Today's Outline

7. First Order Circuits

- unit step response

sequential switching: general procedure

For each time interval of constant input:

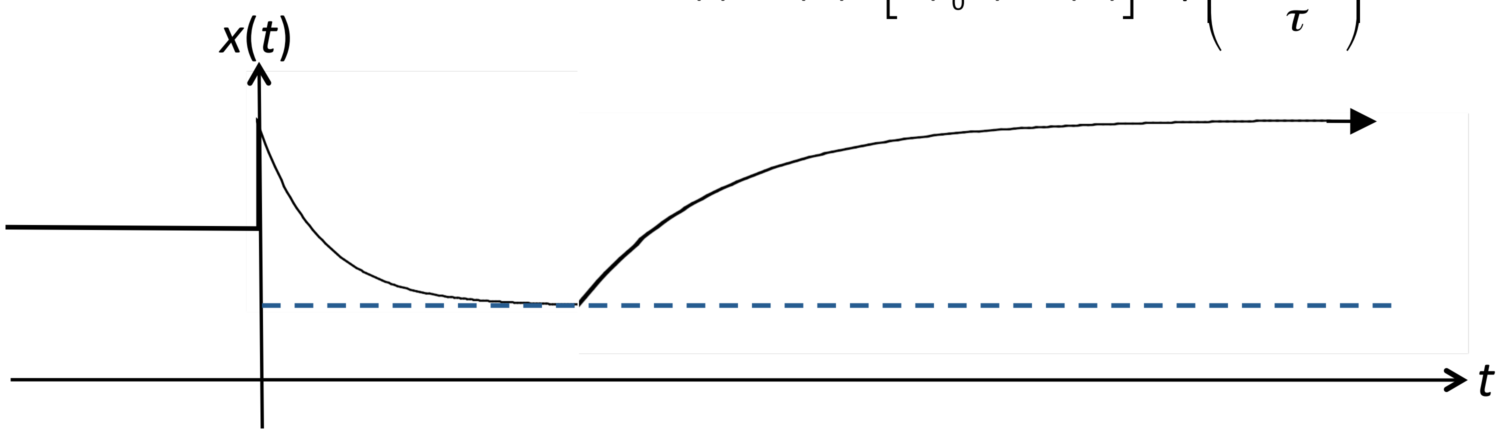
step #1: Find the initial value of the circuit variable of interest, $x(t_0+)$, using circuit analysis and continuity of capacitor voltage or inductor current.

step #2: Find the anticipated final value of the variable of interest, $x(\infty)$, using dc steady state models for the capacitor or inductor.

step #3: Find the Thévenin equivalent resistance R_T as seen from the terminals of the capacitor or inductor. The time constant $\tau = R_T C$ or $\tau = L/R_T$.

step #4: Construct the solution.

$$x(t) = x(\infty) + \left[x(t_0+) - x(\infty)\right]\exp\left(-\frac{t-t_0}{\tau}\right)$$

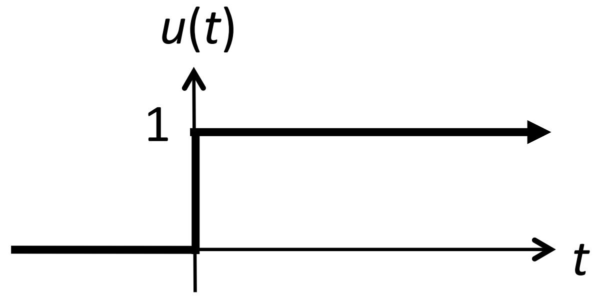


There is an alternative approach using the "unit-step" function.

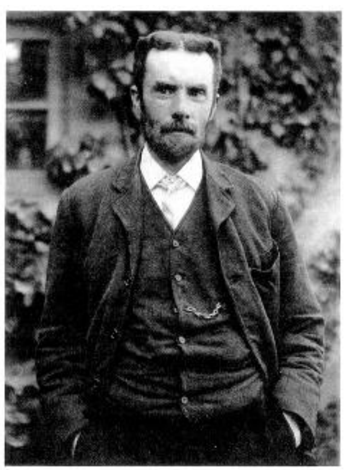
unit step function

The ***unit step function*** $u(t)$ is defined to be:

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$



- $u(t)$ is discontinuous at $t = 0$
- the value of $u(0)$ will not be important in this class

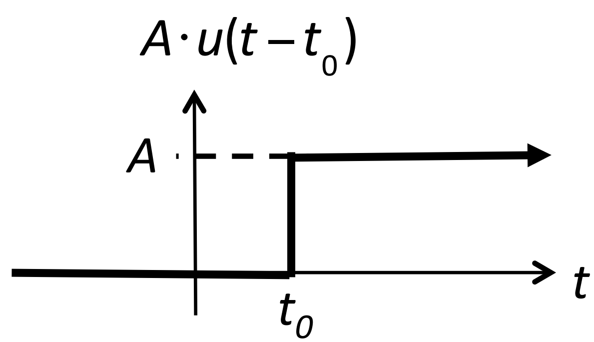


Oliver Heaviside

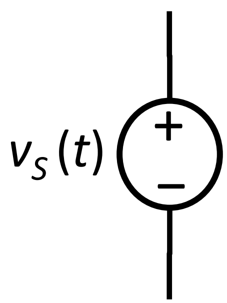
(1850-1924)

