Today's Outline

4. Analysis Methods

- Linearity
- Principle of Superposition

Linearity

Linearity: A *function* f (x) is **linear** in the argument x if and only if:

$$f(ax + by) = af(x) + bf(y)$$

- To evaluate $f(ax + by)$, we can evaluate $f(x)$ and $f(y)$, and then sum appropriately.

- In some cases, it may be easier to evaluate $f(x)$ and $f(y)$ instead of $f(ax + by)$, for example:

  $\mathrm{f}(x) = 2x$

  $\mathrm{f}(179) = \mathrm{f}(170) + \mathrm{f}(9)$ [for evaluation *without* pen or paper]

  $= 340 + 18 = 358$

Linearity (more general)

Linearity: An *operator* $F[\, x(t)\,]$ is **linear** in the function $x(t)$ if and only if:

$$F[\, a\, x(t) + b\, y(t)\,] = a\, F[\, x(t)\,] + b\, F[\, y(t)\,]$$

- To evaluate $F[\, a\, x(t) + b\, y(t)\,]$, we can evaluate $F[x(t)]$ and $F[y(t)]$, and then sum the appropriately.

- In some cases, it may be easier to evaluate $F[x(t)]$ and $F[y(t)]$ instead of $F[\, x(t) + y(t)\,]$, for example:

$$F[x(t)] = \frac{d}{dt}\left[x(t) \right]$$

$$\begin{aligned} F[C + D\exp(-kt)] &= F[C] + F[D\exp(-kt)] \\ &= 0 - Dk\exp(-kt) \end{aligned}$$

Linear Circuits

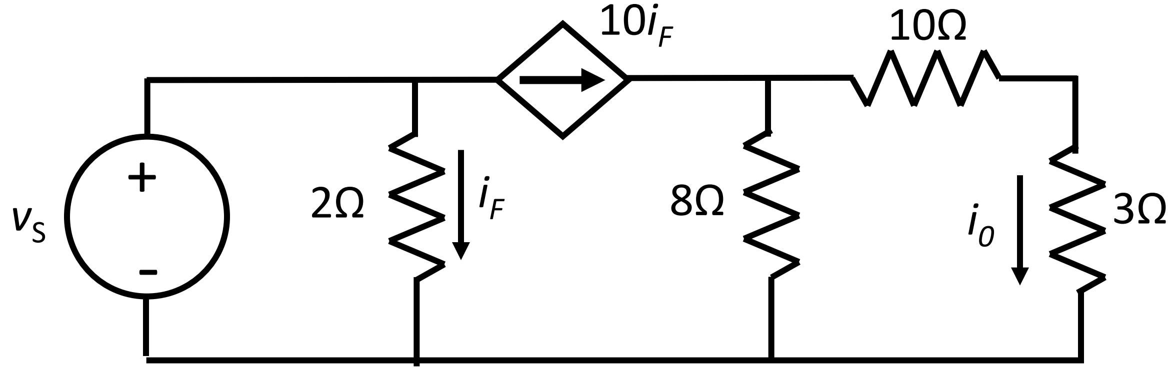
Linear Circuit Element: An element where terminal voltage and current are related to each other by a linear function (or operator). Examples include ideal resistors, dependent sources and ideal op-amps.

Linear Circuit: A circuit composed of independent sources and linear circuit elements.

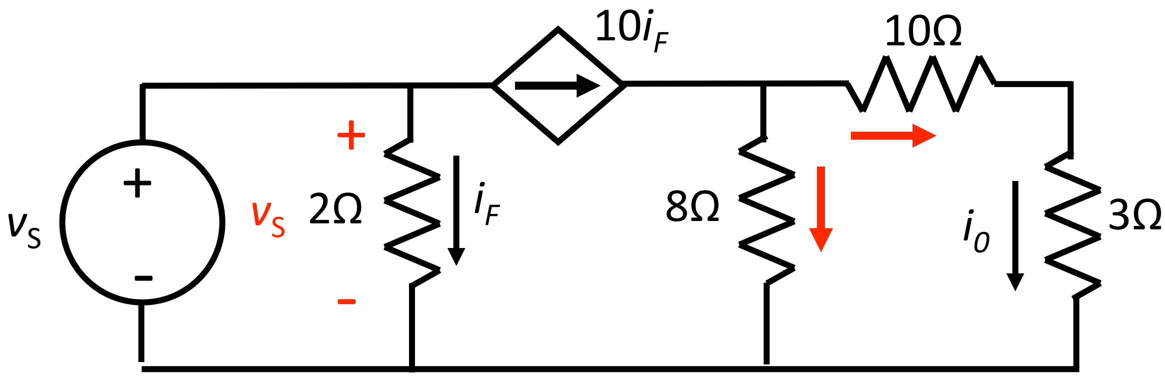
Any voltage or current variable in a linear circuit can always be described by a linear function (or operator) of other voltage and current variables.

Example of Linearity

Verify that the current i_0 is a linear function of the voltage source v_S. Is the power dissipated in the 3Ω resistor a linear function of v_s?



