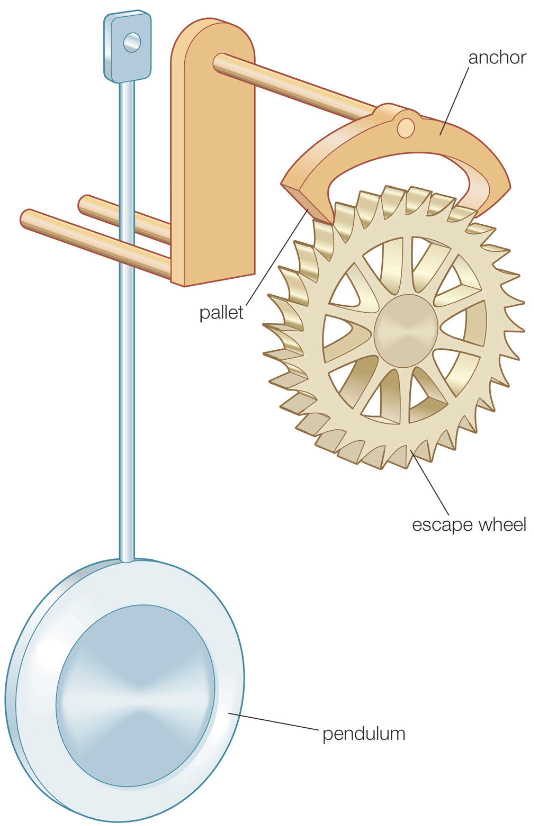
McGill

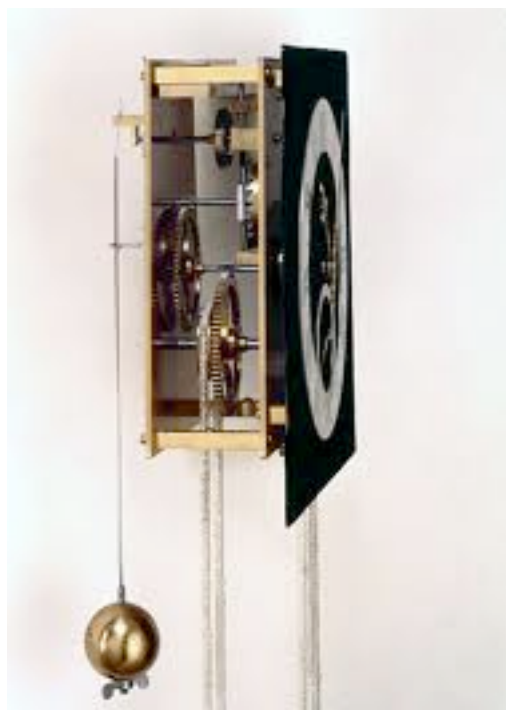
8. Second Order Circuits

- operator method
- the parallel RLC circuit
- underdamped, overdamped, and critically damped natural response

motivation

Second order circuits are of wide importance because they can ***store energy*** and ***oscillate*** at a precise frequency like mechanical oscillators, such as the pendulum and tuning fork. The periodic motion of the pendulum has been used to keep time since Christian Huygen's invention of the pendulum clock in 1656.

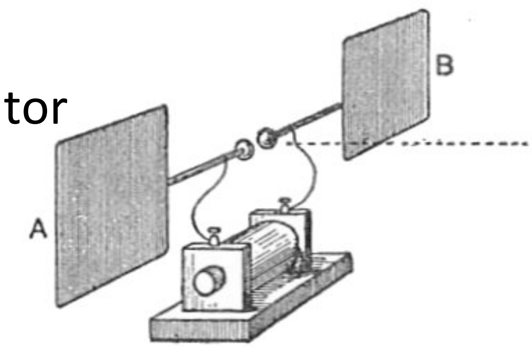




motivation

Second order circuits can produce oscillations of voltage and current. Hertz used an RLC circuit and spark antenna to for the first laboratory demonstration of the transmission of radio waves at a frequency of about 500MHz in 1887.