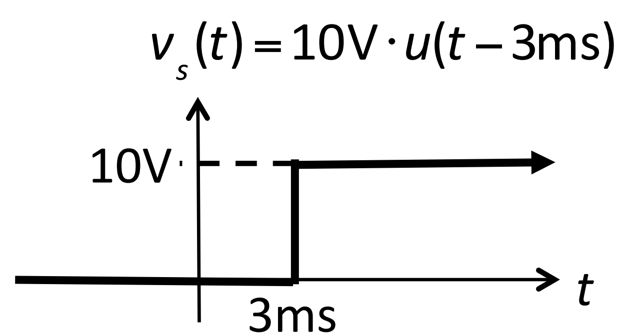
general step function

The unit step function can be used to describe a general step function in voltage or current of amplitude A at a time t_0.



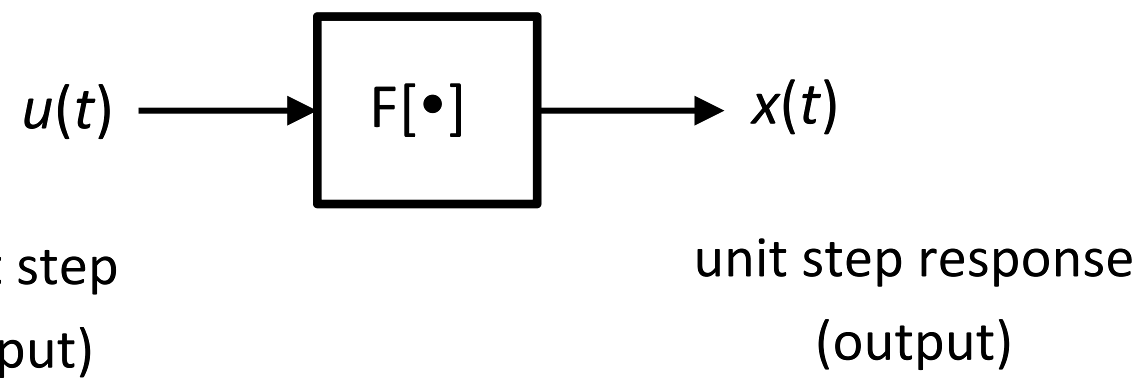
The unit step function is very useful in expressing piece-wise constant signals. For example:





unit step response

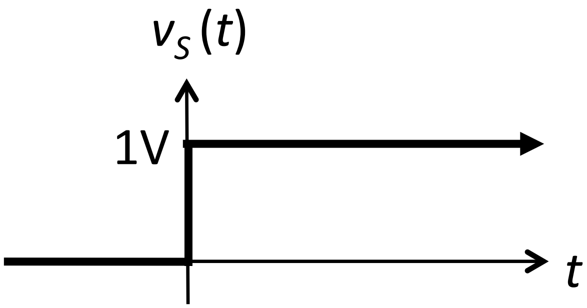
The ***unit step response*** $x(t)$=F[$u(t)$] is the circuit variable response to a unit step voltage source or current source. F[•] is the operator mapping the source function to the circuit variable.



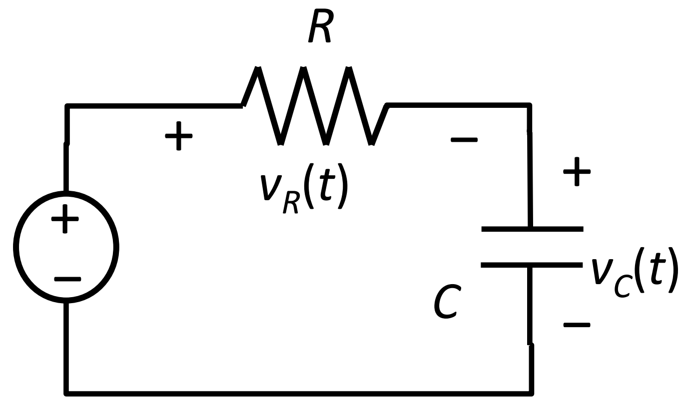
– $x(t)$ can be found by solving the differential equation with a unit step source $u(t)$
– *more sophisticated mathematical methods are required to find explicit expressions for the operator* F[•].

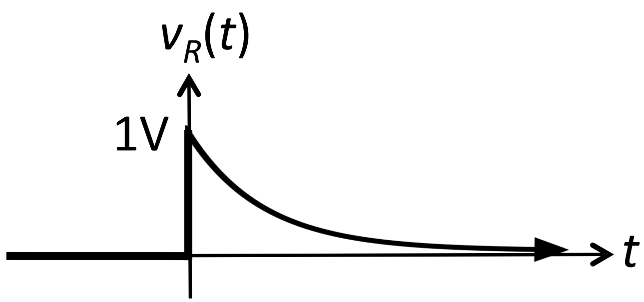
unit step response - example

Consider the example of an RC circuit, solved earlier in this section:



