Today's Outline

6. Energy Storage Elements

- op-amp circuits with energy storage elements

Motivation:
Why is doing calculus with circuits useful?

robotics
efficient, smooth and predictable movement actuated by motors

industrial processes
maintaining temperature (via electrical heating) with minimum deviation from specified target

aerospace
maintaining stability and direction in the presence of perturbations (eg. wind)

Motivation:
Why is doing calculus with circuits useful?

physical response (lift, gravity, drag, turbulence, etc...)

altimeter

z

difference from intended height

$z - z_0$

rate of deviation from target height

$\frac{d(z - z_0)}{dt}$

past deviation of height

$\int (z - z_0)dt$

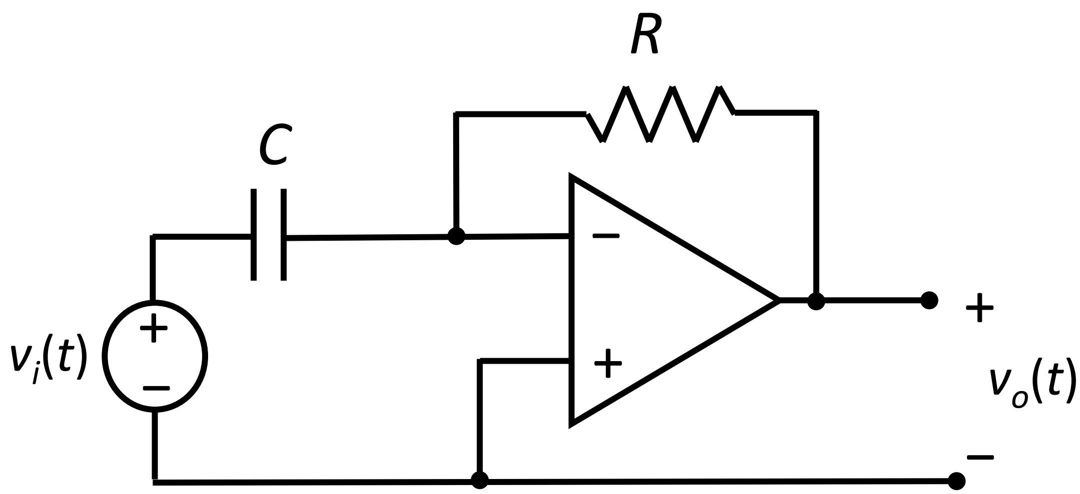
$+$

movement of control surface

$$K_P(z - z_0) + K_I \int (z - z_0)dt + K_D \frac{d(z - z_0)}{dt}$$

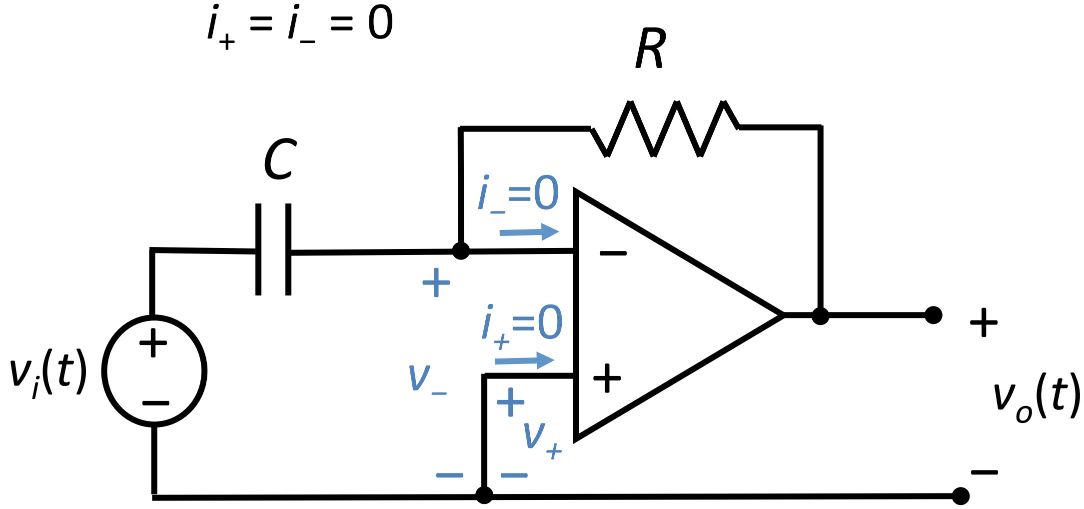
example 3

Assuming ideal op-amp behaviour, what is $v_o(t)$ as a function of $v_i(t)$?



example 3

Ideal op-amp: $v_+ = v_-$
$i_+ = i_- = 0$



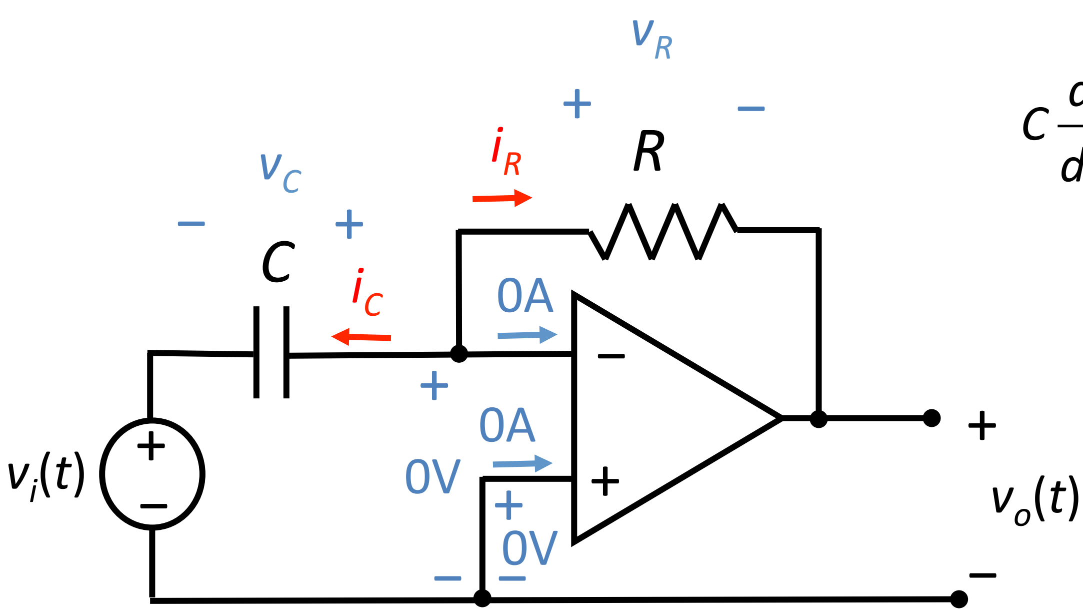
example 3

node-voltage equation at inverting input node:

$$i_C + i_R = 0$$

$$C\frac{dv_C(t)}{dt} + \frac{v_R(t)}{R} = 0$$

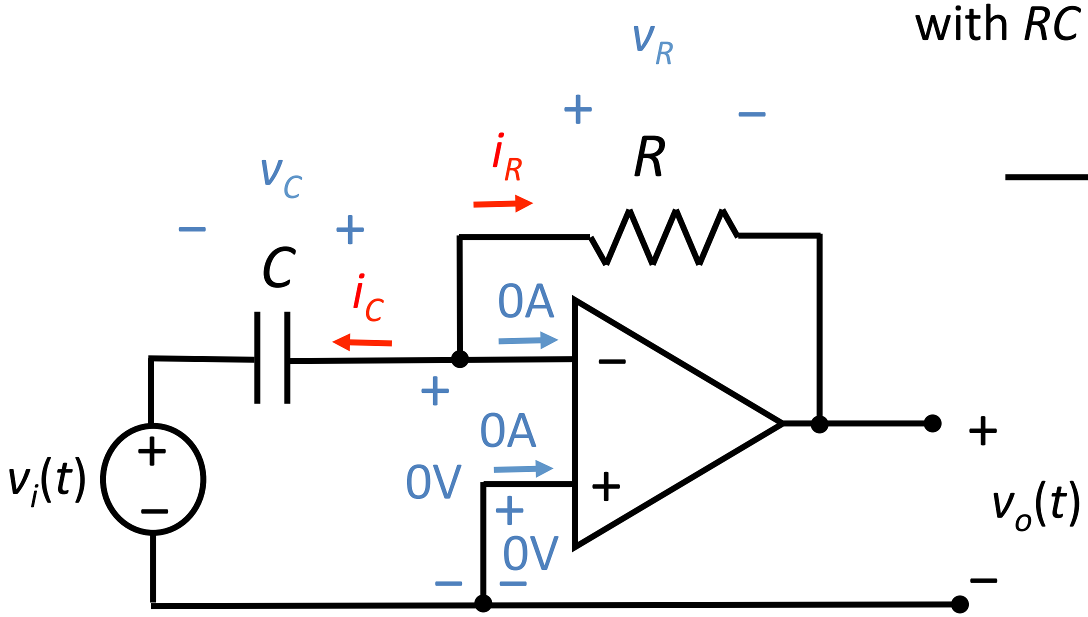
$$C\frac{d}{dt}\left(0\text{V} - v_i(t)\right) + \frac{0\text{V} - v_o(t)}{R} = 0$$



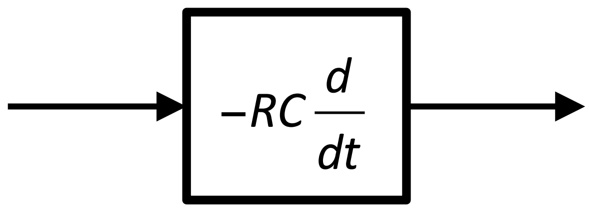
example 3

simplifying the equation: $C\frac{d}{dt}\left(0\text{V}-v_i(t)\right)+\frac{0\text{V}-v_o(t)}{R}=0$

$$\boxed{v_o(t)=-RC\frac{d}{dt}v_i(t)}$$

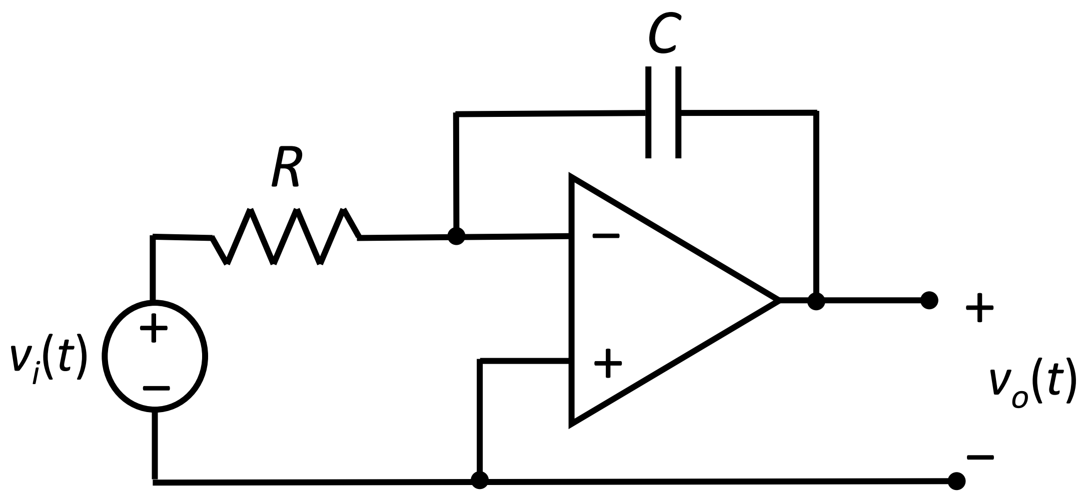


This circuit acts as a ***differentiator***, with *RC* setting the time scale.



