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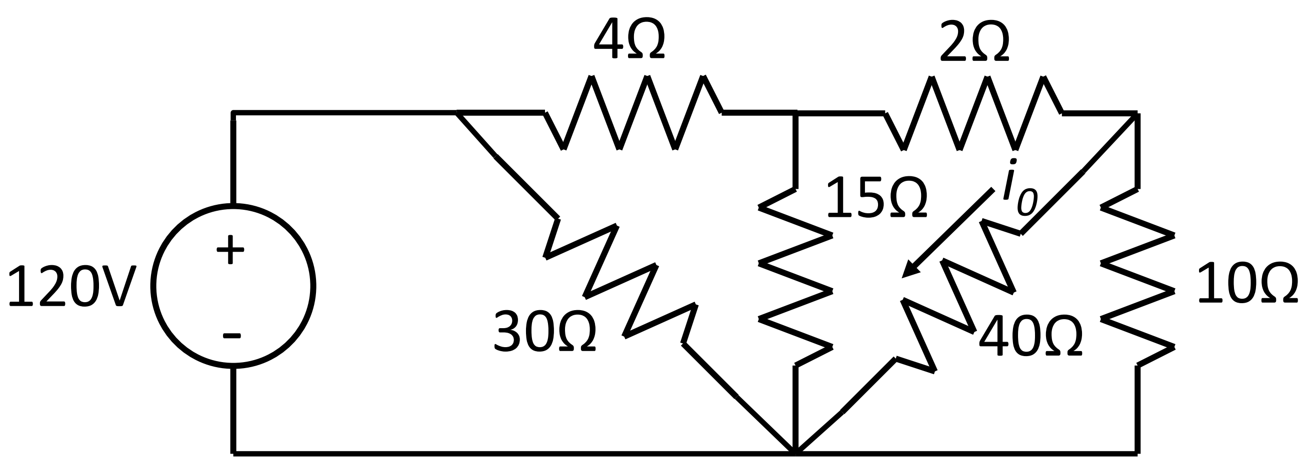
3. Basic Circuit Analysis

- Node Voltage Method
- Mesh Current Method

Motivation

- Consider a circuit with 7 elements:
  - 7 voltage variables + 7 current variables = 14 variables total
  - 14 independent equations required for a solution
- How do we organize KVL, KCL and element law equations?
- What is the most compact way to write out the equations?



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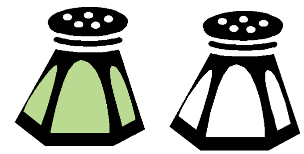
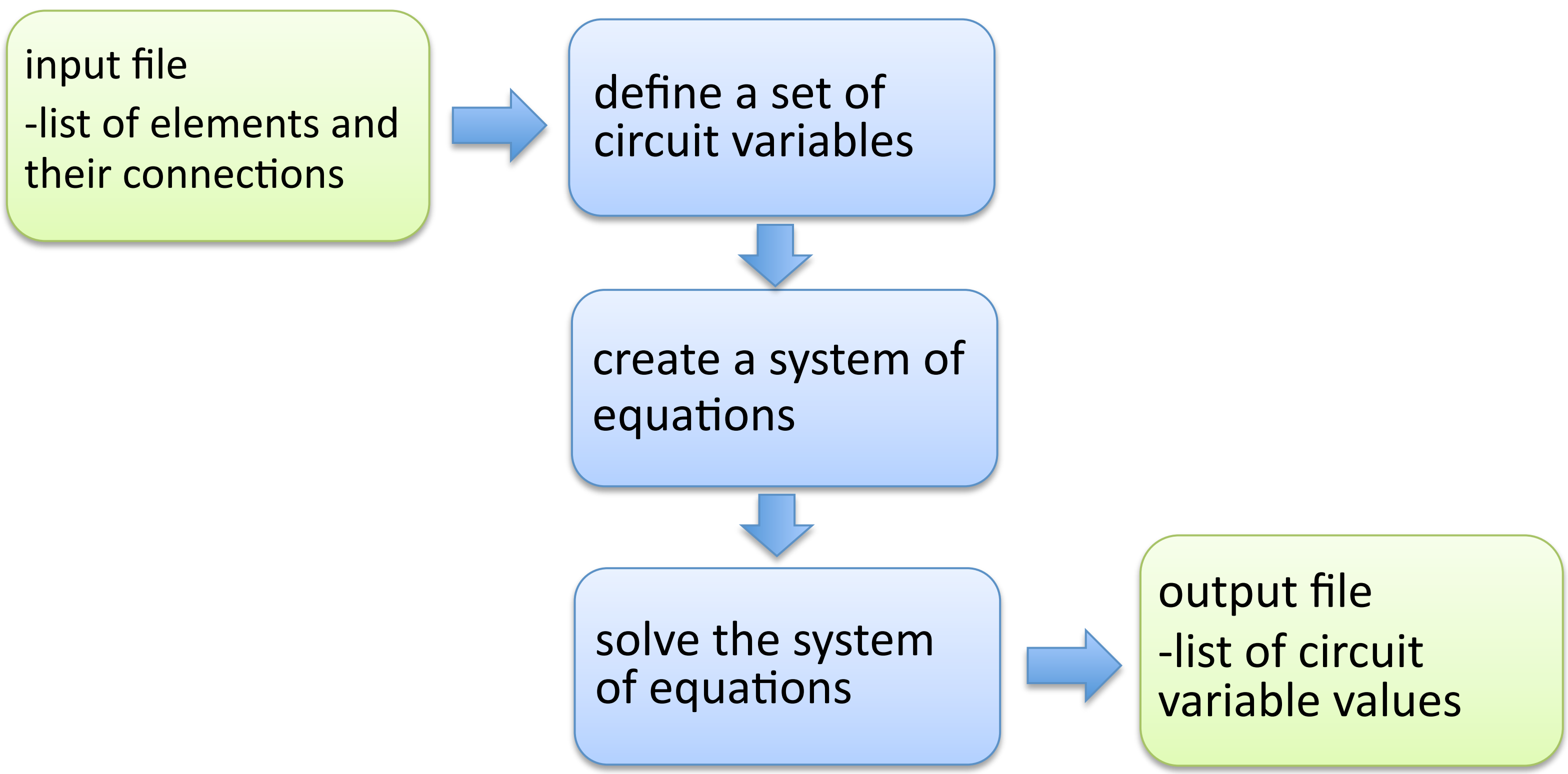
Motivation