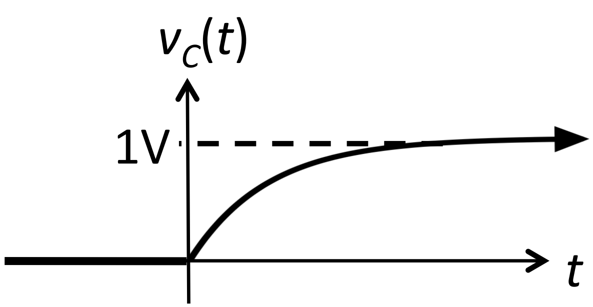
$$v_s(t) = 1\text{V} \cdot u(t)$$





Unit step response of resistor voltage:

$$v_R(t) = 1\text{V} \cdot \exp(-t/\tau) \cdot u(t)$$

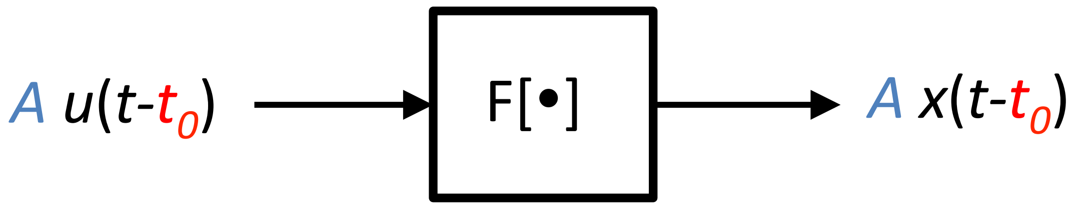


Unit step response of capacitor voltage:

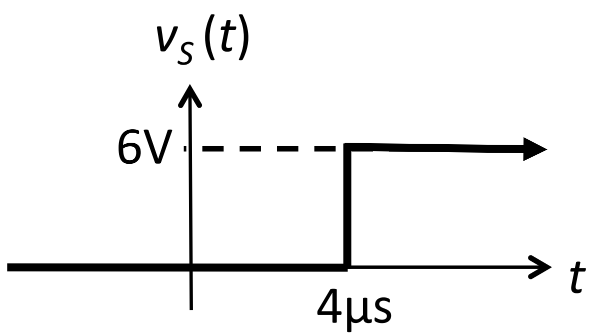
$$v_C(t) = 1\text{V} \cdot \left[1 - \exp(-t/\tau)\right] \cdot u(t)$$

general step response

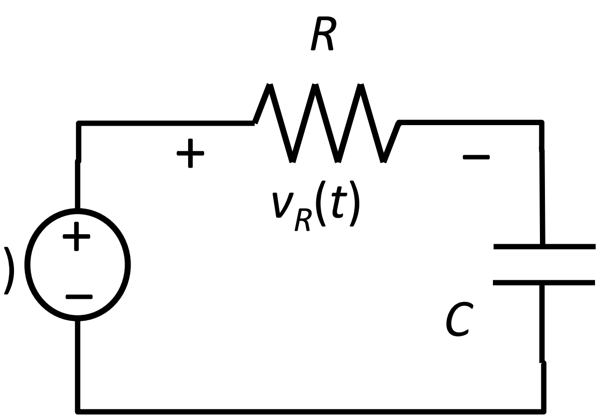
The *general* step response is simply expressed in terms of the unit step response for a ***linear*** and ***time invariant*** circuit.

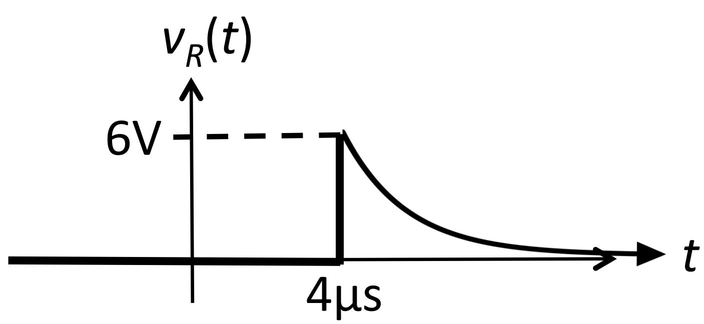


Consider again an RC circuit example:



$$v_s(t) = 6\text{V} \cdot u(t - 4\mu\text{s})$$





Response of resistor voltage:

$$v_R(t) = 6\text{V} \cdot \exp(-(t - 4\mu\text{s}) / \tau) \cdot u(t - 4\mu\text{s})$$

sequential step response

The response to sequential steps is simply expressed in terms of the unit step response for a ***linear*** and ***time invariant*** circuit.

$$A_1\, u(t-t_1) + A_2\, u(t-t_2) + A_3\, u(t-t_3) + \ldots \longrightarrow \boxed{F[\bullet]} \longrightarrow A_1\, x(t-t_1) + A_2\, x(t-t_2) + A_3\, x(t-t_3) + \ldots$$

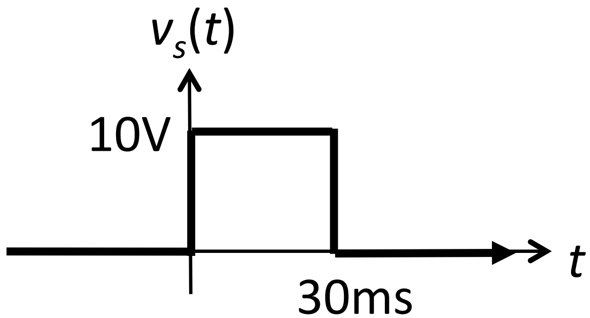
- ***linearity*** ensures that the principle of superposition can be applied
- ***time invariance*** ensures the step response has the same form, independent of the time at which the step occurs

