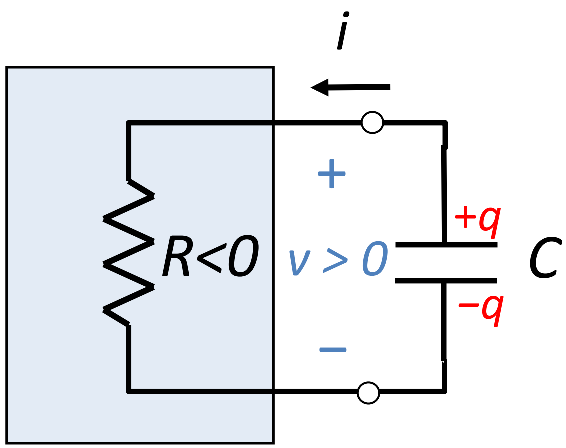
Today's Outline

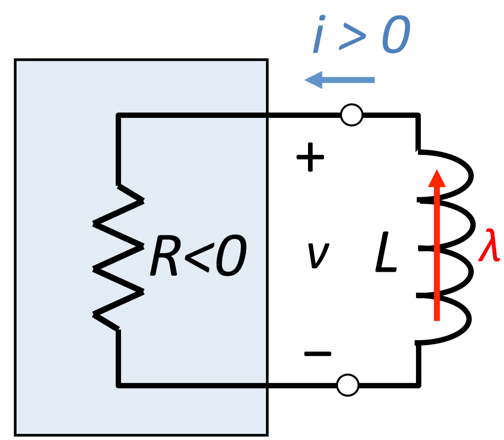
7. First Order Circuits

- stability
- response to non-constant input
  - sinusoidal input

Stability

Question: What is the behaviour of a first order circuit which presents a *negative* Thévenin resistance to the energy storage element?





$$\frac{dq}{dt} = -i = -\frac{v}{R} > 0$$

$$\frac{dv}{dt} = \frac{-i}{C} = -\frac{v}{RC} > 0$$

$$\tau = RC < 0$$

Charge separation and voltage ***grows*** with time.

$$\frac{d\lambda}{dt} = -v = -iR > 0$$

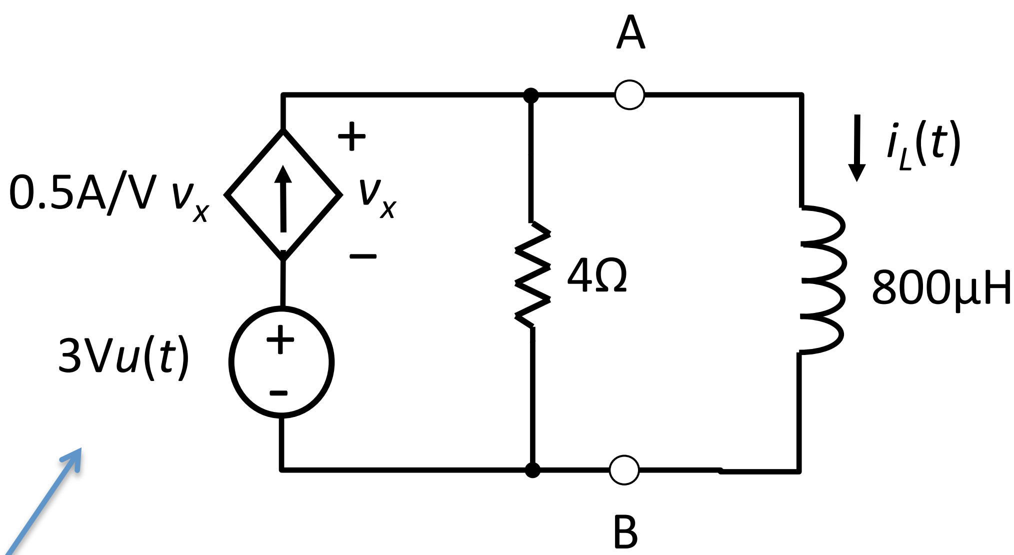
$$\frac{di}{dt} = \frac{-v}{L} = -\frac{iR}{L} = -\frac{i}{L/R} > 0$$

$$\tau = L/R < 0$$

Flux linkage and current ***grows*** with time.

Stability

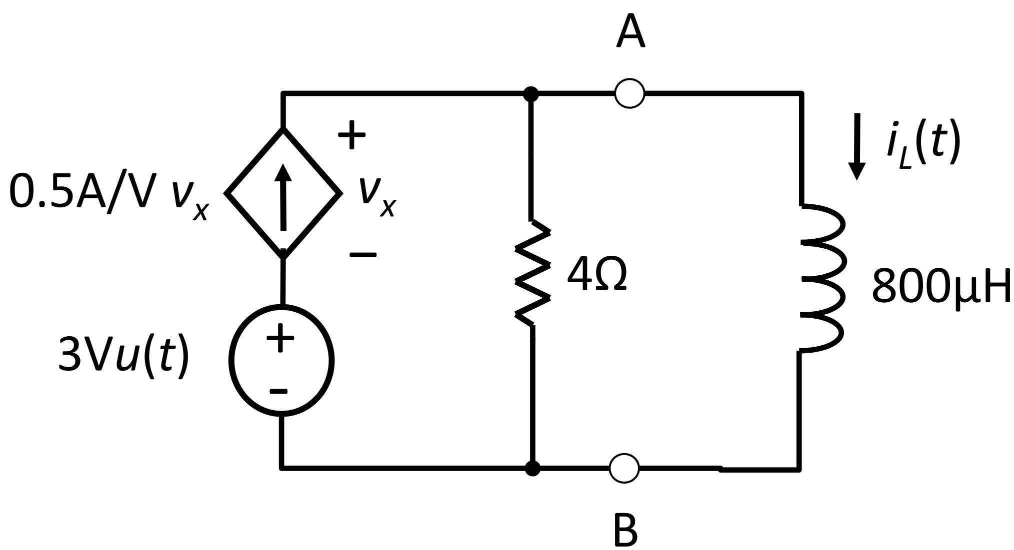
Consider the following circuit. What is the time dependence of the inductor current assuming dc steady state conditions for t<0?



Note the use of the unit step function $u(t)$.

Stability

Replace the circuit seen by the inductor with a Thévenin equivalent (ie. with respect to the terminals labelled A and B). Notice that R_T is independent of time, but v_{oc} and i_{sc} are time dependent.



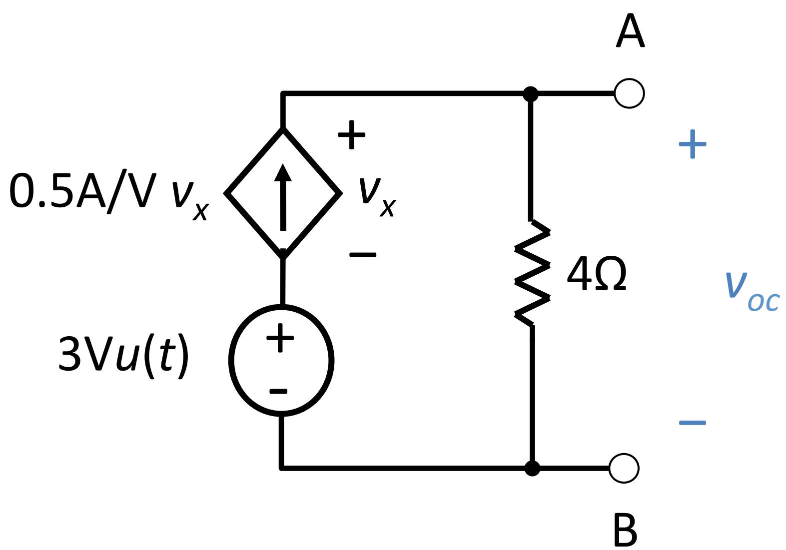
Stability

Find v_{oc}, i_{sc}, and $R_T = v_{oc}/i_{sc}$.

