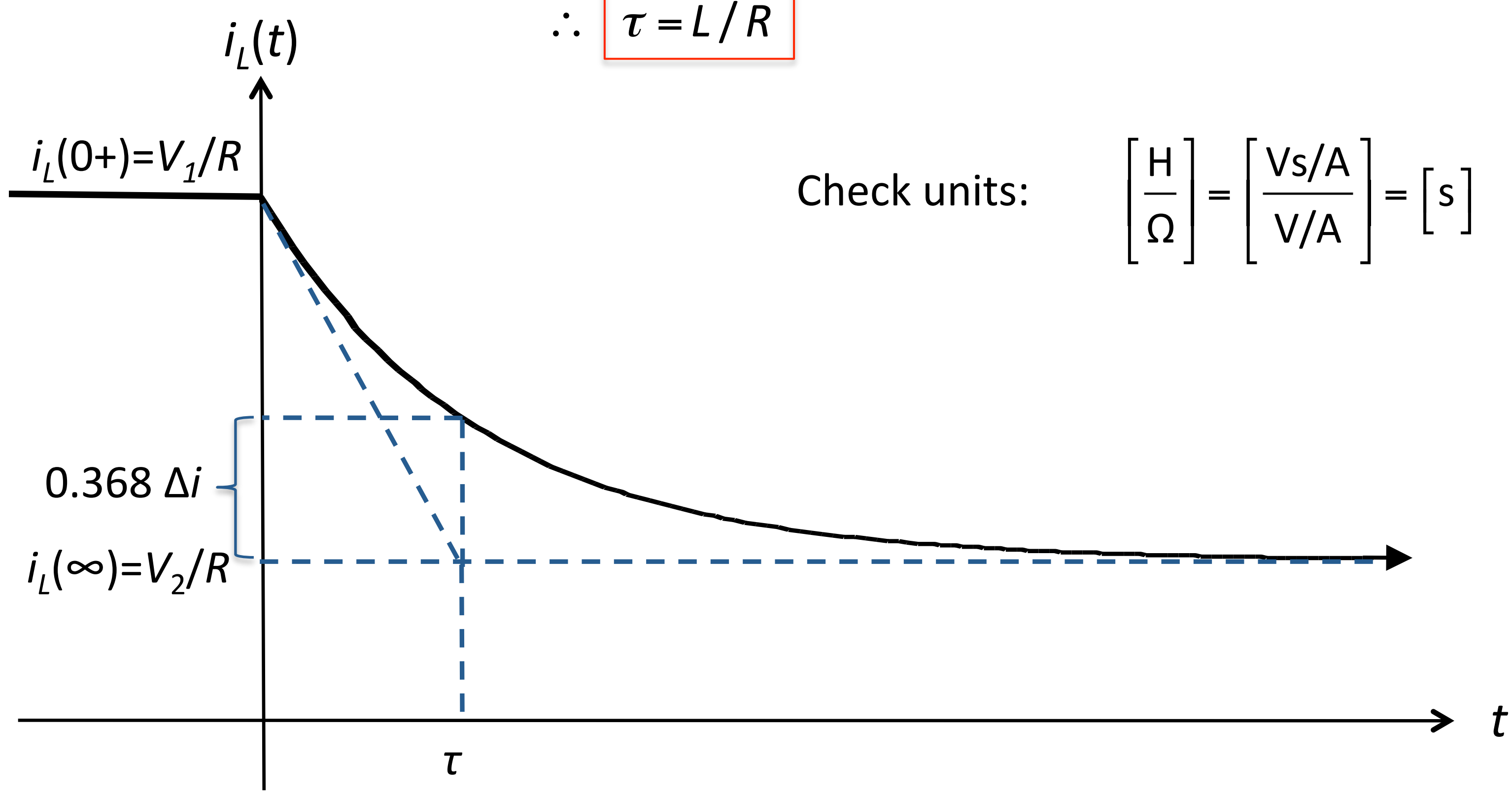
* We assume $V_1 > V_2$ in this graph.

time constant

The time constant τ of an RL circuit is thus easily identified:

$$i_{natural}(t) = \Delta i \exp\left(-\frac{t}{L/R}\right) = \Delta i \exp\left(-\frac{t}{\tau}\right)$$

$$\therefore \quad \boxed{\tau = L/R}$$

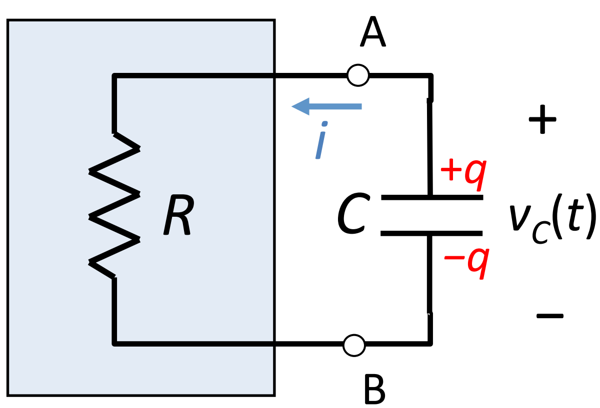


Check units: $\left[\frac{\text{H}}{\Omega}\right] = \left[\frac{\text{Vs/A}}{\text{V/A}}\right] = [\text{s}]$