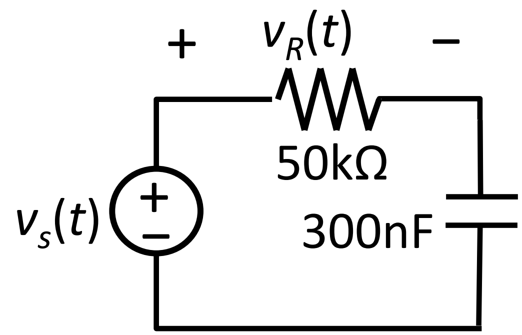
$$F[\, A_1\, u(t-t_1) + A_2\, u(t-t_2)\,] = A_1\, x(t-t_1) + A_2\, x(t-t_2)$$

McGill

example

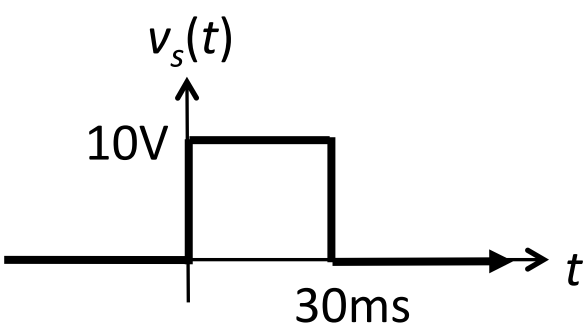
Reconsider finding $v_R(t)$ for sequential switching in an RC circuit.

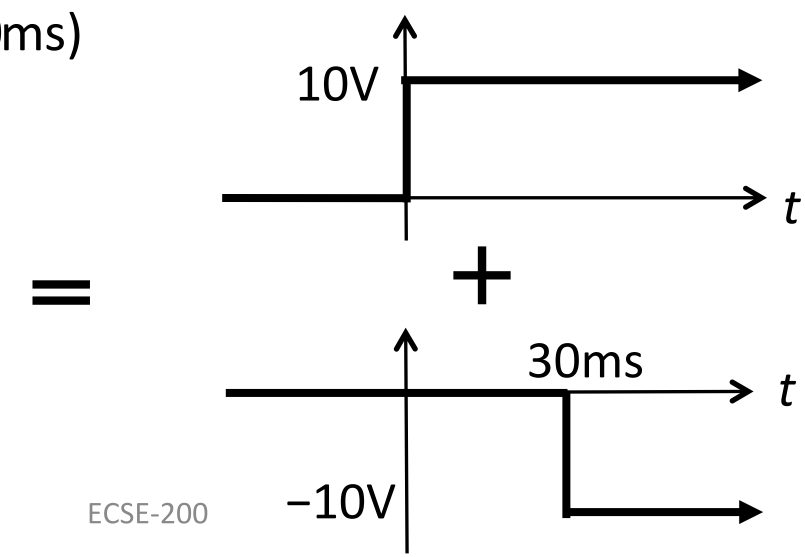




Step #1: We first resolve the input into a sum of steps using $u(t)$.

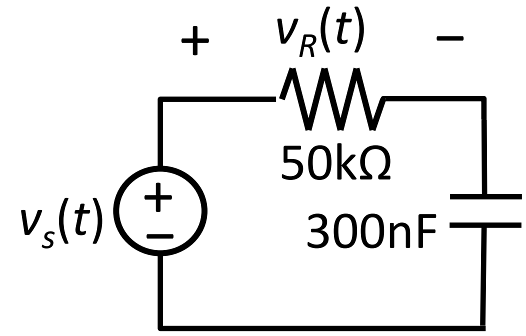
$$v_s(t) = 10\text{V} \cdot u(t) - 10\text{V} \cdot u(t - 30\text{ms})$$

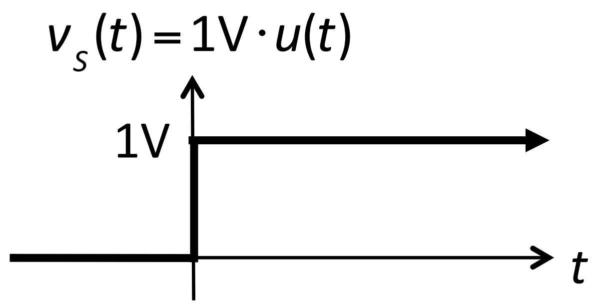


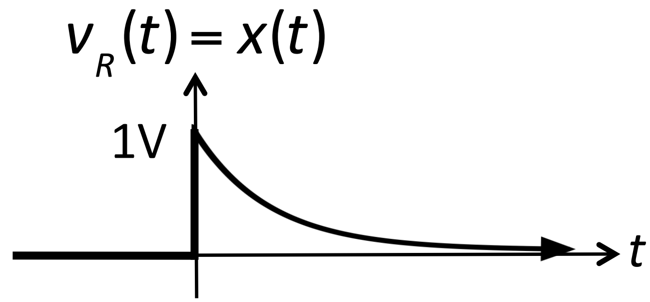


example

Step #2: We find the step response $x(t)$ of the desired variable ($v_R(t)$) to the unit step input $u(t)$.