Example of Linearity



Ohm's Law: $v_S = 2\Omega i_F$

Current Divider:

$$i_0 = 10 i_F \frac{8\Omega}{8\Omega + (10\Omega + 3\Omega)} = 10\left(\frac{v_S}{2\Omega}\right)\frac{8\Omega}{21\Omega} = \frac{40}{21}\Omega^{-1} v_S = f(v_S)$$

Power Absorbed:

$$P_{abs} = i_0^2 3\Omega == \left(\frac{40}{21}\Omega^{-1} v_S\right)^2 3\Omega = \frac{1600}{147}\Omega^{-1} v_s^2 = g(v_S)$$

Example of Linearity

Linearity of $i_0 = f(v_S)$: $$f(ax+by)=\frac{40}{21}\Omega^{-1}\left(ax+by\right)$$

$$f(x)=\frac{40}{21}\Omega^{-1}x,\quad f(y)=\frac{40}{21}\Omega^{-1}y$$

$$af(x)+bf(y)=a\frac{40}{21}\Omega^{-1}x+b\frac{40}{21}\Omega^{-1}y=f(ax+by)$$

Thus, i_0 is a linear function of $v_{S.}$

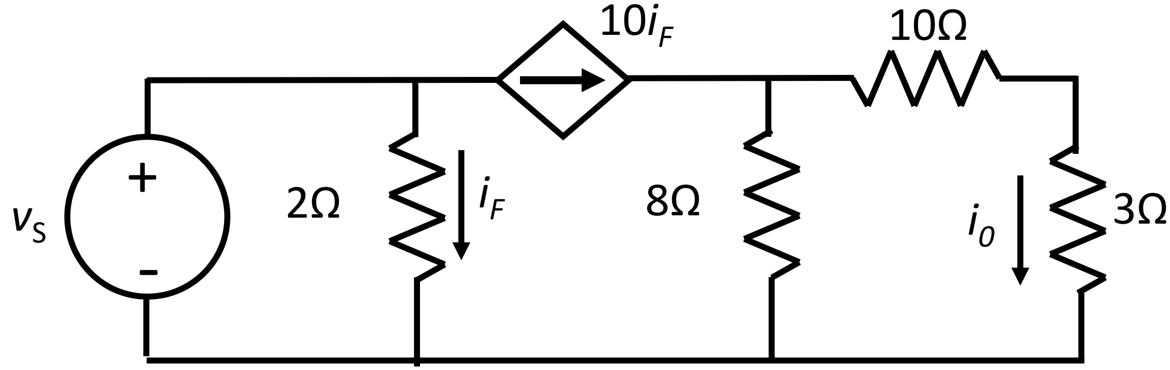
Linearity of $P_{abs} = g(v_S)$: $$g(ax+by)=\frac{1600}{147}\Omega^{-1}\left(ax+by\right)^2=\frac{1600}{147}\Omega^{-1}\left(a^2x^2+b^2y^2+2axby\right)$$

$$g(x)=\frac{1600}{147}\Omega^{-1}x^2,\quad g(y)=\frac{1600}{147}\Omega^{-1}y^2$$

$$ag(x)+bg(y)=a\frac{1600}{147}\Omega^{-1}x^2+b\frac{1600}{147}\Omega^{-1}y^2\neq g(ax+by)$$

Thus, P_{abs} is not a linear function $v_{S.}$

Example of Linearity



In this linear circuit example, the algebraic circuit variable i_0 is a linear function of the voltage source value v_S.

The power dissipated in the 3Ω resistor is not a linear function of the voltage source value v_S.

Voltage and current variables are linear functions of other voltage and current variables, ***but this is not generally true for power***.

Principle of Superposition

Principle of Superposition: Any current or voltage in a **linear circuit** that contains *multiple independent sources* can be calculated as the *algebraic sum* of all the individual contributions due to each independent source acting alone

- Superposition allows a complicated analysis to be performed as a series of simpler analyses
- Superposition will sometimes *but not always* lead to a simpler analysis

Equations of a Linear Circuit

Kirchoff's Current Law: $$0 = \sum_m i_m(t) \quad \rightarrow \quad 0 = i_1 + i_2 + i_3 + \ldots$$

Kirchoff's Voltage Law: $$0 = \sum_m v_m(t) \quad \rightarrow \quad 0 = v_1 + v_2 + v_3 + \ldots$$

Ohm's Law: $$0 = v_x - i_x R_x$$

Dependent Sources:
$$0 = v_x - \alpha v_y \qquad 0 = i_x - \beta i_y$$
$$0 = v_x - r i_y \qquad 0 = i_x - g v_y$$

Independent Sources: $$V_0 = v_x \qquad I_0 = i_x$$

We see that every equation is linear. The independent source equations have a different form than the other equations.

Equations of a Linear Circuit

If all the independent sources are "turned off", meaning each V_0–>0 and I_0–>0, the linear system of equations takes the ***homogenous*** form:

$$\begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \\ A_{n1} & & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad x_m = v_m \text{ or } i_m$$

$$\mathbf{0} = \mathbf{A} \cdot \mathbf{x}$$

$$\mathbf{x}_0 = \mathbf{A}^{-1} \cdot \mathbf{0} = \mathbf{0} \qquad \text{if } \det(\mathbf{A}) \neq 0$$

In other words, all voltages (or currents) are zero.

Equations of a Linear Circuit

If only one independent source is turned on (call it source #1), the linear system of equations takes the ***inhomogeneous*** form:

$$\begin{bmatrix} b_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \\ A_{n1} & & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad x_m = v_m \text{ or } i_m$$

$$\mathbf{b}_1 = \mathbf{A} \cdot \mathbf{x}$$

$$\mathbf{x}_1 = \mathbf{A}^{-1} \cdot \mathbf{b}_1 \qquad \text{if } \det(\mathbf{A}) \neq 0$$

In other words $\mathbf{x}_1$ is the circuit voltage (or current) with independent source #1 turned on and all others turned off.

