Today's Outline

4. Circuit Theorems

- Maximum Power Transfer Theorem

Motivation

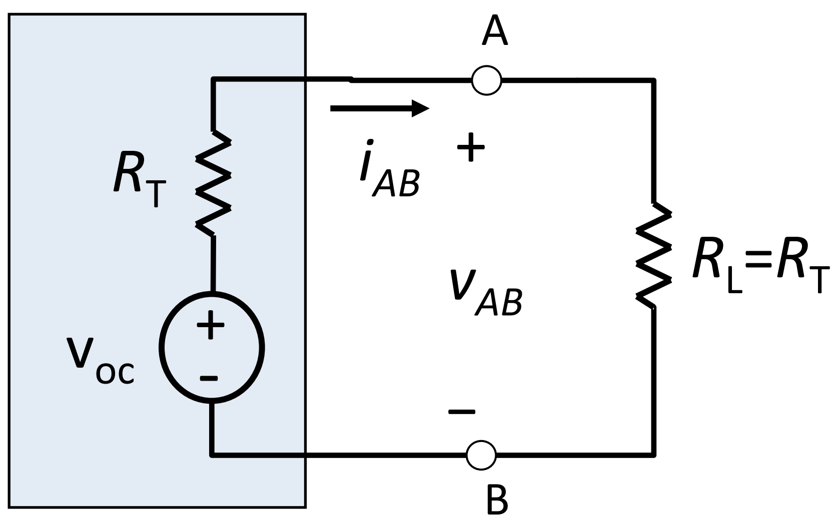
Maximizing power transfer from a source to a load is a commonly encountered problem:

- a solar cell delivering power to a bank of batteries
- a battery powering the motor of an electric vehicle
- a portable electronic device delivering (absorbing) signal power to (from) an antenna

The maximum power transfer theorem gives insight into how a source and load should be chosen.

Maximum Power Transfer

Maximum Power Transfer Theorem: The maximum power that can be delivered by a *Thévenin equivalent* or *Norton equivalent* circuit to a load resistance R_L occurs under the condition when the load resistance is equal to the Thévenin resistance: $R_L = R_T$.

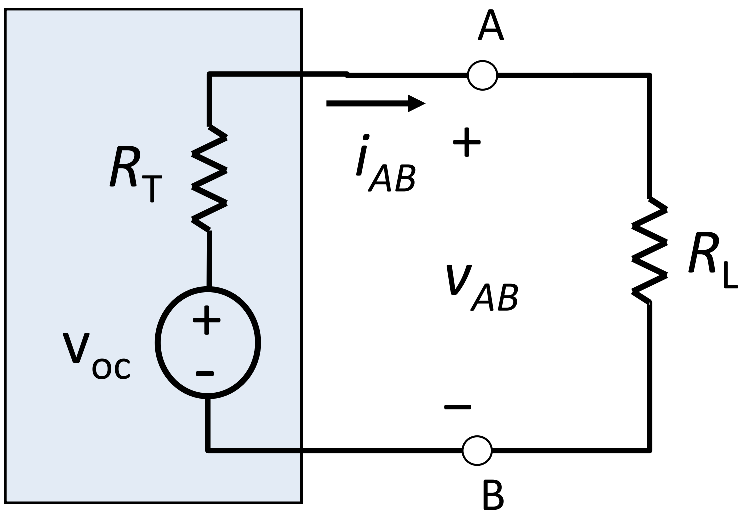


Sir Isaac Newton

1643-1727

Maximum Power Transfer

Proof #1 (Newtonian calculus): Calculate P_L, the power absorbed by the load and maximize P_L versus the load resistance R_L (set $dP_L/dR_L = 0$).