capacitor

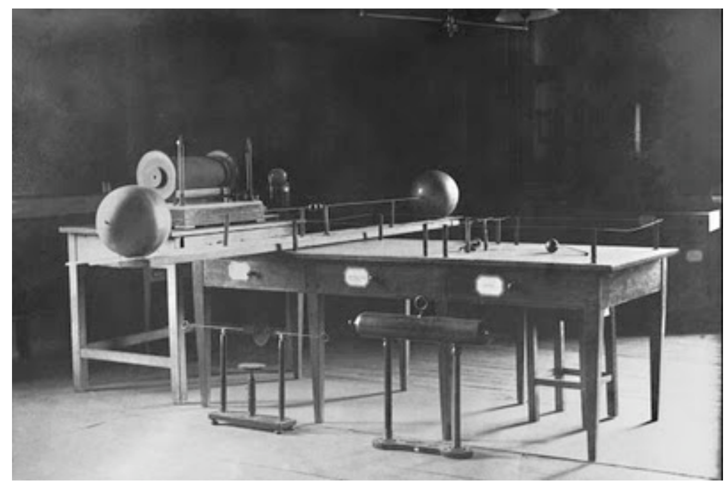


spark gap antenna

inductor

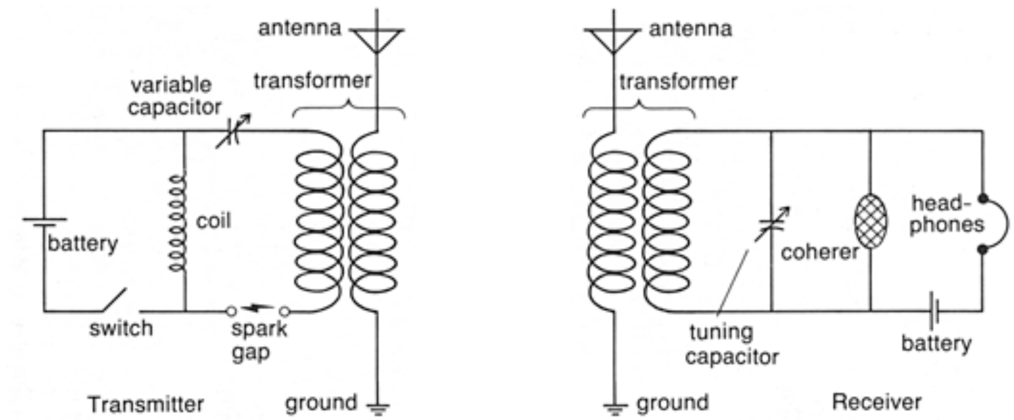
Heinrich Hertz

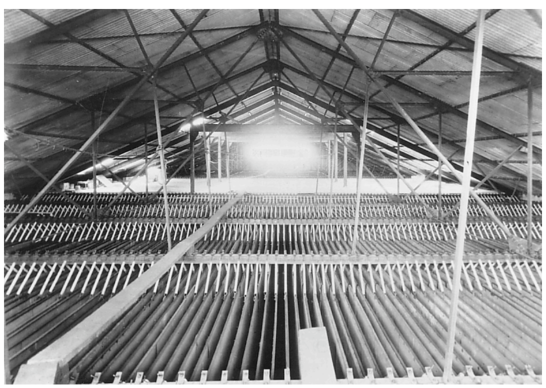
1857-1894



motivation

Marconi demonstrated transatlantic wireless communication in 1902 and setup the first transatlantic wireless link in 1907 using RLC circuits and antennas, integrating the inventions of others.





Marconi towers
Glace Bay, NS

hanging capacitor plates
288 units, C_{total} = 1.7μF

Guglielmo Marconi
1874-1937

McGill

motivation

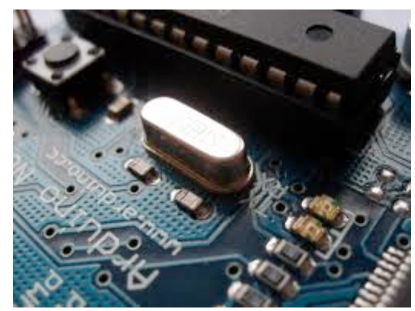
Energy storage and oscillation in second order circuits, and higher order circuits, is found in a wide range of applications.

radio, television, radar and wireless communications

wired communication between computers

industrial machines

power supplies



electronic timing

Today's Outline

8. Second Order Circuits

- operator method

operator method

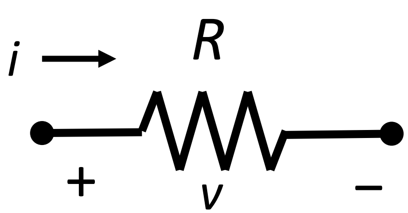
The ***operator method*** is a technique to write out ***differential equations*** for circuits with capacitors and inductors. We identify differentiation versus time with the symbol *s = d/dt* that we manipulate as an algebraic quantity.

Write out circuit equations in terms of *s.*

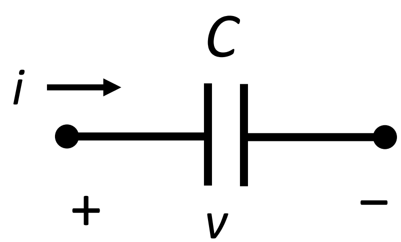
Construct an equation so that *s* is only found in the numerator.

Replace *s* by *d/dt*. The result is a differential equation.