* We assume $V_1 > V_2$ in this graph.

time constant

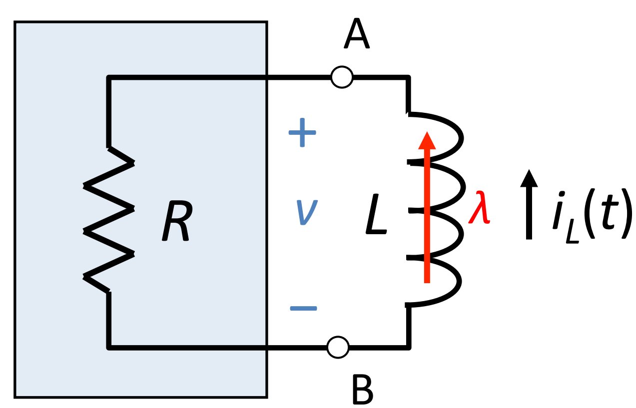
Question: The time constant $\boxed{\tau = RC}$ for an RC circuit and $\boxed{\tau = L/R}$ for an RL circuit, so that a larger *R* lengthens the transient response for an RC circuit, while a larger resistance shortens the transient response of an RL circuit. To see why, consider two circuits converting stored energy to heat.



$$\frac{dq}{dt} = -i = -\frac{v_c}{R}$$

$$\frac{dv_C}{dt} = \frac{-i}{C} = -\frac{v_c}{RC}$$

increasing *R* ***decreases*** the rate at which charge separation decays



$$\frac{d\lambda}{dt} = -v = -i_L R$$

$$\frac{di_L}{dt} = \frac{-v}{L} = -\frac{i_L R}{L} = -\frac{i_L}{L/R}$$

increasing *R* ***increases*** the rate at which flux linkage decays

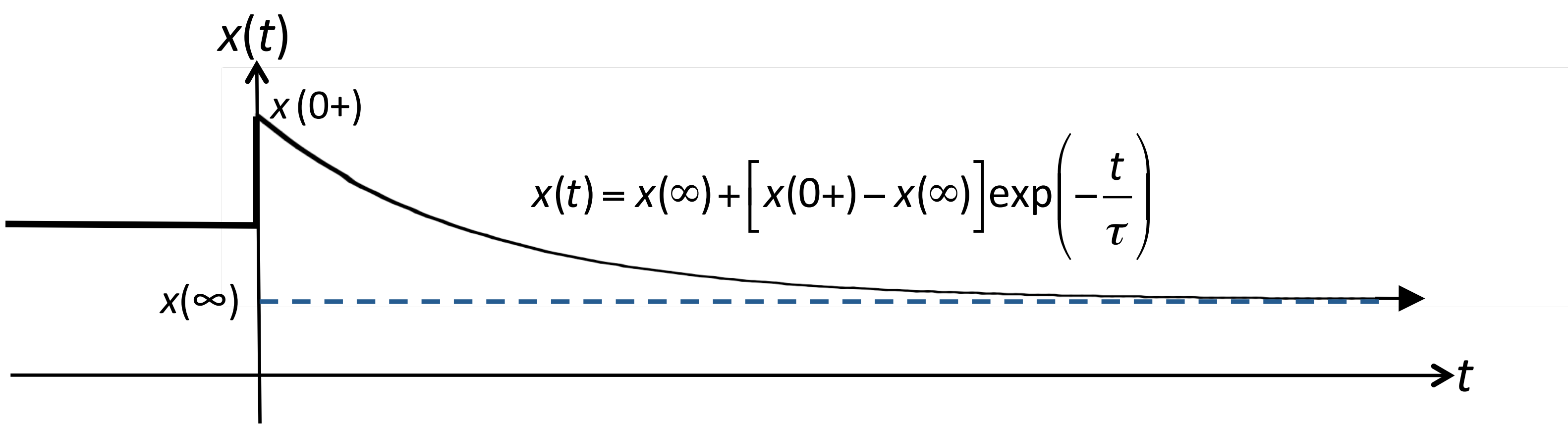
constant input: general procedure

step #1: Find the initial value of the circuit variable of interest, $x(0+)$, using circuit analysis and continuity of *capacitor voltage* or *inductor current*.

step #2: Find the final value of the variable of interest, $x(\infty)$, using dc steady state models for the capacitor or inductor.