- How can we program a computer to take a circuit as input, and produce the circuit variables (voltages, currents) as output ?
- SPICE (Simulation Program with Integrated Circuit Emphasis) is one such common program; if curious, check: http://bwrc.eecs.berkeley.edu/classes/icbook/spice/





Today’s Outline

3. Analysis Methods

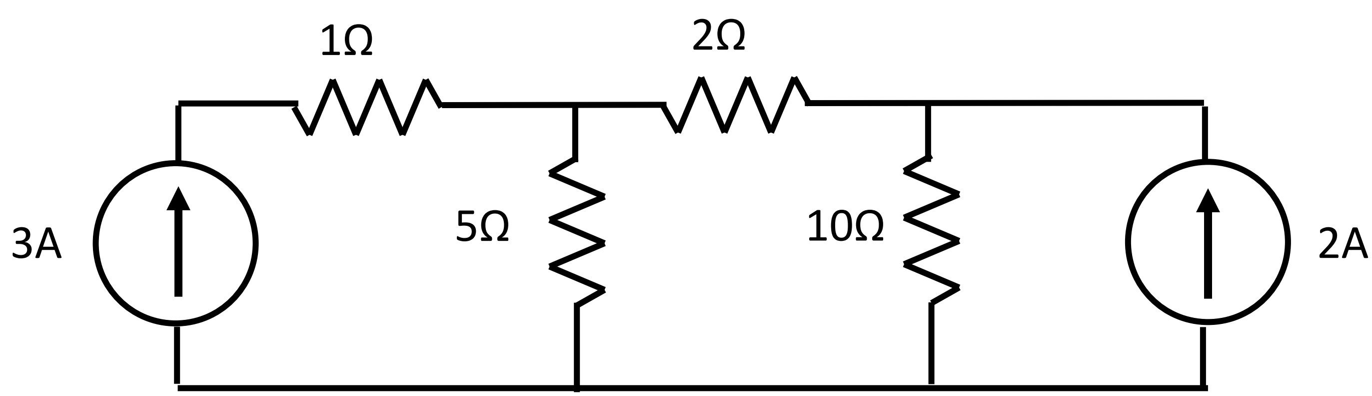
- Nodal Analysis

Node Voltage Method

- A *systematic* way to apply KVL, KCL, Ohm's law (or other terminal laws) to solve for the variables in a circuit
- Also known as **nodal analysis**
- *Not* necessarily the most efficient way to solve a circuit
- Ideally suited for solutions on computer because the procedure is systematic (ie. can be automated)

Node Voltage Method

We illustrate the procedure by finding the power delivered by the 2A source in the following circuit.