The terminal laws for circuit components can be re-written as:

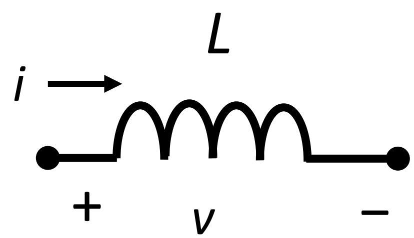


$$v = R \cdot i$$



$$i = C\frac{dv}{dt} \rightarrow i = sC \cdot v$$

$$v = \frac{1}{sC} \cdot i$$

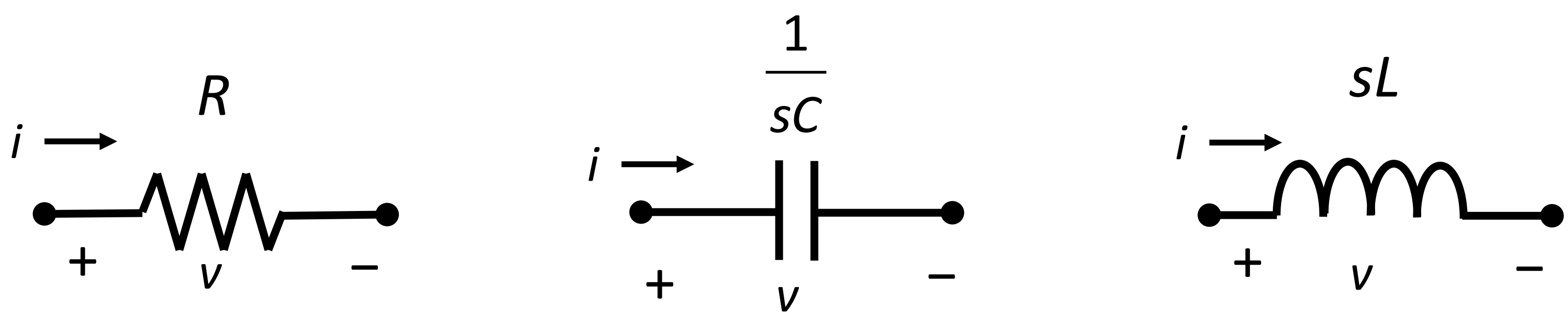


$$v = L\frac{di}{dt}$$

$$v = sL \cdot i$$

operator method

We can therefore give the passive circuit elements a simple *i*-*v* terminal law in terms of operator *s*.



$$\boxed{v = R \cdot i} \qquad \boxed{v = \frac{1}{sC} \cdot i} \qquad \boxed{v = sL \cdot i}$$

The operator method is a compact way of keeping track of differentiation and integration in equations.

McGill

operator method

Oliver Heaviside

(1850-1924)

dropped out of school in order to learn, telegraph operator, developed transmission line theory, co-inventor of vector calculus, invented operator method, predicted Čerenkov radiation, proposed the existence of ionosphere to explain long distance radio communication

" Shall I refuse my dinner because I do not fully understand the process of digestion? " Heaviside in response to criticism at the lack of mathematical proof behind the operator method

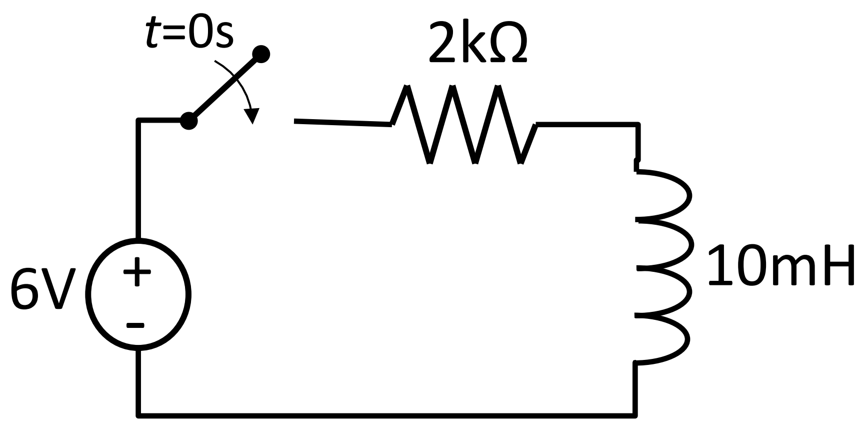
Pierre-Simon Laplace

(1749-1827)

It was only later determined that the operator method is equivalent to the application of Laplace transforms *(you will apply Laplace transforms to circuits in ECSE-210)*.

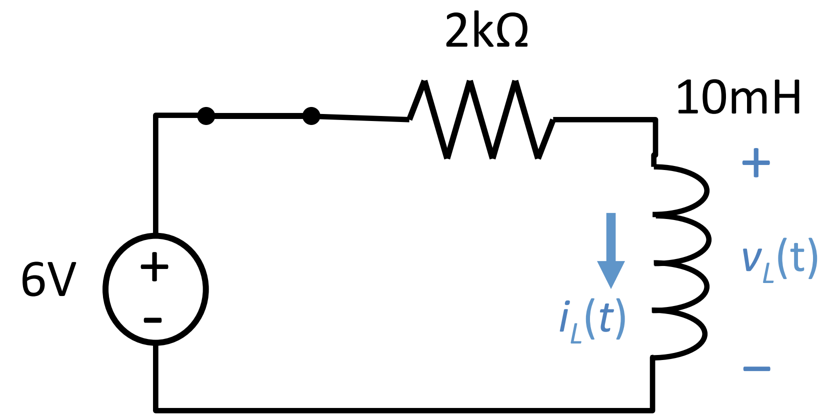
example 1

Use the operator method to write out the differential equation for the inductor current and inductor voltage at $t > 0$.