$$\frac{v_{AB}}{v_{oc}} = \frac{R_L}{R_L + R_T}$$

$$P_L = i_{AB} v_{AB} = v_{AB}^2 / R_L$$

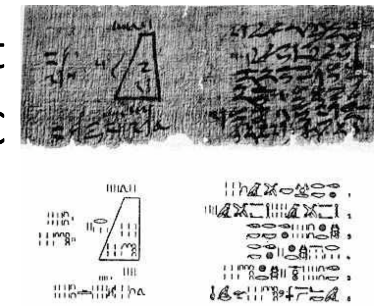
$$= \left(\frac{R_L}{R_L + R_T} \right)^2 \frac{v_{oc}^2}{R_L} = v_{oc}^2 \frac{R_L}{\left(R_L + R_T \right)^2}$$

$$\frac{dP_L}{dR_L} = v_{oc}^2 \frac{\left(R_L + R_T \right)^2 \cdot 1 - R_L \cdot 2\left(R_L + R_T \right)}{\left(R_L + R_T \right)^4}$$

$$0 = \frac{dP_L}{dR_L} = v_{oc}^2 \frac{-R_L + R_T}{\left(R_L + R_T \right)^3} \quad \rightarrow \quad R_L = R_T$$

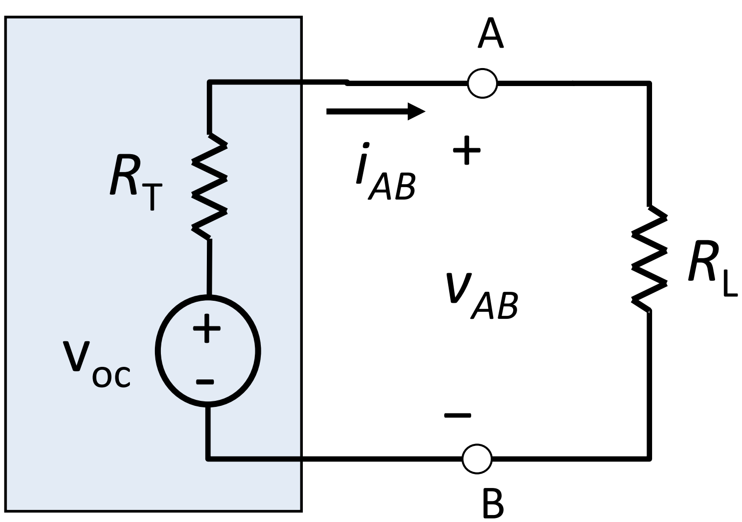
ancient Egypt

1850BC

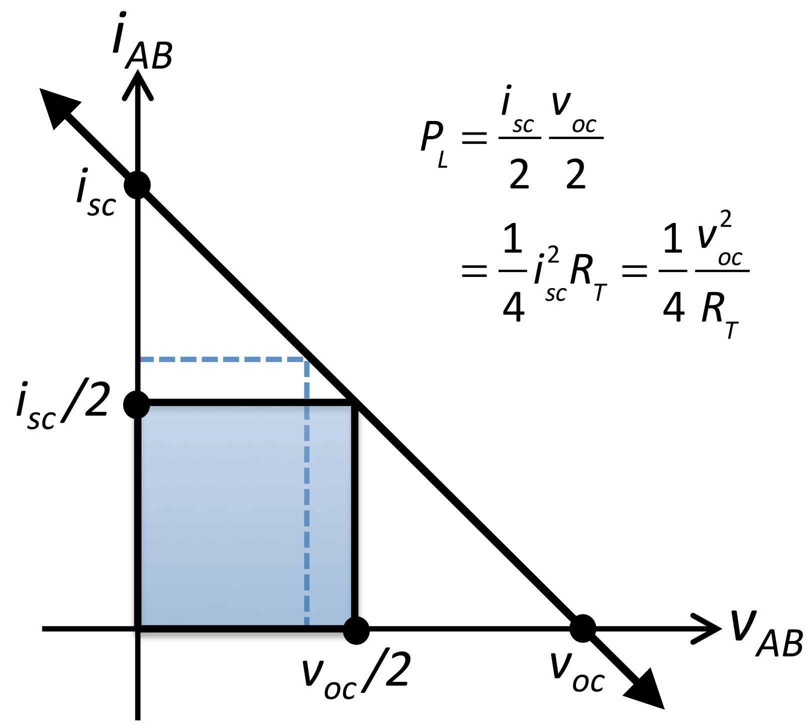


Maximum Power Transfer

Proof #2 (ancient Egyptian calculus): The power absorbed by the load, $P_L = i_{AB}\, v_{AB}$, is the area of the box bounded by i_{AB} and v_{AB}, which is maximal when $i_{AB} = i_{sc}/2$ and $v_{AB} = v_{oc}/2$.



$$\frac{v_{AB}}{v_{oc}} = \frac{R_L}{R_L + R_T}$$



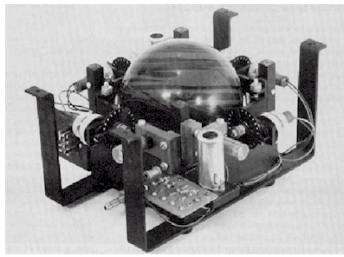
