A warm-up problem…

A spark plug requires *several thousand* to *several tens of thousand* volts to spark and ignite the air-fuel mixture of an internal combustion engine.

A typical battery provides 12V.

How are spark plugs fired in an automobile, motor boat and other instances of internal combustion engines?

7. First Order Circuits

- overview of RC and RL circuits
- response to a constant input
- sequential switching
- stability
- unit step response
- response to a non-constant input
- operator method

Today's Outline

7. First Order Circuits

- overview of RC and RL circuits
- response to a constant input
  - RC circuit

Overview

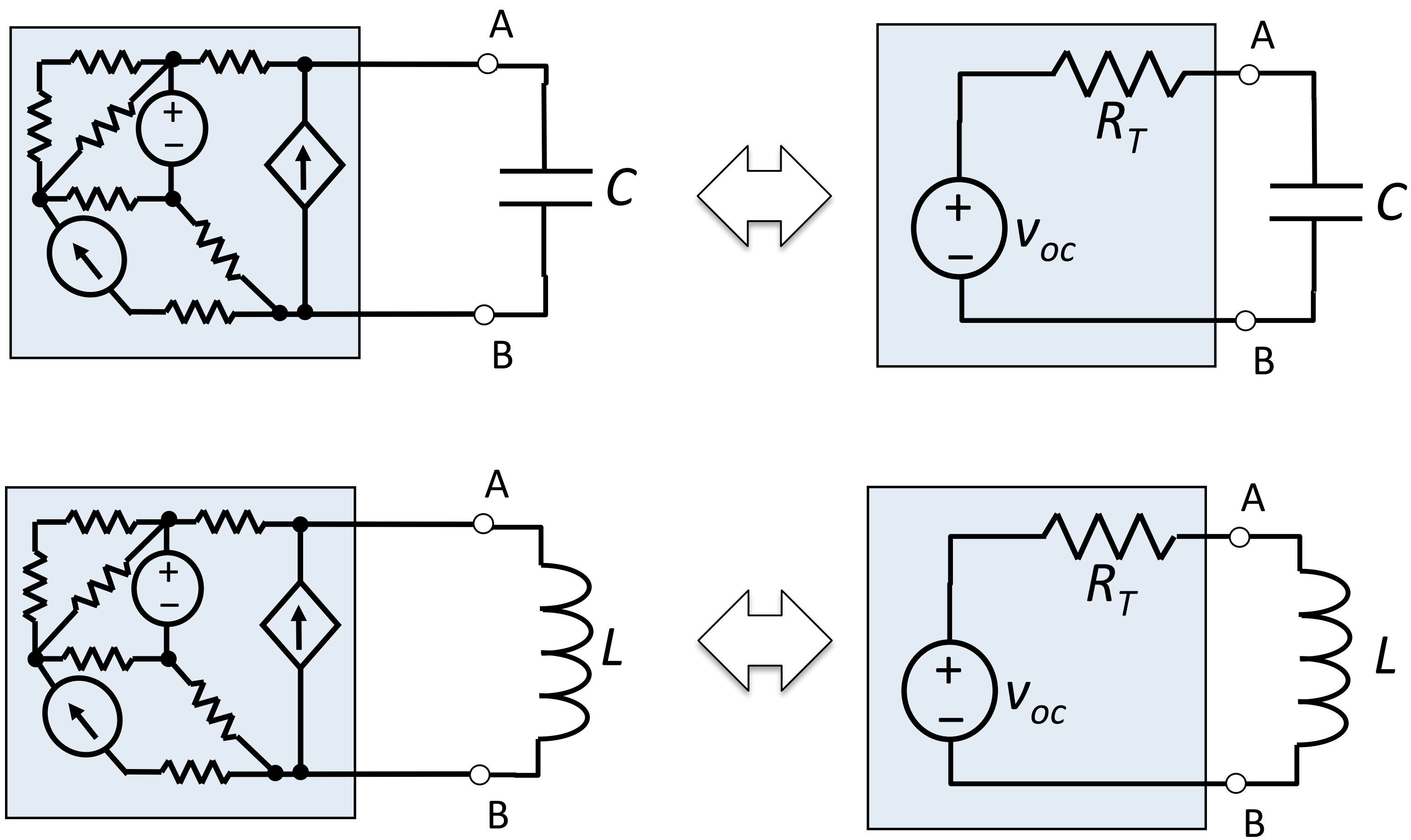
First order circuit: a circuit composed of resistors, sources and either one capacitor (an **RC circuit**) or one inductor (an **RL circuit**).

– “first order” refers to the **first order linear differential equations** that describe the time evolution of the voltage and current variables $v(t)$ and $i(t)$

– *RL* and *RC* circuits are extremely useful because of the *time dependence* of voltages and currents in such circuits

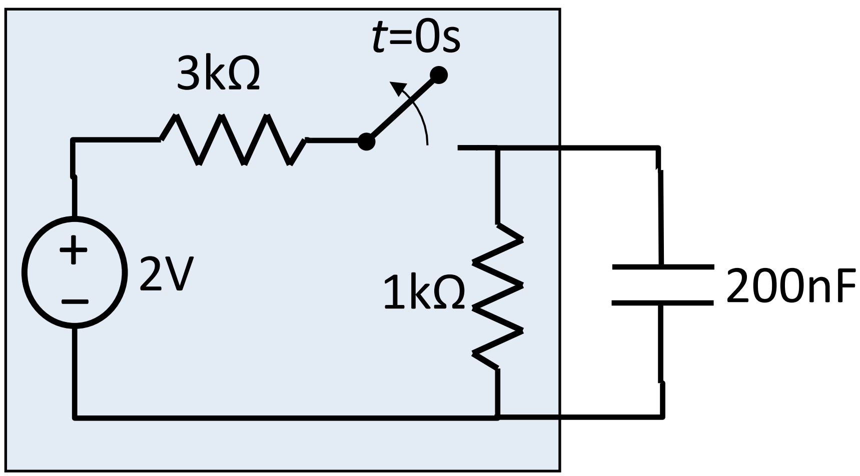
Overview

Using Thévenin (or Norton) equivalence, *RC* and *RL* circuits can be reduced to the following simple forms.