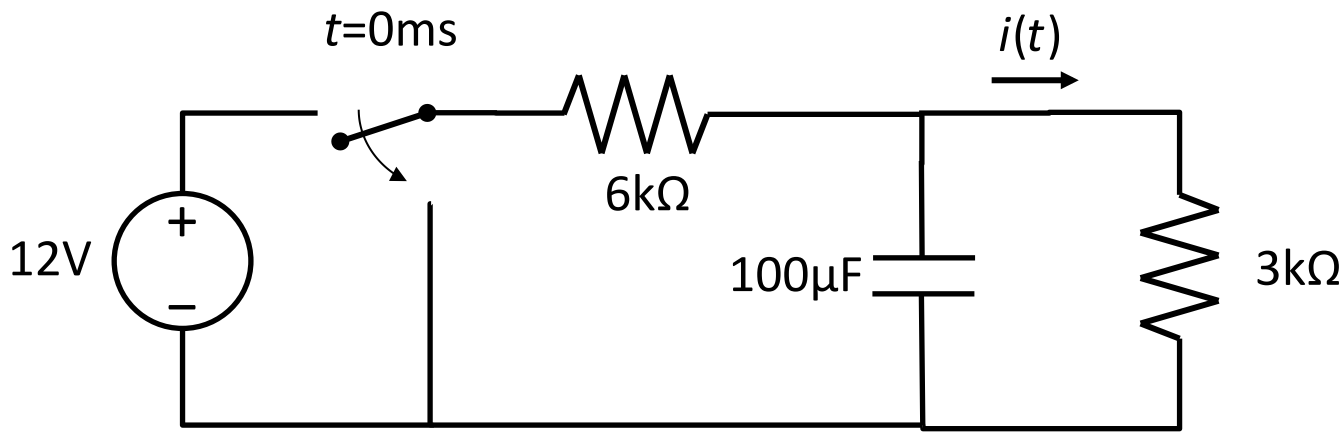
step #3: Find the Thévenin equivalent resistance R_T as seen from the terminals of the capacitor or inductor. The time constant $\tau = R_T C$ or $\tau = L/R_T$.

step #4: Construct the solution.



example 1

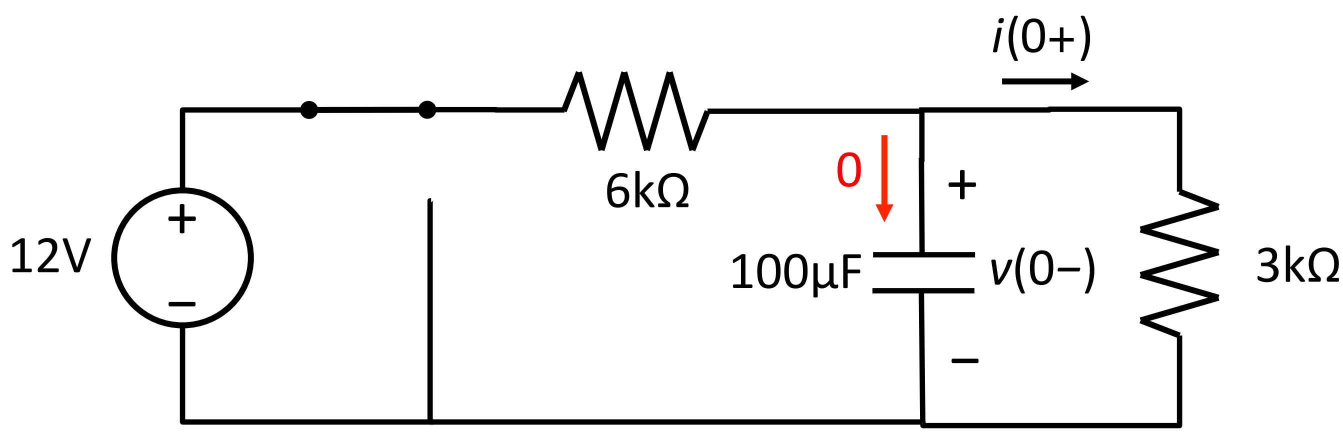
The circuit is in dc steady state at t=0−. At t = 0, the switch changes position. Find $i(t)$ for t > 0.



example 1

step #1: At $t < 0$, the circuit is in dc steady state, so the capacitor acts as an open and the capacitor voltage is found by voltage division.

$$v(0-) = 12\text{V}\,\frac{3\text{k}\Omega}{3\text{k}\Omega + 6\text{k}\Omega} = 4\text{V}$$

