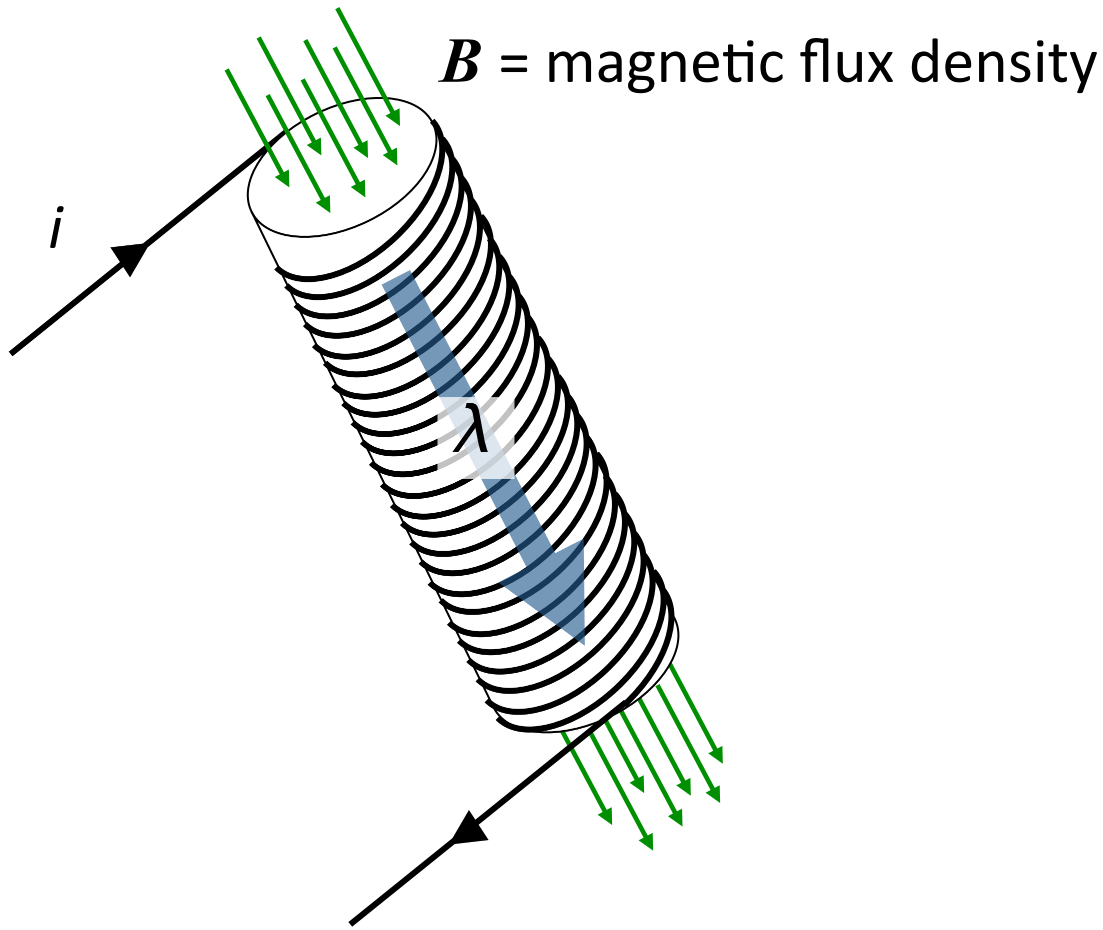
Today’s Outline

6. Energy Storage Elements

- the Inductor
- Coupled Inductors

the inductor

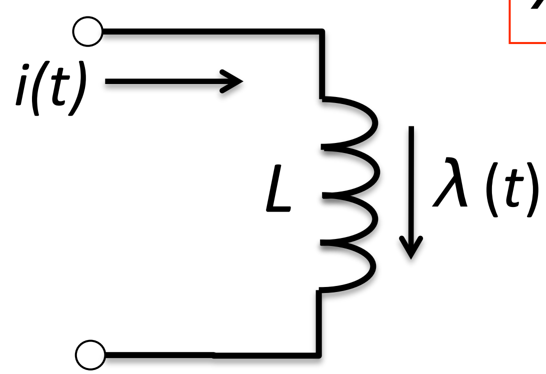
Ideal inductor: physically consists of a coiled conductor, with a magnetic flux, called the **flux linkage** λ, threading the coil

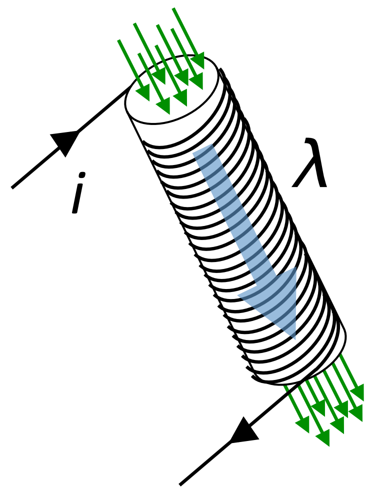


the inductor

Ideal Inductor: the **flux linkage** λ through an ideal inductor is proportional to the current i through the inductor

$$\boxed{\lambda = L\, i}$$



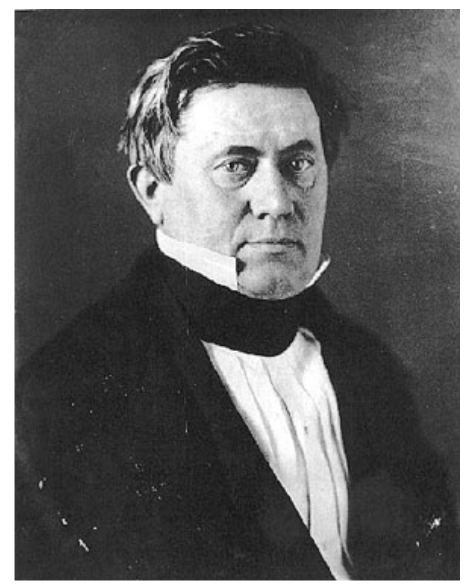


– the inductor is a *passive* circuit element

– the constant of proportionality between flux and current is the **inductance**, given the symbol L