Node Voltage Method

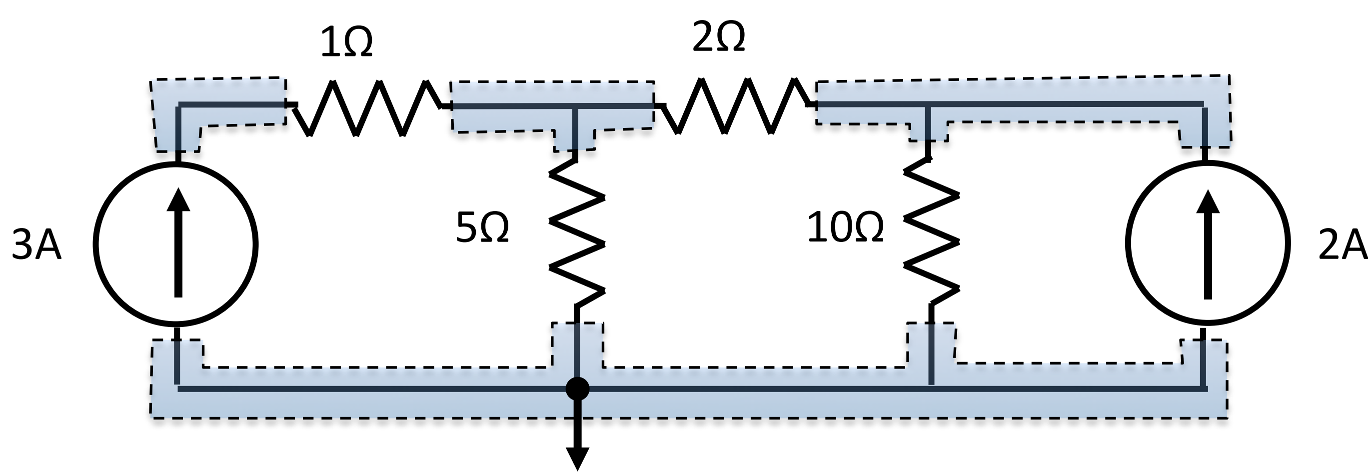
Step #1:

Choose a node in the circuit as a *reference node*, and indicate your choice with the symbol " ". Usually, the node with the most branches connected to it is a good choice.

It is good practice to use the reference node of an op-amp circuit to avoid confusion over multiple references. For the same reason, it is good practice to take as a reference a "ground" node (connection to the Earth) if it exists.

Node Voltage Method

Nodes are identified below.



The reference node is taken as the node with most branches connected to it.

Node Voltage Method

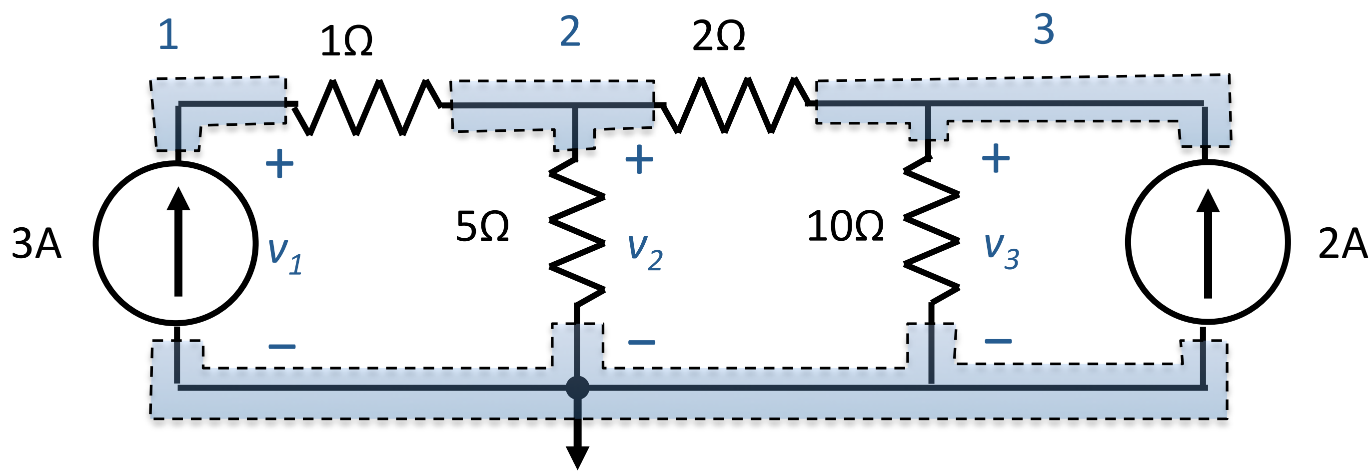
Step #2:

Label all remaining nodes and define algebraic voltage variables between each node (+) and the reference node (−).