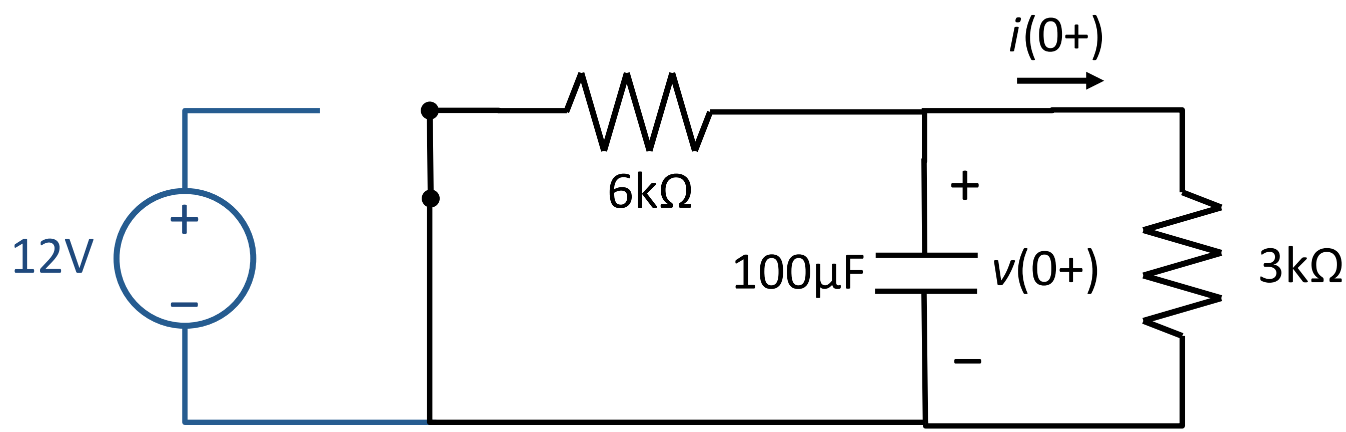


example 1

step #1: At t = 0+ the switch has just moved, and we find the current i(0+) using capacitor voltage continuity.

$$\text{continuity:} \quad v(0+) = v(0-) = 4\text{V}$$

$$\text{Ohm's Law:} \quad i(0+) = \frac{v(0+)}{3\text{k}\Omega} = 1.33\text{mA}$$



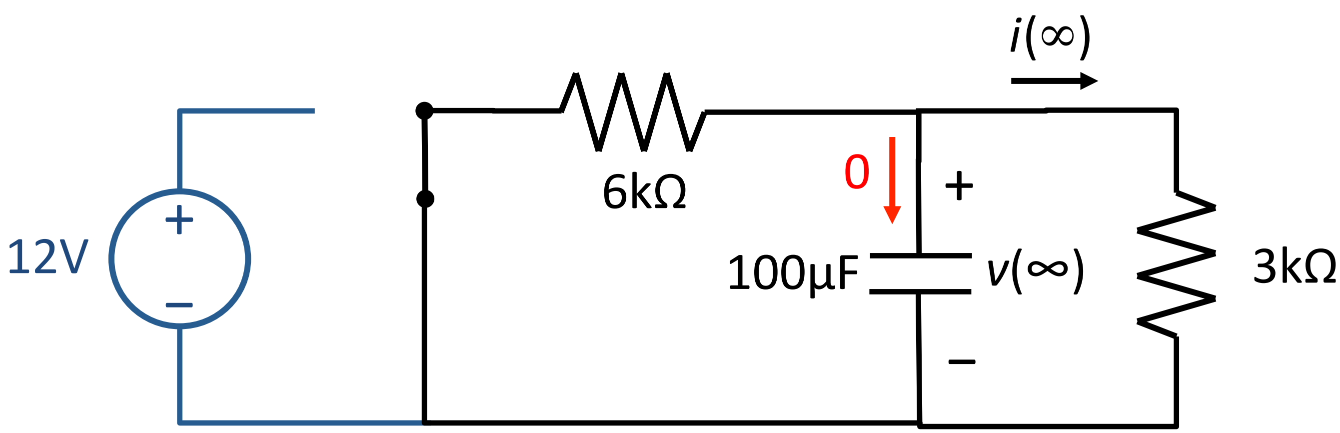
example 1

step #2: As *t* -> ∞, the circuit again reaches dc steady state. We find the current *i*(∞).

$$\text{node equation:} \quad 0 = \frac{v(\infty)}{3\text{k}\Omega} + 0 + \frac{v(\infty)}{6\text{k}\Omega}$$