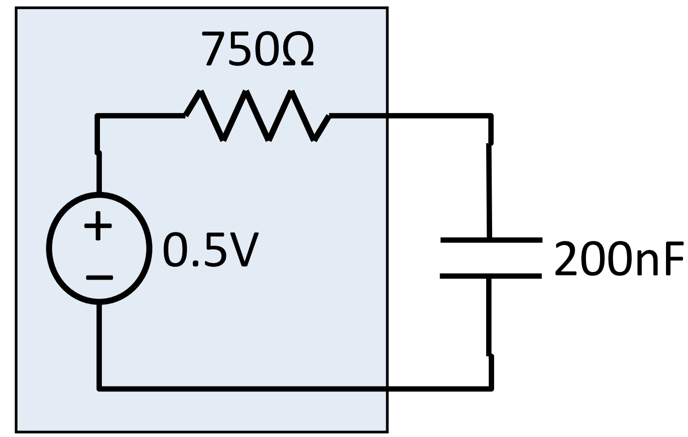


Overview

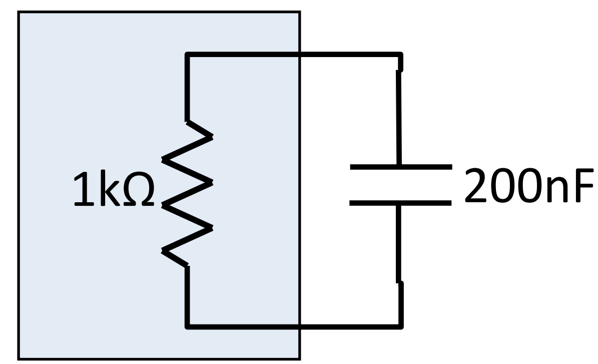
The time dependence of voltages and currents are of interest in circuits that include **switches.** The Thévenin equivalent circuit in each time interval should be considered, for example:



t < 0s



t ≥ 0s



t < 0s calculation details:

$$v_{oc} = 2\text{V}\frac{1\text{k}\Omega}{1\text{k}\Omega + 3\text{k}\Omega} = 0.5\text{V}$$

$$R_T = 1\text{k}\Omega \,||\, 3\text{k}\Omega = 0.75\text{k}\Omega$$

Overview

It is useful to divide the response of a first order circuit in two ways:

total response = transient response + steady state response

we distinguish the “short-lived” transient and “long-lived” steady-state portions of the circuit response

total response = natural response + forced response

we distinguish unforced (natural) and forced portions of the circuit response

mathematical review 1

Consider the ***homogeneous linear first order differential equation,*** which corresponds to ***unforced (natural)*** behaviour:

$$\frac{dx}{dt} + kx = 0$$

The unforced solution is: $\boxed{x_n(t) = X\exp(-kt)}$

(**natural response**)

Check:

$$\frac{dx}{dt} + kx = \frac{d}{dt}\Big(X\exp(-kt)\Big) + k\cdot X\exp(-kt)$$

$$= -k\cdot X\exp(-kt) + k\cdot X\exp(-kt)$$

$$= 0 \quad \checkmark$$

The value of X is specified by initial conditions of the circuit variable $x(t)$.

mathematical review 2

Consider the ***inhomogeneous linear first order differential equation,*** which corresponds to ***forced*** behaviour:

$$\frac{dx}{dt} + kx = G$$

The particular solution is: $x_p(t) = G/k$

(**forced response**)

Check: $$\frac{dx}{dt} + kx = 0 + k \cdot \frac{G}{k} = G \quad ✔$$

The complete solution is: $x(t) = x_p(t) + x_n(t) = G/k + X\exp(-kt)$

(**forced + natural response**) $= \boxed{c_1 + c_2\exp(-kt)}$

Check: $$\frac{dx}{dt} + kx = -kX\exp(-kt) + k\frac{G}{k} + kX\exp(-kt)$$

$$= G \quad ✔$$

mathematical review 3

Consider the general ***inhomogeneous linear first order differential equation,*** which corresponds to ***forced*** behaviour:

$$\frac{dx}{dt} + kx = g(t)$$

The particular solution is: $x_p(t) = \exp(-kt)\int \exp(kt')g(t')\,dt'$

Check:

$$\frac{dx}{dt} = \frac{d}{dt}\Big(\exp(-kt)\Big)\cdot\int \ldots dt' + \exp(-kt)\cdot\frac{d}{dt}\Big(\int \ldots dt'\Big)$$

$$= -k\cdot x_p(t) + \exp(-kt)\cdot\exp(+kt)g(t)$$

$$= -k\cdot x_p(t) + g(t)$$

$$kx = +k\cdot x_p(t)$$

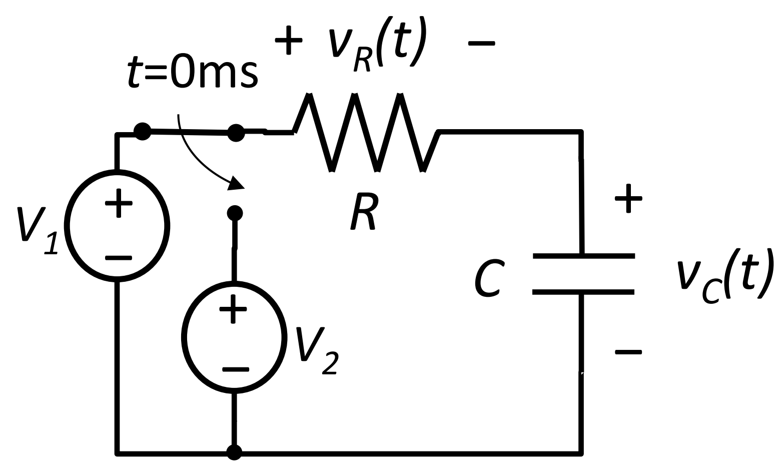
$$\frac{dx}{dt} + kx = g(t) \quad \checkmark$$

It is easy to show that the complete solution is:

$$x(t) = x_p(t) + x_n(t)$$

RC circuit with a (switched) constant input

Consider an *RC* circuit being switched at t=0 between two different open circuit voltages. Assume steady state has been reached for t<0. We would like to know $v_C(t)$ and $v_R(t)$.

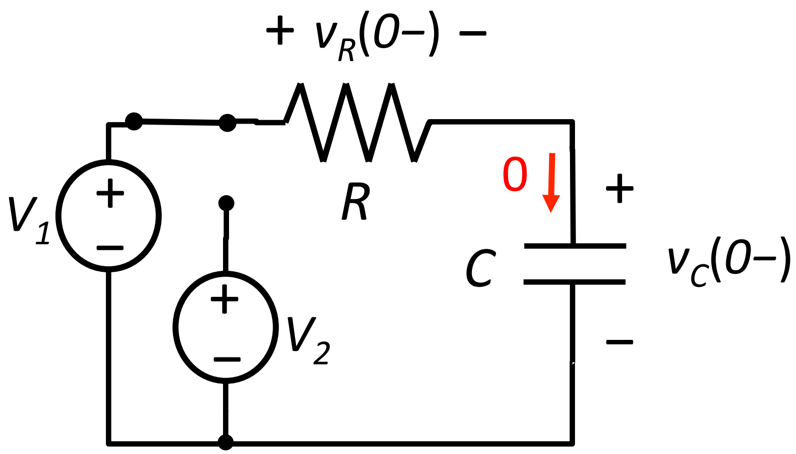


Note: The solution to this problem is found with the mathematics of differential equations. However, every feature of the solution can be easily explained by physics (KCL, KVL, Ohm's Law, and q=Cv).

constant input: capacitor voltage

Consider first the steady state conditions for $t < 0$.