If you label nodes with numbers 1,2,3… or letters A,B,C…, then create corresponding voltage variables v_1, v_2, v_3… or v_A, v_B, v_C…

Node Voltage Method

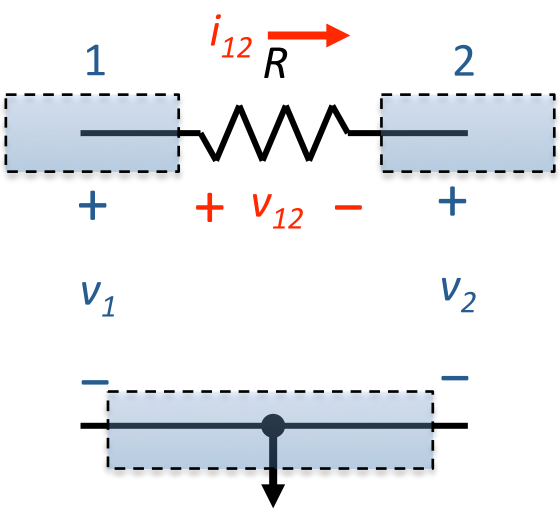
Non-reference nodes are labeled and voltage variables are added.



Node Voltage Method

Step #3:

Apply KCL at each node (except the reference). Express each branch current using the voltage variables defined earlier (by KVL and terminal laws of elements). For example:



v_{12} and i_{12} = temporary variables

KVL: $-v_1 + v_{12} + v_2 = 0$

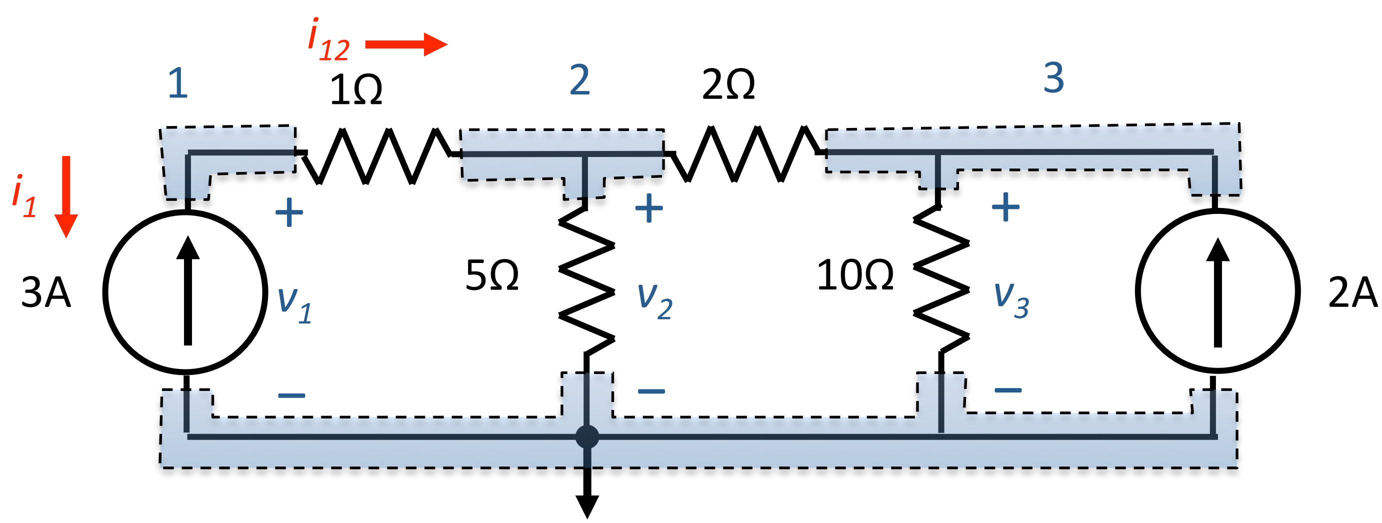
$$v_{12} = v_1 - v_2$$

Ohm's Law: $i_{12} = v_{12} / R$

$$= \underbrace{(v_1 - v_2) / R}_{\textit{current term in KCL equation}}$$

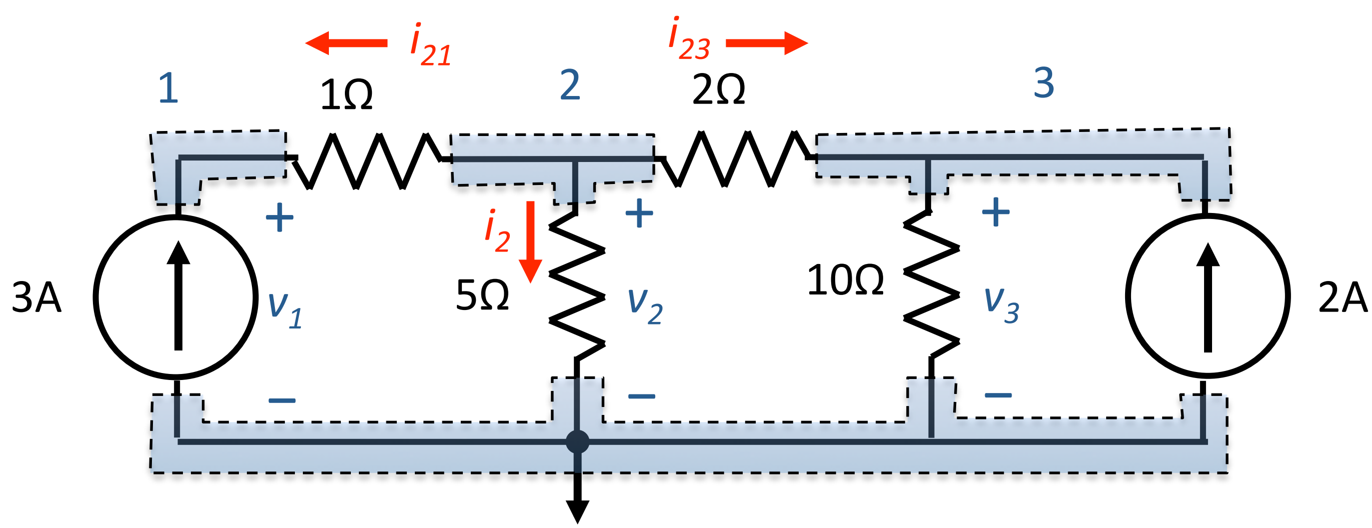
Node Voltage Method

KCL at node 1: $$0 = \underbrace{-3A}_{i_1} + \underbrace{\frac{v_1 - v_2}{1\Omega}}_{i_{12}}$$



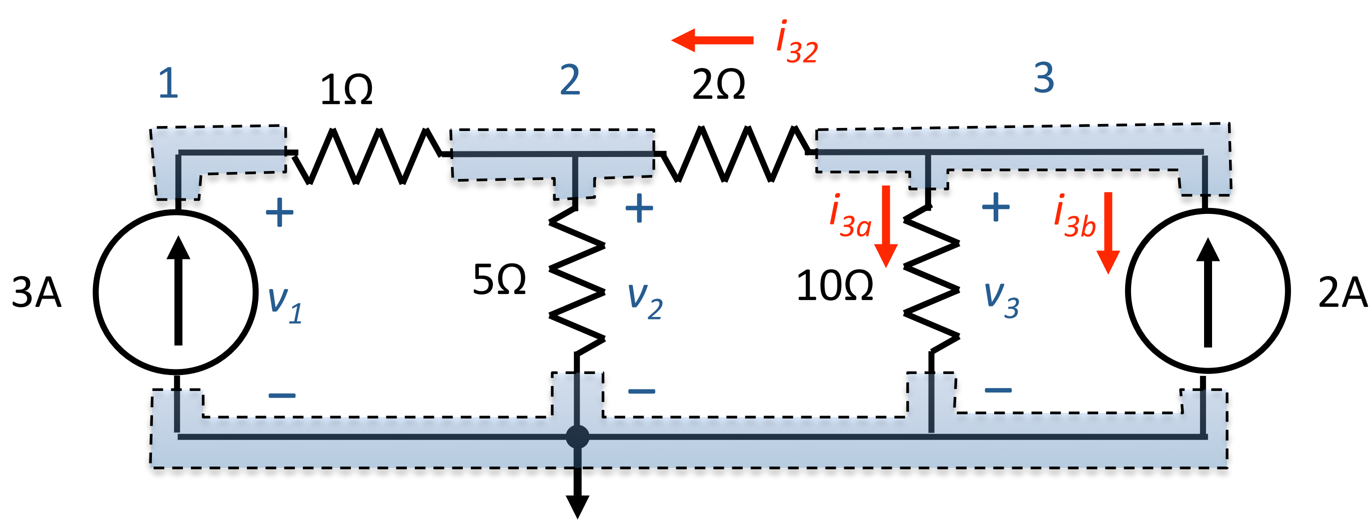
Node Voltage Method

KCL at node 2: $$0 = \underbrace{\frac{v_2 - v_1}{1\Omega}}_{i_{21}} + \underbrace{\frac{v_2}{5\Omega}}_{i_2} + \underbrace{\frac{v_2 - v_3}{2\Omega}}_{i_{23}}$$