$$v(\infty) = 0$$

$$i(\infty) = \frac{v(\infty)}{3\text{k}\Omega} = 0$$

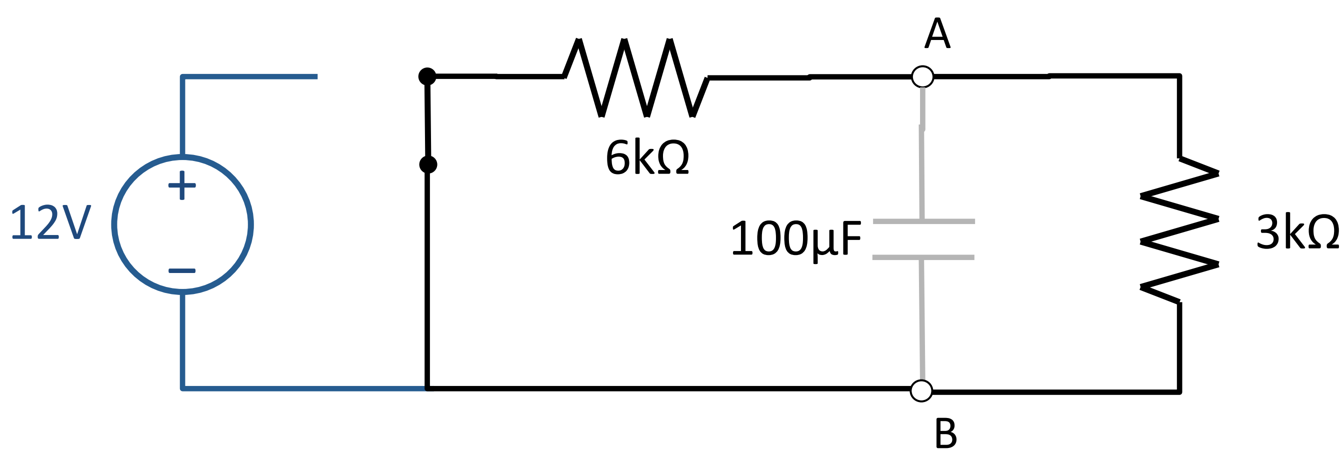


example 1

step #3: Find R_T as seen from the capacitor terminals (identified A and B). Find the time constant $\tau = R_T C$.

$$R_T = 3\text{k}\Omega \,||\, 6\text{k}\Omega = \frac{3\cdot 6}{3+6}\text{k}\Omega = 2\text{k}\Omega$$

$$\tau = R_T C = 2\text{k}\Omega \cdot 100\,\mu\text{F}$$
$$= 200\text{ms}$$



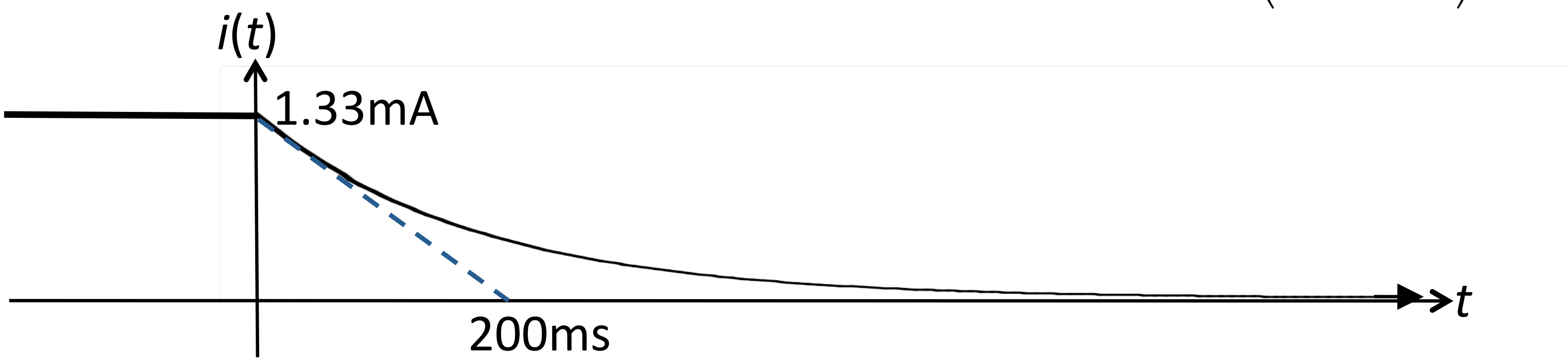
example 1

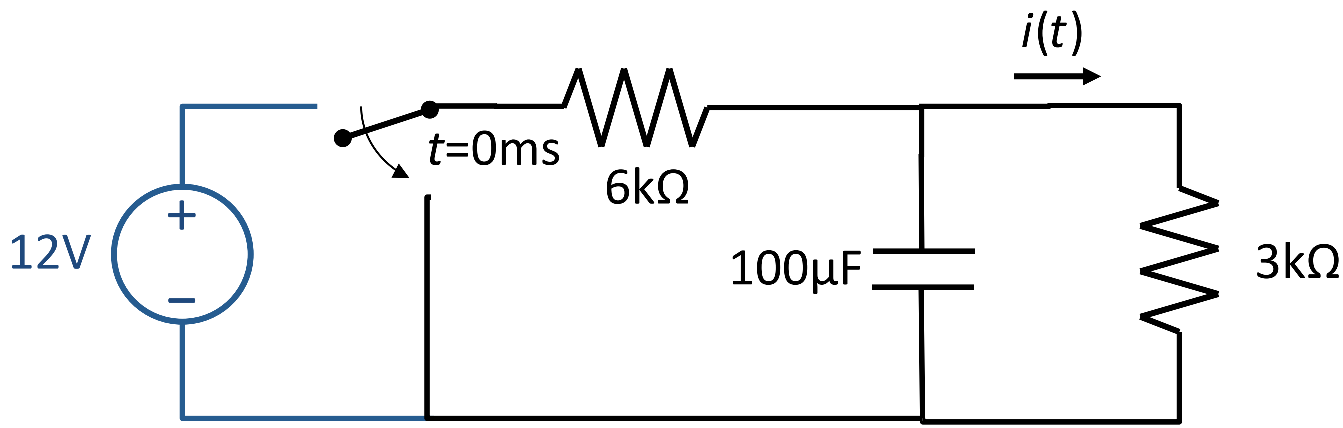
step #4: Assemble the solution.