$t < 0$



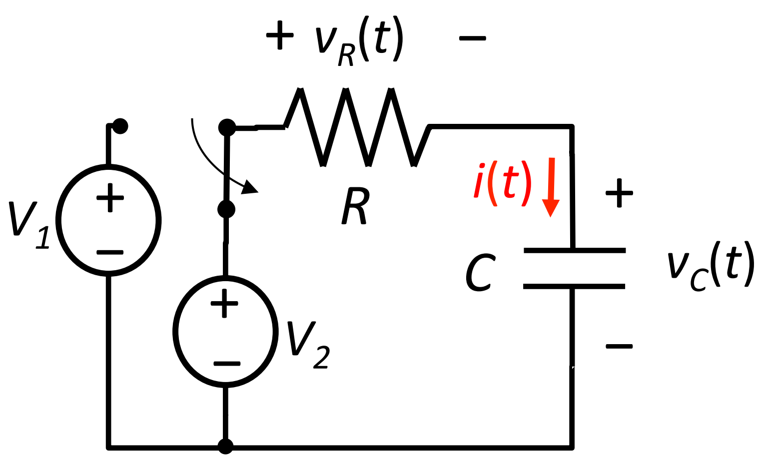
KVL: $$0 = -V_1 + v_R + v_C(0-)$$
$$= -V_1 + 0 \cdot R + v_C(0-)$$
$$v_C(0-) = V_1$$

Steady state: $i = C\dfrac{dv_C}{dt} = 0$

constant input: capacitor voltage

Consider the circuit equations for $t > 0$.

$t > 0$



KVL: $0 = -V_2 + v_R + v_C$

$= -V_2 + R \cdot i + v_C$

$= -V_2 + R \cdot C \dfrac{dv_C}{dt} + v_C$

$$\boxed{\frac{dv_C}{dt} + \frac{1}{RC} v_C = \frac{V_2}{RC} \qquad t > 0}$$

continuity of capacitor voltage:

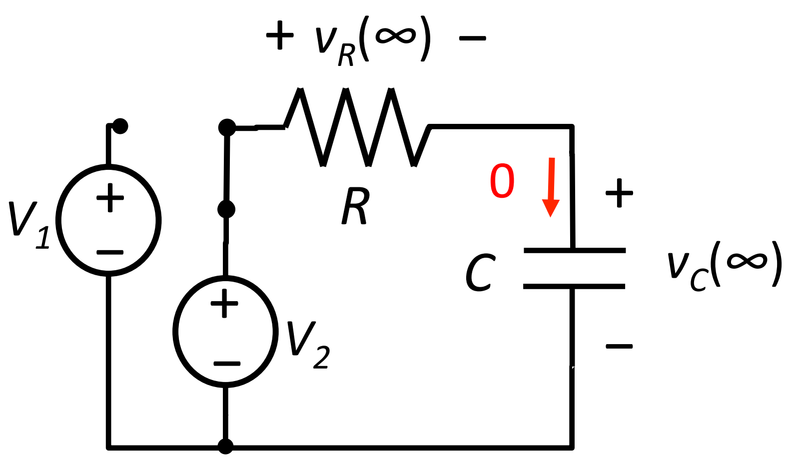
$$\boxed{v_C(0+) = v_C(0-) = V_1}$$

We have a first-order linear differential equation with initial conditions.

constant input: capacitor voltage

Consider steady state as t -> ∞.

t -> ∞



Steady state: $i = C\dfrac{dv_C}{dt} = 0$

KVL: $$\begin{aligned} 0 &= -V_2 + v_R(\infty) + v_C(\infty) \\ &= -V_2 + R\cdot 0 + v_C(\infty) \end{aligned}$$

$$\boxed{v_C(\infty) = V_2}$$

This can also be concluded from the circuit equation for t > 0 :

$$\left.\frac{dv_C}{dt}\right|_{t\to\infty} + \frac{1}{RC}v_C(\infty) = \frac{V_2}{RC}$$

$$0 \quad + \frac{1}{RC}v_C(\infty) = \frac{V_2}{RC}$$

$$v_C(\infty) = V_2$$

constant input: capacitor voltage

Solve the differential equation.

$$\frac{dv_C}{dt} + \frac{1}{RC} v_C = \frac{V_2}{RC} \qquad t > 0$$

$$v_C(0+) = V_1$$

Recall:

$$\frac{dx}{dt} + kx = G$$

$$x(t) = c_1 + c_2 \exp(-kt)$$

The form of the solution is: $v_c(t) = c_1 + c_2 \exp\left(-\frac{t}{RC}\right)$

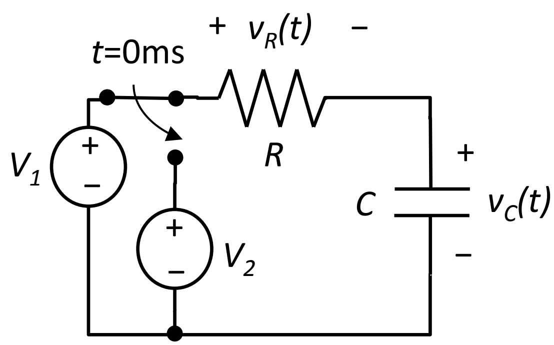
Use our initial and final conditions: $v_c(\infty) = \lim_{t \to \infty} \left[c_1 + c_2 \exp\left(-\frac{t}{RC}\right)\right] = c_1$

$$\therefore c_1 = v_c(\infty) = V_2$$

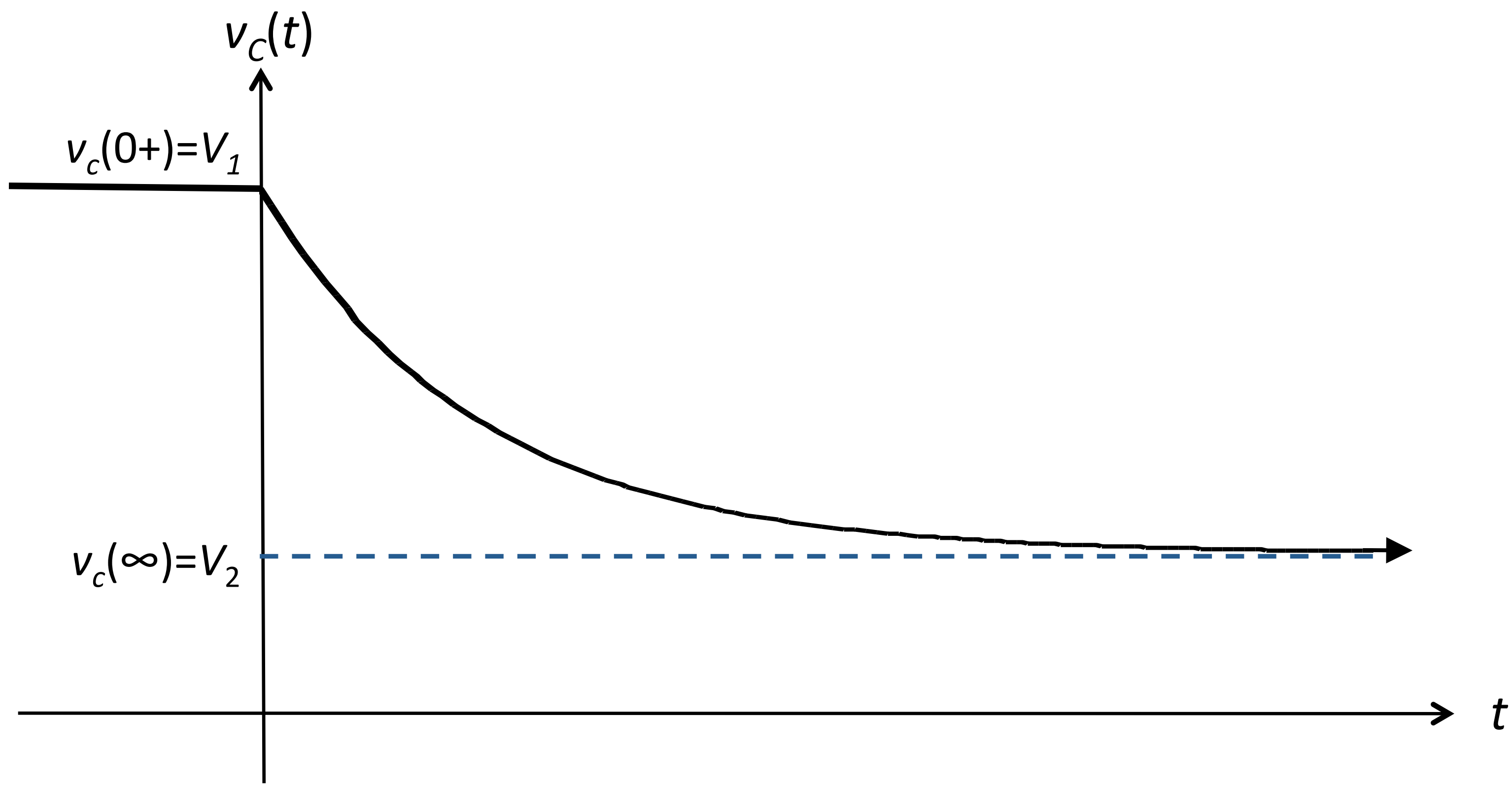
$$v_c(0+) = c_1 + c_2 \exp\left(0\right) = c_1 + c_2$$