$$0 = -3\text{V}u(t) - v_x + 0.5\text{A/V} \cdot v_x \cdot 4\Omega$$

$$v_x = 3\text{V}u(t)$$

$$v_{oc} = 0.5\text{A/V} \cdot v_x \cdot 4\Omega = 6\text{V}\,u(t)$$



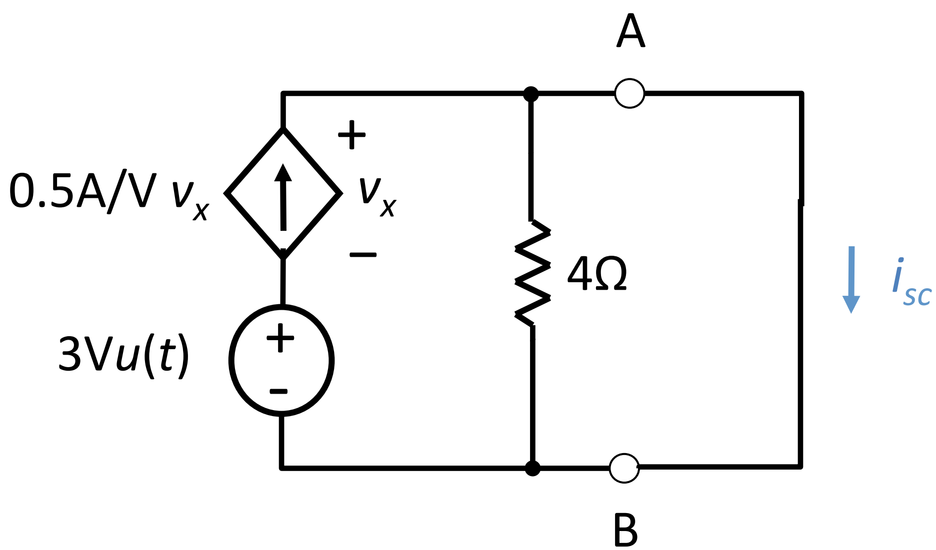
Stability

Find v_{oc}, i_{sc}, and $R_T = v_{oc}/i_{sc}$.

$$0 = -3\mathrm{V}u(t) - v_x$$

$$v_x = -3\mathrm{V}u(t)$$

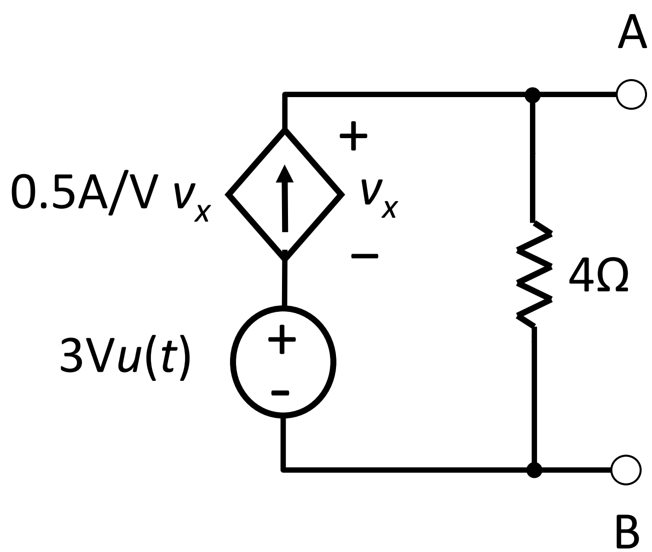
$$i_{sc} = 0.5\mathrm{A/V} \cdot v_x = -1.5\mathrm{A}\, u(t)$$

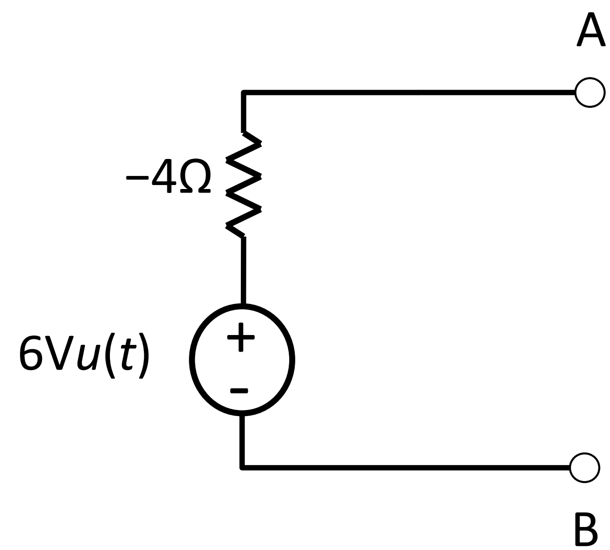


Stability

Find v_{oc}, i_{sc}, and $R_T = v_{oc}/i_{sc}$ on the time interval $t > 0$.

$$R_T = \frac{v_{oc}}{i_{sc}} = \frac{6\text{V}}{-1.5\text{A}} = -4\Omega \qquad t > 0$$





Stability

First, we find the initial conditions using the fact that dc steady state applies for t<0, and thus the inductor acts as a short circuit.

KVL, KCL, Ohm:

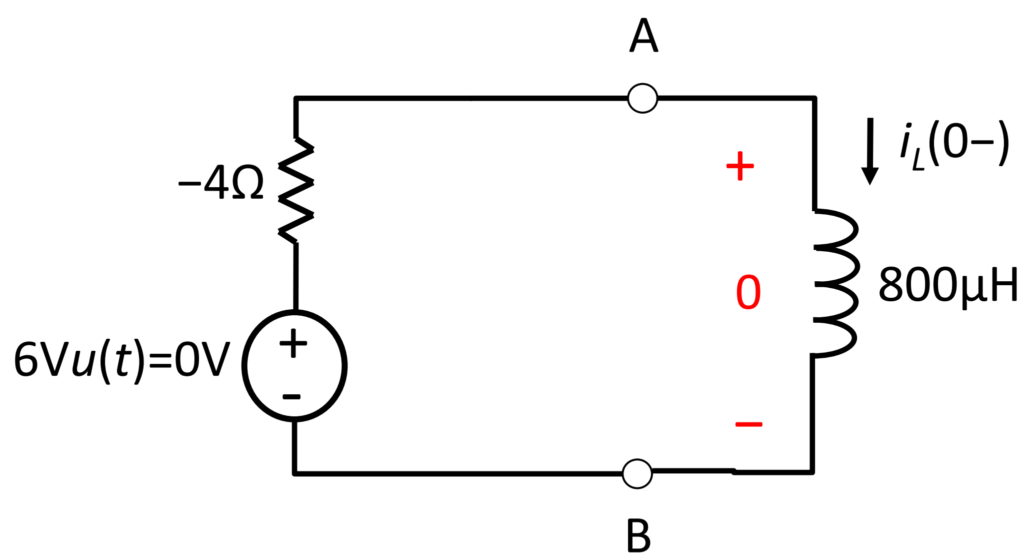
$$0 = 0\text{V} + (-4\Omega)\cdot i_L(0-) + 0\text{V}$$

$$i_L(0-) = 0\text{A}$$

inductor current continuity:

$$i_L(0+) = i_L(0-) = 0$$

$t < 0$

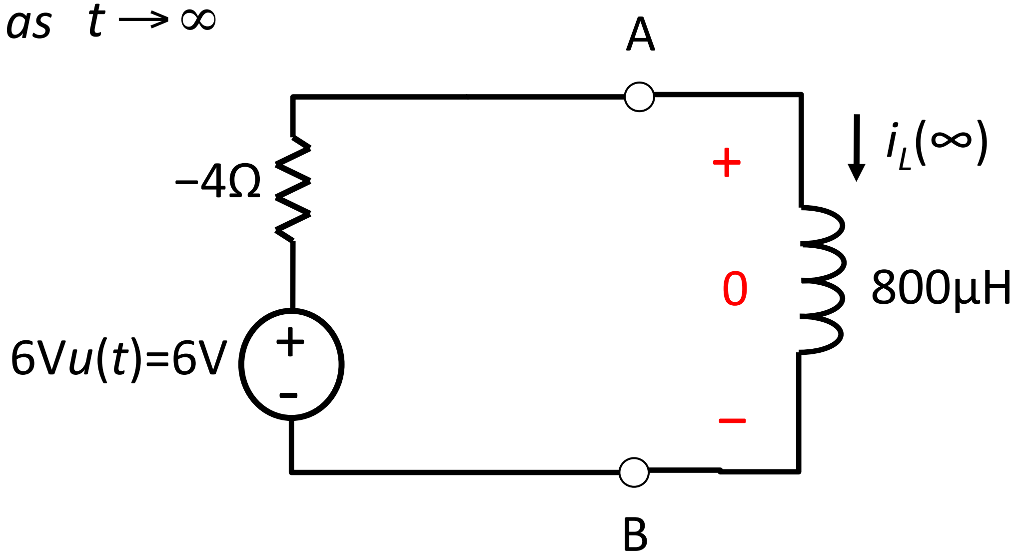


Stability

We find the response in dc steady state. At dc steady state, the inductor acts as a short, and draws the short circuit current from the Thévenin equivalent circuit at terminals A and B.

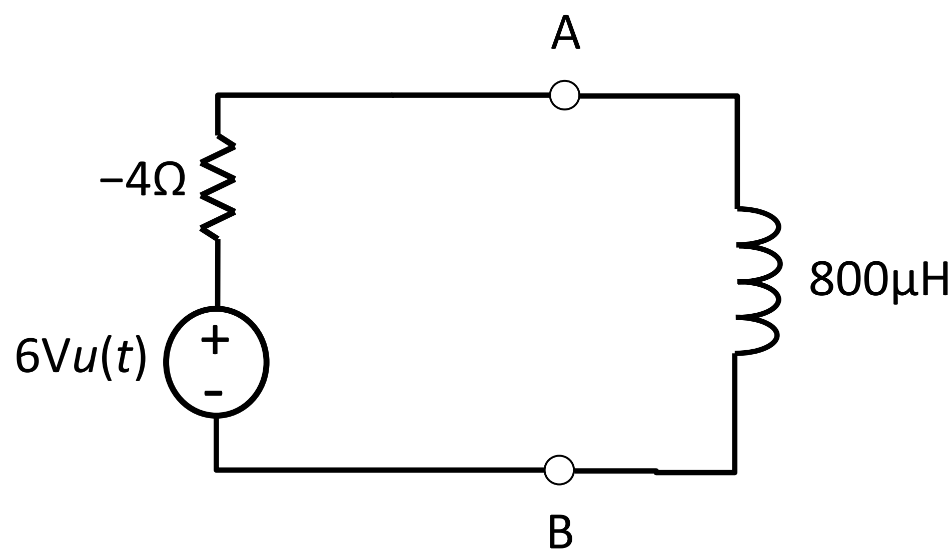
$$i_L(\infty) = i_{sc}(\infty) = -1.5\text{A}$$

dc steady state as $t \rightarrow \infty$



Stability

We find the time constant: $\tau = \frac{L}{R_T} = \frac{800\mu H}{-4\Omega} = -200\mu s$

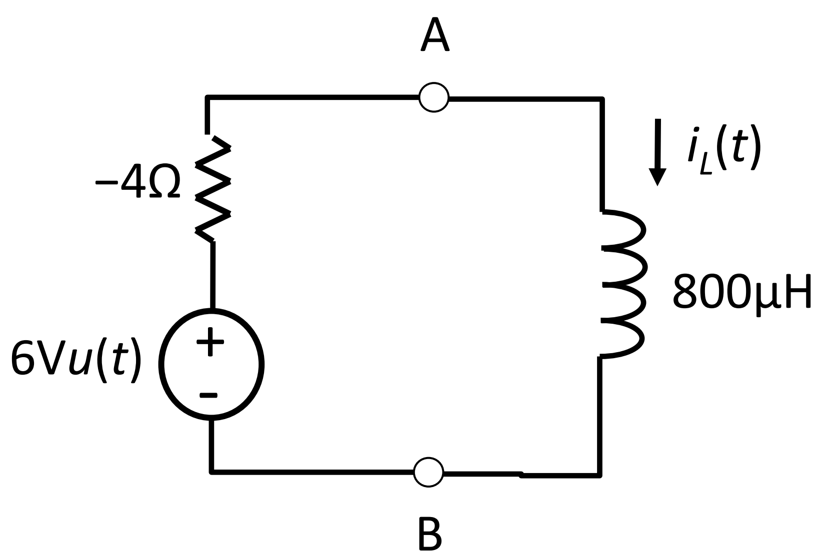


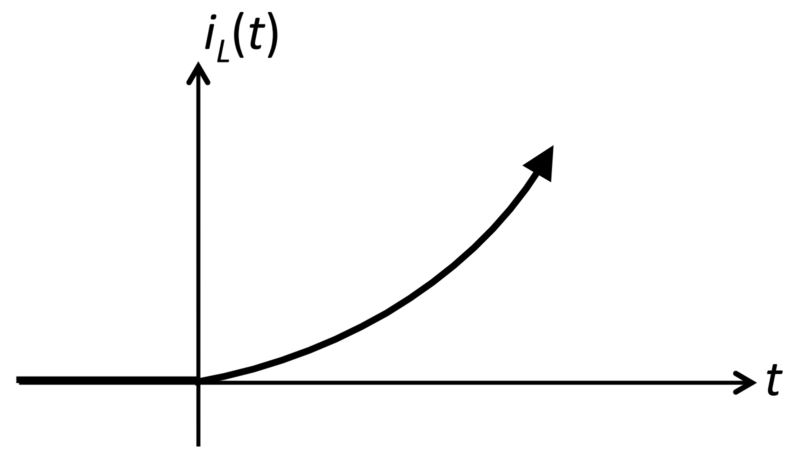
Stability

We assemble the solution.