Node Voltage Method

KCL at node 3: $$0 = \underbrace{\frac{v_3 - v_2}{2\Omega}}_{i_{32}} + \underbrace{\frac{v_3}{10\Omega}}_{i_{3a}} \underbrace{- 2A}_{i_{3b}}$$

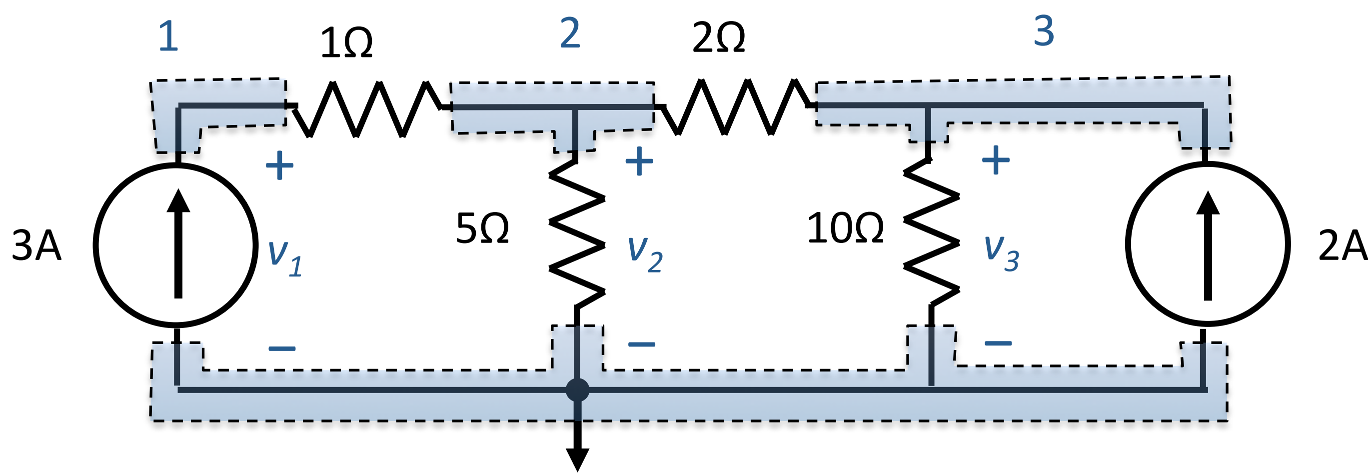


Node Voltage Method

KCL at node 1: $$0 = -3\text{A} + \frac{v_1 - v_2}{1\Omega}$$

KCL at node 2: $$0 = \frac{v_2 - v_1}{1\Omega} + \frac{v_2}{5\Omega} + \frac{v_2 - v_3}{2\Omega}$$

KCL at node 3: $$0 = \frac{v_3 - v_2}{2\Omega} + \frac{v_3}{10\Omega} - 2\text{A}$$