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Equations of a Linear Circuit

If a different independent source is turned on (calling it #2, and leaving all other independent sources off), the linear system of equations gives:

$$\begin{bmatrix} \vdots \\ 0 \\ b_2 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \\ A_{n1} & & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \qquad x_m = v_m \text{ or } i_m$$

$$\mathbf{b}_2 = \mathbf{A} \cdot \mathbf{x}$$

$$\mathbf{x}_2 = \mathbf{A}^{-1} \cdot \mathbf{b}_2 \qquad \text{if } \det(\mathbf{A}) \neq 0$$

In other words $\mathbf{x}_2$ is the circuit voltage (or current) with independent source #2 turned on and all others turned off.

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Equations of a Linear Circuit

If *all* independent sources are turned on, linear algebra gives us:

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \\ A_{n1} & & A_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$$\mathbf{b} = \mathbf{A} \cdot \mathbf{x}$$

$$\begin{aligned} \mathbf{x} &= \mathbf{A}^{-1} \cdot \mathbf{b} \qquad \text{if } \det(\mathbf{A}) \neq 0 \\ &= \mathbf{A}^{-1} \cdot \left(\mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3 + \ldots \right) \\ &= \mathbf{A}^{-1} \cdot \mathbf{b}_1 + \mathbf{A}^{-1} \cdot \mathbf{b}_2 + \mathbf{A}^{-1} \cdot \mathbf{b}_3 + \ldots \\ &= \mathbf{x}_1 + \mathbf{x}_2 + \mathbf{x}_3 + \ldots \end{aligned}$$

In other words, the total response $\mathbf{x}$ is the algebraic sum of responses $\mathbf{x}_1$, $\mathbf{x}_2$, … to each independent source.

Principle of Superposition

1. Keep one ***independent*** source “on” and “turn-off” all other ***independent*** sources.

2. Calculate the current or voltage variable(s) of interest.

3. Repeat steps 1 and 2 for each remaining ***independent*** source.

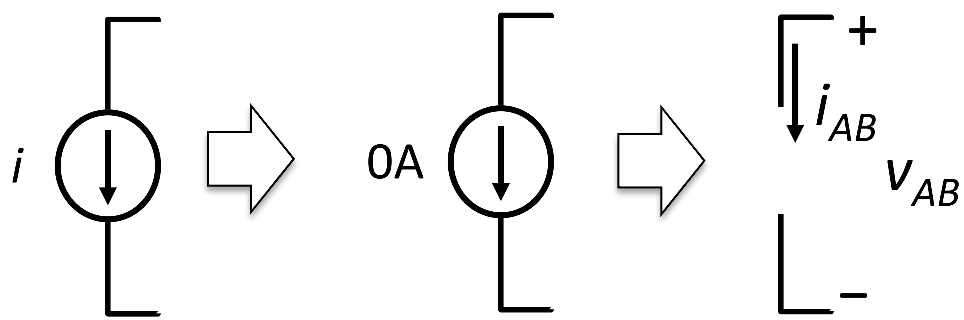
4. Add algebraically the individual contributions for the total response.

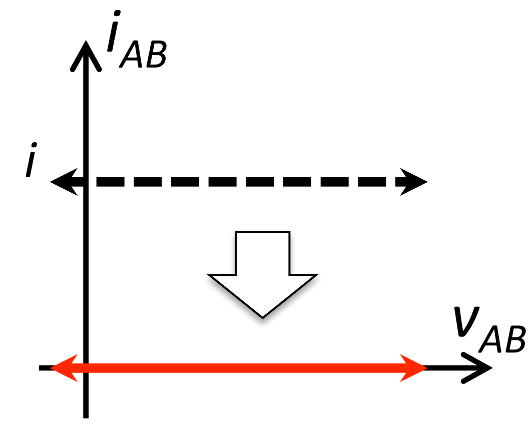
Note: Retain all ***dependent*** sources, unchanged, through each step of the analysis.

“Turning Off” Sources

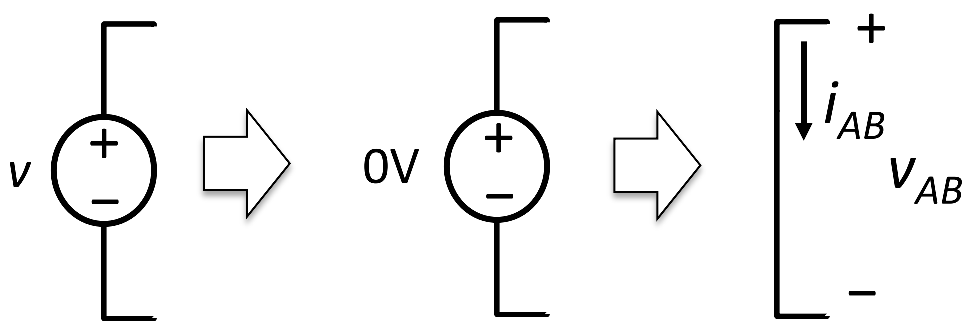
When turning off independent sources, we make the following replacements.