example 1

$t > 0$



We find the diff. equation for i_L:
(mesh equation)

$$0 = -6\text{V} + 2\text{k}\Omega \cdot i_L + s \cdot 10\text{mH} \cdot i_L$$

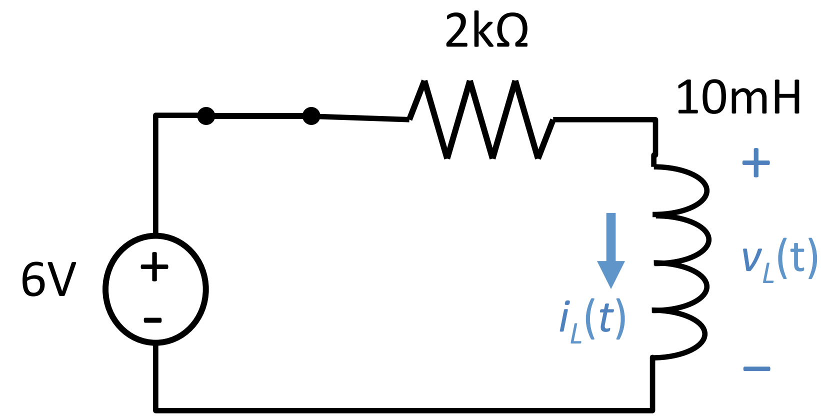
$$0 = -6\text{V} + 2\text{k}\Omega \cdot i_L + 10\text{mH} \cdot \frac{di_L}{dt}$$

Note that the first equation is equivalent to an equation for series combination:

$$i_L = \frac{6\text{V}}{2\text{k}\Omega + s \cdot 10\text{mH}}$$

example 1

$t > 0$



We find the diff. equation for v_L:
(node equation)

$$0 = \frac{v_L - 6\text{V}}{2\text{k}\Omega} + \frac{v_L}{s \cdot 10\text{mH}}$$

$$0 = \frac{s \cdot 10\text{mH}}{2\text{k}\Omega}\left(v_L - 6\text{V}\right) + \frac{s \cdot 10\text{mH}}{s \cdot 10\text{mH}} v_L$$

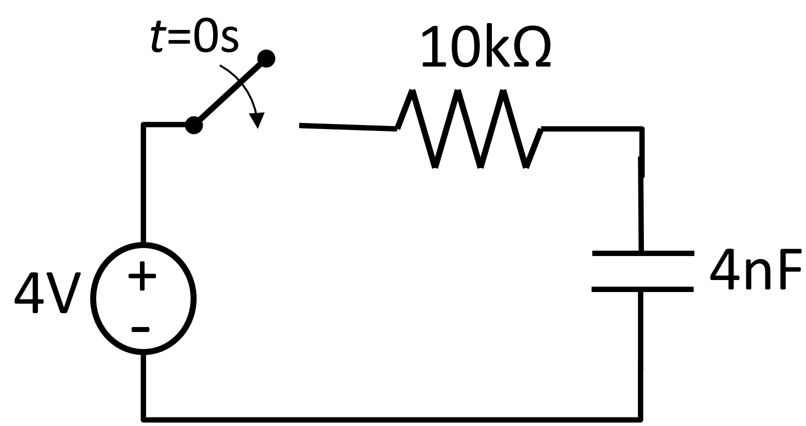
$$0 = \frac{10\text{mH}}{2\text{k}\Omega}\frac{dv_L}{dt} + v_L$$

Note that the first equation is equivalent to an equation for a potential divider:

$$v_L = 6\text{V}\frac{s \cdot 10\text{mH}}{2\text{k}\Omega + s \cdot 10\text{mH}}$$

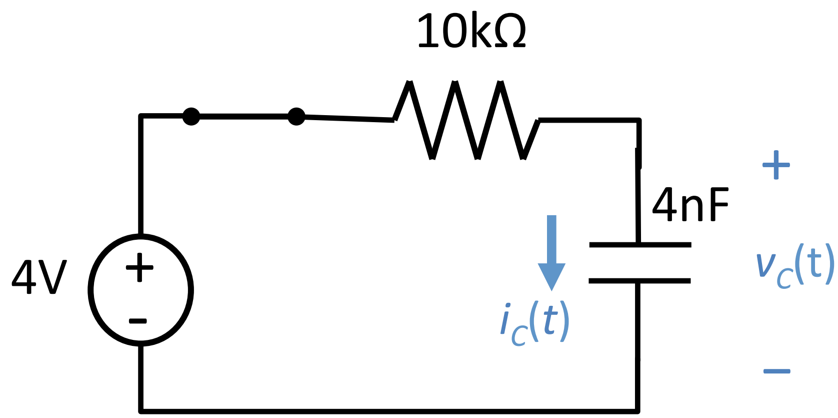
example 2

Use the operator method to write out the differential equation for the capacitor voltage and capacitor current at $t > 0$.