Negative R_T leads to ***exponential growth*** of the response. The energy stored in the inductor grows with time.

$$i_L(t) = c_1 + c_2 \exp\left(-t/\tau\right)$$
$$= i_L\left(\infty\right) + \left[i_L\left(0+\right) - i_L\left(\infty\right)\right]\exp\left(-t/\tau\right)$$
$$= -1.5\text{A} + 1.5\text{A}\exp\left(+t/200\mu s\right)$$

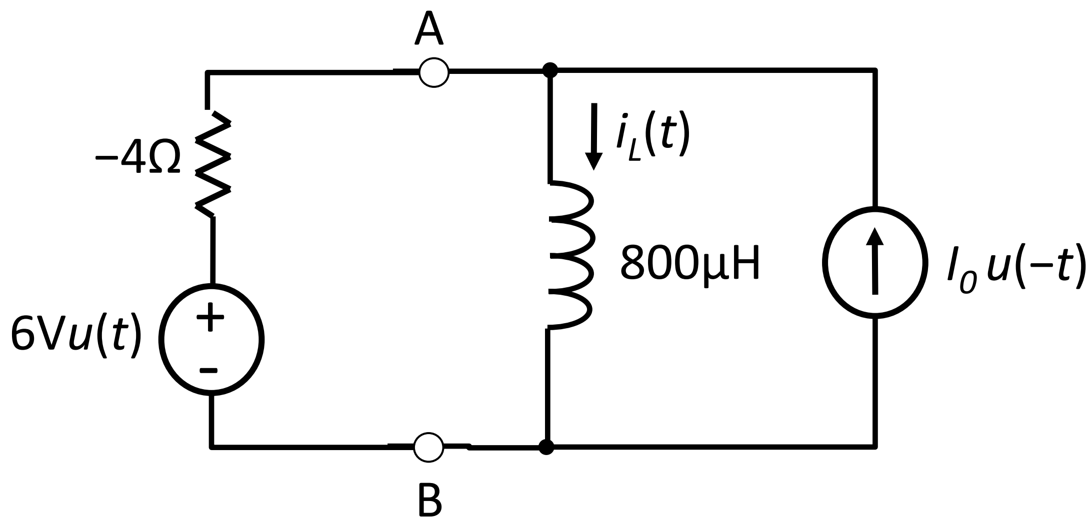


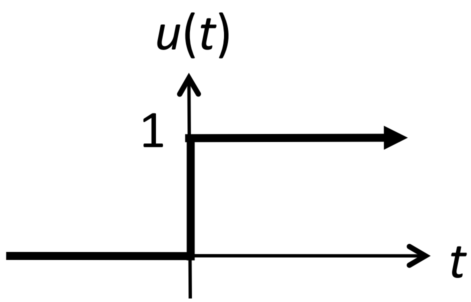


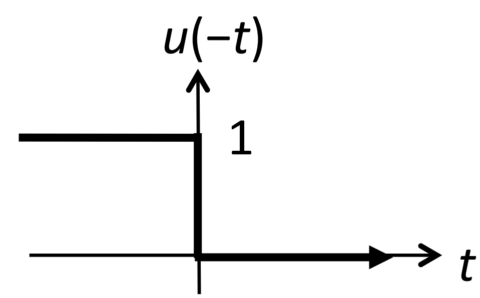
The dc steady state value $i_L(\infty)$=−1.5A is ***never*** reached for t>0.

Stability

Let's add a current source to control the initial inductor current $i_L(0+)$. For $t < 0$, the current source imposes the current I_0 in the inductor at dc steady state.







For $t < 0$, the Thévenin equivalent circuit is -4Ω alone.

For $t > 0$, the current source is 0A, equivalent to an open.

McGill

Stability

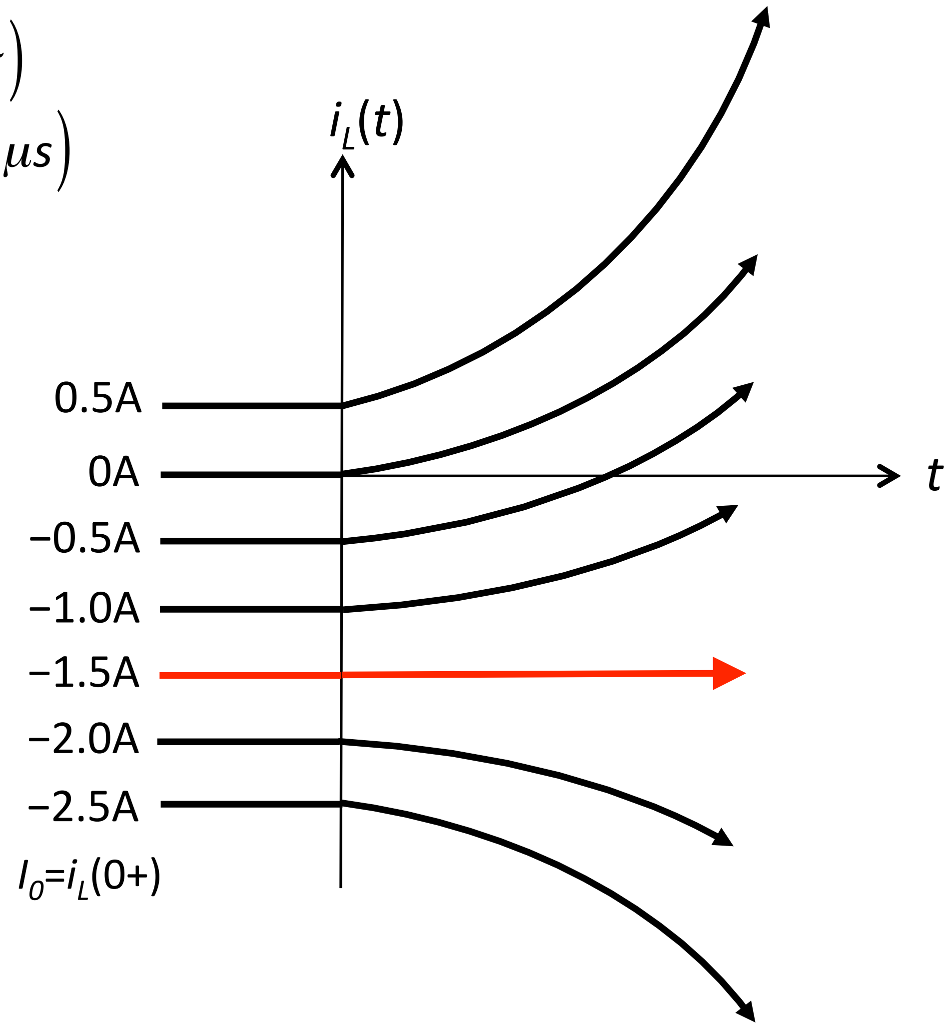
Inductor current response:

$$i_L(t) = i_L(\infty) + \left[i_L(0+) - i_L(\infty)\right]\exp\left(-t/\tau\right)$$
$$= -1.5\text{A} + \left[I_0 + 1.5\text{A}\right]\exp\left(+t/200\mu s\right)$$

The response is finite if and only if $i_L(0+)=i_L(\infty)=-1.5$A.