– SI unit of inductance is the Henry (abbreviated H)

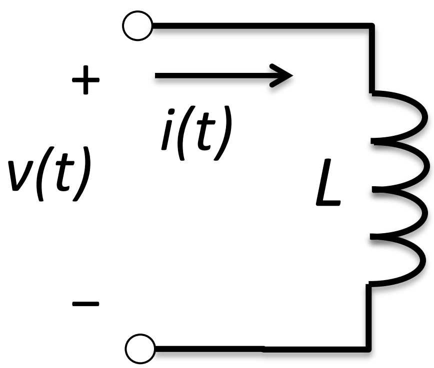
$$1\ \text{H} = 1\ \text{Wb} / \text{A} = 1\ \text{V s} / \text{A}$$



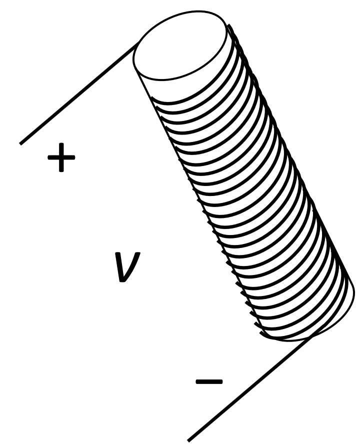
Joseph Henry
(1797-1878)

the inductor

Although made of an ideal conductor, a voltage v equal to the time rate of change of flux linkage λ can develop "across" the inductor (Faraday's Law).

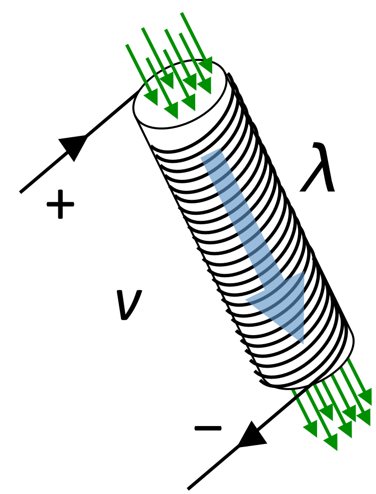


$$\boxed{v = \frac{d\lambda}{dt} = L\frac{di}{dt}}$$



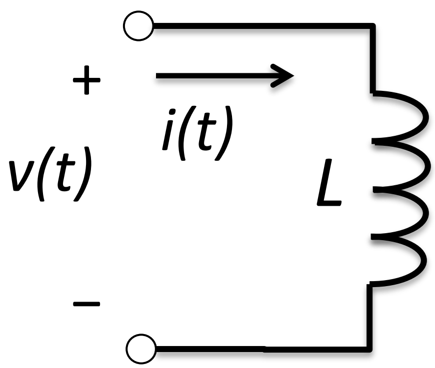
– the voltage v and current i are defined above to satisfy ***passive sign convention***

– the voltage v and current i are related to each other by a linear operator (differentiation / integration)



the inductor

There are alternative but equivalent forms of the equations describing terminal behaviour of ideal inductors.



differential form:

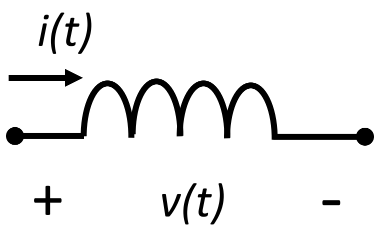
$$v = L\frac{di}{dt} = \frac{d\lambda}{dt}$$

integral form:

$$i(t) - i(t_0) = \frac{1}{L}\int_{t_0}^{t} v(\tau)\,d\tau = \frac{\lambda(t) - \lambda(t_0)}{L}$$

the inductor

Consider the energy *stored* in an inductor. The instantaneous power *absorbed* (note the *passive sign convention*) by an inductor is:



$$p(t) = i(t) \cdot v(t) = Li(t)\frac{di(t)}{dt}$$

The energy absorbed by the inductor from time t_0 to time t is:

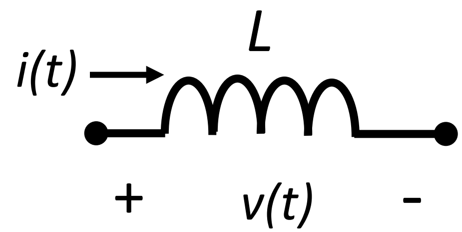
$$W_{t_0 \to t} = \int_{t_0}^{t} p(t')dt' = \int_{t_0}^{t} Li(t')\frac{di(t')}{dt'}dt' = \int_{i(t_0)}^{i(t)} Li(t')di(t') = \frac{1}{2}Li^2(t) - \frac{1}{2}Li^2(t_0)$$

The energy absorbed is *stored* as magnetic potential energy *U(t)*:

$$\boxed{U(t) = \frac{1}{2}Li^2(t) = \frac{1}{2}\frac{\lambda^2(t)}{L}} \qquad W_{t_0 \to t} = U(t) - U(t_0)$$

continuity of inductor current

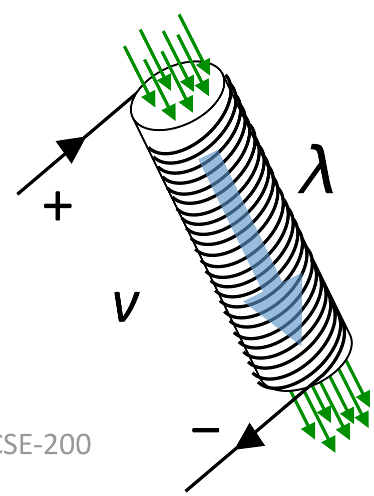
The current flow through a inductor is:



$$v = L\frac{di}{dt} = L\lim_{\Delta t \to 0}\frac{i(t+\Delta t)-i(t)}{\Delta t}$$

where we restate the definition of the derivative.