$$\therefore c_2 = v_c(0+) - c_1 = V_1 - V_2$$

constant input: capacitor voltage



solution for $v_c(t)$ for t>0 : $$v_c(t) = V_2 + \left[V_1 - V_2\right]\exp\left(-\frac{t}{RC}\right)$$

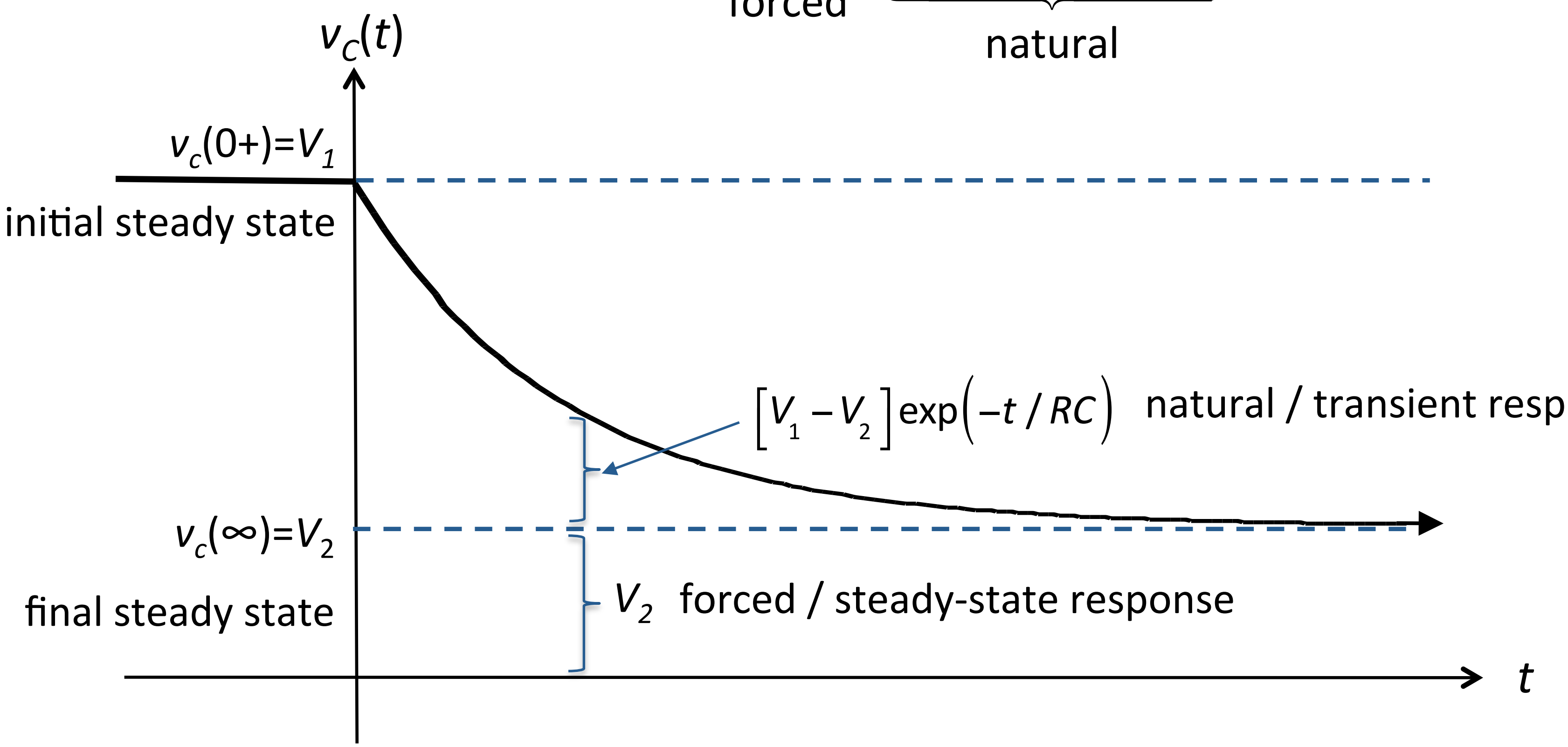


* We assume $V_1 > V_2$ in this graph.

natural and forced response

The response can be considered as the sum of a natural / transient response and a forced / steady-state response.

$$v_c(t) = \underbrace{V_2}_{\text{forced}} + \underbrace{\left[V_1 - V_2\right]\exp\left(-\frac{t}{RC}\right)}_{\text{natural}}$$



* We assume $V_1 > V_2$ in this graph.

time constant

Time constant: the characteristic time τ of the natural response of a first order circuit. Many important quantities are determined by the time constant.