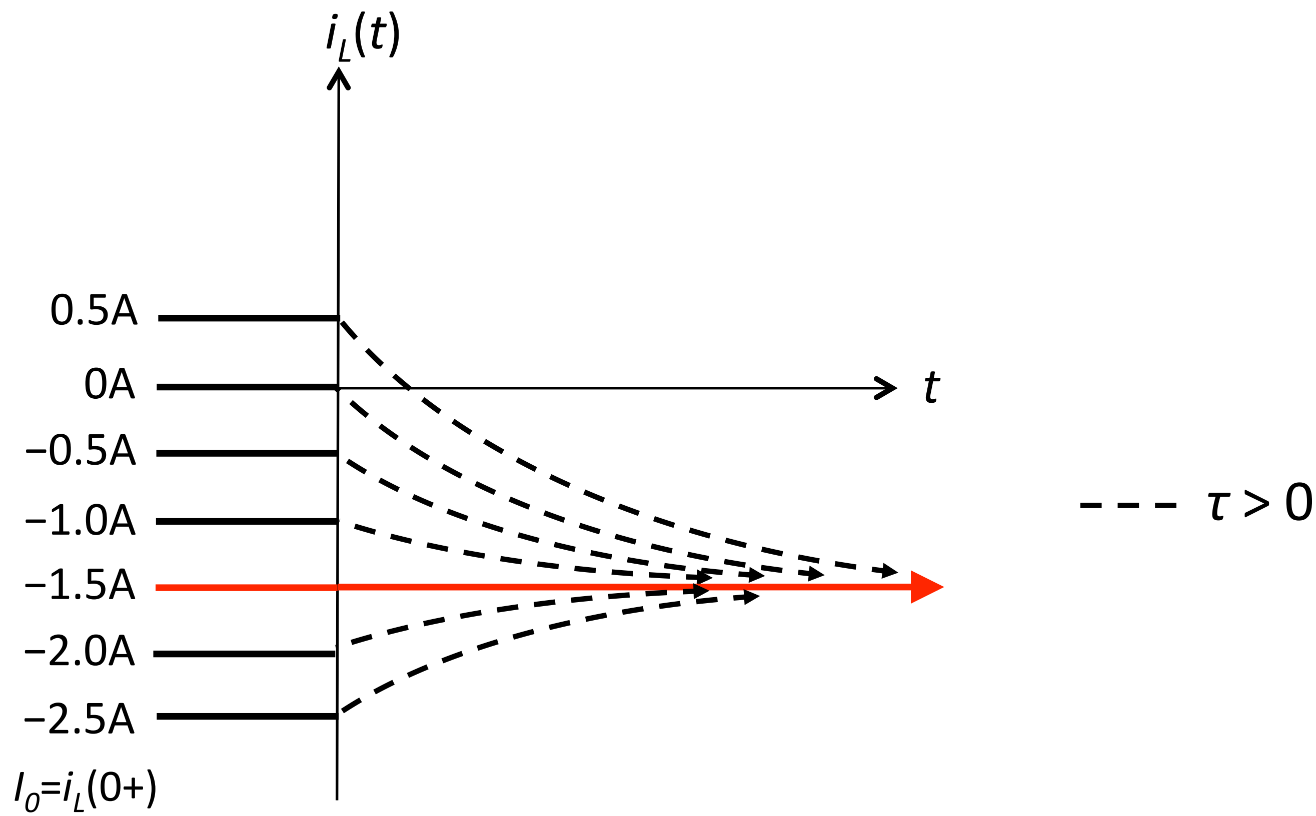
Any other initial condition leads leads to unlimited growth.

The circuit is ***unstable***.



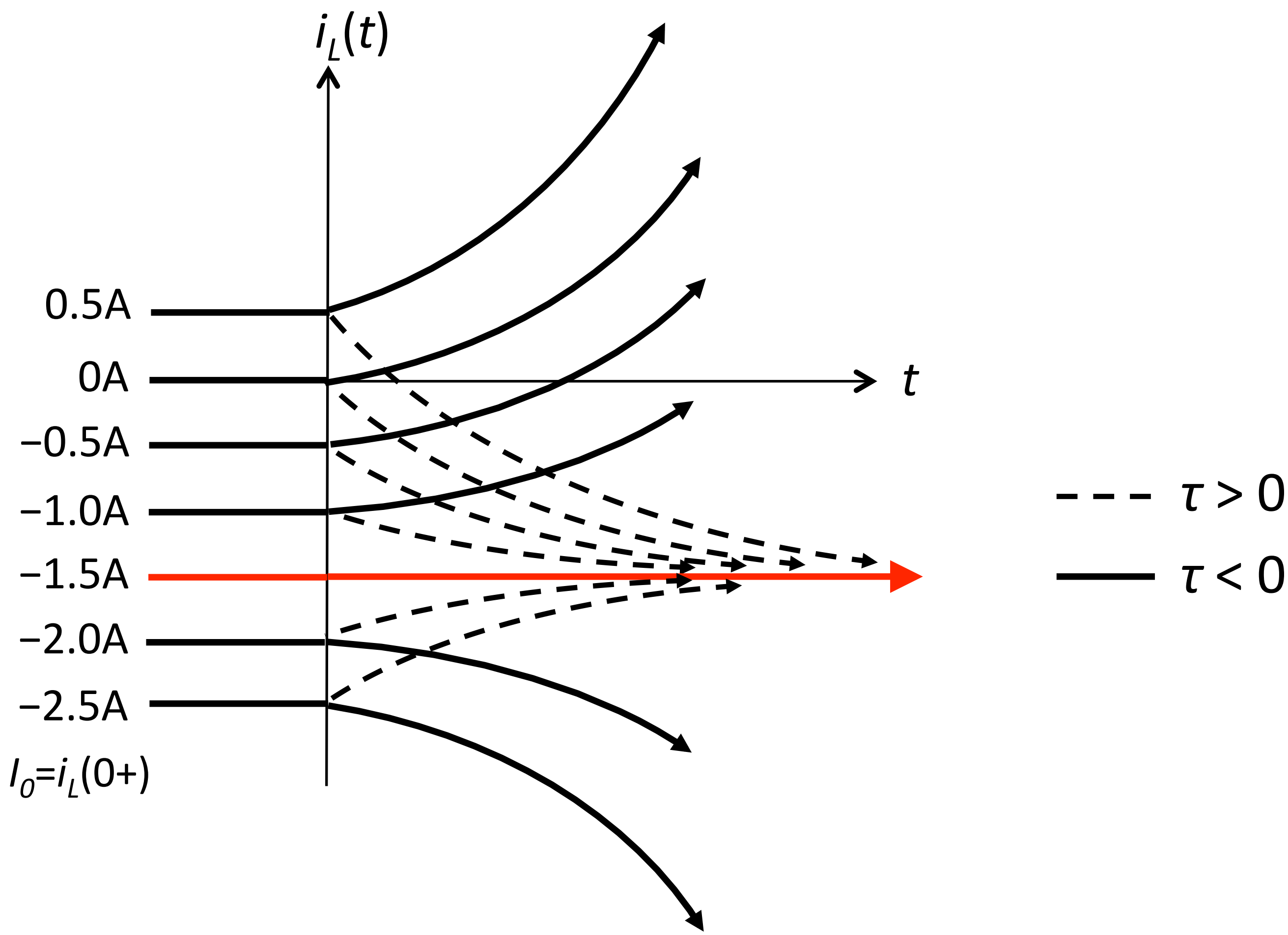
Stability

$\tau > 0$: $i_L(t)$ tends **towards** $i_L(\infty)$ for $t > 0$, giving ***stable*** response.



Stability

$\tau > 0$: $i_L(t)$ tends **towards** $i_L(\infty)$ for $t > 0$, giving ***stable*** response.
$\tau < 0$: $i_L(t)$ tends **away** from $i_L(\infty)$ for $t > 0$, giving ***unstable*** response.