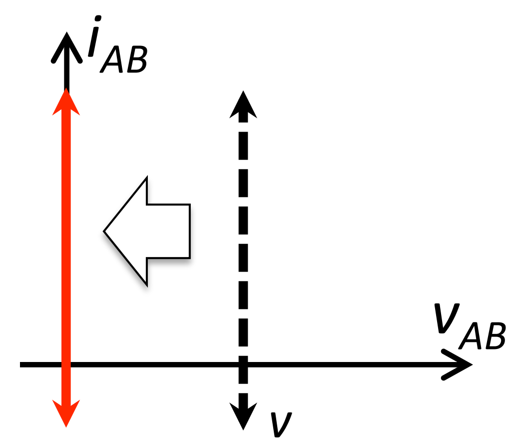
Current sources set to 0A = **open circuit**





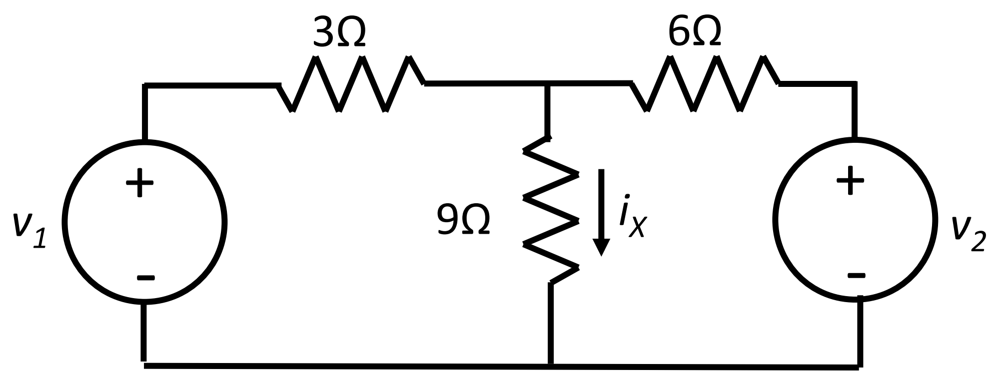
Voltage sources set to 0V = **short circuit**





Example 1

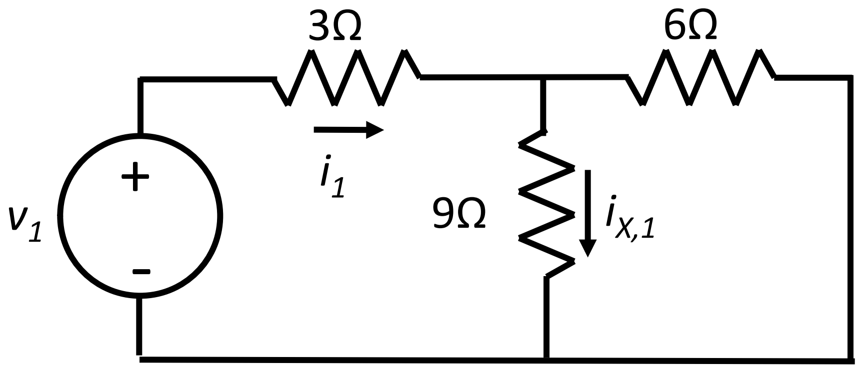
Find the current i_X and the power dissipated in the 9Ω resistor, as functions of v_1 and v_2.

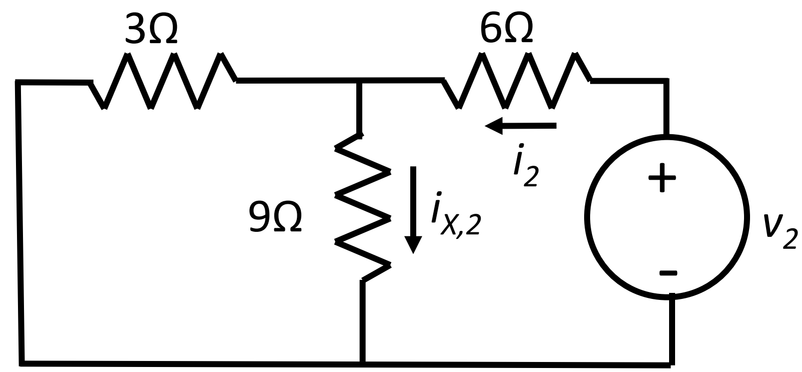


Strategy:

- apply superposition, ie. solve for i_X with v_2=0, then solve for i_X with v_1=0, add the solutions for i_X
- use the solution for i_X to find the power absorbed in the 9Ω resistor

Example 1





$$i_1 = \frac{v_1}{3\Omega + 6\Omega || 9\Omega} = \frac{5}{33}\Omega^{-1} v_1$$

$$i_{X,1} = i_1 \frac{6\Omega}{6\Omega + 9\Omega} = \frac{2}{5}\frac{5}{33}\Omega^{-1} v_1 = \frac{2}{33}\Omega^{-1} v_1$$

