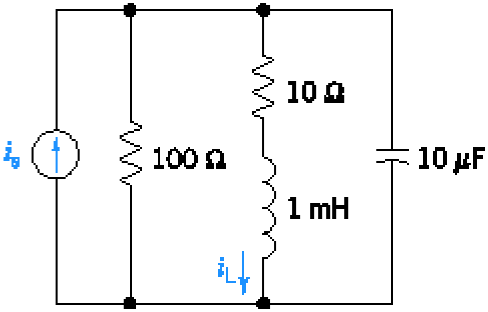
ECSE 200 - Electric Circuits 1
Tutorial 12

ECE Dept., McGill University

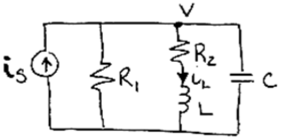
Problem P 9.2-2

Find the differential equation for the circuit shown in the figure using the operator method.



Problem P 9.2-2 Solution

Solution:



$$\text{KCL: } i_s = \frac{v}{R_1} + i_L + Csv$$

$$\text{KVL: } v = R_2 i_L + Ls i_L$$

Solving Cramer's rule for i_L:

$$i_L = \frac{i_s}{\frac{R_2}{R_1} + \frac{Ls}{R_1} + R_2 Cs + LCs^2 + 1}$$

$$\left[1 + \frac{R_2}{R_1}\right] i_L + \left[\frac{L}{R_1} + R_2 C\right] s i_L + [LC] s^2 i_L = i_s$$

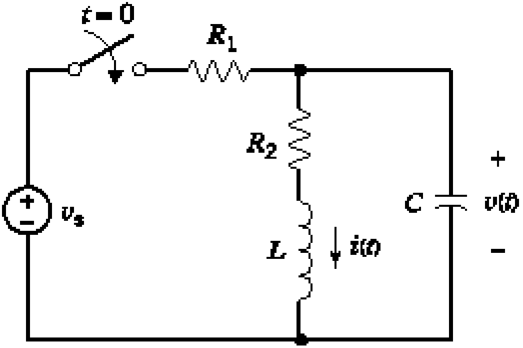
$$R_1 = 100\Omega,\ R_2 = 10\Omega,\ L = 1\text{mH},\ C = 10\mu\text{F}$$

$$1.1 i_L + .00011 s i_L + 1\times 10^{-8} s^2 i_L = i_s$$

$$\underline{1.1\times 10^8 i_L + 11000 s i_L + s^2 i_L = 1\times 10^8 i_s}$$

Problem P 9.2-6

The input to the circuit shown in the figure is the voltage of the voltage source, v_s. The output is the inductor current $i(t)$. Represent the circuit by a second-order differential equation that shows how the output of this circuit is related to the input, for $t > 0$.



Problem P 9.2-6 Solution

Solution:
After the switch closes use KVL to get

$$R_2\, i(t) + L\frac{d}{dt}i(t) = v(t)$$

Use KCL and KVL to get

$$v_s = R_1\left(i(t) + C\frac{d}{dt}v(t)\right) + v(t)$$

Substitute to get

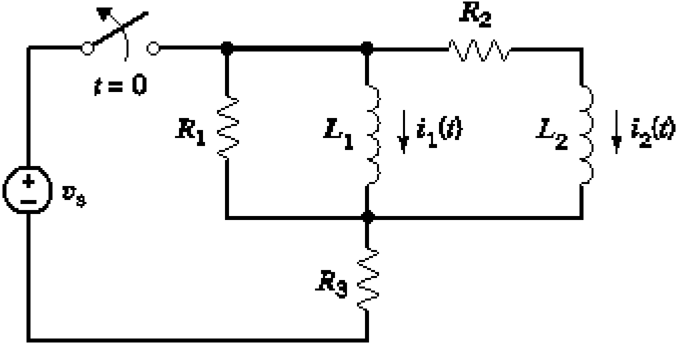
$$\begin{aligned} v_s &= R_1 i(t) + R_1 C R_2 \frac{d}{dt}i(t) + R_1 C L\frac{d^2}{dt^2}i(t) + R_2 i(t) + L\frac{d}{dt}i(t) \\ &= R_1 C L\frac{d^2}{dt^2}i(t) + \left(R_1 R_2 C + L\right)\frac{d}{dt}i(t) + \left(R_1 + R_2\right)i(t) \end{aligned}$$