$$i(t) = i(\infty) + \left[i(0+) - i(\infty)\right]\exp\left(-\frac{t}{\tau}\right)$$

$i(\infty) = 0\text{mA} \quad i(0+) = 1.333\text{mA} \quad \tau = 200\text{ms}$

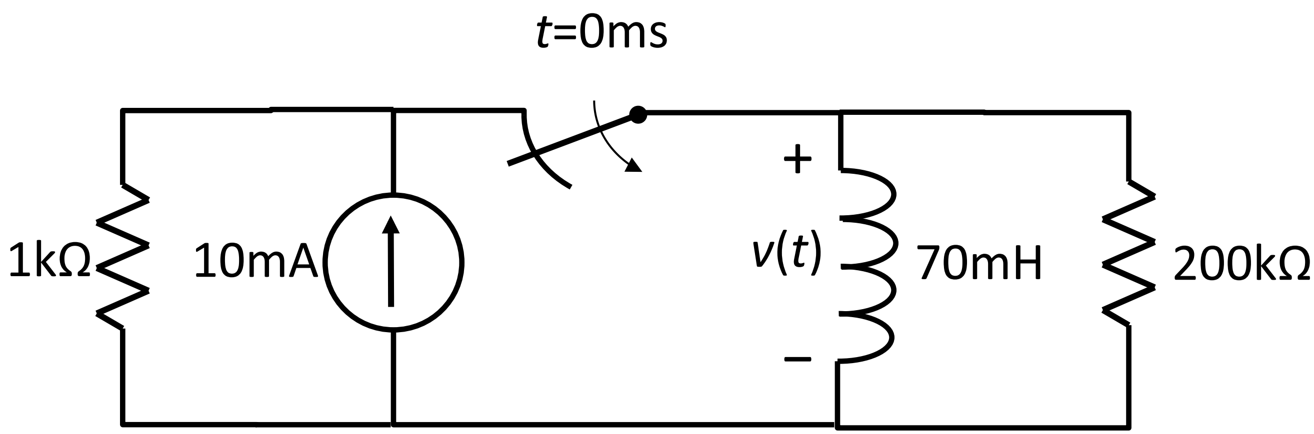
$$i(t) = 1.333\text{mA}\exp\left(-\frac{t}{200\text{ms}}\right)$$





example 2

The circuit is in dc steady state at t=0−. At t = 0, the switch changes position. Find $v(t)$ for $t > 0$.

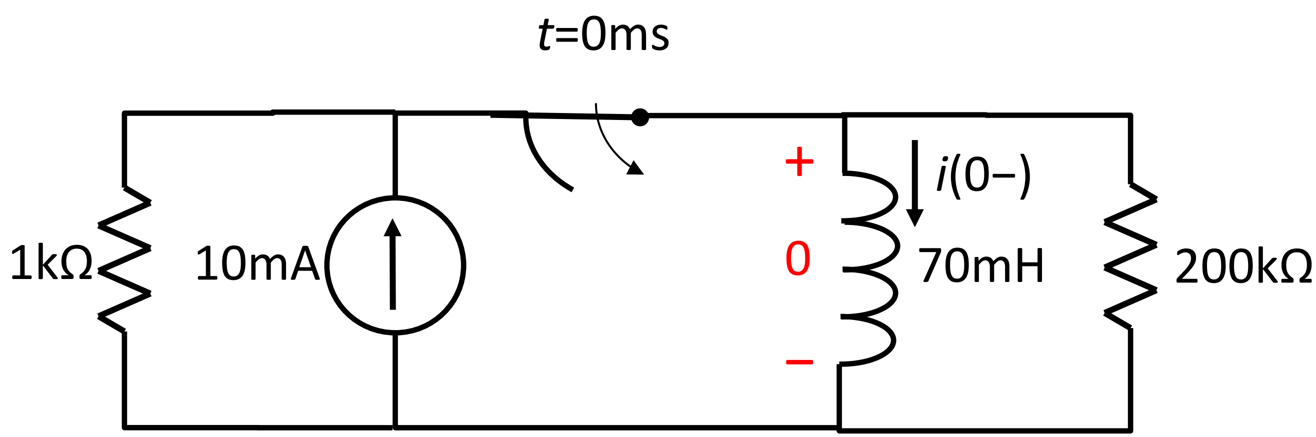


example 2

step #1: At $t < 0$, the circuit is in dc steady state, so the inductor acts as a short.