$$P_L = \frac{i_{sc}}{2}\frac{v_{oc}}{2}$$

$$= \frac{1}{4} i_{sc}^2 R_T = \frac{1}{4}\frac{v_{oc}^2}{R_T}$$

Note that one cannot increase the area with a change in the rectangle!

Example

Your computer has a terminal with an open circuit voltage of 1.7V, and an internal resistance of 6Ω. You wish to use this port to deliver power to a peripheral located 5 meters away. The wire resistance to your device is 1Ω, each way.



first trackball
1952

first mouse
1963

a) What terminal resistance should you design to maximize the power delivered to your peripheral, and what is the maximum power that is delivered under these conditions?

b) If you require a minimum of 1.2V at your device, what is the maximum power that can be delivered, and what terminal resistance must you design?

Strategy:

Find the Thévenin equivalent circuit for the computer port + wires.

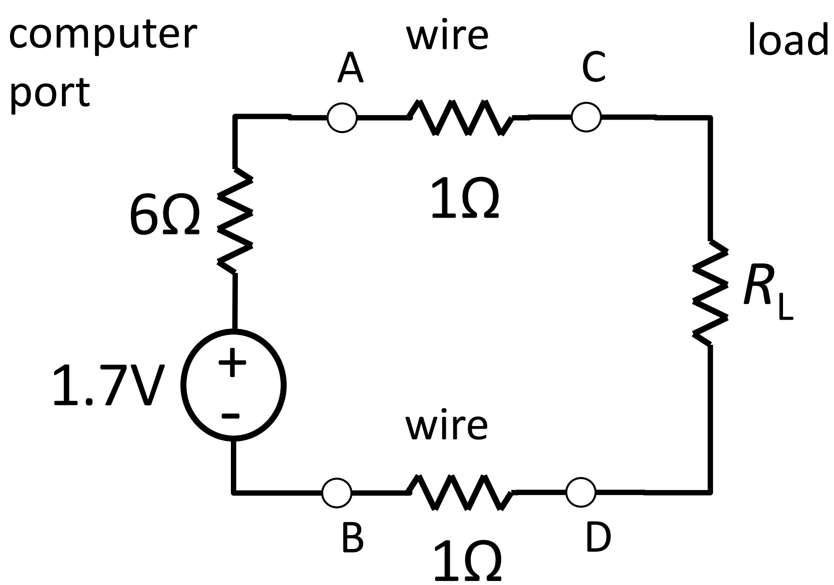
a) Apply the maximum power transfer theorem.

b) Solve the optimization problem.