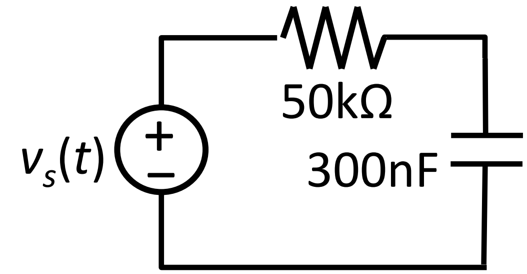


$v_R(0+) = 1\text{V}$ $\quad$ $v_R(\infty) = 0\text{V}$ $\quad$ $\tau = 15$ ms

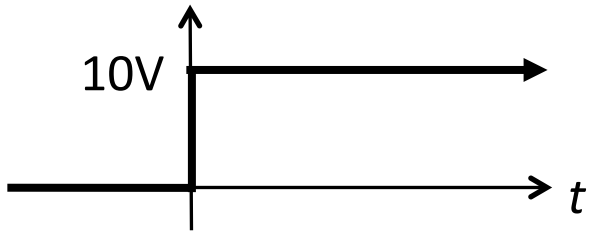
$$v_R(t) = x(t) = 1\text{V} \cdot \exp(-t / 15\text{ms}) \cdot u(t)$$

example

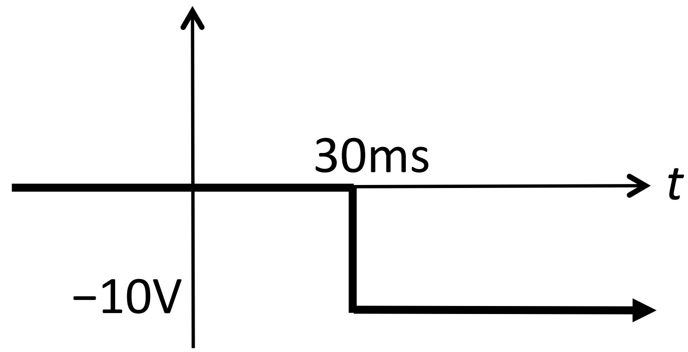
Step #3: Add the response to each step.



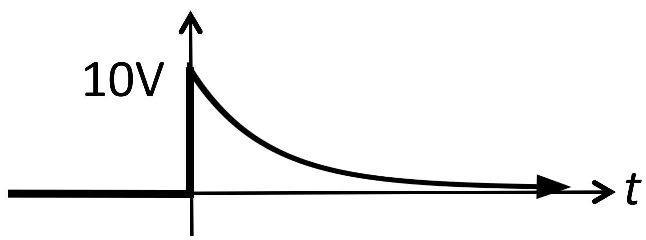
$$v_{s,1}(t) = 10\text{V} \cdot u(t)$$



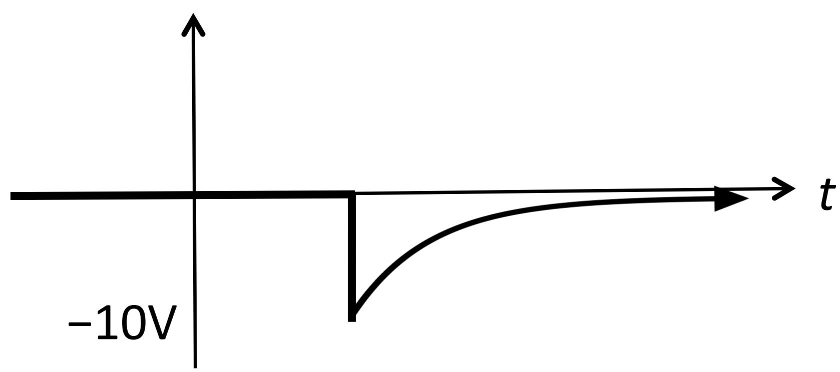
$$v_{s,2}(t) = -10\text{V} \cdot u(t - 30\text{ms})$$