Finally

$$\frac{v_s}{R_1 C L} = \frac{d^2}{dt^2}i(t) + \left(\frac{R_2}{L} + \frac{1}{R_1 C}\right)\frac{d}{dt}i(t) + \frac{R_1 + R_2}{R_1 C L}i(t)$$

Problem P 9.2-7

The input to the circuit shown in the figure is the voltage of the voltage source, v_s. The output is the inductor current $i_2(t)$. Represent the circuit by a second-order differential equation that shows how the output of this circuit is related to the input, for $t > 0$.



Problem P 9.2-7 Solution

Solution:
After the switch opens, KVL gives

$$L_1 \frac{d}{dt} i_1(t) = R_2\, i_2(t) + L_2 \frac{d}{dt} i_2(t)$$

KVL and KCL give

$$L_1 \frac{d}{dt} i_1(t) + R_1\left(i_1(t) + i_2(t)\right) = 0$$

Use the operator method to get

$$L_1 s\, i_1 = R_2\, i_2 + L_2 s\, i_2$$

$$L_1 s\, i_1 + R_1\left(i_1 + i_2\right) = 0$$

$$L_1 s^2 i_1 + R_1 s\, i_1 + R_1 s\, i_2 = 0$$

$$s\left(R_2 i_2 + L_2 s\, i_2\right) + \frac{R_1}{L_1}\left(R_2 i_2 + L_2 s\, i_2\right) + R_1 s\, i_2 = 0$$

$$L_2 s^2\, i_2 + \left(R_2 + R_1 \frac{L_2}{L_1} + R_1\right) s\, i_2 + \frac{R_1 R_2}{L_1} i_2 = 0$$

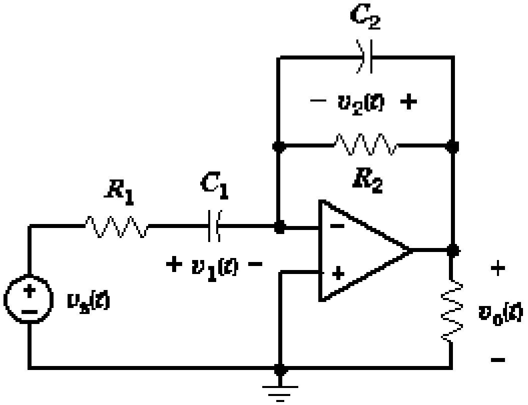
$$s^2 i_2 + \left(\frac{R_2}{L_2} + \frac{R_1}{L_2} + \frac{R_1}{L_1}\right) s\, i_2 + \frac{R_1 R_2}{L_1 L_2} i_2 = 0$$

So,

$$\frac{d^2}{dt^2} i_2(t) + \left(\frac{R_2}{L_2} + \frac{R_1}{L_2} + \frac{R_1}{L_1}\right) \frac{d}{dt} i_2(t) + \frac{R_1 R_2}{L_1 L_2} i_2(t) = 0$$

Problem P 9.2-12

The input to the circuit shown in the figure is the voltage of the voltage source, $v_s(t)$. The output is the voltage $v_o(t)$. Derive the second-order differential equation that shows how the output of this circuit is related to the input.