An instantaneous change in inductor current (and flux linkage) requires an infinite (unphysical) voltage. For a finite terminal voltage, we require that the inductor current *i*(*t*) is ***continuous***.



continuity of inductor current

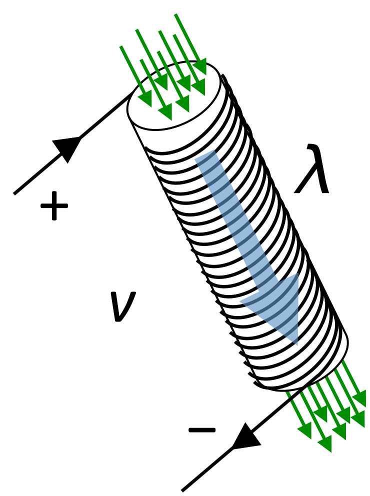
Continuity of inductor current ensures that:

– the voltage v is finite

– the power absorbed $p = iv$ by the inductor is finite

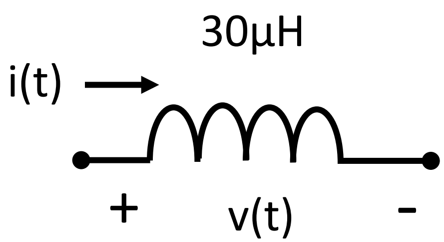
– the flux linkage λ is continuous

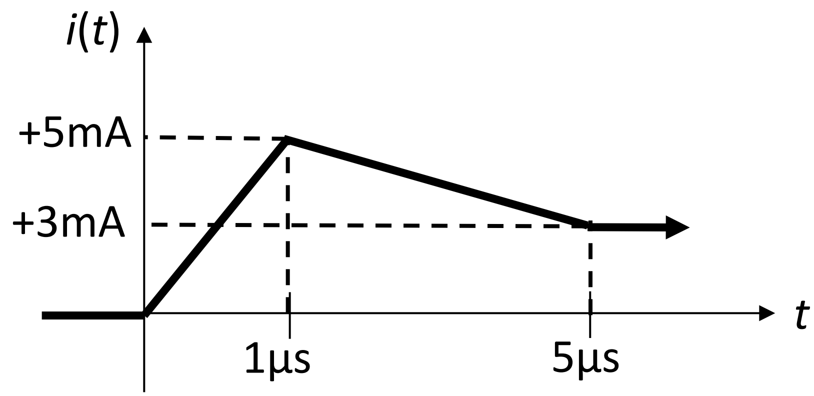
– the magnetic energy stored $U = \frac{1}{2} Li^2$ is continuous, satisfying the **conservation of energy**



Example 1

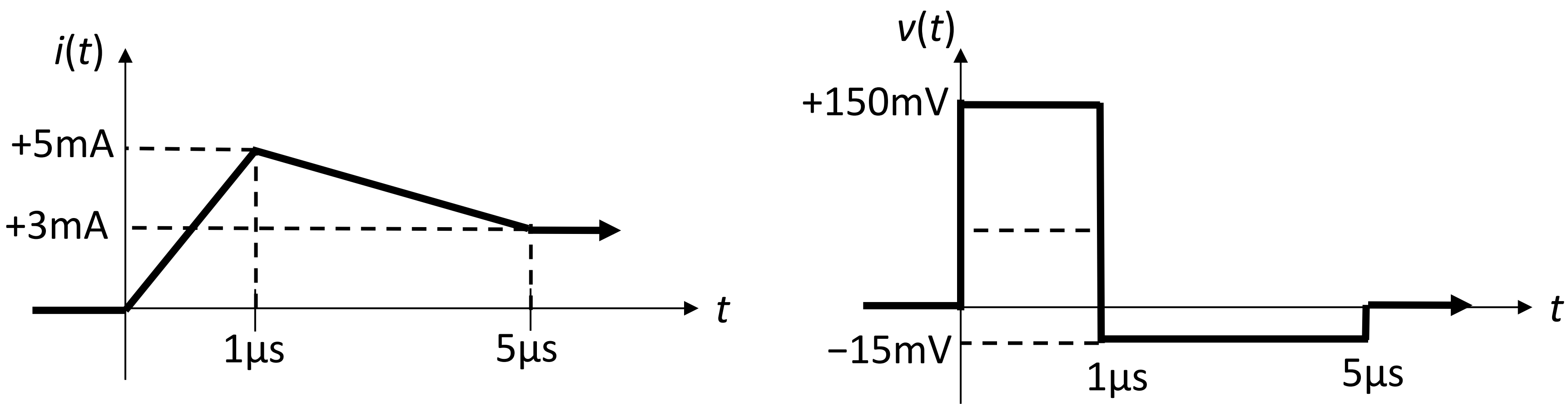
An ideal 30μH inductor has a current *i*(*t*) flow through its terminals, as specified below. Plot the voltage *v*(*t*) as a function of time. How much energy is stored in the inductor at *t* = 5μs?