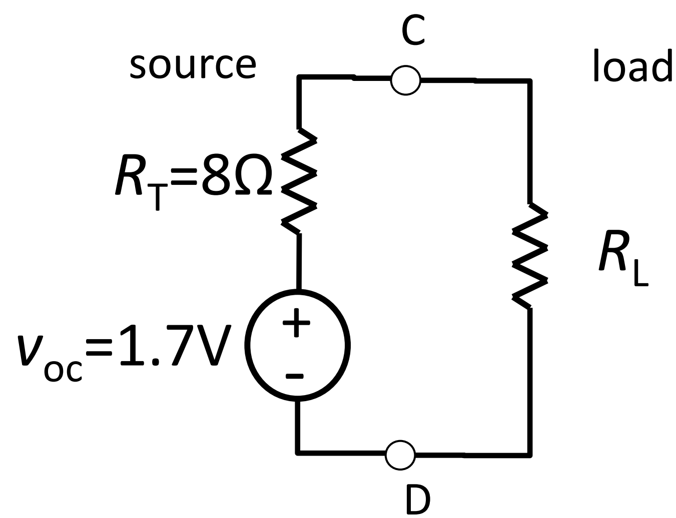
McGill

Example (part a)

Find the Thévenin equivalent circuit.

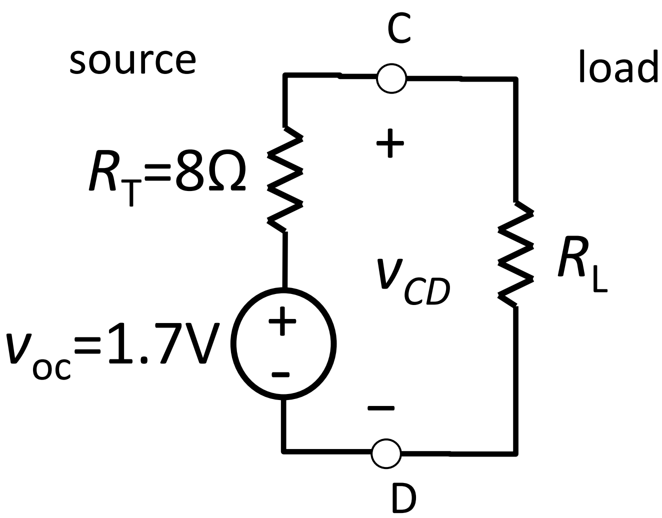


electrical model



electrical model divided
into source and load

Example (part a)



electrical model divided into source and load

Apply the maximum power transfer theorem. Match the load resistor to the Thévenin resistance, find the power delivered.

$$R_L = R_T = 8\Omega$$

$$v_{CD} = v_{oc} / 2 = 1.7\text{V} / 2 = 0.85\text{V}$$

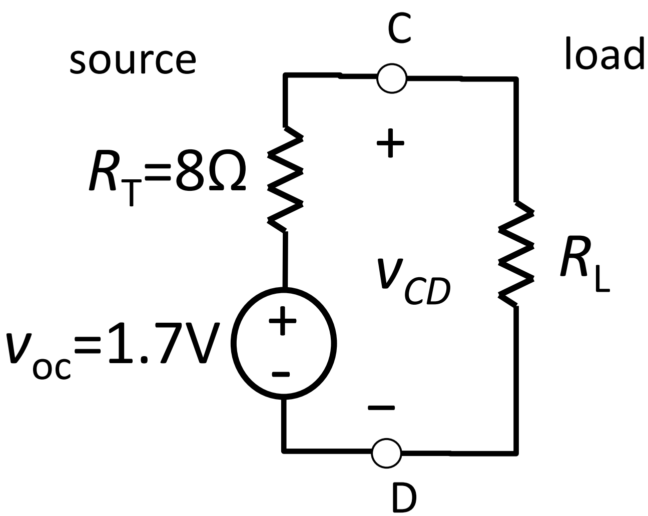
$$i_{CD} = v_{oc} / 2R_T = 1.7\text{V} / (2 \cdot 8\Omega) = 106.25\text{mA} \quad (= i_{sc} / 2)$$