Problem P 9.2-12 Solution

KVL gives $$v_s(t) = R_1C_1\frac{d}{dt}v_1(t) + v_1(t)$$

KCL gives $$C_1\frac{d}{dt}v_1(t) + C_2\frac{d}{dt}v_2(t) + \frac{v_2(t)}{R_2} = 0$$

KVL gives $$v_o(t) = v_2(t)$$

Using the operator method

$$v_s = R_1C_1sv_1 + v_1$$

$$C_1sv_1 + C_2sv_2 + \frac{v_2}{R_2} = 0$$

Solving

$$v_1 = -\left(\frac{C_2}{C_1}v_2 + \frac{1}{R_2C_1s}v_2\right)$$

$$sv_s = \left(sR_1C_1 + 1\right)\left(\frac{C_2}{C_1}s + \frac{1}{R_2C_1}\right)v_o$$

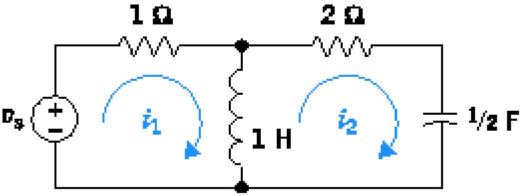
$$\frac{1}{R_1C_2}sv_s = s^2v_o + \left(\frac{1}{R_1C_1} + \frac{1}{R_2C_2}\right)sv_o + \frac{1}{R_1R_2C_1C_2}v_o$$

The corresponding differential equation is

$$\frac{1}{R_1C_2}\frac{d}{dt}v_s(t) = \frac{d^2}{dt^2}v_o(t) + \left(\frac{1}{R_1C_1} + \frac{1}{R_2C_2}\right)\frac{d}{dt}v_o(t) + \frac{1}{R_1R_2C_1C_2}v_o(t)$$

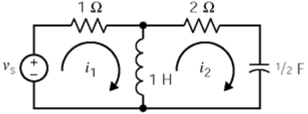
Problem P 9.2-15

Find the second-order differential equation for i_2 for the circuit of the figure using the operator method. Recall that the operator for the integral is $1/s$.



Problem P 9.2-15 Solution

Solution:



Apply KVL to the left mesh : $\underline{i_1 + s(i_1 - i_2) = v_s}$ (1)

where $s = d/dt$

Apply KVL to the right mesh : $2i_2 + 2\left(\frac{1}{s}\right)i_2 + s(i_2 - i_1) = 0$

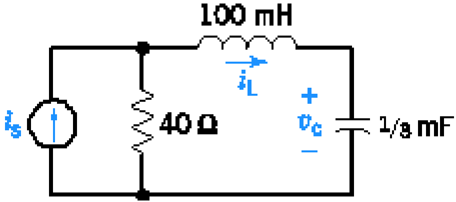
$$\Rightarrow \underline{i_1 = 2\left(\frac{1}{s}\right)i_2 + 2\left(\frac{1}{s^2}\right)i_2 + i_2} \qquad (2)$$

Plugging (2) into (1) yields

$$3s^2 i_2 + 4si_2 + 2i_2 = s^2 v_s \quad or \quad \underline{3\frac{d^2 i_2}{dt^2} + 4\frac{di_2}{dt} + 2i_2 = \frac{d^2 v_s}{dt^2}}$$

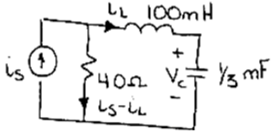
Problem P 9.3-2

Find the characteristic equation and its roots for the circuit of the figure.



Problem P 9.3-2 Solution

Solution:



$$\text{KVL}: 40(i_s - i_L) = 100\times10^{-3}\frac{di_L}{dt} + v_c$$

$$i_L = i_c = \left(\frac{1}{3}\times10^{-3}\right)\frac{dv_c}{dt}$$

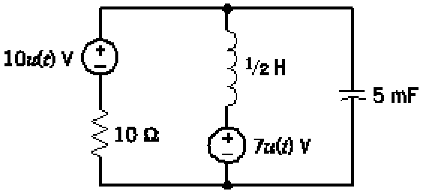
$$i_L = \frac{40}{3}\times10^{-3}\frac{di_s}{dt} - \frac{40}{3}\times10^{-3}\frac{di_L}{dt} - \frac{100}{3}\times10^{-6}\frac{d^2 i_L}{dt^2}$$

$$\frac{d^2 i_L}{dt^2} + 400\frac{di_L}{dt} + 30000 i_L = 400\frac{di_s}{dt}$$

$$\underline{s^2 + 400s + 30000 = 0} \Rightarrow (s+100)(s+300) = 0 \Rightarrow \underline{s_1 = -100, \quad s_2 = -300}$$