example 4

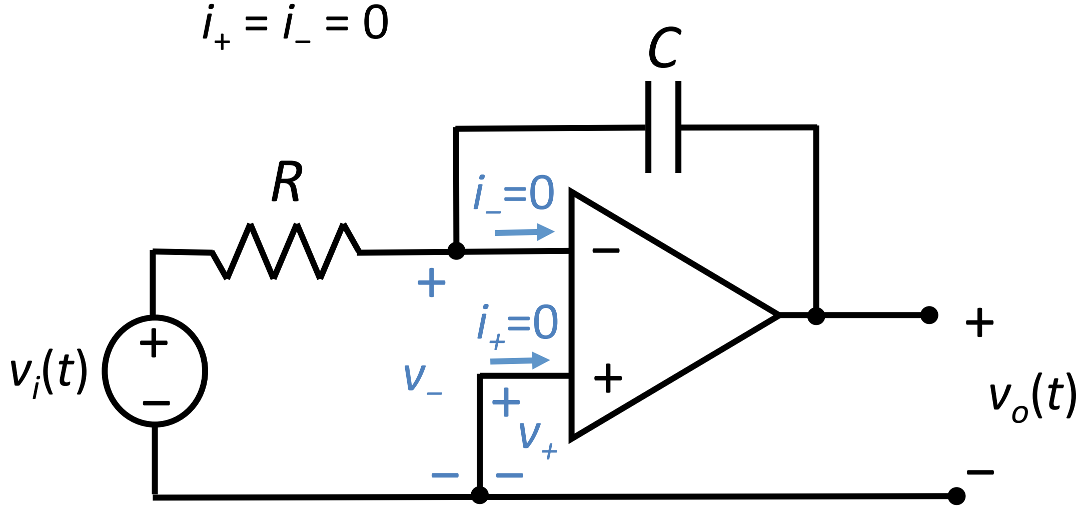
Assuming ideal op-amp behaviour, what is $v_o(t)$ as a function of $v_i(t)$?