$$i_2 = \frac{v_1}{6\Omega + 3\Omega || 9\Omega} = \frac{4}{33}\Omega^{-1} v_2$$

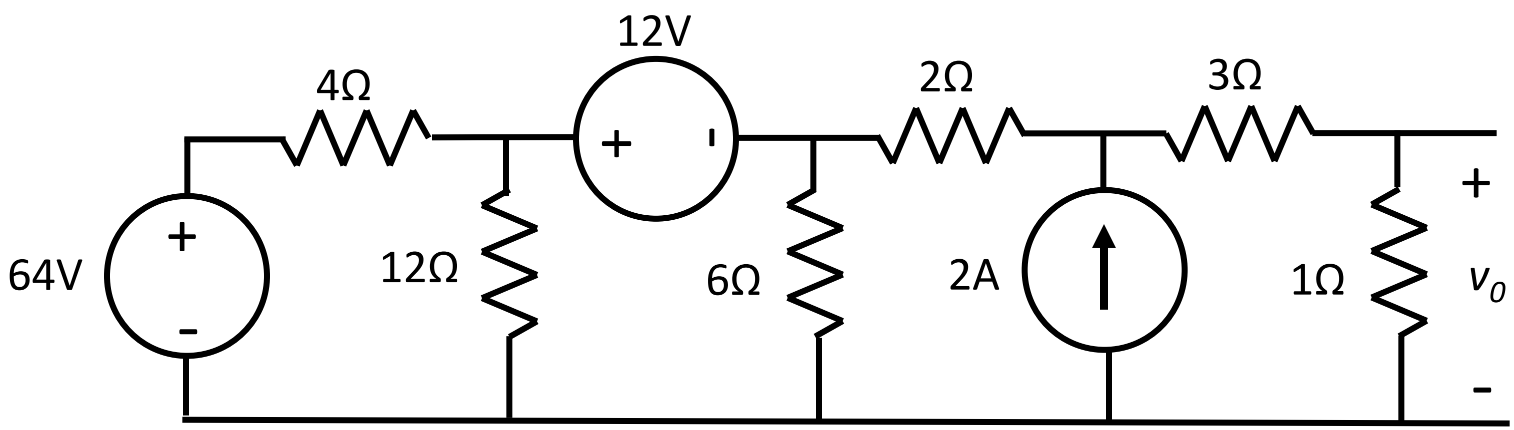
$$i_{X,2} = i_2 \frac{3\Omega}{3\Omega + 9\Omega} = \frac{1}{4}\frac{4}{33}\Omega^{-1} v_2 = \frac{1}{33}\Omega^{-1} v_2$$

Thus: $$i_X = i_{X,1} + i_{X,2} = \frac{2}{33}\Omega^{-1} v_1 + \frac{1}{33}\Omega^{-1} v_2$$

$$P_{abs} = i_X^2 9\Omega = \frac{\left(2v_1 + v_2\right)^2}{121\Omega}$$

Example 2

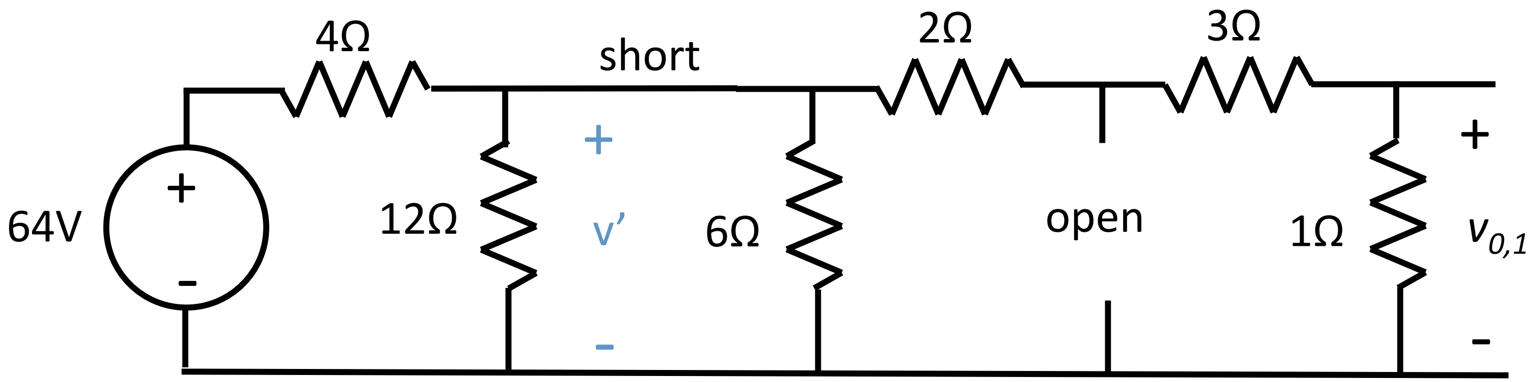
Find the voltage v_0.

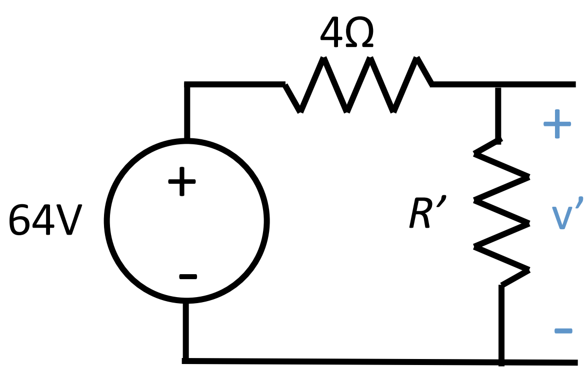


Strategy:

- apply superposition to solve for $v_{0,}$ ie. add the contributions to v_0 arising from each source