Example 1

Voltage is given by *v = L di/dt,* meaning in proportion to the slope of *i(t)* vs *t.*



$t<0\mu s$: $L\,di/dt = 0V$

$0<t<1\mu s$: $L\,di/dt = 30\mu H \times 5mA/1\mu s = 150\ mV$

$1\mu s<t<5\mu s$: $L\,di/dt = 30\mu H \times (-2mA)/4\mu s = -15\ mV$

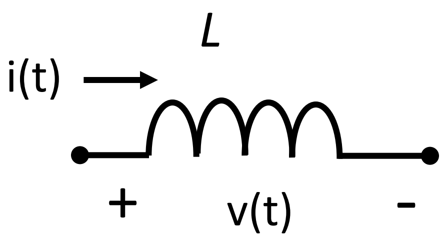
$5\mu s<t$: $L\,di/dt = 0V$

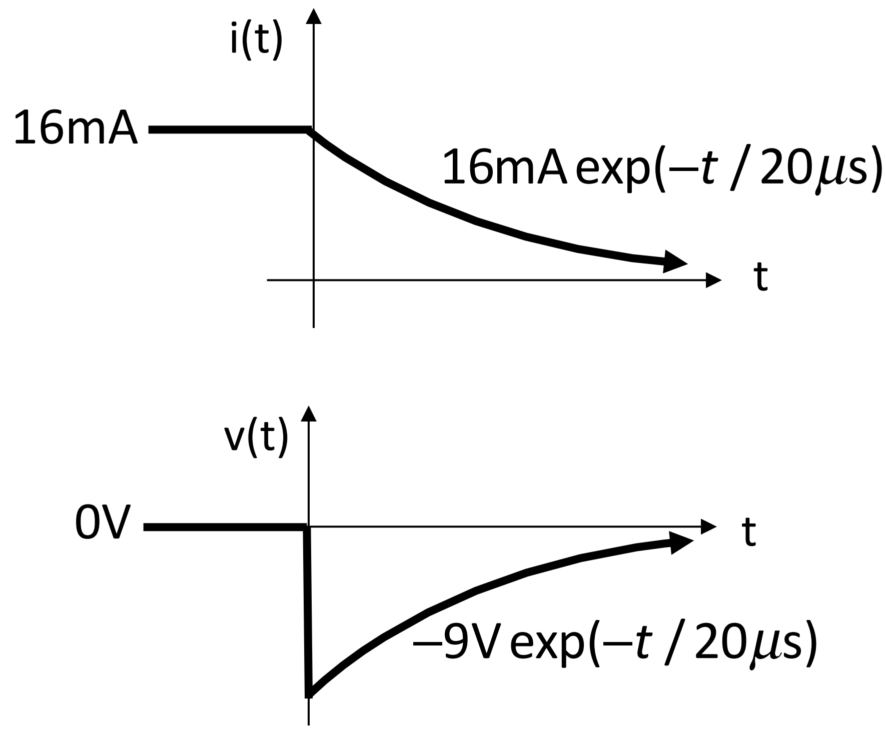
The stored energy at t=5μs is: $U(5\mu s) = \frac{1}{2} L i^2 (5\mu s) = \frac{1}{2} \cdot 30mH \cdot \left(3mA\right)^2 = 135nJ$

McGill

Example 2

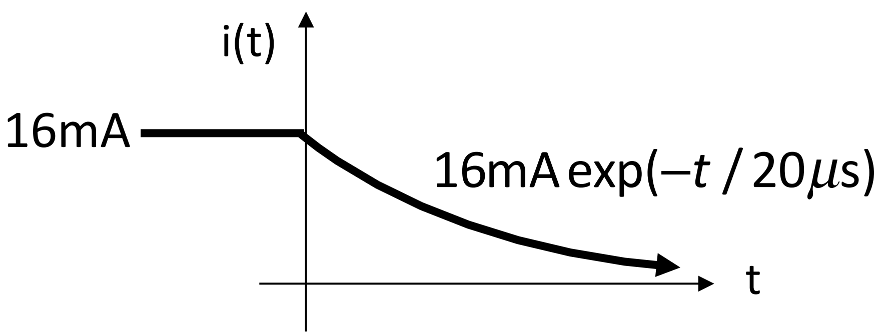
An ideal inductor has a voltage and current as plotted below. What is the value of the inductance *L*?

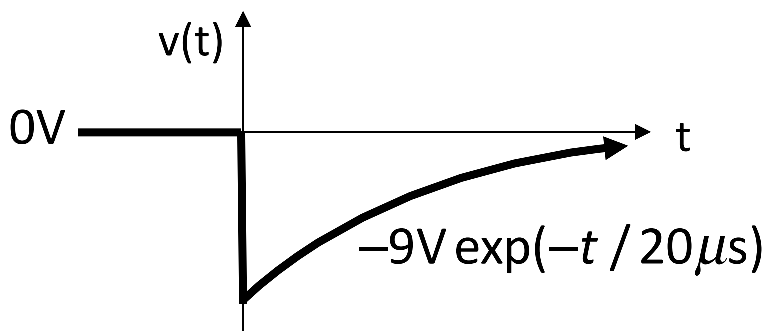




Example 2

The inductance is given by the ratio $L = v / [\ di/dt\ ]$, which is indeed a constant for the given curves.