$$P_L = \frac{i_{sc}}{2} \frac{v_{oc}}{2}$$

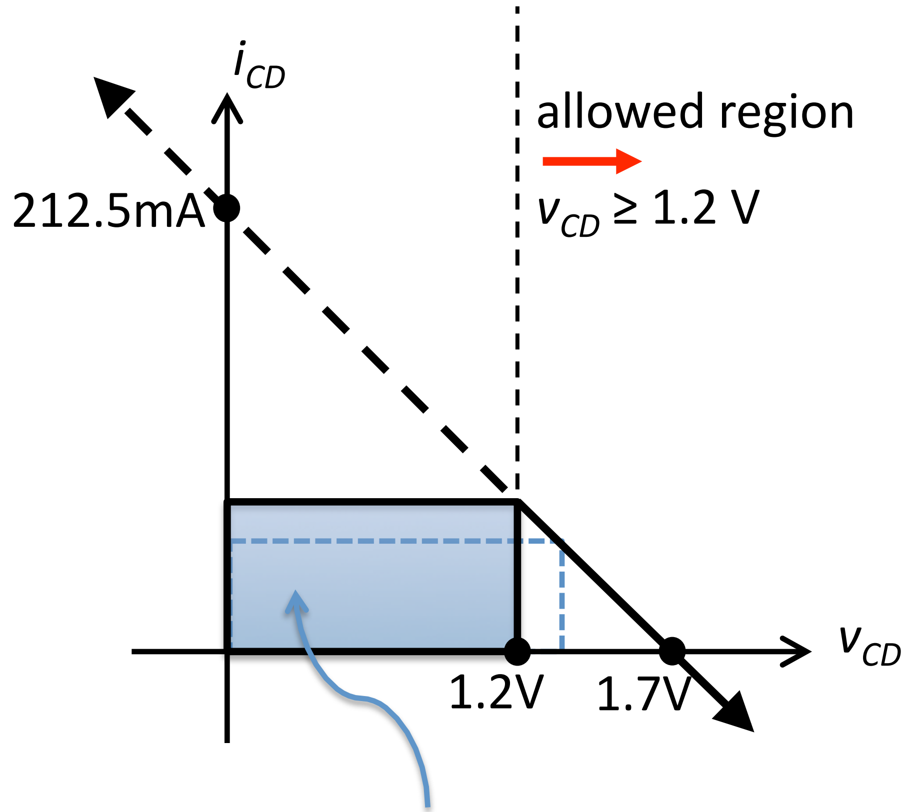
$$= 106.25\text{mA} \times 0.85\text{V}$$

$$= 90.3\text{mW}$$

Example (part b)