Example 2



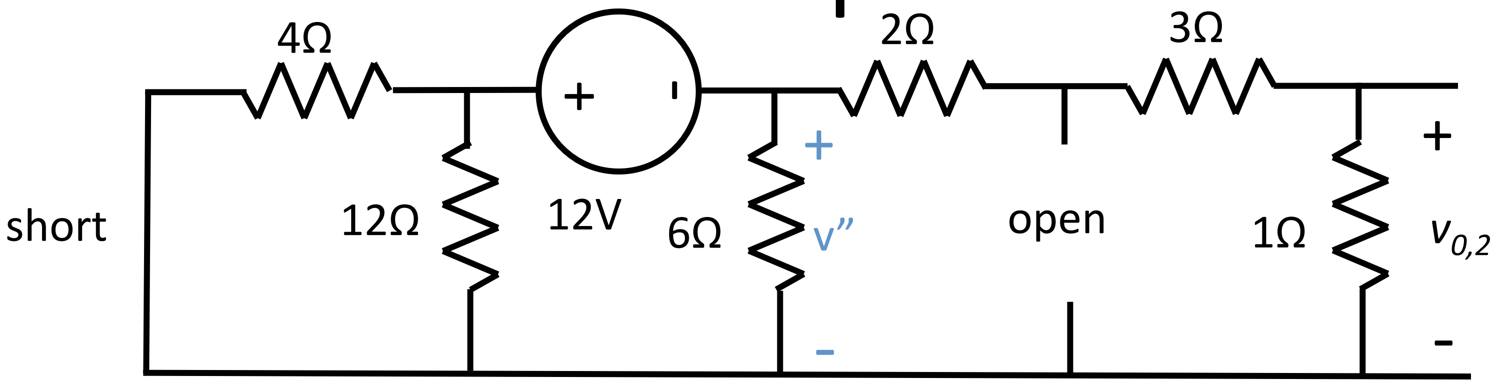


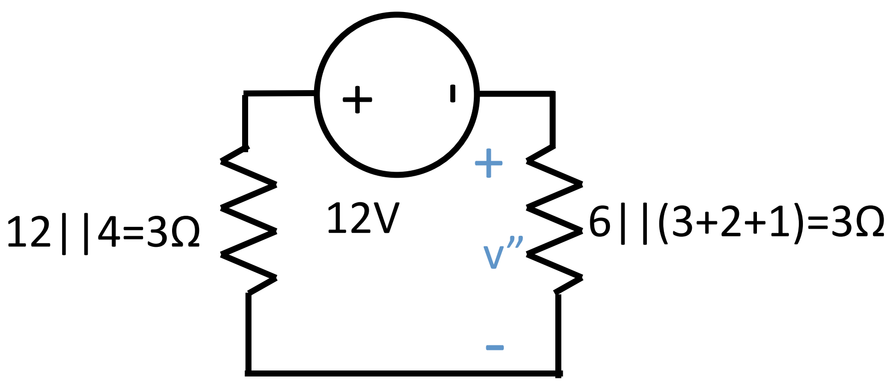
$$
\begin{aligned}
R' &= 12\Omega\,||\,6\Omega\,||\,(2\Omega+3\Omega+1\Omega) \\
&= 12\Omega\,||\,6\Omega\,||\,6\Omega \\
&= 12\Omega\,||\,3\Omega \\
&= 12/5\Omega
\end{aligned}
$$

voltage divider: $$v' = 64\text{V}\frac{R'}{R'+4\Omega} = 64\text{V}\frac{12/5\Omega}{12/5\Omega+4\Omega} = 24\text{V}$$

voltage divider: $$v_{0,1} = v'\frac{1\Omega}{1\Omega+2\Omega+3\Omega} = 4\text{V}$$