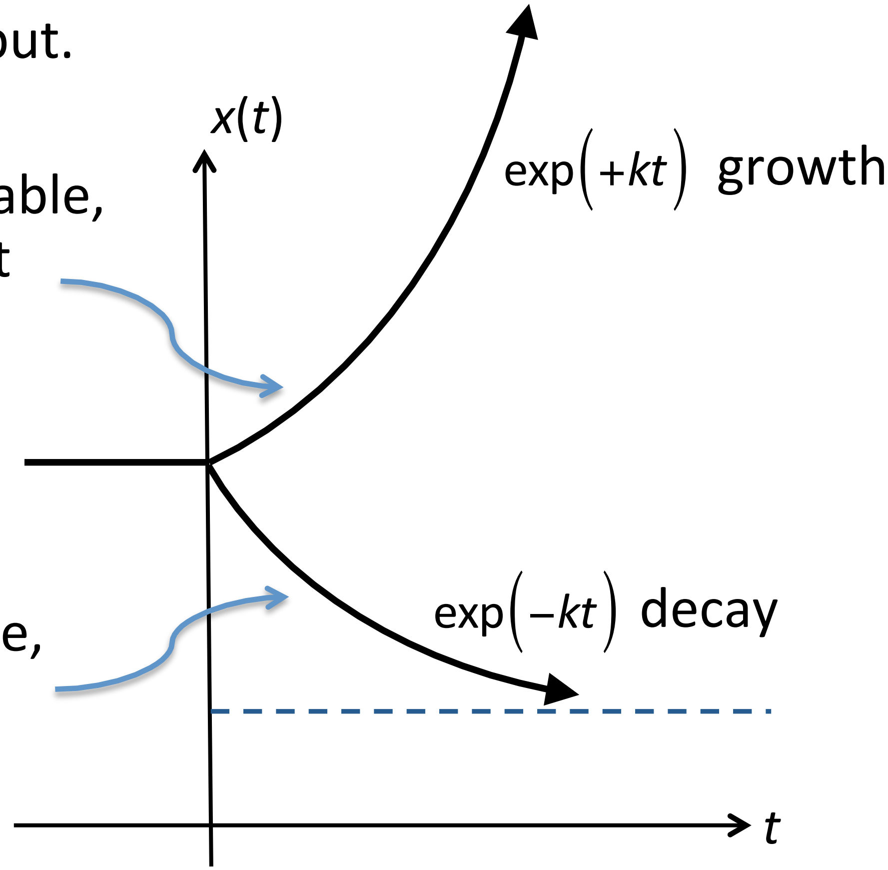


Stability

The sign of the Thévenin resistance determines the stability of the output.

$R_T < 0$ gives $\tau < 0$, and leads to unstable, exponential growth of the transient response.

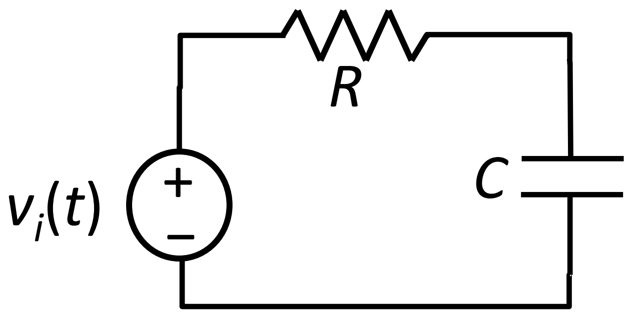
$R_T > 0$ gives $\tau > 0$, and leads to stable, exponential decay of the transient response.



Unstable response is generally undesirable in electric circuits.

response to a non-constant input

In the general case of a non-constant input, the particular / forced solution must be found by integration.



Differential equation for t>0:

$$\frac{dx}{dt} + \frac{x}{\tau} = g(t)$$

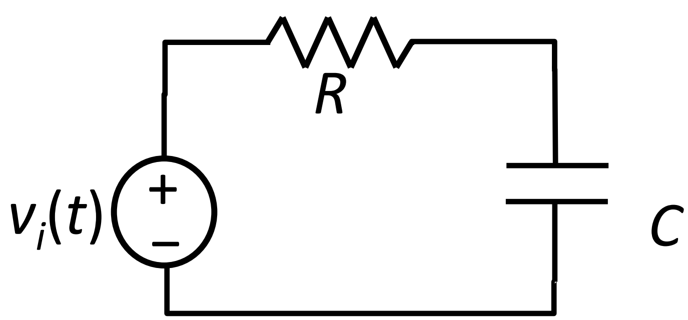
Solution:

$$x_p(t) = \exp(-t/\tau)\int \exp(t'/\tau)g(t')dt'$$

$$x(t) = K\exp(-t/\tau) + x_p(t)$$

response to a non-constant input

The most common forms of input and corresponding responses are:



$g(t)$	forced solution
M	$x_p(t) = N$
$M\sin(\omega t + \theta)$	$x_p(t) = A\cos(\omega t) + B\sin(\omega t)$

The general solution is of the form: $x(t) = K\exp(-t/\tau) + x_p(t)$

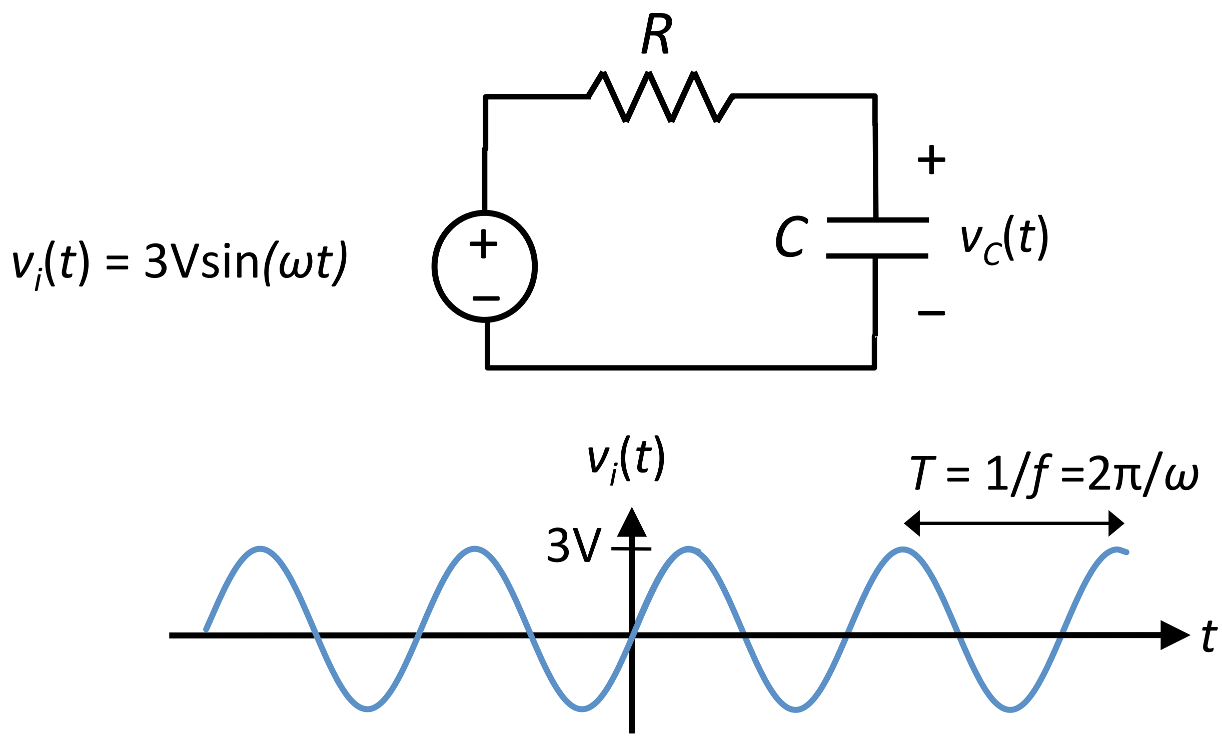
If the transient response has decayed away, only the forced solution remains:

$$x(t) = x_p(t)$$

Substitution into the differential equation determines the constants of the particular solution.