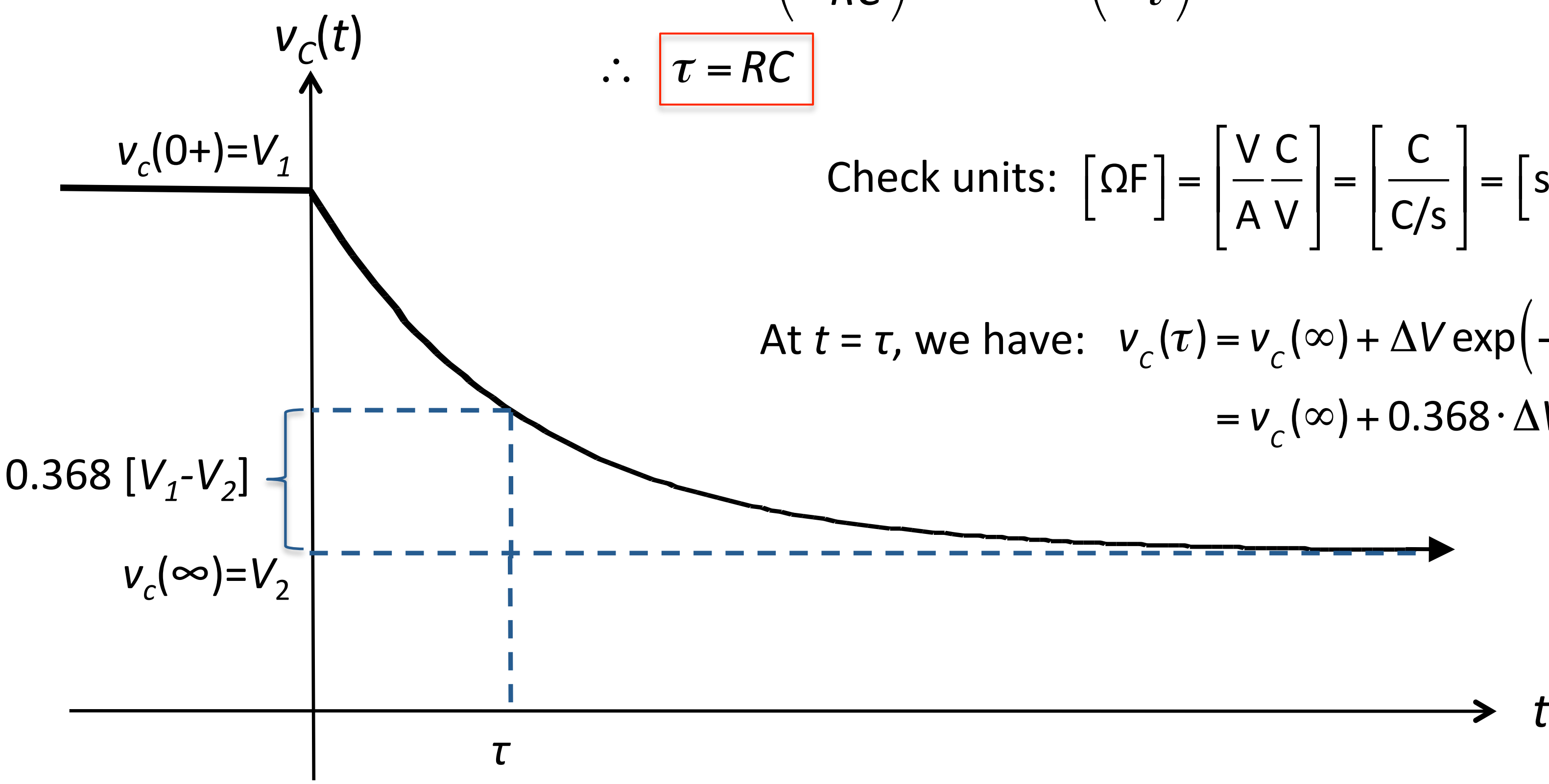
$$v_{natural}(t) = \Delta V \exp\left(-\frac{t}{RC}\right) = \Delta V \exp\left(-\frac{t}{\tau}\right)$$

$$\therefore \boxed{\tau = RC}$$

Check units: $\left[\Omega \mathrm{F}\right] = \left[\frac{\mathrm{V}}{\mathrm{A}}\frac{\mathrm{C}}{\mathrm{V}}\right] = \left[\frac{\mathrm{C}}{\mathrm{C/s}}\right] = \left[\mathrm{s}\right]$

At $t = \tau$, we have: $v_c(\tau) = v_c(\infty) + \Delta V \exp(-1)$

$$= v_c(\infty) + 0.368 \cdot \Delta V$$



* We assume $V_1 > V_2$ in this graph.

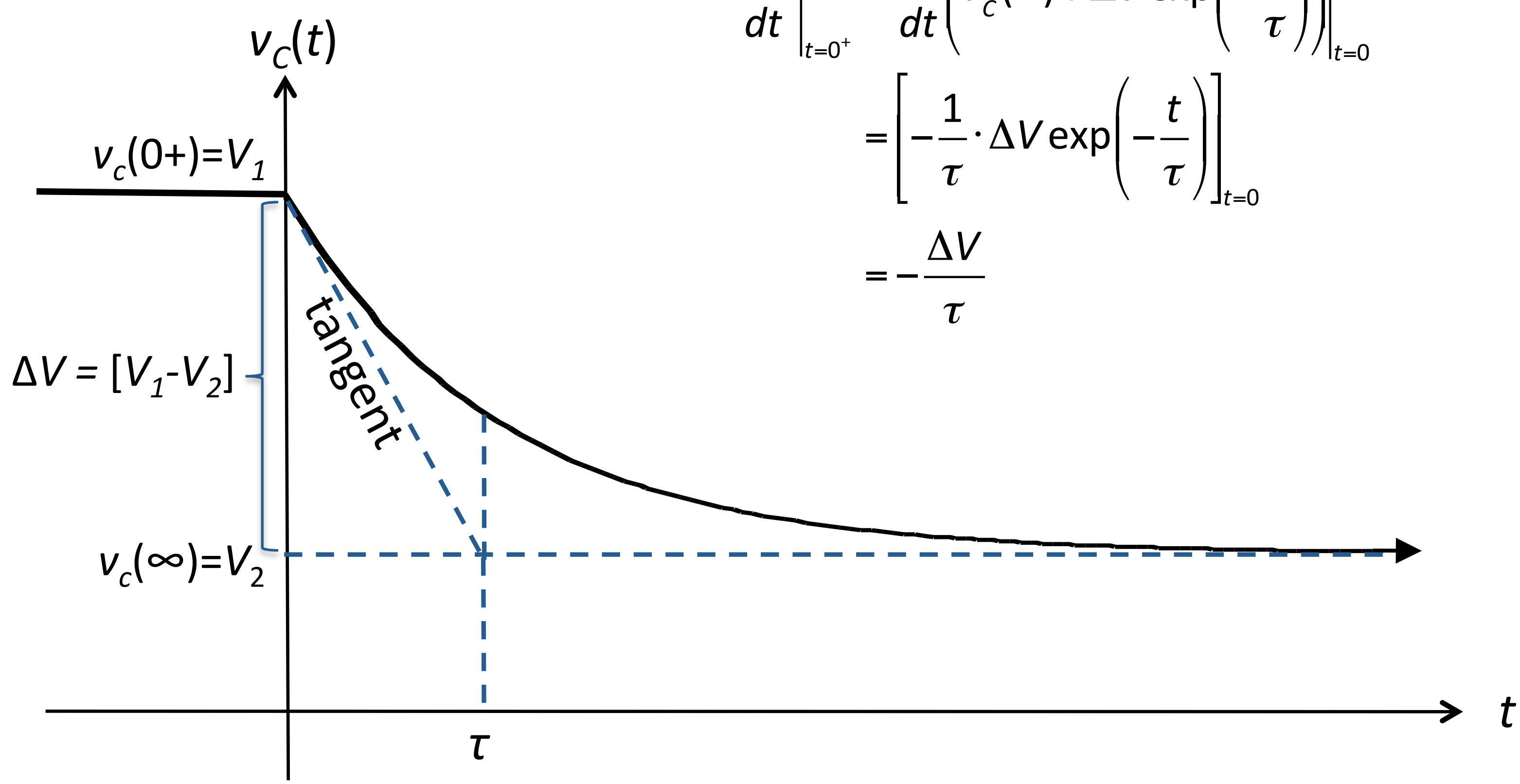
time constant

The time constant τ determines the initial rate of change of voltage, as does the amplitude of the transient response $\Delta V = v_C(0) - v_C(\infty) = V_1 - V_2$.