$$v_{R,1}(t) = 10\text{V} \cdot \exp(-t/15\text{ms}) \cdot u(t)$$

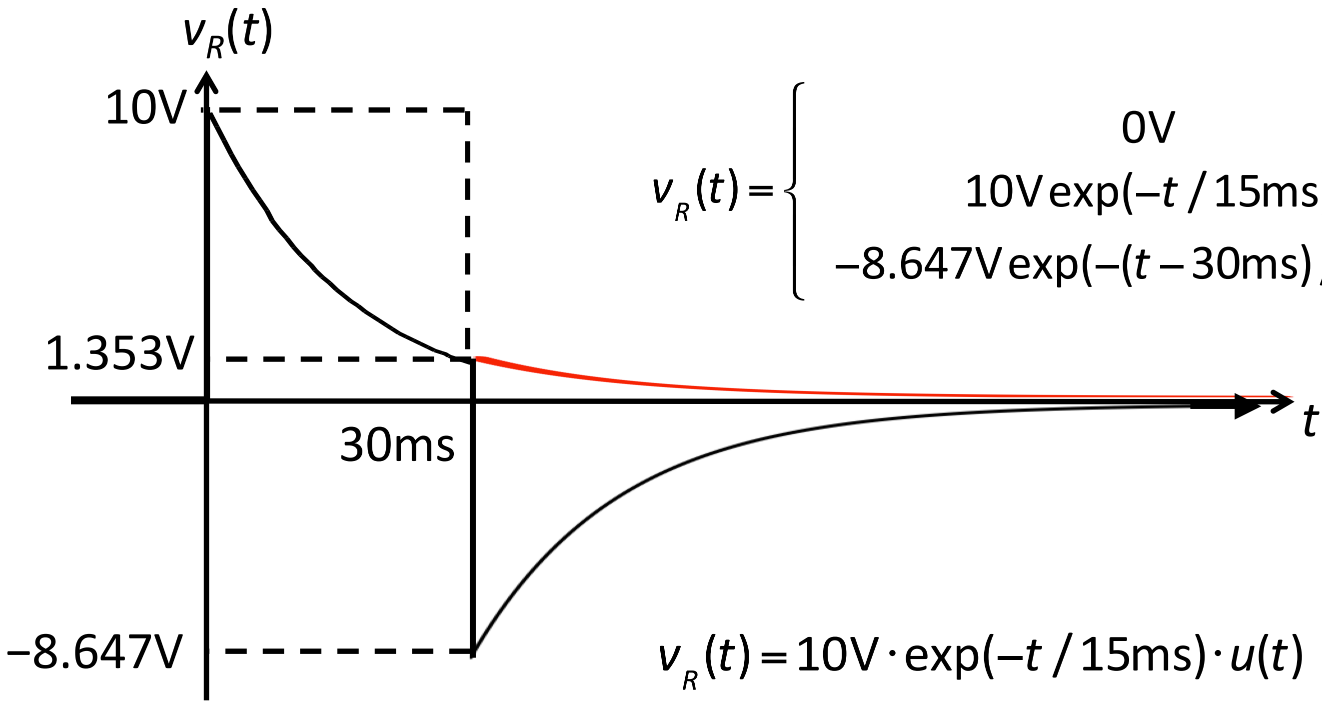


$$v_{R,2}(t) = -10\text{V} \cdot \exp(-(t - 30\text{ms})/15\text{ms}) \cdot u(t - 30\text{ms})$$



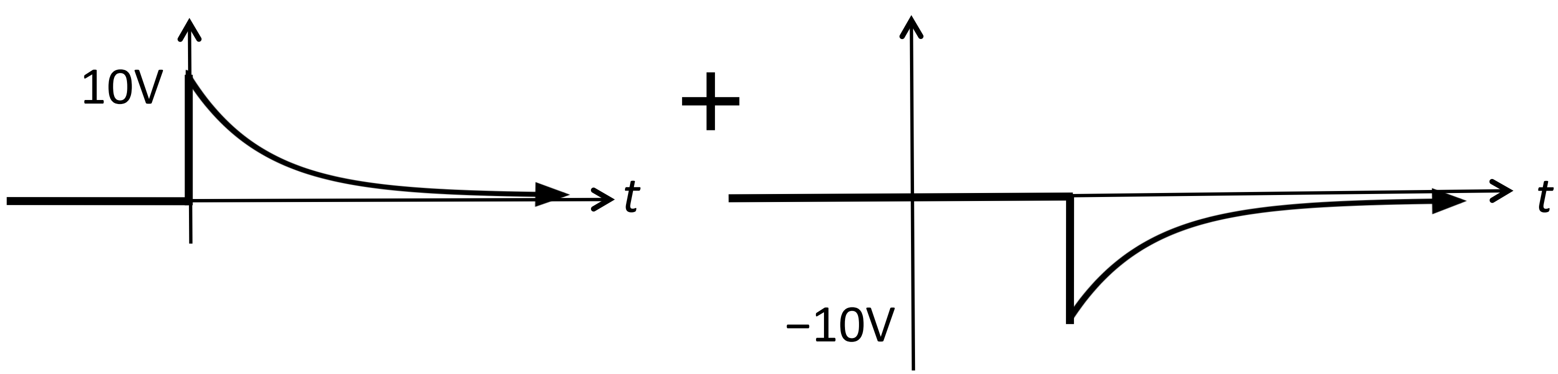
$$v_R(t) = v_{R,1}(t) + v_{R,2}(t)$$

example : compare solutions



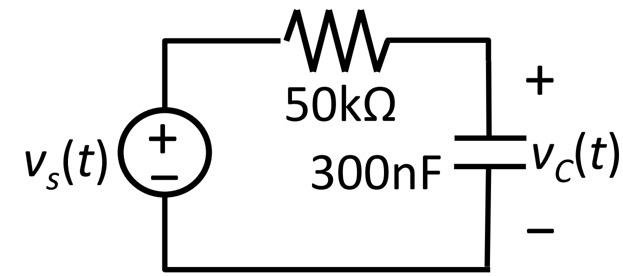
$$v_R(t) = \begin{cases} 0\text{V} & t < 0 \\ 10\text{V}\exp(-t/15\text{ms}) & 0 < t < 30\text{ms} \\ -8.647\text{V}\exp(-(t-30\text{ms})/15\text{ms}) & 30\text{ms} < t \end{cases}$$

$$v_R(t) = 10\text{V}\cdot\exp(-t/15\text{ms})\cdot u(t)$$
$$-10\text{V}\cdot\exp(-(t-30\text{ms})/15\text{ms})\cdot u(t-30\text{ms})$$



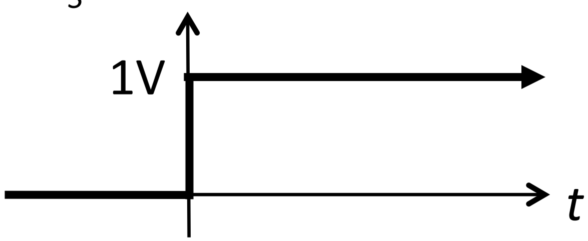
example

Consider now $v_C(t)$ for the same RC circuit.