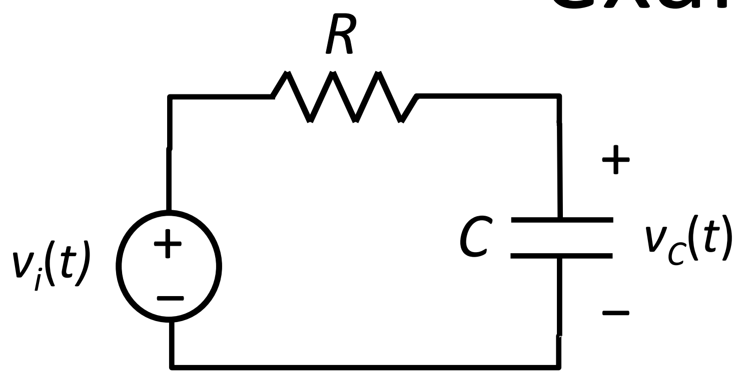
example

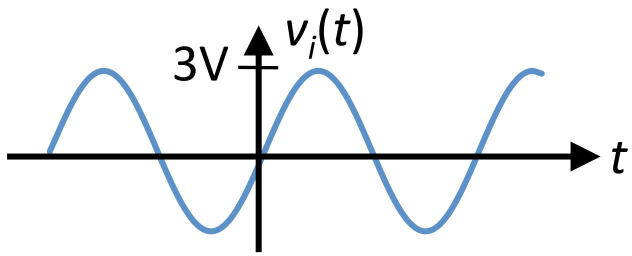
What is the capacitor voltage $v_C(t)$? Assume that the transient response has decayed away (the circuit is in ***ac steady state***).



Note: *There are more elegant ways to find the solution to this inhomogeneous (forced) differential equation problem. One such technique is the Laplace transform, which you will use in ECSE-210.*

example





KVL (can also use node eqn.):

$$0 = -v_i + \left(C \frac{dv_C}{dt} \right) \cdot R + v_C$$

differential equation:

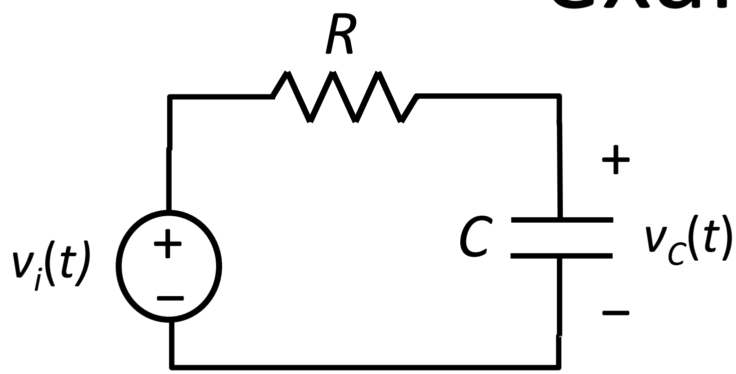
$$\frac{dv_C}{dt} + \frac{1}{RC} v_C = \frac{1}{RC} 3\mathrm{V} \sin(\omega t)$$

The solution takes the form:

$$v_C(t) = A\cos(\omega t) + B\sin(\omega t)$$

We make use of the differential equation to determine the constants *A* and *B*.

example



$$v_c = A\cos(\omega t) + B\sin(\omega t)$$

$$\frac{dv_c}{dt} = -\omega A\sin(\omega t) + \omega B\cos(\omega t)$$

Substitution into the differential equation gives:

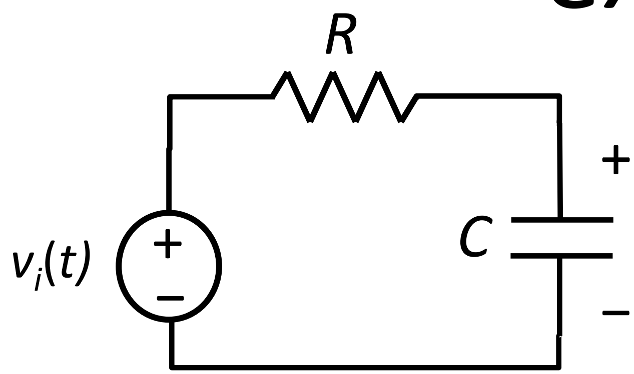
$$\frac{dv_c}{dt} + \frac{1}{RC}v_c = -\omega A\sin(\omega t) + \omega B\cos(\omega t) + \frac{A}{RC}\cos(\omega t) + \frac{B}{RC}\sin(\omega t)$$

$$\frac{3\text{V}}{RC}\sin(\omega t) = \left(\omega B + \frac{A}{RC}\right)\cos(\omega t) + \left(-\omega A + \frac{B}{RC}\right)\sin(\omega t)$$

Equating coefficients of sin(•) and cos(•) on left and right gives *A* and *B* :

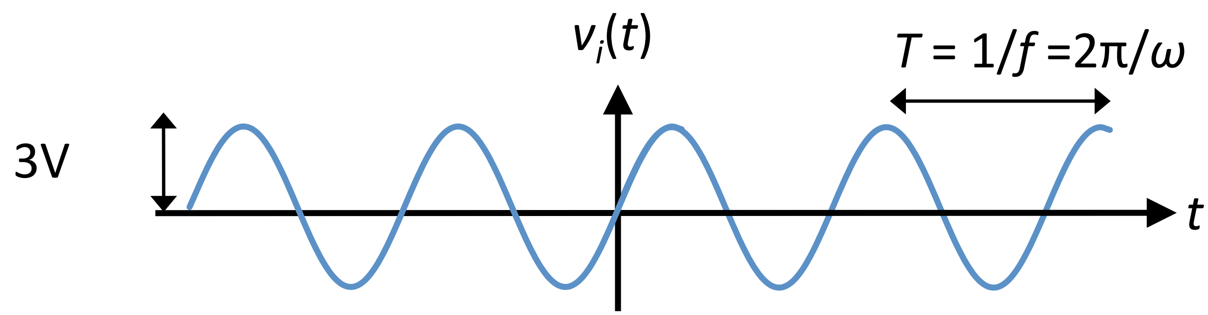
$$\left.\begin{aligned} \omega B + \frac{A}{RC} &= 0 \\ -\omega A + \frac{B}{RC} &= \frac{3\text{V}}{RC} \end{aligned}\right\} \quad \begin{aligned} A &= -3\text{V}\cdot\frac{\omega RC}{1+(\omega RC)^2} \\ B &= 3\text{V}\cdot\frac{1}{1+(\omega RC)^2} \end{aligned}$$