$$\left.\frac{dv_c}{dt}\right|_{t=0^+} = \left.\frac{d}{dt}\left(v_C(\infty) + \Delta V \exp\left(-\frac{t}{\tau}\right)\right)\right|_{t=0}$$

$$= \left[-\frac{1}{\tau}\cdot \Delta V \exp\left(-\frac{t}{\tau}\right)\right]_{t=0}$$

$$= -\frac{\Delta V}{\tau}$$

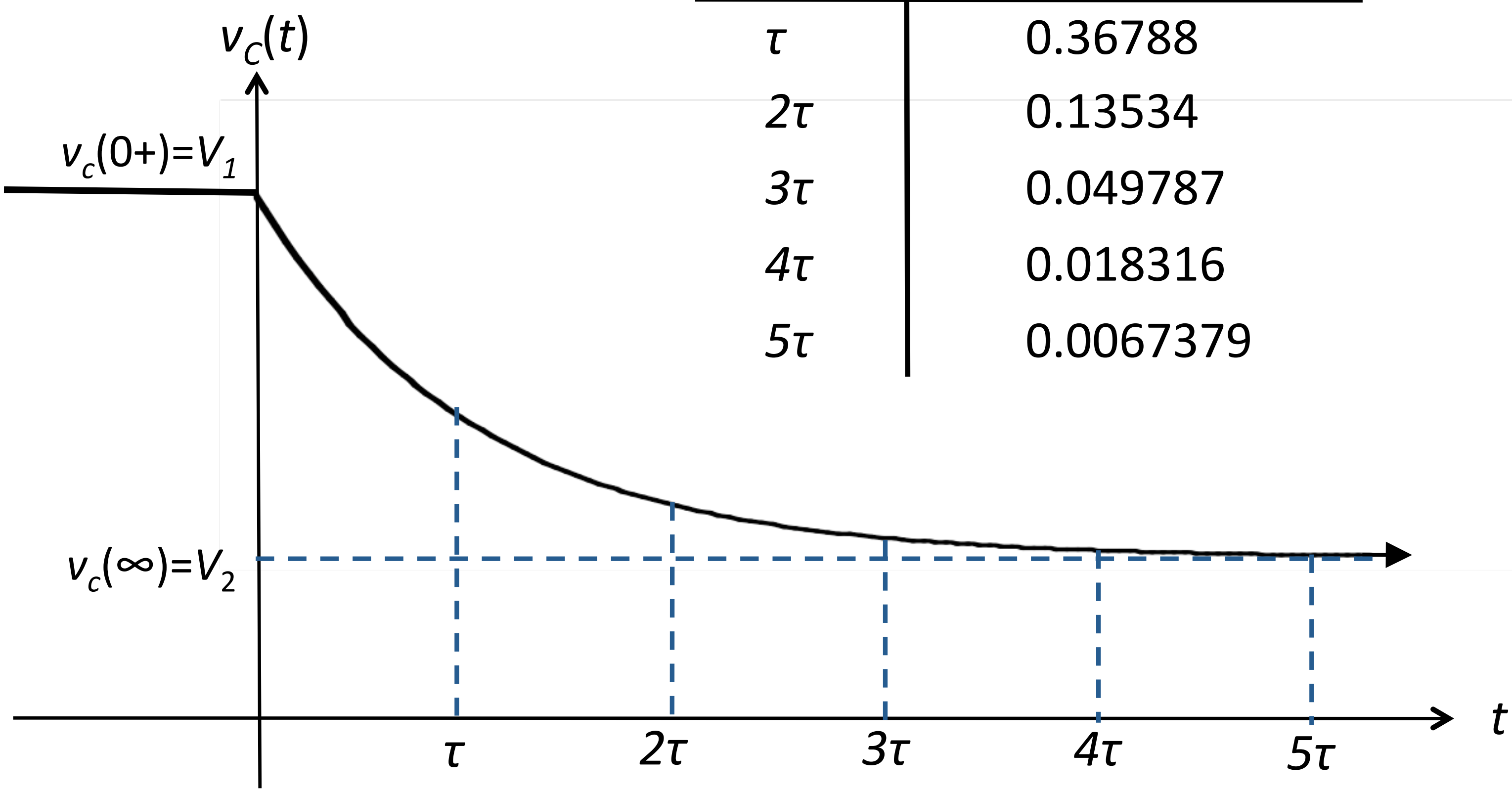


* We assume $V_1 > V_2$ in this graph.

time constant

After 5τ, the voltage is within 1% of its final, steady state value. One *often* approximates 5τ as the "long time" required for a first order circuit to reach steady state conditions.

t	$\exp(-t/\tau)$
τ	0.36788
2τ	0.13534
3τ	0.049787
4τ	0.018316
5τ	0.0067379

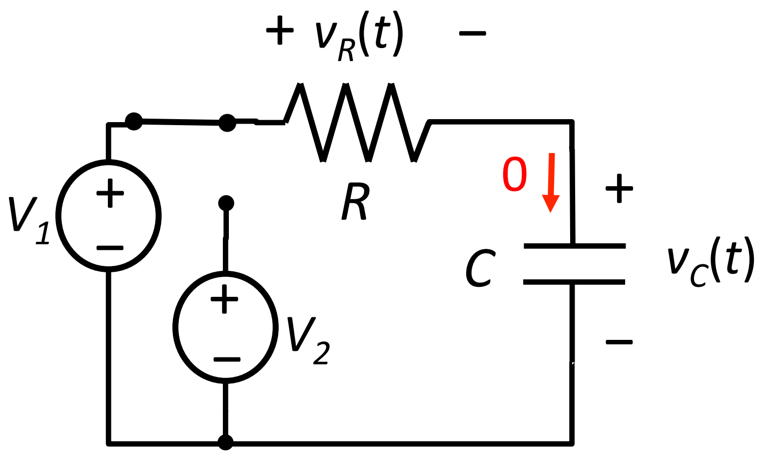


* We assume $V_1 > V_2$ in this graph.

constant input: resistor voltage

We now consider the resistor voltage $v_R(t)$. First consider the interval $t < 0$.