example 2

$t > 0$



We find the diff. equation for v_C:
(node equation)

$$0 = \frac{v_c - 4\text{V}}{10\text{k}\Omega} + \frac{v_c}{1/s \cdot 4\text{nF}} = \frac{v_c - 4\text{V}}{10\text{k}\Omega} + s \cdot 4\text{nF} \cdot v_c$$

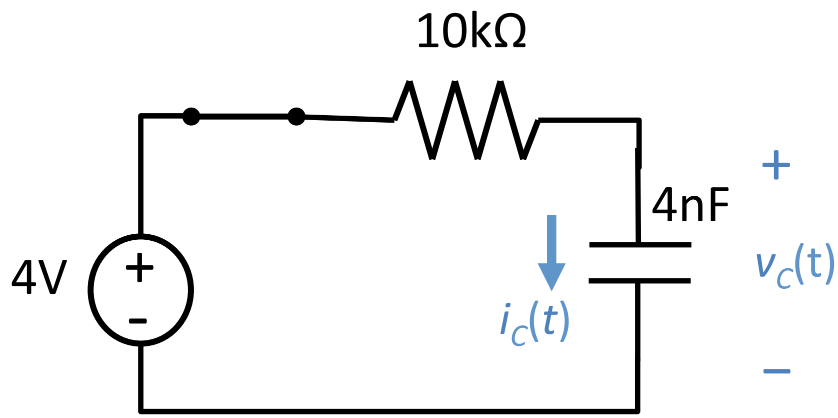
$$0 = v_c - 4\text{V} + 10\text{k}\Omega \cdot 4\text{nF}\frac{dv_c}{dt}$$

Note that the first equation is equivalent to an equation for a potential divider:

$$v_c = 4\text{V} \cdot \frac{1/s \cdot 4\text{nF}}{10\text{k}\Omega + 1/s \cdot 4\text{nF}}$$

example 2

$t > 0$



We find the diff. equation for i_C:
(mesh equation)

$$0 = -4\text{V} + 10\text{k}\Omega \cdot i_c + \frac{1}{s \cdot 4\text{nF}} i_c$$

$$0 = s \cdot 4\text{nF} \cdot \left(-4\text{V}\right) + s \cdot 4\text{nF} \cdot 10\text{k}\Omega \cdot i_c + \frac{s \cdot 4\text{nF}}{s \cdot 4\text{nF}} i_c$$

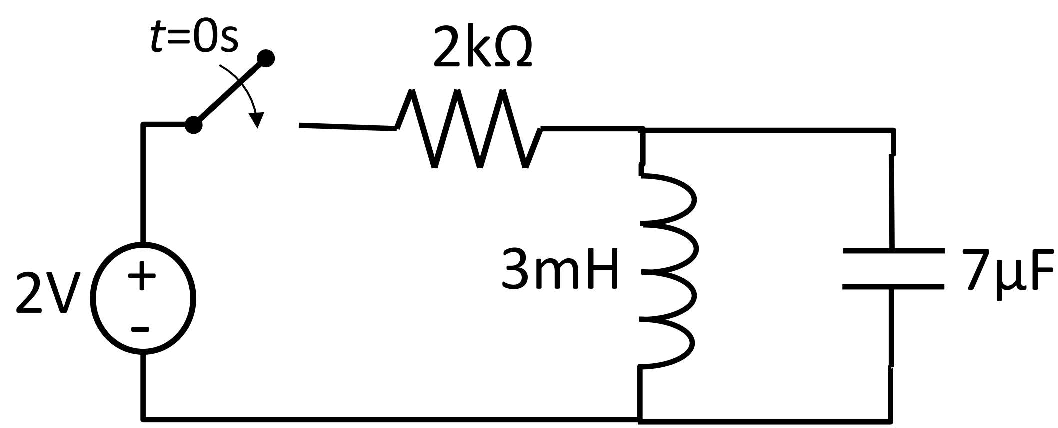
$$0 = 4\text{nF} \cdot 10\text{k}\Omega \cdot \frac{di_c}{dt} + i_c$$