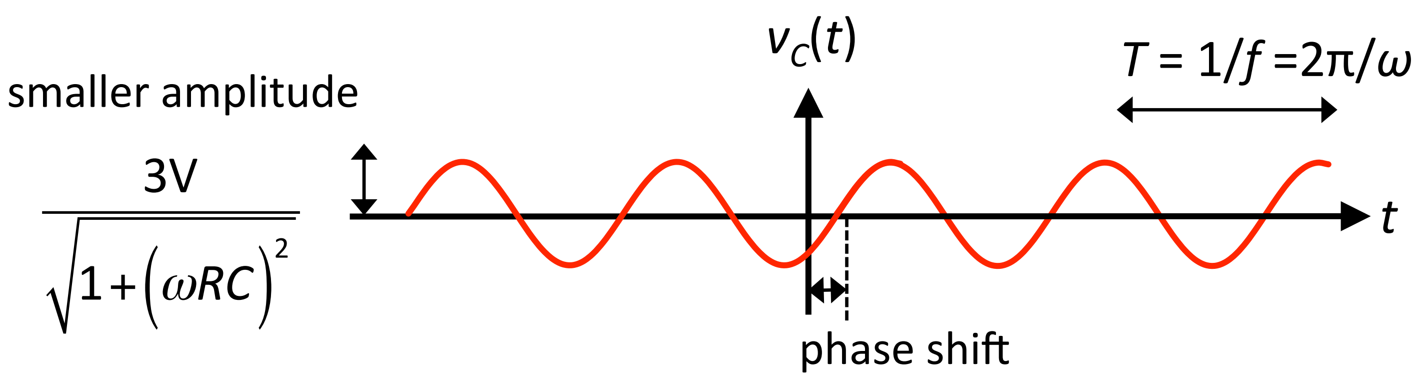
example



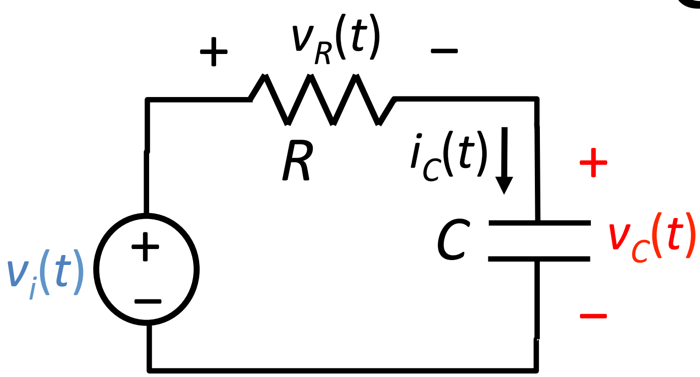
The solution is given by:

$$v_c = 3\mathrm{V}\frac{-\omega RC}{1+(\omega RC)^2}\cos(\omega t) + 3\mathrm{V}\frac{1}{1+(\omega RC)^2}\sin(\omega t)$$



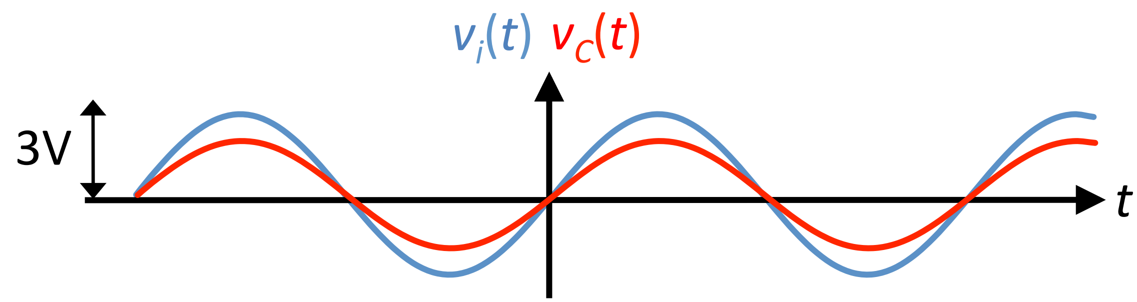


example



We can understanding the response of $v_C(t)$ when the frequency ω of the input voltage $v_i(t)$ is very small compared to $1/\tau = 1/RC$.

$\omega << 1 / \tau = 1/RC$

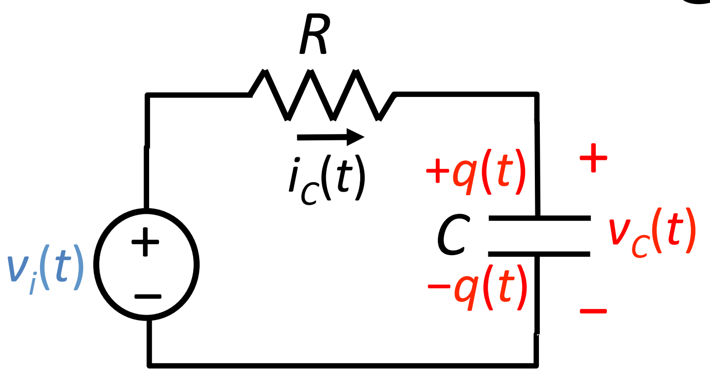


$$v_C = 3\text{V}\frac{-\omega RC}{1+(\omega RC)^2}\cos(\omega t) + 3\text{V}\frac{1}{1+(\omega RC)^2}\sin(\omega t) \approx 3V\sin(\omega t)$$

For very low frequencies, the capacitor voltage follows the input voltage since the voltage drop on the resistor is very small. The capacitor is acting almost as an open circuit.

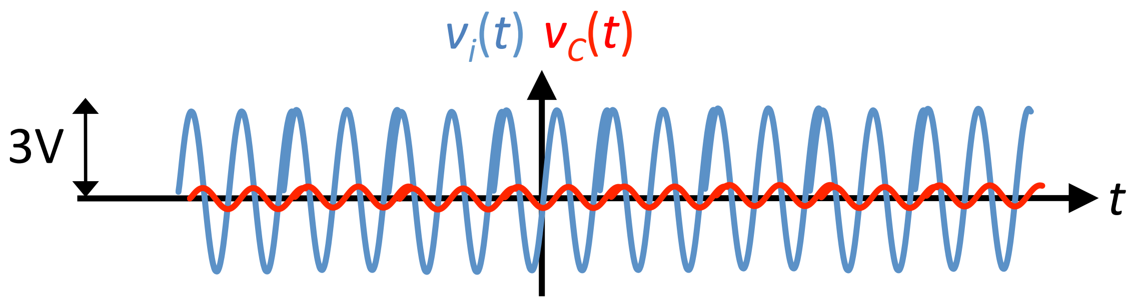
$$v_R = Ri_C = R \cdot C\frac{dv_c}{dt} \rightarrow 0 \quad \text{as} \quad \omega \rightarrow 0 \qquad\qquad \therefore v_C = v_i - v_R \rightarrow v_i \quad \text{as} \quad \omega \rightarrow 0$$

example



We can understanding the response of $v_C(t)$ when the frequency ω of the input voltage $v_i(t)$ is very large compared to $1/\tau = 1/RC$.

$\omega >> 1 / \tau = 1/RC$



$$v_C = 3\text{V}\frac{-\omega RC}{1+(\omega RC)^2}\cos(\omega t) + 3\text{V}\frac{1}{1+(\omega RC)^2}\sin(\omega t) \approx -\frac{3V}{\omega RC}\cos(\omega t)$$

For very high frequencies, the capacitor charge separation remains small since the current reverses direction rapidly (recall that q is the area under the i_C versus t graph). The voltage on the capacitor becomes small.

$$q(t) = \int i_c(t')dt' \to 0 \quad \text{as} \quad \omega \to \infty \qquad \therefore v_C = \frac{q}{C} \to 0 \quad \text{as} \quad \omega \to \infty$$

Section 7 Summary

First-order circuit response: Composed of a forced response and natural response.

Response to constant input: Find the initial conditions (using capacitor voltage or inductor current continuity), final conditions, time constant, and assemble the first order response.

Time constant: Either RC or L/R, where R is the Thevenin resistance seen by the capacitor or inductor. The characteristic time of the natural response

Sequential switching: General technique is applied to each time interval of constant input.

Unit step response: The principle of superposition can be used to determine the response to a sequence of steps at the input.

Section 7 Summary

Stability: Time constant must be positive for stable behaviour. A negative Thévenin resistance gives rise to a negative time constant, and unstable response.

Response to non-constant input: The particular solution is found by integration. The value of undetermined constants for the particular solution are found by substitution into the differential equation. Combining with the natural solution and initial conditions specifies the solution completely.