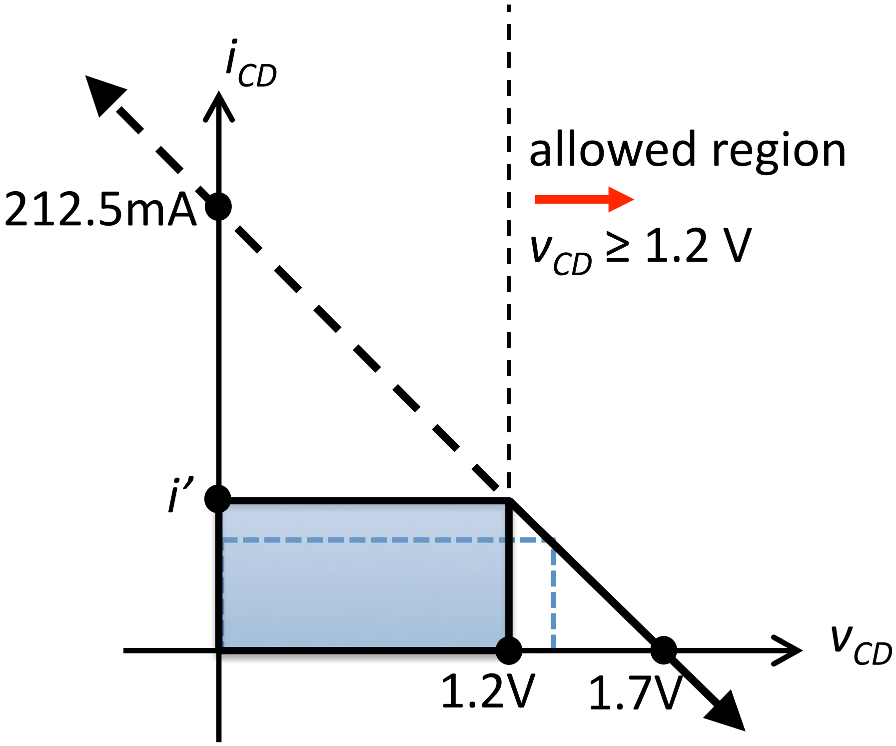


electrical model divided into source and load



maximum area of $i_{CD}v_{CD}$ that satisfies $v_{CD} \geq 1.2$ V

Example (part b)



By the geometry of triangles (ratios):

$$\frac{i'}{212.5\text{mA}} = \frac{1.7\text{V} - 1.2\text{V}}{1.7\text{V}}$$

$$i' = 62.5\text{mA}$$

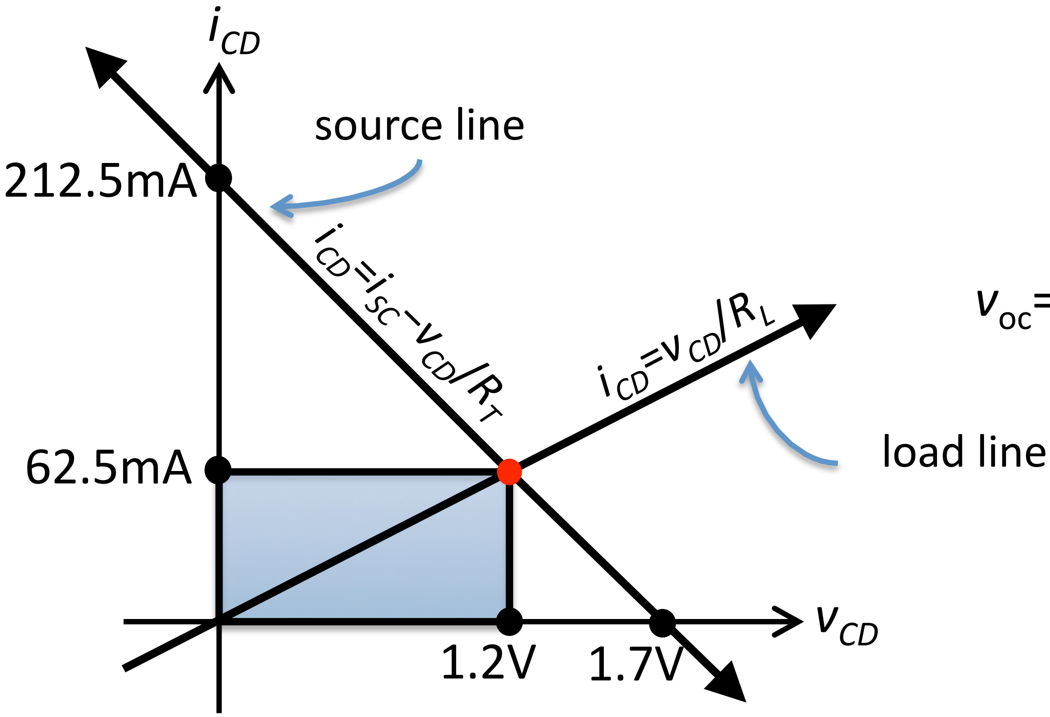
The maximum power absorbed while satisfying $v_{CD} \geq 1.2$ V is:

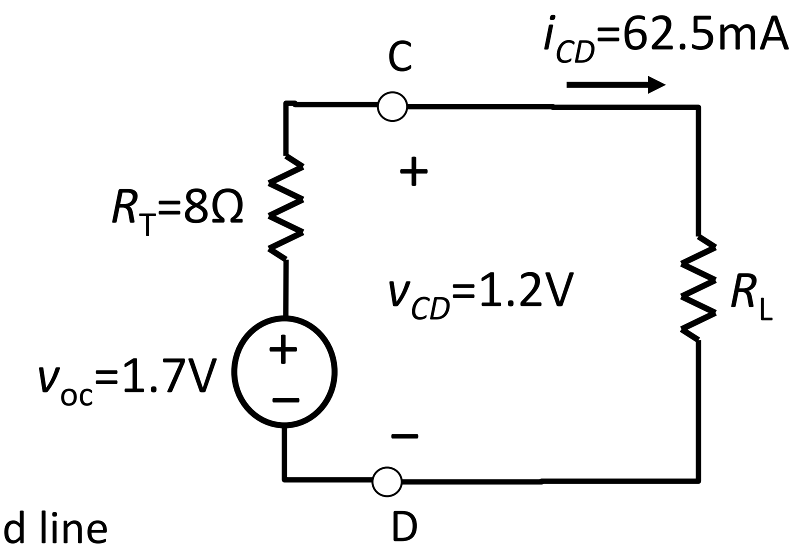
$$P_{del} = i_{CD} v_{CD}$$

$$= 62.5\text{mA} \times 1.2\text{V}$$

$$= 75\text{mW}$$

Example (part b)





Geometrically, the operating point of the circuit is the intersection of the ***source line*** and the ***load line*** on an *i-v* diagram.

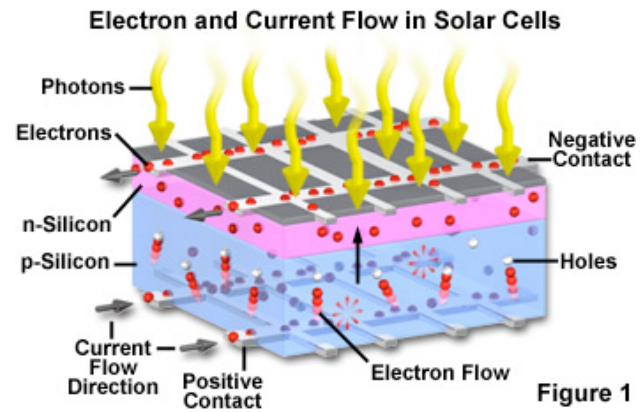
The load resistance must be:

$$R_L = \frac{v_{CD}}{i_{CD}} = \frac{1.2\text{V}}{62.5\text{mA}}$$

$$= 19.2\Omega$$

a diversion: photovoltaics

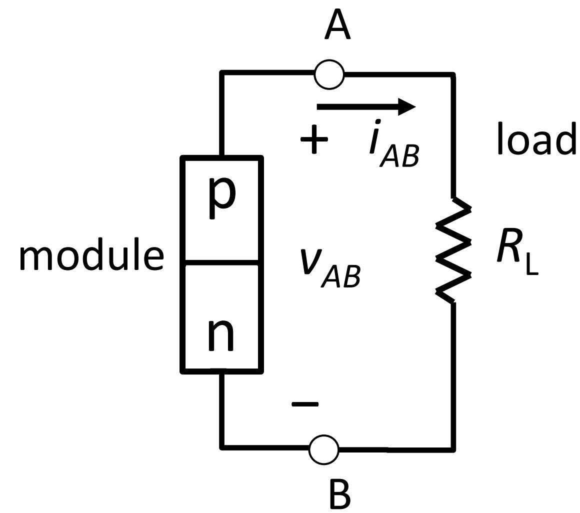
A photovoltaic device converts photon energy into the potential energy of electric charges within a semiconductor. Solar panels are assembled from modules composed of individual cells. The cells are usually made of the semiconductor silicon. The efficiency of the energy conversion process is important in determining the economic cost ($) of the energy produced.