Note that the first equation is equivalent to an equation for series combination:

$$i_c = \frac{4\text{V}}{10\text{k}\Omega + 1/s \cdot 4\text{nF}}$$

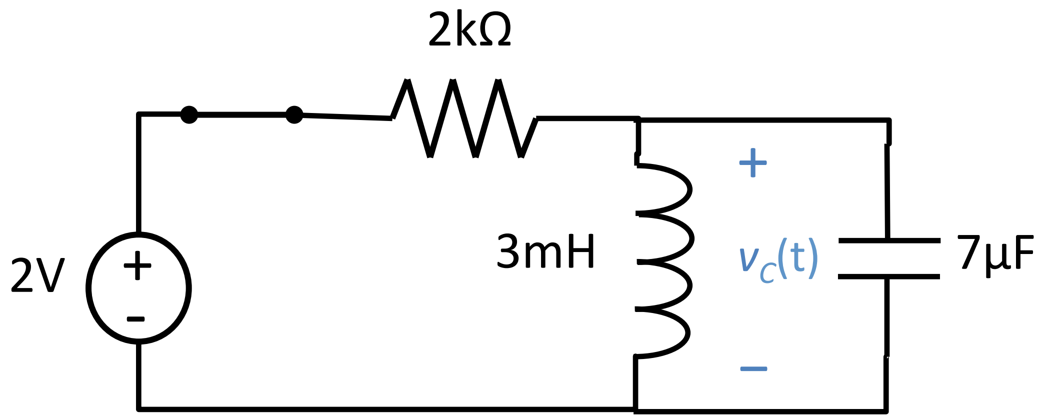
example 3

Use the operator method to write out the differential equation for the capacitor voltage at $t > 0$.



example 3

$t > 0$



We find the diff. equation for v_C:
(node equation)

$$0 = \frac{v_c - 2\text{V}}{2\text{k}\Omega} + \frac{v_c}{s \cdot 3\text{mH}} + s \cdot 7\mu\text{F} \cdot v_c$$

$$0 = s \cdot 3\text{mH} \cdot \frac{v_c - 2\text{V}}{2\text{k}\Omega} + v_c + s \cdot 3\text{mH} \cdot s \cdot 7\mu\text{F} \cdot v_c$$

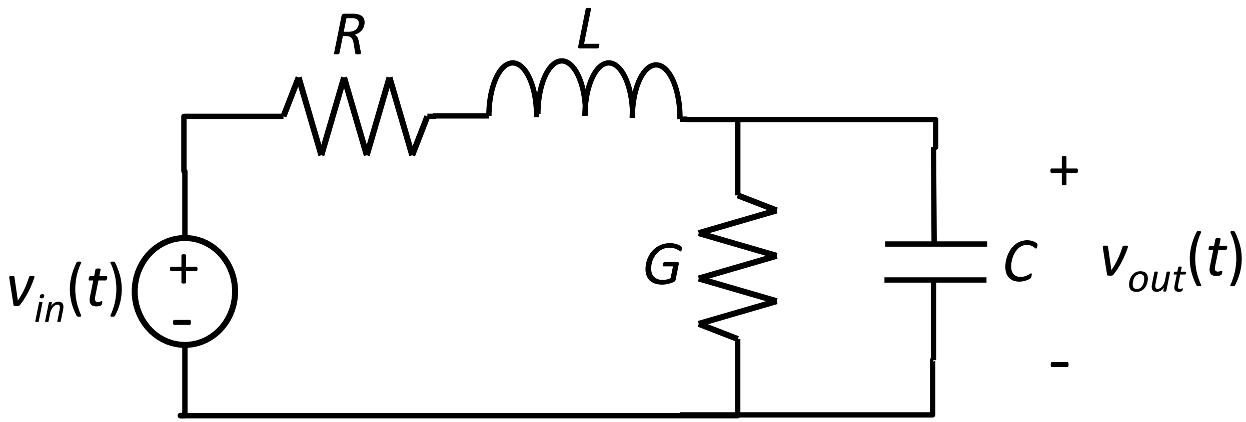
$$0 = s \cdot \frac{3\text{mH}}{2\text{k}\Omega} \cdot v_c + v_c + s^2 \cdot 3\text{mH} \cdot 7\mu\text{F} \cdot v_c$$

Note that we have a 2nd order differential equation.

$$0 = v_c + \frac{3\text{mH}}{2\text{k}\Omega} \cdot \frac{dv_c}{dt} + 3\text{mH} \cdot 7\mu\text{F} \cdot \frac{d^2 v_c}{dt^2}$$

example 4

A ***transmission line*** for signal transmission, such as a ***twisted pair*** of wires, can often be modelled by the circuit below. Find the differential equation relating the input voltage v_{in} to the output voltage v_{out}.