$$\frac{di}{dt} = 16\text{mA} \cdot \frac{-1}{20\mu s}\exp(-t/20\mu s)$$

$$= -\frac{4}{5}\frac{\text{A}}{\text{ms}}\exp(-t/20\mu s)$$

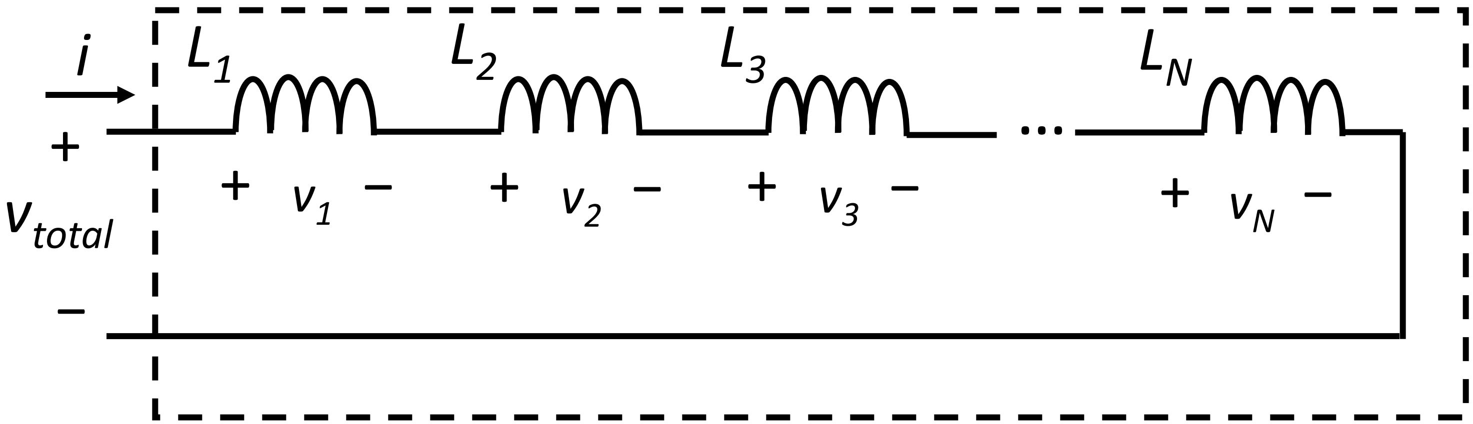
$$L = \frac{v}{di/dt}$$

$$= \frac{-9\text{V}\exp(-t/20\mu s)}{-\frac{4}{5}\frac{\text{A}}{\text{ms}}\exp(-t/20\mu s)}$$

$$= 11.25\text{mH}$$

inductors in series

A series combination of inductors has an equivalent inductance L_{eq}.



Voltage across each inductor: $v_m = L_m\, di/dt$

Total voltage (KVL):

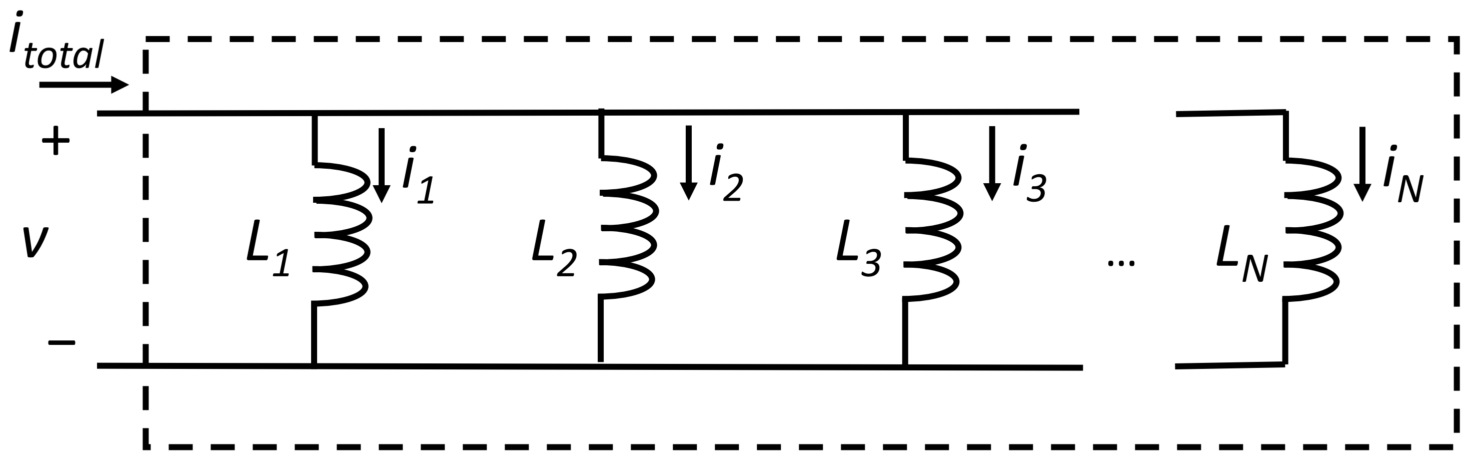
$$v_{total} = v_1 + v_2 + \ldots + v_N = \left(L_1 + L_2 + \ldots + L_3\right)\frac{di}{dt}$$

Equivalent inductance:

$$\frac{v_{total}}{di/dt} = \boxed{L_{eq} = L_1 + L_2 + \ldots + L_N}$$

inductors in parallel

A parallel combination of inductors has an equivalent inductance L_{eq}.



Voltage across each inductor: $v = L_m \, di_m/dt$

Total current:

(time derivative of KCL)

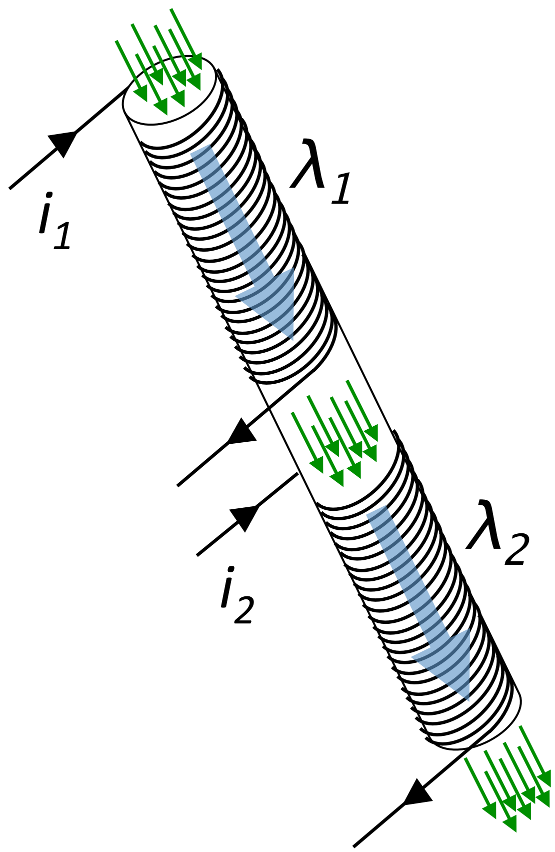
$$\frac{di_{total}}{dt} = \frac{di_1}{dt} + \frac{di_2}{dt} + \ldots + \frac{di_N}{dt} = v\left(\frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_N}\right)$$

Equivalent inductance:

$$\frac{di_{total}/dt}{v} = \boxed{\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \ldots + \frac{1}{L_N}}$$

coupled inductors

coupled inductors: physically consists of ideal inductors whose physical arrangement leads to a *sharing of flux linkage*.