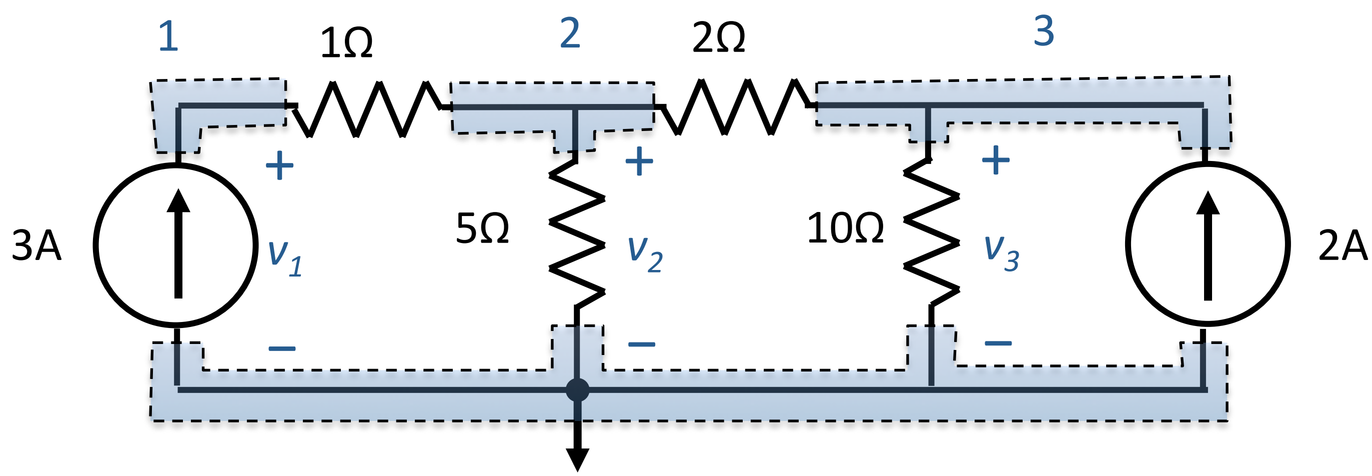
Node Voltage Method

KCL at node 1: $0 = -3\text{A} + \boxed{\dfrac{v_1 - v_2}{1\Omega}}$

KCL at node 2: $0 = \boxed{\dfrac{v_2 - v_1}{1\Omega}} + \dfrac{v_2}{5\Omega} + \boxed{\dfrac{v_2 - v_3}{2\Omega}}$

KCL at node 3: $0 = \boxed{\dfrac{v_3 - v_2}{2\Omega}} + \dfrac{v_3}{10\Omega} - 2\text{A}$

We see branch currents with opposite signs in different KCL equations. We do not know in advance the physical current flows (or equivalently, which node voltage is the greatest).



Node Voltage Method

Step #4:

Solve for the node voltage variables, using any linear algebra technique of your preference. The simplest method is repeated substitution (but there are many techniques developed for solution by computer).

Any quantity can be found in terms of the solved node voltages.

Node Voltage Method

Use repeated substitution to find the value of v_3, and then v_2 and v_1.

node 1:

$$0 = -3\text{A} + \frac{v_1 - v_2}{1\Omega}$$

$$v_1 = 3\text{V} + v_2$$

node 3:

$$0 = \frac{v_3 - v_2}{2\Omega} + \frac{v_3}{10\Omega} - 2\text{A}$$

$$v_2 = 2\Omega\left(\frac{1}{2\Omega} + \frac{1}{10\Omega}\right)v_3 - 4\text{V} = \frac{6}{5}v_3 - 4\text{V}$$

node 2:

$$0 = \frac{v_2 - v_1}{1\Omega} + \frac{v_2}{5\Omega} + \frac{v_2 - v_3}{2\Omega}$$

$$0 = \frac{v_2 - (3\text{V} + v_2)}{1\Omega} + \frac{v_2}{5\Omega} + \frac{v_2 - v_3}{2\Omega}$$

substitution of node 1 equation

$$0 = -3\text{A} + \frac{v_2}{5\Omega} + \frac{v_2 - v_3}{2\Omega}$$

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Node Voltage Method

cont.: $$0=-3\text{A}+\frac{v_2}{5\Omega}+\frac{v_2-v_3}{2\Omega}$$

$$0=-3\text{A}+\frac{(\frac{6}{5}v_3-4\text{V})}{5\Omega}+\frac{(\frac{6}{5}v_3-4\text{V})-v_3}{2\Omega}$$ substitution of node 3 equation