$$0 = \frac{0}{1\text{k}\Omega} - 10\text{mA} + i(0-) + \frac{0}{200\text{k}\Omega}$$

$$i(0-) = 10\text{mA}$$



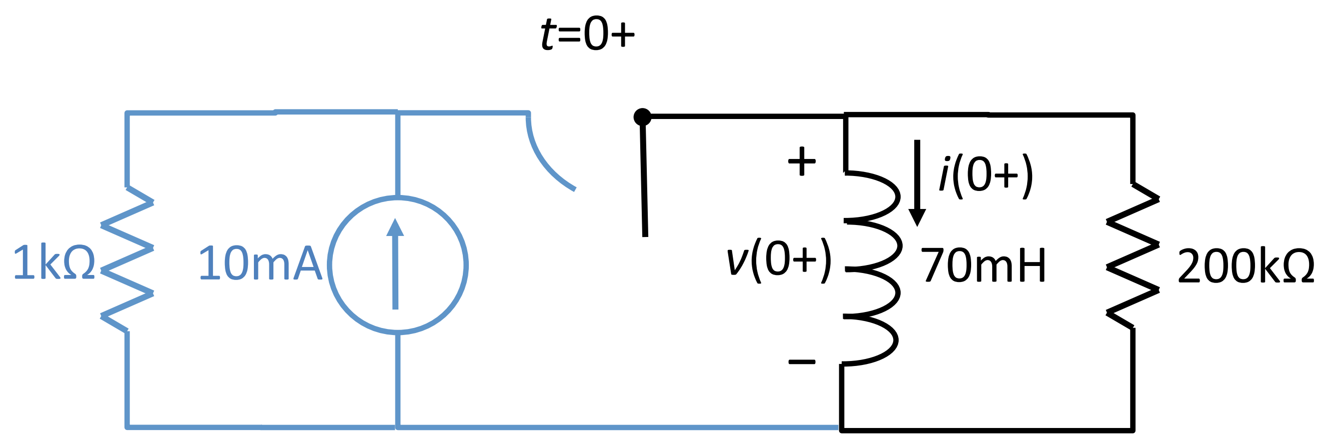
example 2

step #1: At t = 0+, the switch moves, and we find v(0+) using inductor current continuity (the inductor supplies current to the 200kΩ resistor).

continuity: $i(0+) = i(0-) = 10\text{mA}$

node eqn.: $0 = i(0+) + \dfrac{v(0+)}{200\text{k}\Omega}$

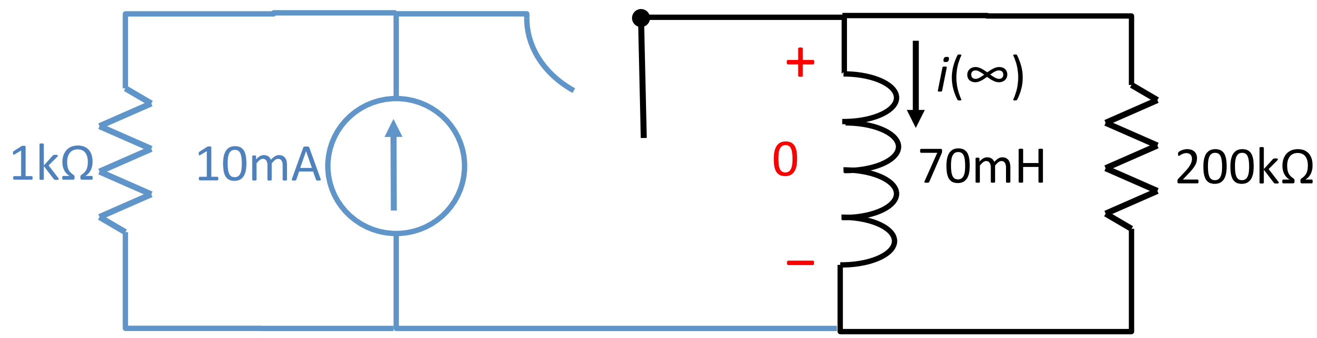
$$v(0+) = -i(0+)\cdot 200\text{k}\Omega = -2\text{kV}$$



example 2

step #2: As $t \to \infty$, the circuit approaches dc steady state so the inductor acts as a short.