$t < 0$



Ohm: $v_R(t) = 0 \cdot R$

$$= 0 \qquad t < 0$$

Steady state: $i = C\dfrac{dv_C}{dt} = 0$

It will also be useful to recall our capacitor voltage:

$$0 = -V_1 + v_R + v_C(0-)$$

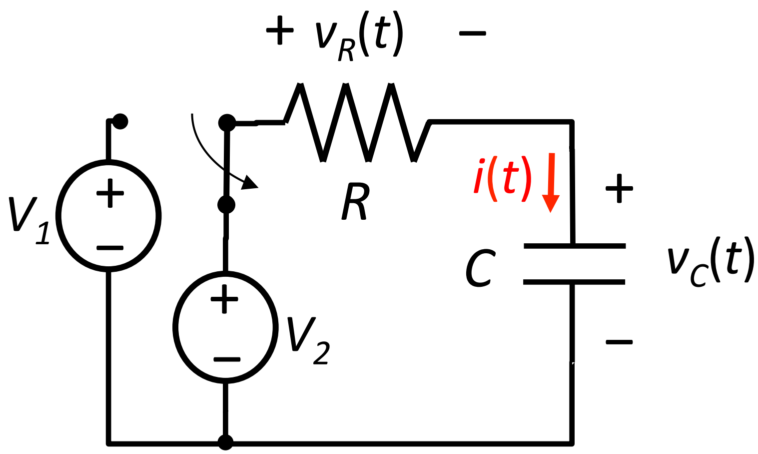
$$= -V_1 + 0 + v_C(0-)$$

$$v_C(0-) = V_1$$

constant input: resistor voltage

Next, consider the circuit equations for $t > 0$.

$t > 0$



KVL: $$\begin{aligned} 0 &= -V_2 + v_R + v_C \\ &= -V_2 + v_R + \frac{1}{C}\int i\, dt' \\ &= -V_2 + v_R + \frac{1}{C}\int \frac{v_R}{R}\, dt' \end{aligned}$$

Differentiate the entire equation:

$$\boxed{0 = \frac{dv_R}{dt} + \frac{1}{RC} v_R}$$

The initial value is given by considering *capacitor* voltage continuity:

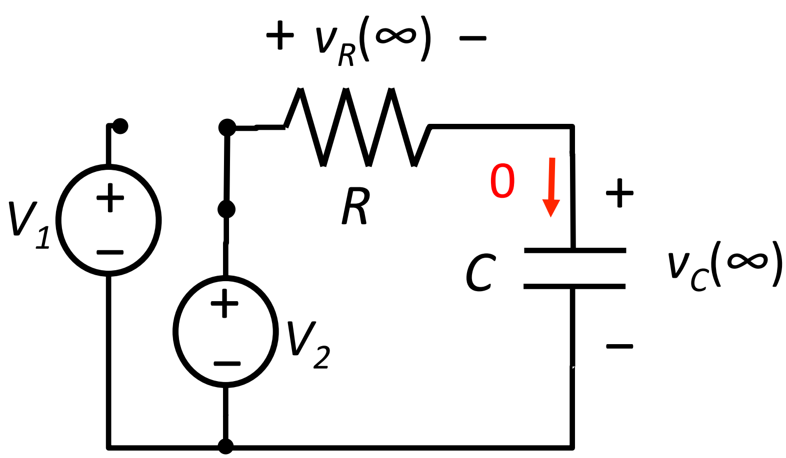
$$\begin{aligned} 0 &= -V_2 + v_R(0+) + v_C(0+) \\ &= -V_2 + v_R(0+) + V_1 \end{aligned}$$

$$\boxed{v_R(0+) = V_2 - V_1}$$

constant input: resistor voltage

Consider steady state as *t* -> ∞.

t -> ∞



Steady state: $i = C\dfrac{dv_C}{dt} = 0$

Ohm: $$v_R(\infty) = 0 \cdot R$$
$$= 0$$

This can also be concluded from the circuit equation for t > 0 :

$$\left.\frac{dv_R}{dt}\right|_{t\to\infty} + \frac{1}{RC}v_R(\infty) = 0$$

$$0 \quad + \frac{1}{RC}v_R(\infty) = 0$$

$$v_R(\infty) = 0$$

constant input: resistor voltage

Solve the differential equation.

$$\frac{dv_R}{dt} + \frac{1}{RC} v_R = 0 \qquad t > 0$$

$$v_R(0+) = V_2 - V_1$$

Recall:

$$\frac{dx}{dt} + kx = 0$$

$$x(t) = c_2 \exp(-kt)$$

The form of the solution is: $$v_R(t) = c_2 \exp\left(-\frac{t}{RC}\right)$$

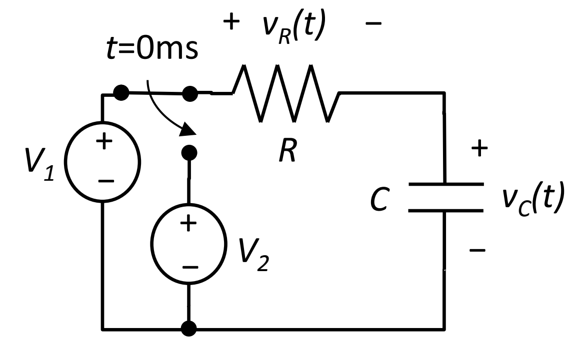
Use our initial conditions: $$v_R(0+) = c_2 \exp\left(0\right) = c_2$$

$$\therefore c_2 = v_R(0+) = V_2 - V_1$$

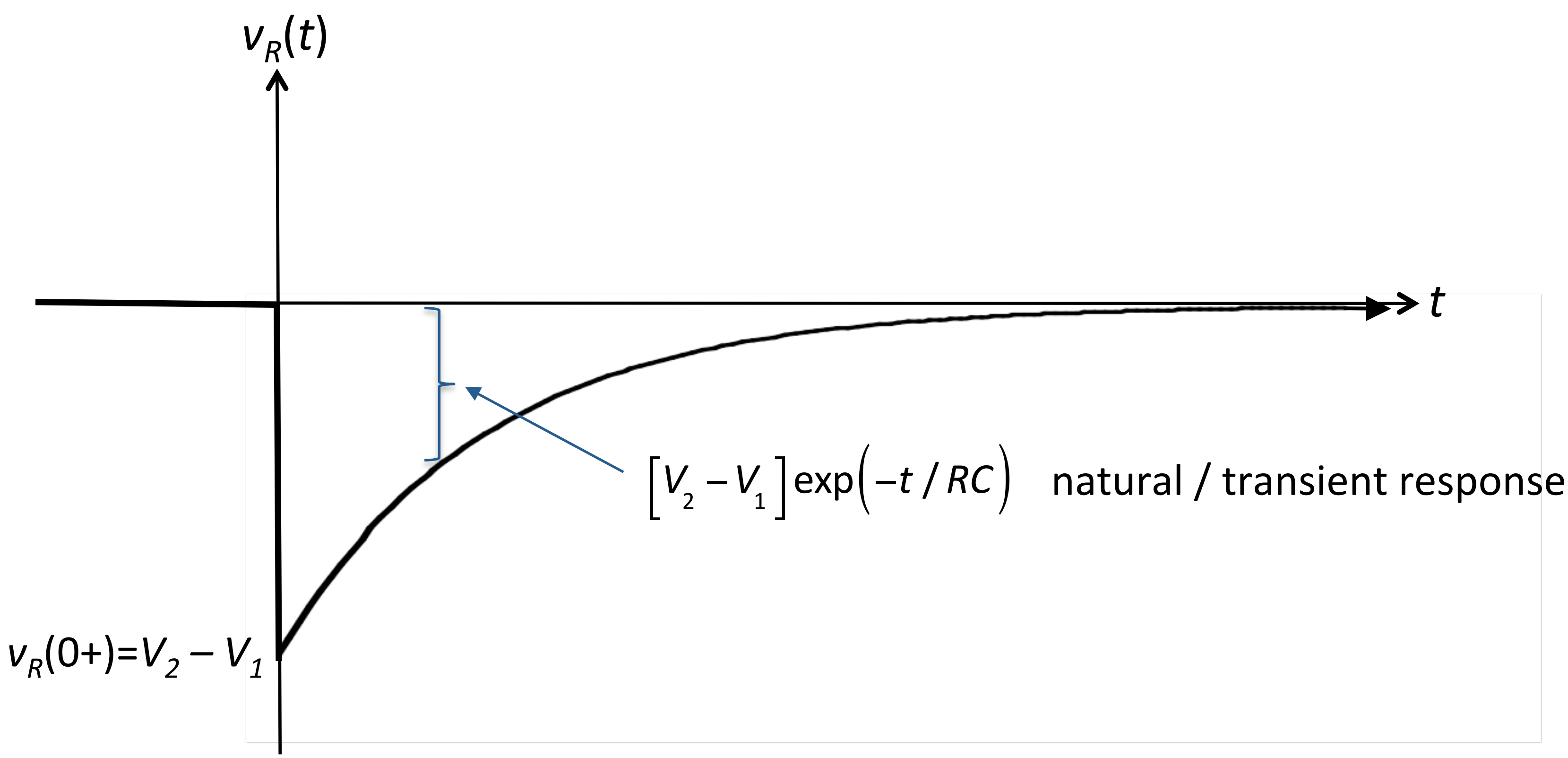
***** Important exercise***: Show that you can find $v_R(t)$ by