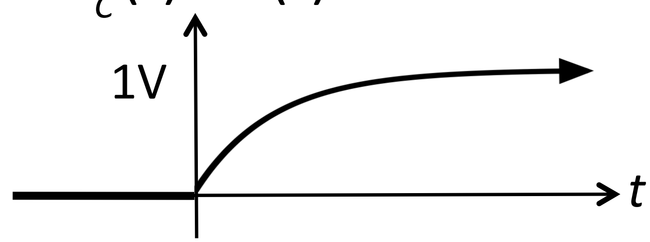


Step #2: We find the step response $x(t)$ of the desired variable ($v_C(t)$) to the unit step input $u(t)$.

$$v_s(t) = 1\text{V} \cdot u(t)$$



$$v_C(t) = x(t)$$

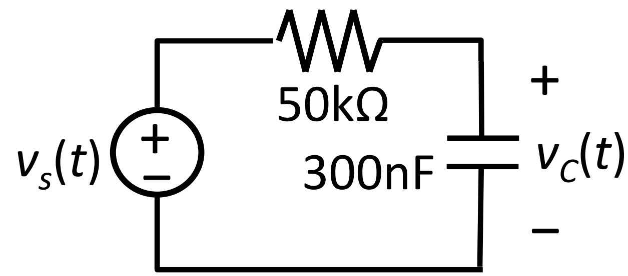


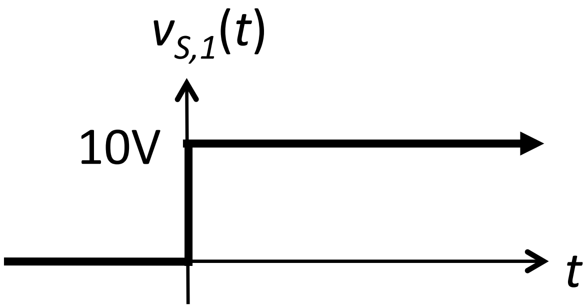
$v_C(0+) = 0\text{V}$ $\quad$ $v_C(\infty) = 1\text{V}$ $\quad$ $\tau = 15\text{ ms}$

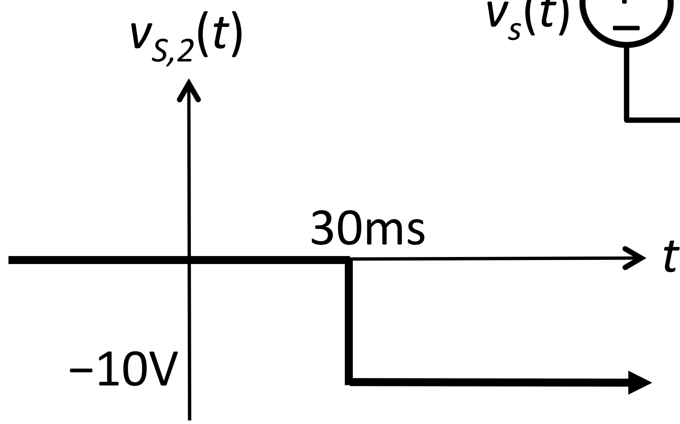
$$v_C(t) = x(t) = 1\text{V} \cdot [1 - \exp(-t/15\text{ms})] \cdot u(t)$$

example

Step #3: Add the response to each step.

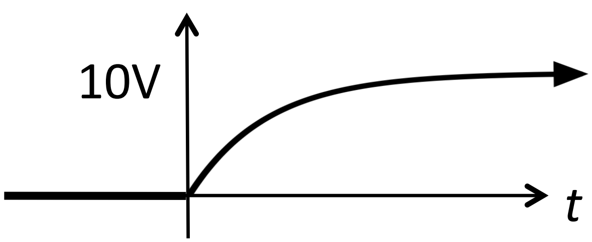


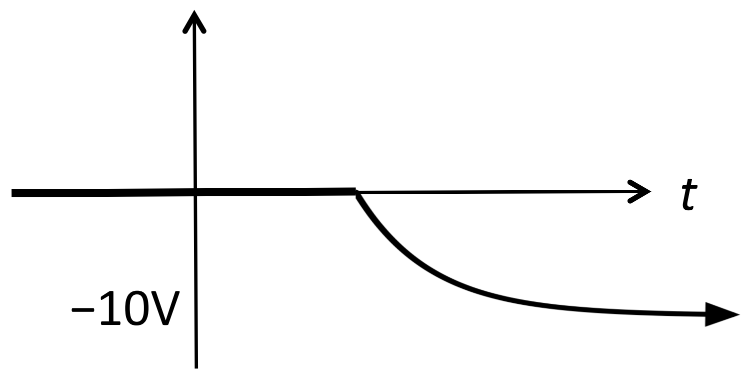




$$v_{C,1}(t) = 10\text{V} \cdot \left[1 - \exp(-t/15\text{ms})\right] \cdot u(t)$$