$$0=-30\text{V}+\frac{12}{5}v_3-8\text{V}+6v_3-20\text{V}-5v_3$$ multiply by 10Ω

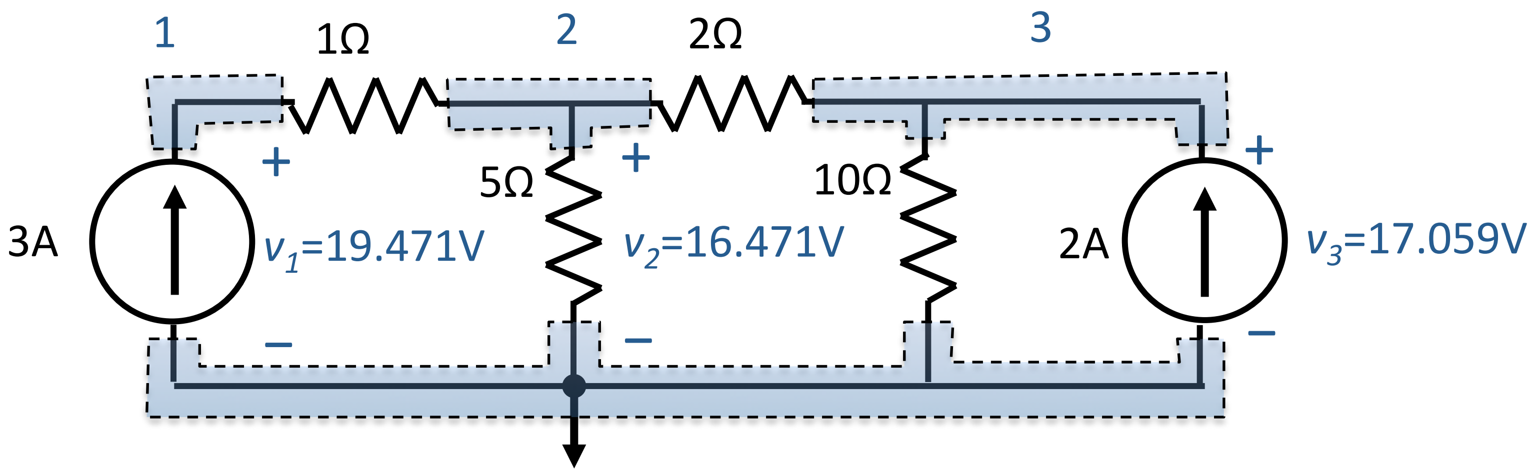
$$v_3=\frac{(30\text{V}+8\text{V}+20\text{V})}{\frac{12}{5}+6-5}=\frac{58\text{V}}{17/5}=17.059\text{V}$$

node 3: $$v_2=\frac{6}{5}v_3-4\text{V}=\frac{6}{5}(17.059\text{V})-4\text{V}=16.471\text{V}$$

node 1: $$v_1=3\text{V}+v_2=3\text{V}+(16.471\text{V})=19.471\text{V}$$

Node Voltage Method

We can now easily calculate the power delivered by the 2A current source (the original question).



P_{2A}= power delivered by 2A current source

$= (2A)(v_3)$

$= 2A \times 17.059 = +34.118W$

Summary of Node Voltage Method

Step #1: Define a reference node.

Step #2: Label remaining nodes, and define node voltage variables with respect to reference node.

Step #3: Write KCL equations for each node using node voltage variables only, by intrinsically using KVL and terminal laws (such as Ohm's law).

Step #4: Solve the linear system of equations, and use the node voltages to calculate the desired quantity.

Addendum on Cramer's Rule

Cramer's Rule

- method for solving a linear system of equations
- can be easily done by hand for small systems of equations, eg. 2 or 3 variables
- can be simpler than substitution

Gabriel Cramer
(1704-1752)

Cramer's Rule for 2 variables

Given the linear system of equations for unknowns x_1 and x_2:

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

If $A_{11}A_{22}-A_{12}A_{21} \neq 0$, the solution is the ratio of determinants:

$$x_1 = \frac{\begin{vmatrix} b_1 & A_{12} \\ b_2 & A_{22} \end{vmatrix}}{\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}} = \frac{b_1 A_{22} - A_{12} b_2}{A_{11}A_{22} - A_{12}A_{21}} \qquad x_2 = \frac{\begin{vmatrix} A_{11} & b_1 \\ A_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}} = \frac{A_{11} b_2 - b_1 A_{21}}{A_{11}A_{22} - A_{12}A_{21}}$$

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Cramer's Rule for 3 variables

Given the linear system of equations for unknowns x_1, x_2, x_3:

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \text{or} \qquad \mathbf{b} = \mathbf{A} \cdot \mathbf{x}$$

If $|\mathbf{A}| \neq 0$, the solution is also a ratio of determinants:

$$x_1 = \frac{\left|\tilde{\mathbf{A}}(1)\right|}{|\mathbf{A}|} \quad x_2 = \frac{\left|\tilde{\mathbf{A}}(2)\right|}{|\mathbf{A}|} \quad x_3 = \frac{\left|\tilde{\mathbf{A}}(3)\right|}{|\mathbf{A}|} \qquad |\mathbf{A}| = \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$

$$\left|\tilde{\mathbf{A}}(1)\right| = \begin{vmatrix} b_1 & A_{12} & A_{13} \\ b_2 & A_{22} & A_{23} \\ b_3 & A_{32} & A_{33} \end{vmatrix} \qquad \left|\tilde{\mathbf{A}}(2)\right| = \begin{vmatrix} A_{11} & b_1 & A_{13} \\ A_{21} & b_2 & A_{23} \\ A_{31} & b_3 & A_{33} \end{vmatrix} \qquad \left|\tilde{\mathbf{A}}(3)\right| = \begin{vmatrix} A_{11} & A_{12} & b_1 \\ A_{21} & A_{22} & b_2 \\ A_{31} & A_{32} & b_3 \end{vmatrix}$$