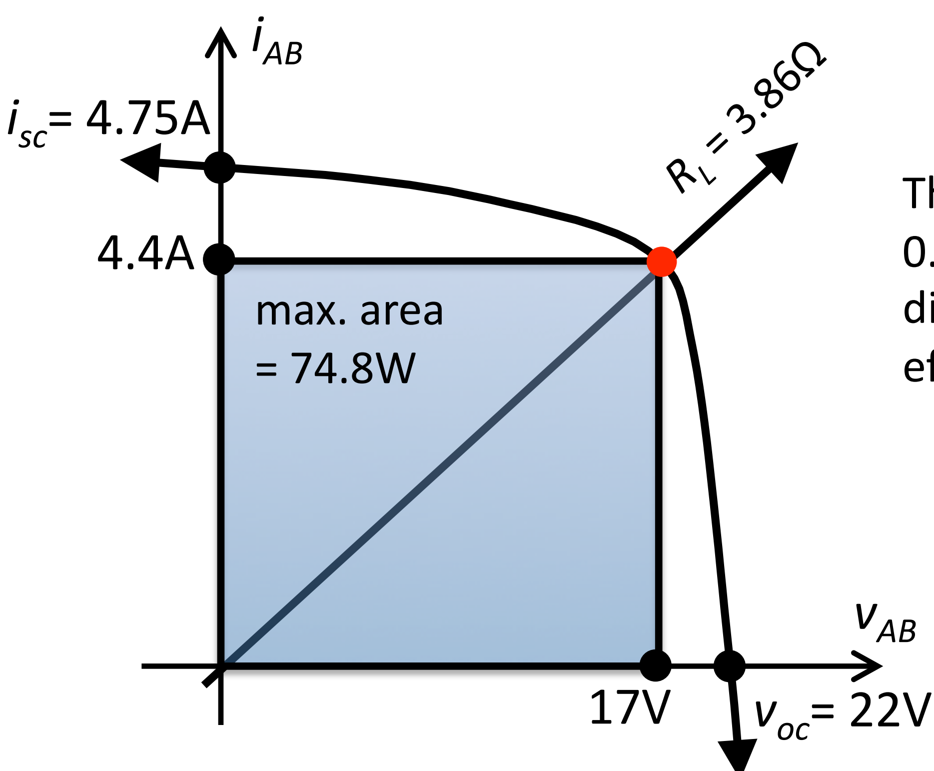


source: Molecular Expressions

a diversion: photovoltaics

The i-v diagram of a solar cell is ***nonlinear***, and so ***cannot*** be modeled with Thévenin or Norton equivalent circuits. However, power transfer can still be interpreted in terms of the geometry of the i-v diagram.





The i-v diagram is for a commercial module of 0.6m² area under 1kW/m² of sunlight (sun directly overhead on a clear day). Note the efficiency η:

$$P_{in} = 0.6\text{m}^2 \times 1\frac{\text{kW}}{\text{m}^2} = 600\text{W}$$

$$P_{out} = 75\text{W}$$

$$\eta = \frac{75\text{W}}{600\text{W}} = 12.5\%$$

Section 4 Summary

Source Transformations: Thévenin equivalent circuits and Norton equivalent circuits can be converted amongst each other using the relation $v_{oc} = i_{sc} R_T$.

Principle of Superposition: A circuit variable can be calculated as the algebraic sum of contributions from each independent source in a linear circuit.

Thévenin's Theorem: Any two terminal circuit composed of independent sources, dependent sources and resistors is equivalent to a Thévenin equivalent circuit.

Norton's Theorem: Any two terminal circuit composed of independent sources, dependent sources and resistors is equivalent to a Norton equivalent circuit.

Section 4 Summary

Maximum Power Transfer Theorem: The maximum power that can be delivered by a Thévenin or Norton equivalent circuit to a load resistor is achieved when the load resistor is equal to the Thévenin equivalent resistance R_T.