- both currents i_1 and i_2 generate non-zero flux linkage in both inductors.

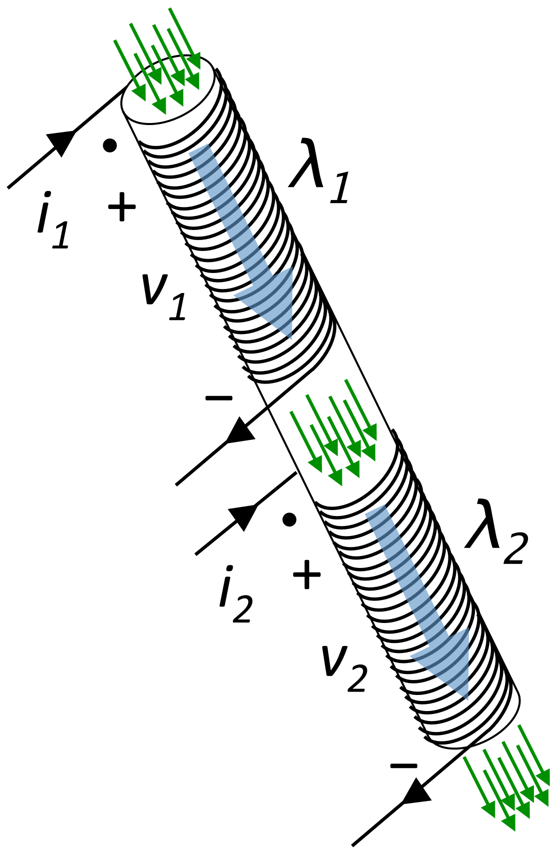
$$\boxed{\begin{aligned} \lambda_1 &= L_1 i_1 + M i_2 \\ \lambda_2 &= L_2 i_2 + M i_1 \end{aligned}}$$

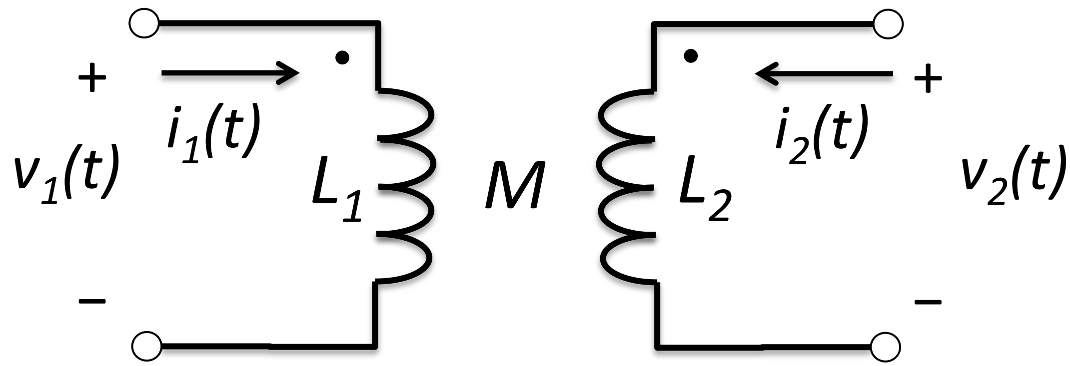
- L_1 and L_2 are ***self-inductances***, and M is the ***mutual inductance***.

- The SI unit for M is the Henry.
  1 H = 1 Wb / A = 1 V s / A

coupled inductors

The **dot convention** indicates the polarity of the mutual flux linkage for coil windings. The • indicates the terminals leading to additive mutual linkage.