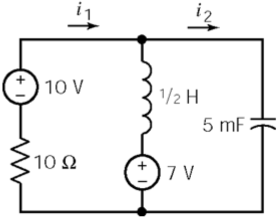
Problem P 9.3-4

German automaker Volkswagen, in its bid to make more efficient cars, has come up with an auto whose engine saves energy by shutting itself off at stoplights. The stopstart system springs from a campaign to develop cars in all its world markets that use less fuel and pollute less than vehicles now on the road. The stopstart transmission control has a mechanism that senses when the car does not need fuel: coasting downhill and idling at an intersection. The engine shuts off, but a small starter flywheel keeps turning so that power can be quickly restored when the driver touches the accelerator. A model of the stopstart circuit is shown in the figure. Determine the characteristic equation and the natural frequencies for the circuit.



Problem P 9.3-4 Solution

Assume zero initial conditions



$$\text{loop 1}: 10i_1 + \frac{1}{2}\frac{di_1}{dt} - \frac{1}{2}\frac{di_2}{dt} = 10-7$$

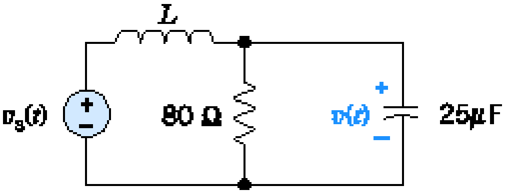
$$\text{loop 2}: -\frac{1}{2}\frac{di_1}{dt} + \frac{1}{2}\frac{di_2}{dt} + 200\int i_2 dt = 7$$

$$\text{determinant}: \begin{bmatrix} \left(10 + \frac{1}{2}s\right) & -\frac{1}{2}s \\ -\frac{1}{2}s & \left(\frac{1}{2}s + \frac{200}{s}\right) \end{bmatrix}$$

$$s^2 + 20s + 400 = 0, \quad \therefore \ s = -10 \pm j\,17.3$$

Problem P 9.4-1

Determine $v(t)$ for the circuit of the figure when $L = 1H$ and $v_s = 0$ for $t \geq 0$. The initial conditions are $v(0) = 6V$ and $dv/dt(0) = 3000V/s$.



Problem P 9.4-1 Solution

Solution:

$$v(0) = 6, \quad \frac{dv(0)}{dt} = -3000$$

Using operators, the node equation is: $Csv + \frac{v}{R} + \frac{(v - v_S)}{sL} = 0 \quad or \quad \left(LCs^2 + \frac{L}{R}s + 1\right) v = v_S$

So the characteristic equation is: $s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$

$$\Rightarrow s_{1,2} = -250 \pm \sqrt{250^2 - 40,000} = -100, \; -400$$

So $v(t) = Ae^{-100t} + Be^{-400t}$

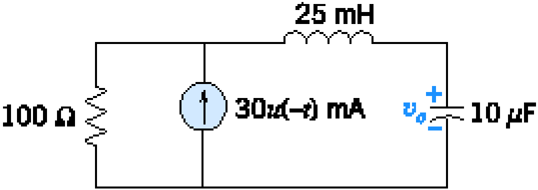
$$v(0) = 6 = A + B$$

$$\left.\frac{dv(0)}{dt} = -3000 = -100A - 400B\right\} \quad \begin{aligned} A &= -2 \\ B &= 8 \end{aligned}$$

$$\therefore \; v(t) = -2e^{-100t} + 8e^{-400t} \qquad t>0$$

Problem P 9.5-1

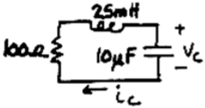
Find $v_c(t)$ for $t > 0$ for the circuit shown in the figure.