The origin of the circuit values are:

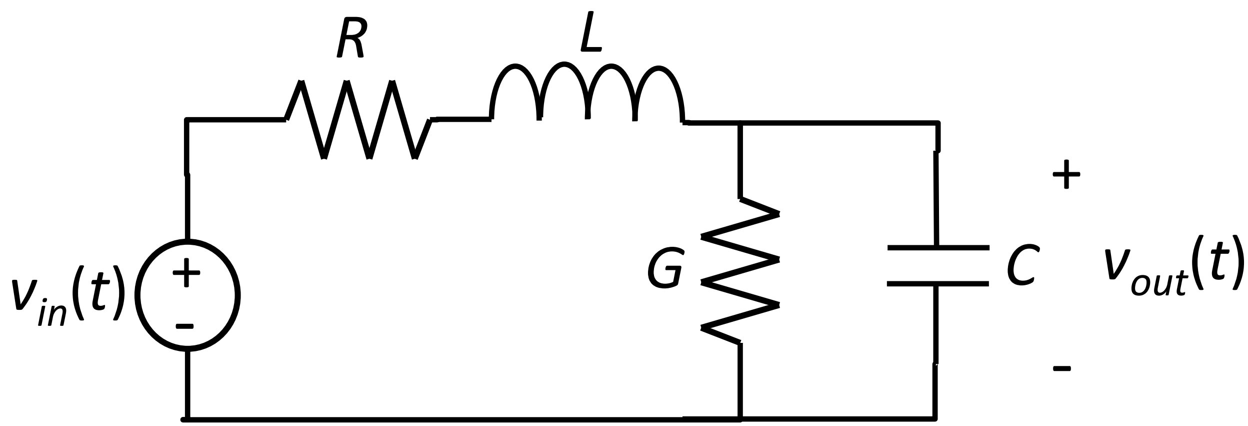
R = resistance along wire length

L = wire inductance

C = capacitance between wires

G = conductance between wires (eg. leaky insulator)

example 4



$$0 = G \cdot v_{out} + s \cdot C \cdot v_{out} + \frac{v_{out} - v_{in}}{R + s \cdot L}$$

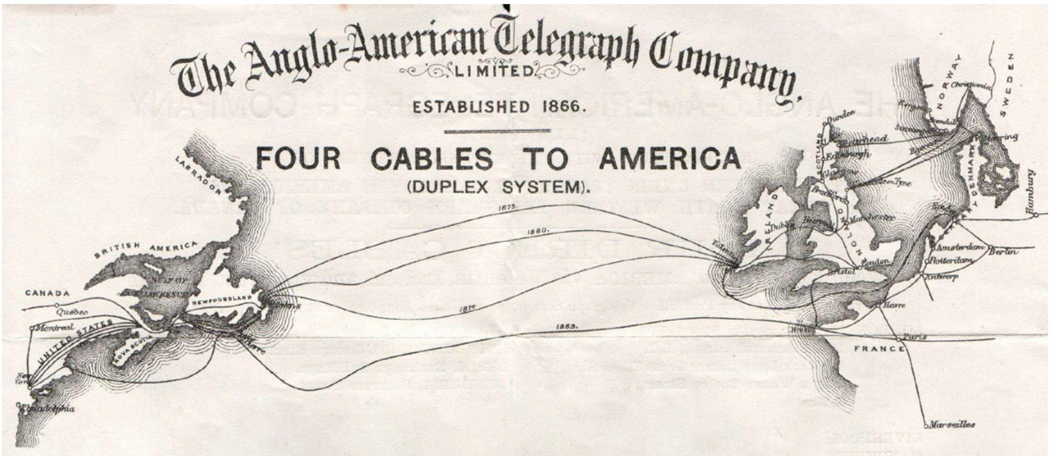
$$0 = \left(G + s \cdot C\right)\left(R + s \cdot L\right)v_{out} + v_{out} - v_{in}$$

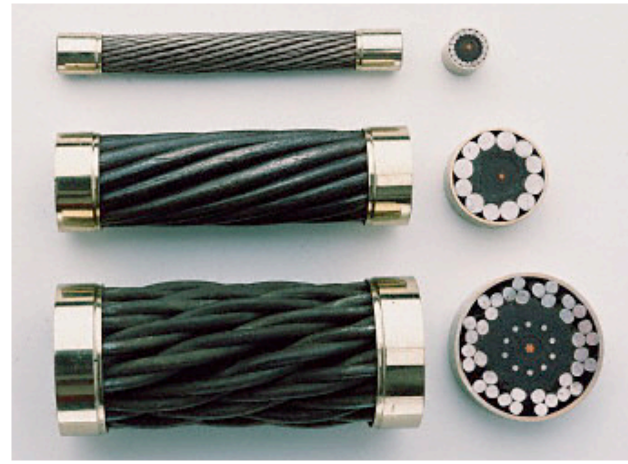
$$v_{in} = \left(1 + RG\right)v_{out} + \left(RC + LG\right)\frac{dv_{out}}{dt} + LC\frac{d^2 v_{out}}{dt^2}$$

The output voltage is related to the input voltage by a 2nd order differential equation. The transmission line distorts the shape of the input voltage signal through this equation, which leads to the so called ***telegrapher's equation***.