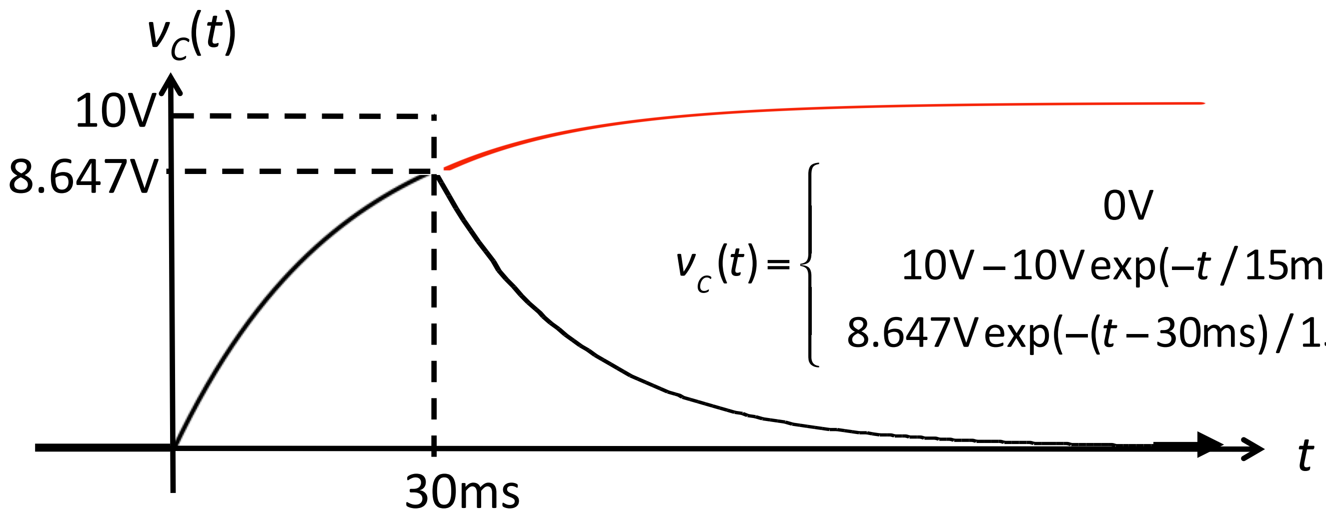
$$v_{C,2}(t) = -10\text{V} \cdot \left[1 - \exp(-(t - 30\text{ms})/15\text{ms})\right] \cdot u(t - 30\text{ms})$$





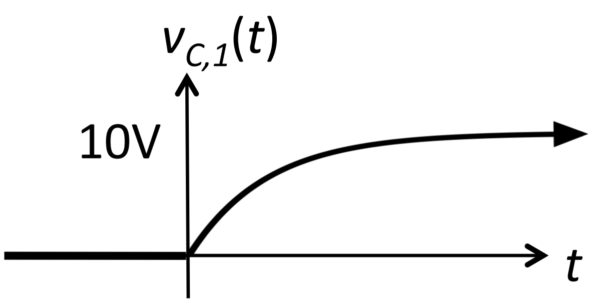
$$v_C(t) = v_{C,1}(t) + v_{C,2}(t)$$

example: compare solutions



$$v_c(t) = \begin{cases} 0\text{V} & t < 0 \\ 10\text{V} - 10\text{V}\exp(-t/15\text{ms}) & 0 < t < 30\text{ms} \\ 8.647\text{V}\exp(-(t-30\text{ms})/15\text{ms}) & 30\text{ms} < t \end{cases}$$

$$v_c(t) = 10\text{V}\cdot\left[1-\exp(-t/15\text{ms})\right]\cdot u(t)$$
$$-10\text{V}\cdot\left[1-\exp(-(t-30\text{ms})/15\text{ms})\right]\cdot u(t-30\text{ms})$$



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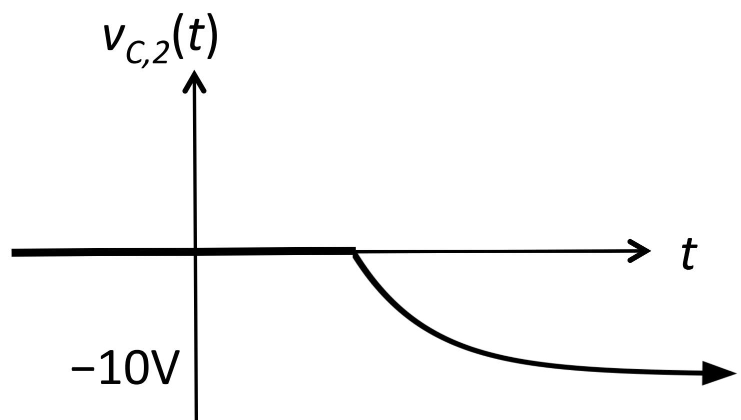

sequential switching: another view

For a piece-wise constant input :

step #1: Resolve the input into a summation of appropriately scaled and delayed unit step functions, $A_1u(t-t_1) + A_2u(t-t_2) + \ldots$

step #2: Find the unit step response $x(t)$ of the circuit variable to a unit step function input $u(t)$, using the technique for finding the constant input response of an RC or RL circuit.

step #3: Add the response to each step function at the input.

$$A_1 \cdot x(t-t_1) + A_2 \cdot x(t-t_2) + \ldots$$