Example 2

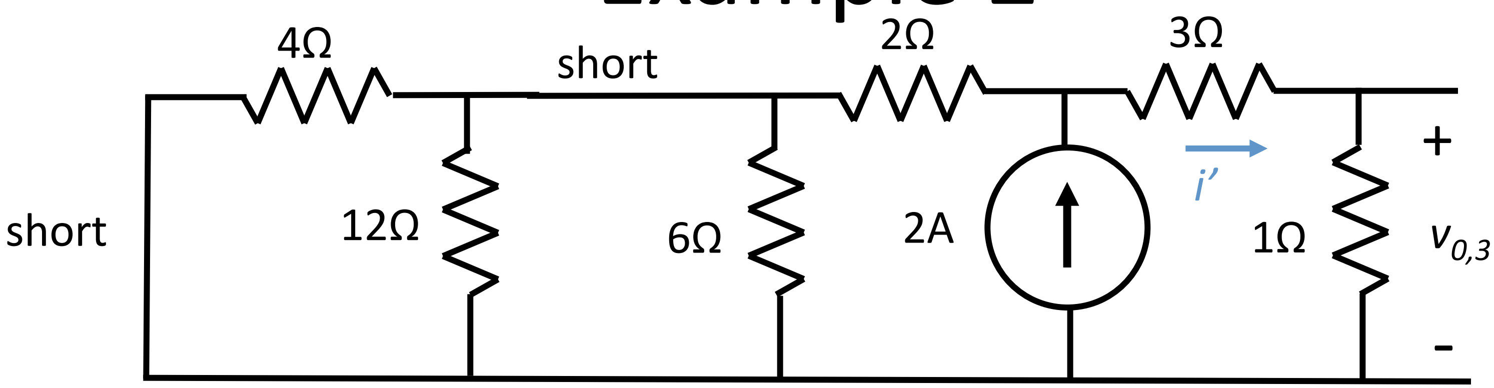


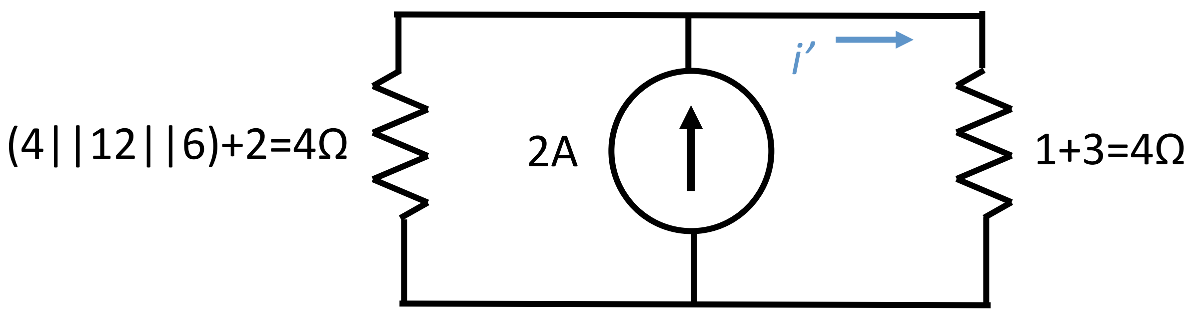


voltage divider: $$v'' = -12\text{V}\,\frac{3\Omega}{3\Omega + 3\Omega} = -6\text{V}$$

voltage divider: $$v_{0,2} = v''\frac{1\Omega}{1\Omega + 2\Omega + 3\Omega} = -1\text{V}$$

Example 2





current divider: $$i' = 2\text{A}\frac{4\Omega}{4\Omega + 4\Omega} = 1\text{A}$$

Ohm's Law: $$v_{0,3} = i' 1\Omega = 1\text{V}$$

Example 2

Summing up the contributions from each source:

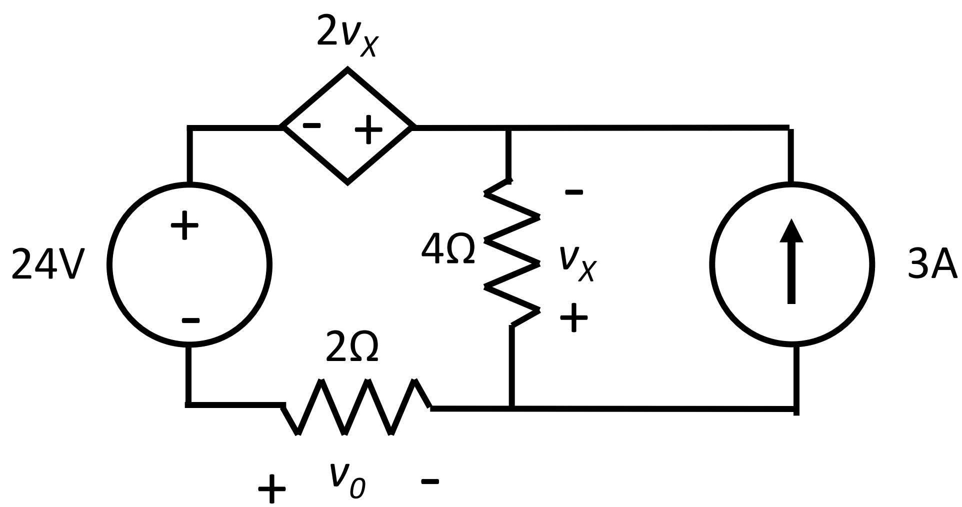
$$v_0 = v_{0,1} + v_{0,2} + v_{0,3}$$
$$= 4V + (-1V) + 1V$$
$$= 4V$$

The problem was reduced to a series of calculations using equivalent resistance, voltage dividers and current dividers.

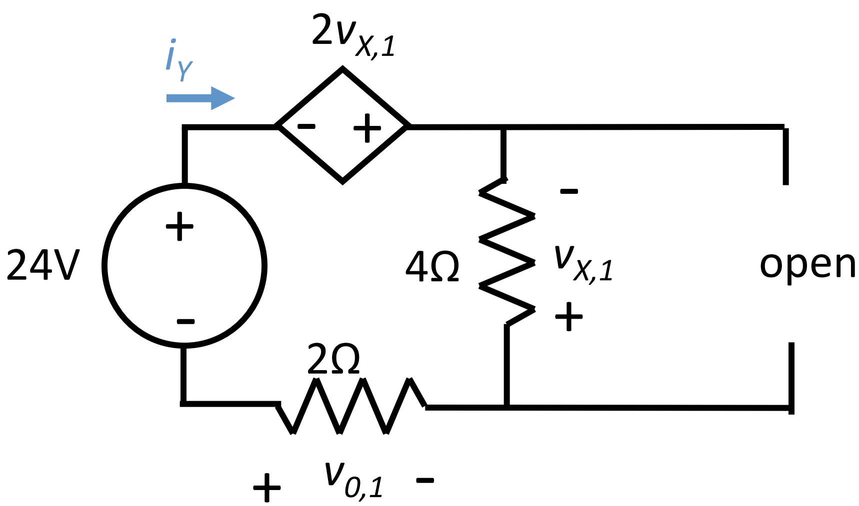
Source transformations could also be used to solve this problem.

Superposition Example 3

Find the voltage v_0.