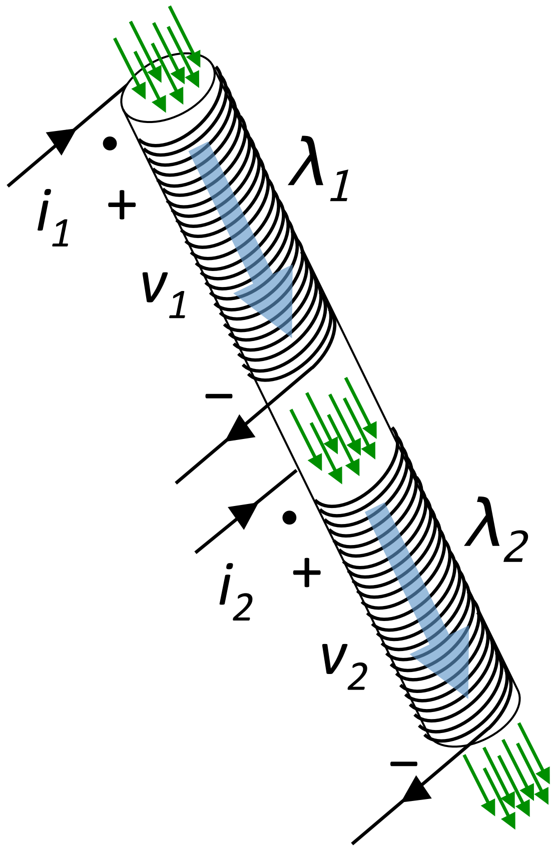


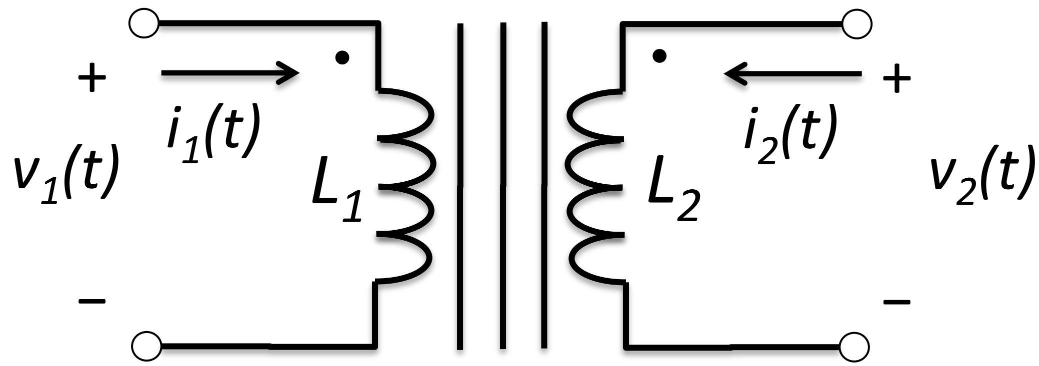
$$v_1 = \frac{d\lambda_1}{dt} = L_1\frac{di_1}{dt} + M\frac{di_2}{dt}$$

$$v_2 = \frac{d\lambda_2}{dt} = L_2\frac{di_2}{dt} + M\frac{di_1}{dt}$$

ideal transformer

An ideal **transformer** is an ideal coupled inductor where the mutual inductance is the maximum physically allowed, $M^2 = L_1 L_2$.





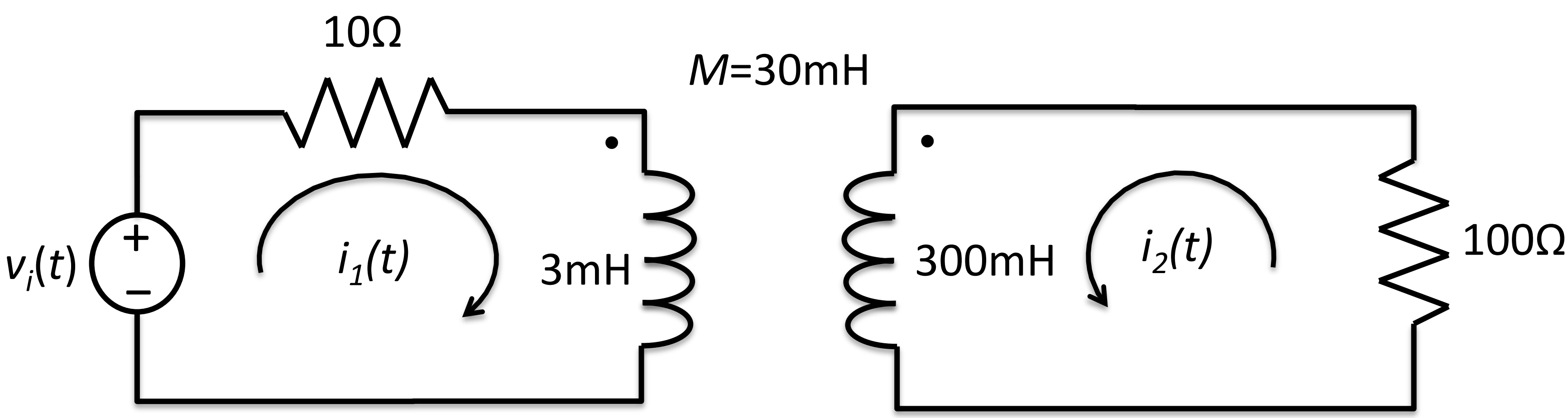
$$v_1 = \frac{d\lambda_1}{dt} = L_1 \frac{di_1}{dt} + M \frac{di_2}{dt}$$

$$v_2 = \frac{d\lambda_2}{dt} = L_2 \frac{di_2}{dt} + M \frac{di_1}{dt}$$

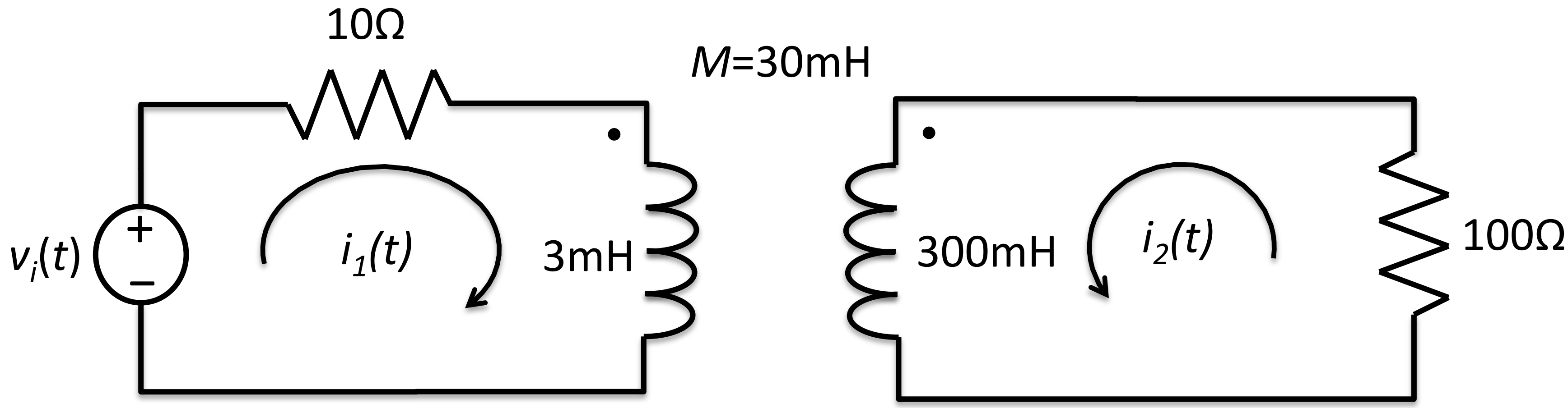
$$\boxed{M = \sqrt{L_1 L_2}}$$

Example 3

What are the mesh current equations for the coupled inductor circuit below?