Problem P 9.5-1 Solution

Solution:

$\underline{t>0}$



$$\text{KVL a: } 100i_c + .025\frac{di_c}{dt} + v_c = 0,\ i_c = 10^{-5}\frac{dv_c}{dt}$$

$$\therefore\ \frac{d^2v_c}{dt^2} + 4000\frac{dv_c}{dt} + 4\times10^6 v_c = 0$$

$$s^2 + 4000s + 4\times10^6 = 0 \Rightarrow s = -2000,\ -2000 \ \therefore\ v_c(t) = A_1e^{-2000t} + A_2te^{-2000t}$$

$\underline{t=0^-}$ (Steady − State)

$$i_L = i_c(0^-) = 0 = i_c(0^+) \Rightarrow \frac{dv_c(0^+)}{dt} = 0$$

$$v_c(0^-) = 3\text{ V} = v_c(0^+)$$

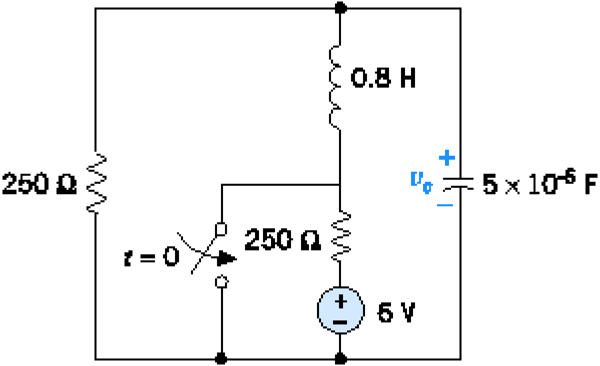
so $v_c(0^+) = 3 = A_1$

$$\frac{dv_c(0^+)}{dt} = 0 = -2000A_1 + A_2 \Rightarrow A_2 = 6000$$

$$\underline{v_c(t) = (3+6000t)e^{-2000t}\text{ V}}$$

Problem P 9.6-1

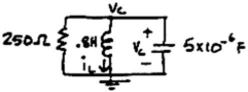
A communication system from a space station uses short pulses to control a robot operating in space. The transmitter circuit is modeled in the figure. Find the output voltage $v_c(t)$ for $t > 0$. Assume steady-state conditions at $t = 0$.



Problem P 9.6-1 Solution

Solution:

$\underline{t>0}$



KCL at v_c: $\quad v_c/250 + i_L + 5\times10^{-6}\dfrac{dv_c}{dt} = 0 \qquad (1)$

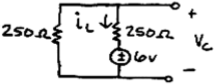
also: $\quad v_c = 0.8\dfrac{di_L}{dt} \qquad (2)$

Solving for i_L in (1) & plugging into (2)

$$\frac{d^2v_c}{dt^2}+800\frac{dv_c}{dt}+2.5\times10^5 v_c = 0 \quad \Rightarrow s^2+800s+250{,}000=0,\ s=-400\pm j\,300$$

$$\therefore\ v_c(t) = e^{-400t}\left[A_1\cos 300t + A_2\sin 300t\right]$$

$\underline{t=0^-}$ (Steady−State)



$$i_L(0^-) = \frac{-6\text{ V}}{500\ \Omega} = -6/500\ \text{A} = i_L(0^+)$$

$$v_c(0^-) = 250\left(-6/500\right)+6 = 3\text{ V} = v_c(0^+)$$

Now from (1): $\dfrac{dv_c(0^+)}{dt} = -\,2\times10^5 i_L(0^+) - 800 v_c(0^+) = 0$

So $v_c(0^+) = 3 = A_1$

$$\frac{dv_c(0^+)}{dt} = 0 = -400A_1+300A_2 \Rightarrow A_2 = 4$$

$$\therefore\ v_c(t) = e^{-400t}\left[3\cos 300t + 4\sin\ 300t\right]\ V$$

Thank you !