Superposition Example 3



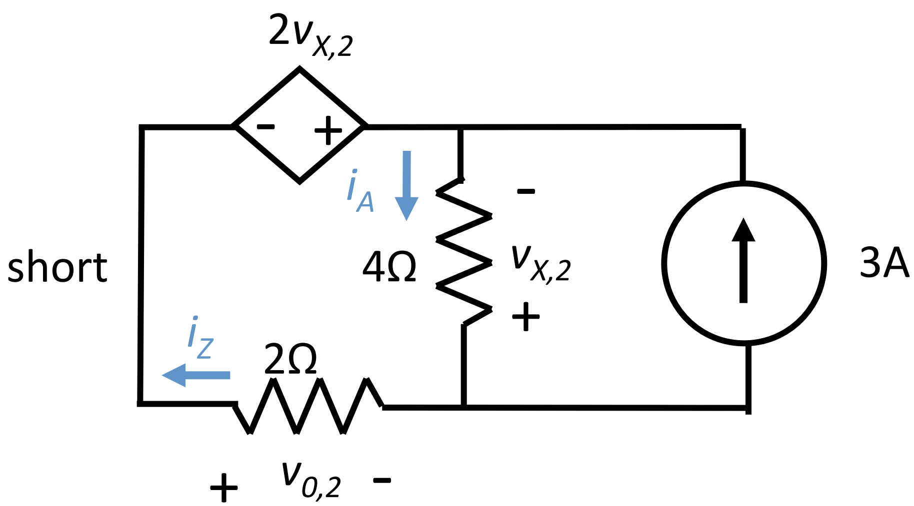
KVL: $-24V - 2v_{X,1} - v_{X,1} - v_{0,1} = 0$

KCL+Ohm's Law: $i_Y = -v_{X,1} / 4\Omega = -v_{0,1} / 2\Omega \rightarrow v_{X,1} = 2v_{0,1}$

Substitution: $-24V - 4v_{0,1} - 2v_{0,1} - v_{0,1} = 0$

$$v_{0,1} = -24/7 \text{ V}$$

Superposition Example 3



KVL: $-2v_{X,2} - v_{X,2} - v_{0,2} = 0$

KCL: $i_A - 3A - i_Z = 0 \rightarrow i_A = 3A + i_Z$

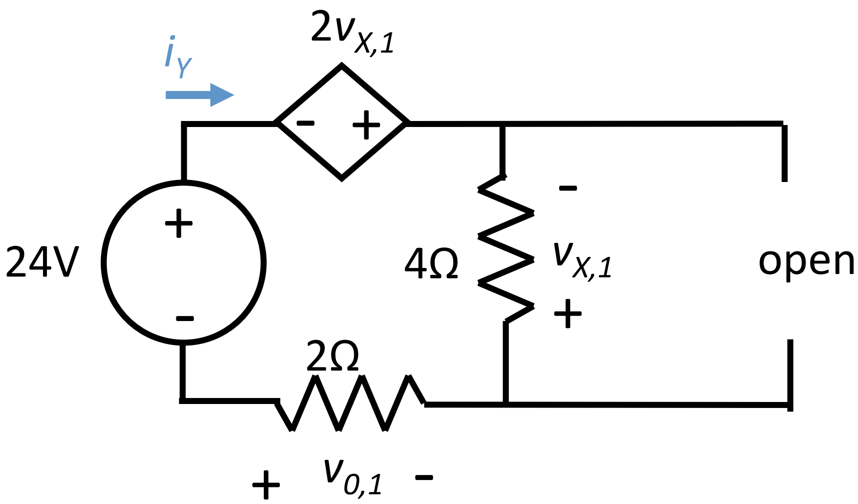
Ohm's Law: $v_{X,2} = -i_A 4\Omega = -(3A + i_Z)\, 4\Omega$, $v_{0,2} = -i_Z\, 2\Omega$

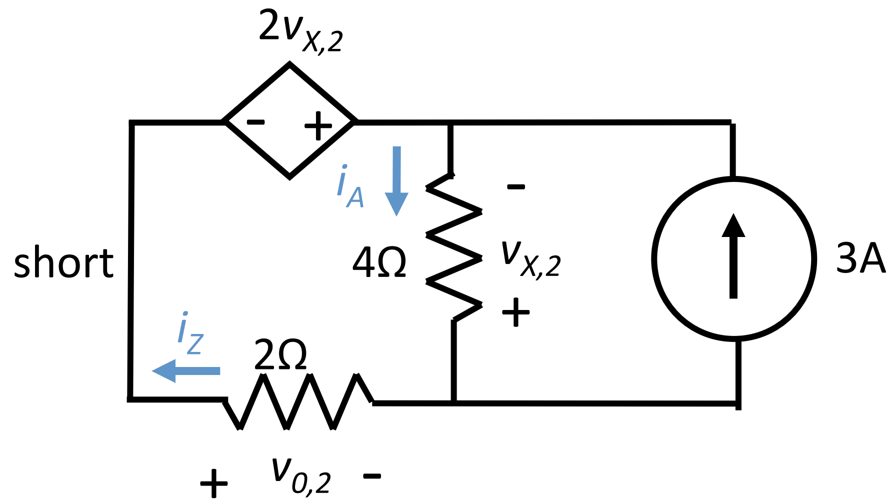
$\rightarrow v_{X,2} = -(3A - v_{0,2}/2\Omega)\, 4\Omega = -12V + 2\, v_{0,2}$

substitution: $-2(-12V + 2\, v_{0,2}) - (-12V + 2\, v_{0,2}) - v_{0,2} = 0$

$v_{0,2} = 36/7\ V$

Superposition Example 3





By principle of superposition:

$v_0 = v_{0,1} + v_{0,2} = -24/7+36/7 = 12/7$ V