$$v(\infty) = L\left.\frac{di}{dt}\right|_{t\to\infty} = 0$$

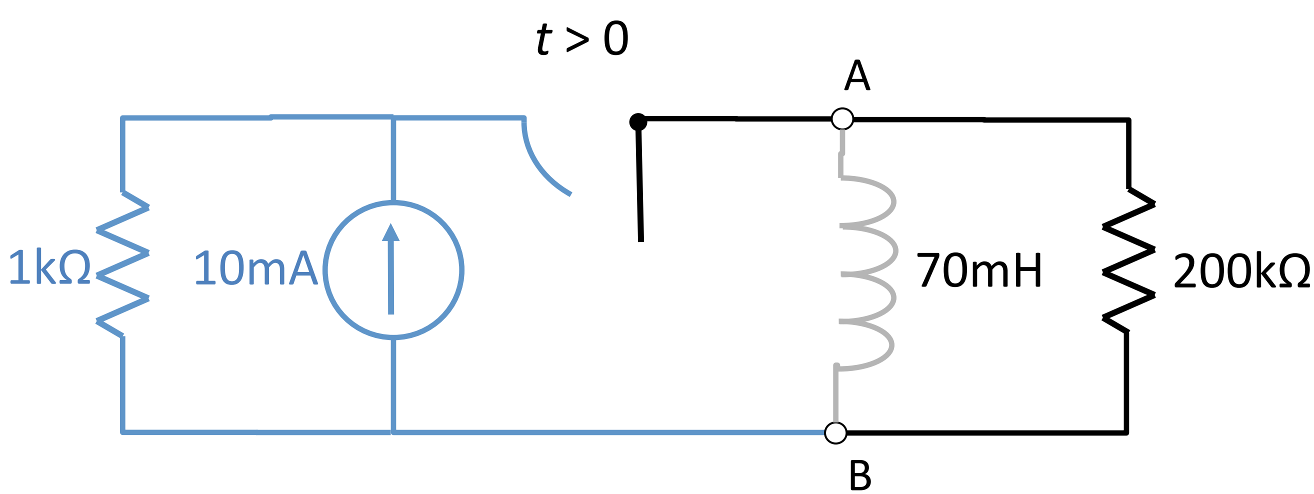


example 2

step #3: Find R_T as seen from the inductor terminals (identified A and B). Find the time constant $\tau = L/R_T$.

$$R_T = 200\text{k}\Omega$$

$$\tau = \frac{L}{R_T} = \frac{70\text{mH}}{200\text{k}\Omega} = 0.35\mu\text{s} = 350\text{ns}$$



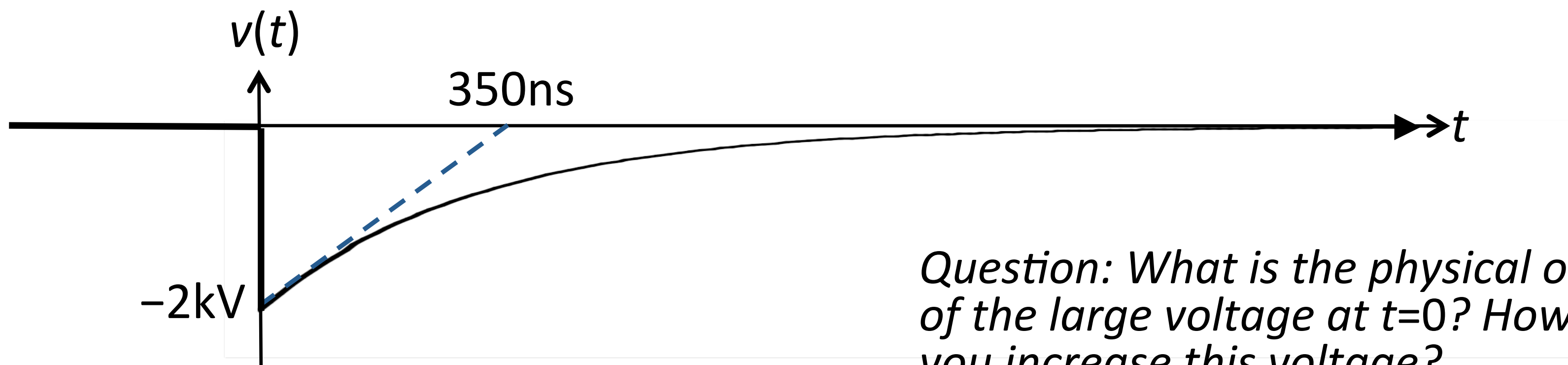
example 2

step #4: Assemble the solution.

$$v(t) = v(\infty) + \left[v(0+) - v(\infty)\right]\exp\left(-\frac{t}{\tau}\right)$$

$v(\infty) = 0\text{V} \quad v(0+) = -2\text{kV} \quad \tau = 350\text{ns}$

$$v(t) = -2\text{kV}\exp\left(-\frac{t}{350\text{ns}}\right)$$



Question: What is the physical origin of the large voltage at t=0? How can you increase this voltage?