1) Using KVL and our solution for $v_C(t)$.
2) Using Ohm's Law and $i = C\, dv_C/dt$ with our solution for $v_C(t)$.

constant input: resistor voltage

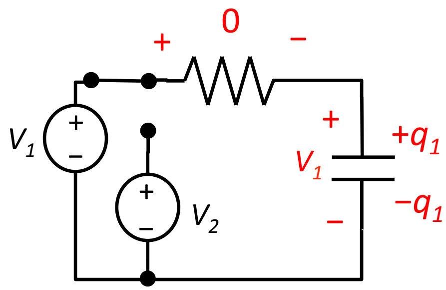


solution for $v_R(t)$ for $t>0$: $v_R(t) = [V_2 - V_1]\exp\left(-\frac{t}{RC}\right)$

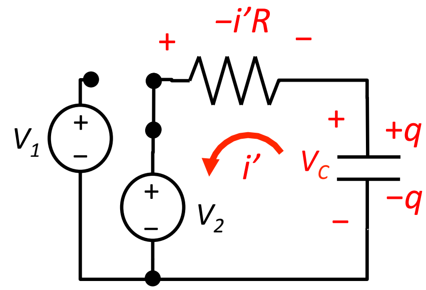


* We assume $V_1 > V_2$ in this graph.

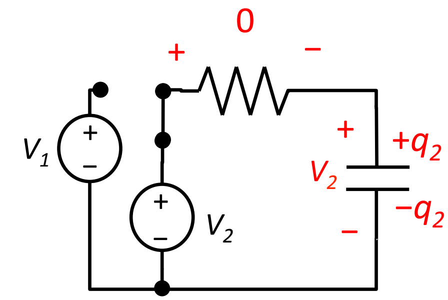
RC circuit: physical explanation



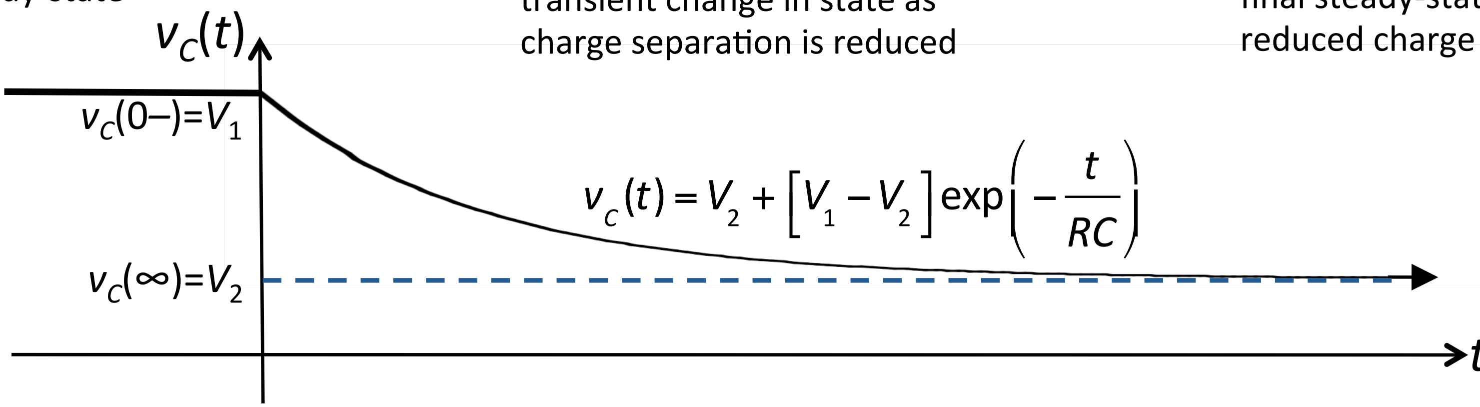
initial steady-state

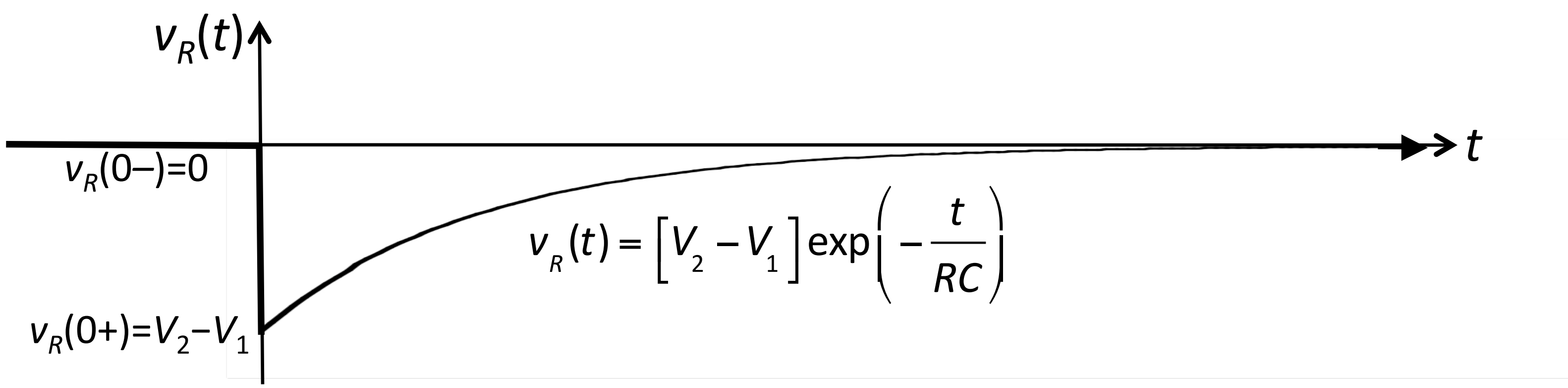


transient change in state as charge separation is reduced



final steady-state with reduced charge separation





* We assume $V_1 > V_2$ in this graph.