McGill

example 4: transmission lines





Transatlantic cable (1900):

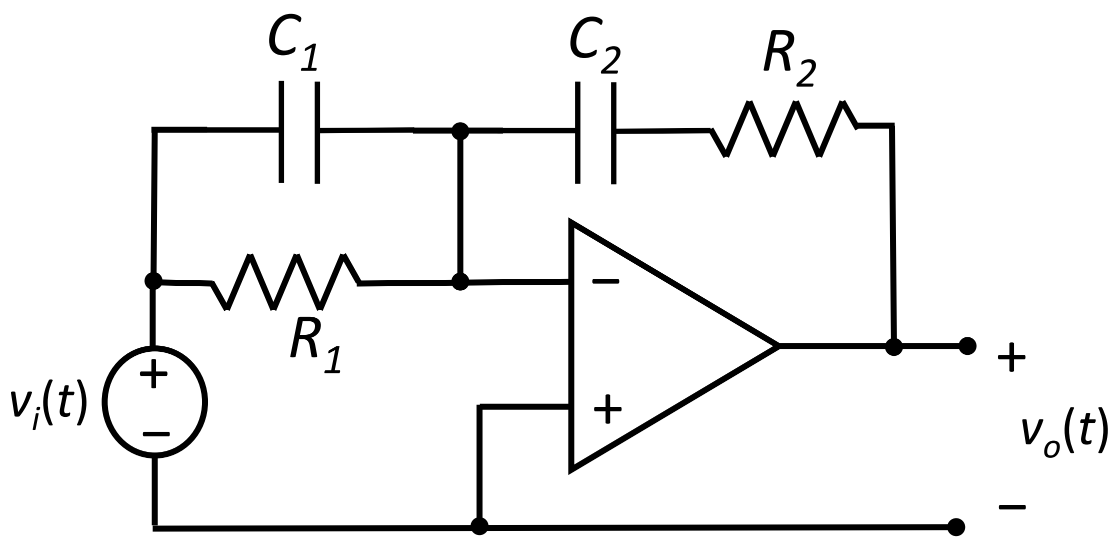
~ 1bit/s over 4000km

Gigabit Ethernet (2000):

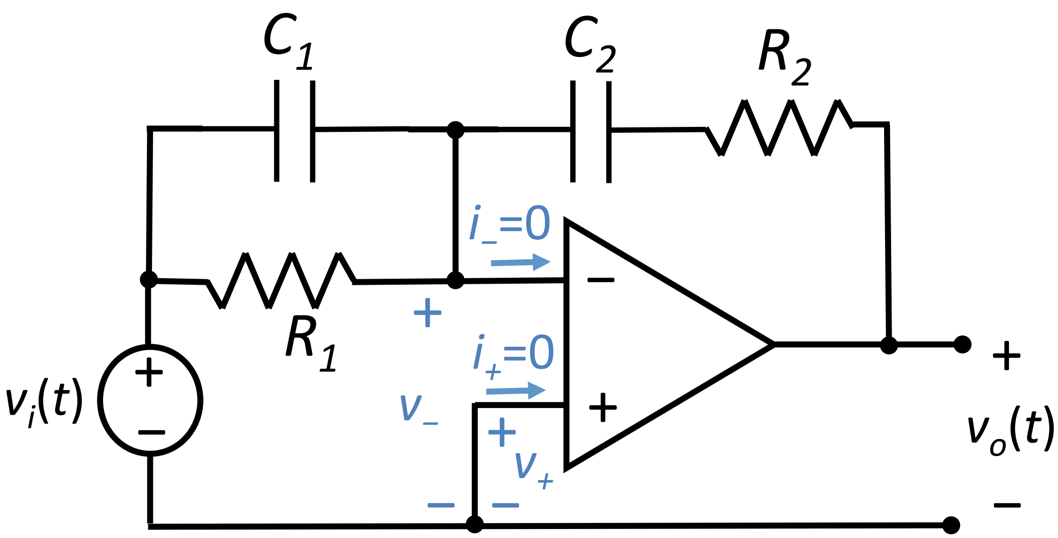
~10^9 bit/s over 25m

example 5

Assuming ideal op-amp behaviour, what is $v_o(t)$ as a function of $v_i(t)$?
Assume the capacitors are discharged at t=0.



example 5



Node equation:

$$0 = \frac{-v_i}{R_1} + \frac{-v_i}{1/sC_1} + \frac{-v_o}{R_2 + 1/sC_2}$$

$$v_o = -\left(R_2 + \frac{1}{sC_2}\right)\left(\frac{1}{R_1} + sC_1\right)v_i$$

$$= -\left[\left(\frac{R_2}{R_1} + \frac{C_1}{C_2}\right) + sR_2C_1 + \frac{1}{sR_1C_2}\right]v_i$$

Converting the operators back, and making use of uncharged state of the capacitors at t=0, gives:

$$v_o(t) = -\left(\frac{R_2}{R_1} + \frac{C_1}{C_2}\right)v_i(t) - R_2C_1\frac{dv_i(t)}{dt} - \frac{1}{R_1C_2}\int_0^t v_i(t')dt'$$

Note: we interpret 1/s as the inverse operation of s, corresponding to integration. Usage of Laplace transforms gives a more rigorous meaning.