Example 3



$$0 = -v_i + 10\Omega \cdot i_1 + 3\text{mH} \cdot \frac{di_1}{dt} + \underline{30\text{mH} \cdot \frac{di_2}{dt}}$$

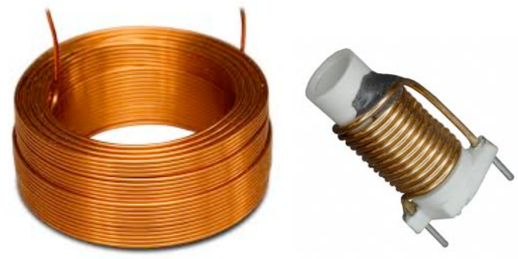
$$0 = 100\Omega \cdot i_2 + 300\text{mH} \cdot \frac{di_2}{dt} + \underline{30\text{mH} \cdot \frac{di_1}{dt}}$$

The current definition with respect to the dots leads to mutual inductance terms with the same sign as self inductance terms.

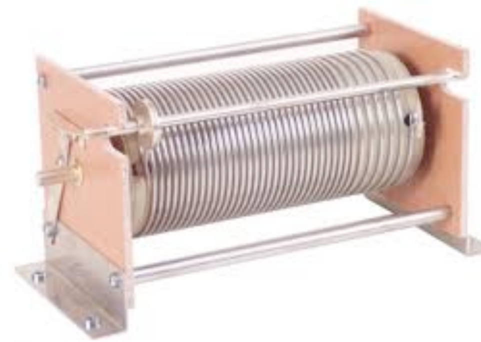
practical inductors

Practical inductors come in a variety of shapes, sizes and materials.

toroids with ferrite cores



air core wire wound

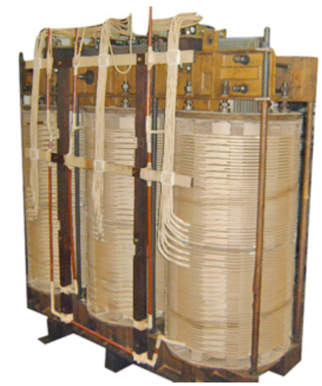


variable roller

ferrite core wire wound

practical coupled inductors

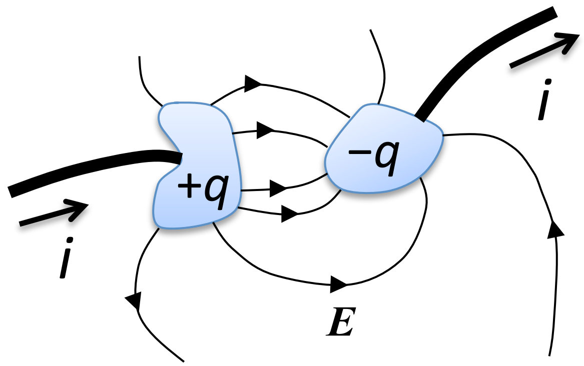
Practical coupled inductors and transformers come in a variety of shapes and sizes as well.

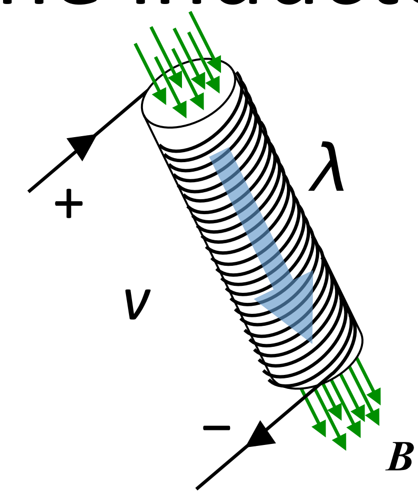


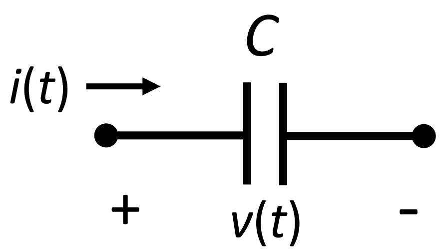
laminated ferrite cores

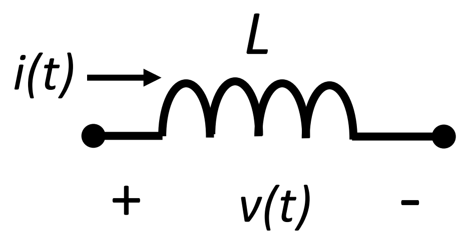
oil cooled power transformers

the capacitor and the inductor





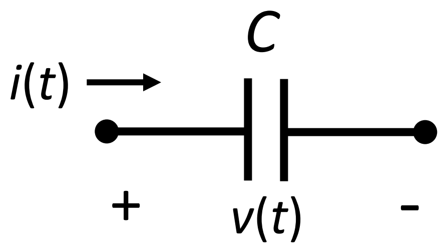




electric potential energy
stored in charge separation q

magnetic potential energy
stored in flux linkage λ

the capacitor and the inductor

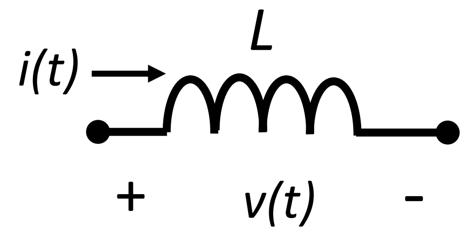


$$q = Cv$$

$$i = C\frac{dv}{dt} = \frac{dq}{dt}$$

$$v(t) - v(t_0) = \frac{1}{C}\int_{t_0}^{t} i(\tau)\,d\tau$$

$$U = \frac{1}{2}Cv^2$$



$$\lambda = Li$$

$$v = L\frac{di}{dt} = \frac{d\lambda}{dt}$$

$$i(t) - i(t_0) = \frac{1}{L}\int_{t_0}^{t} v(\tau)\,d\tau$$

$$U = \frac{1}{2}Li^2$$