example 4

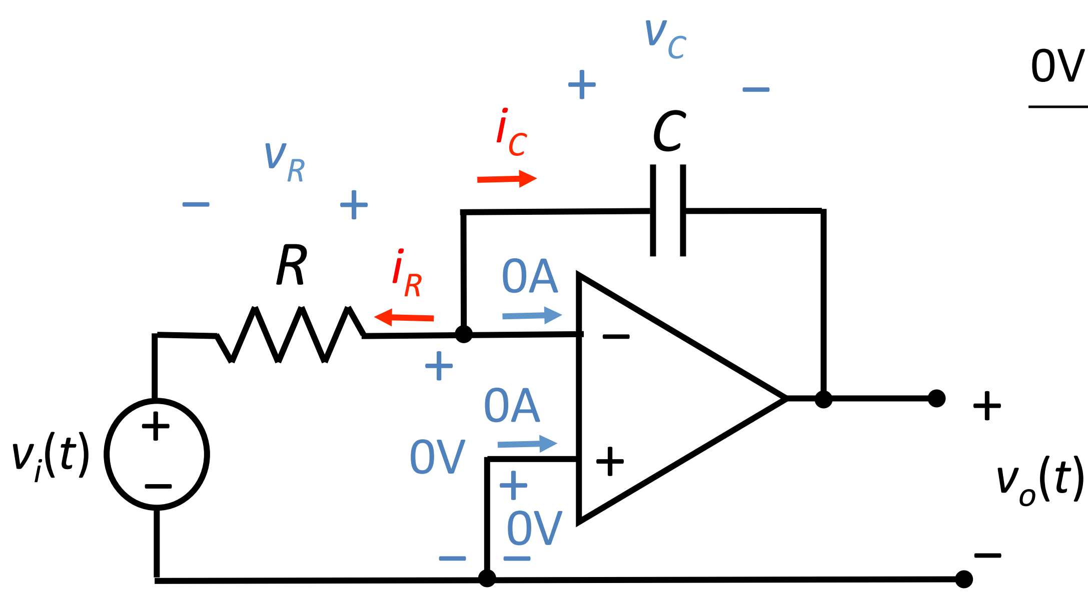
Ideal op-amp: $v_+ = v_-$

$i_+ = i_- = 0$



example 4

node-voltage equation at inverting input node:



$$i_R + i_C = 0$$

$$\frac{v_R(t)}{R} + C\frac{dv_C(t)}{dt} = 0$$

$$\frac{0\text{V} - v_i(t)}{R} + C\frac{d}{dt}\Big(0\text{V} - v_o(t)\Big) = 0$$

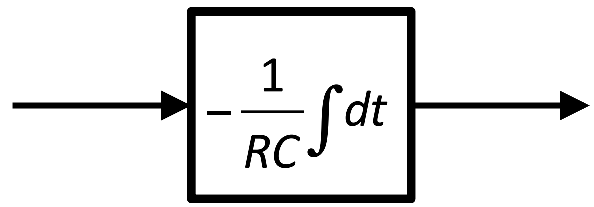
example 4

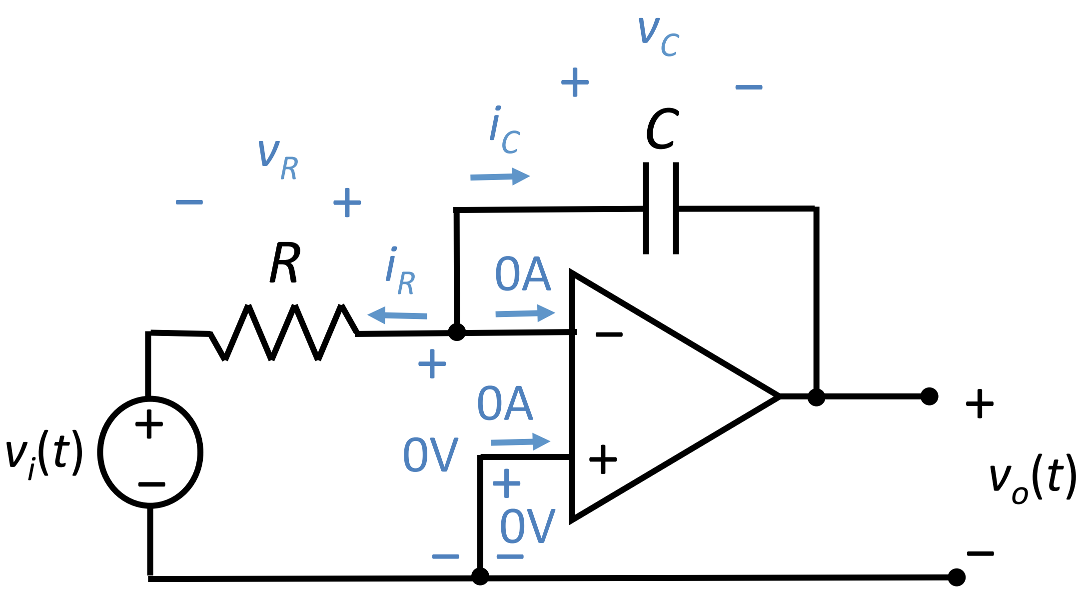
$$\frac{0\text{V} - v_i(t)}{R} + C\frac{d}{dt}\big(0\text{V} - v_o(t)\big) = 0$$

$$v_i(t) = -RC\frac{d}{dt}v_o(t)$$

$$\boxed{v_o(t_2) - v_o(t_1) = -\frac{1}{RC}\int_{t_1}^{t_2} v_i(t)\,dt}$$

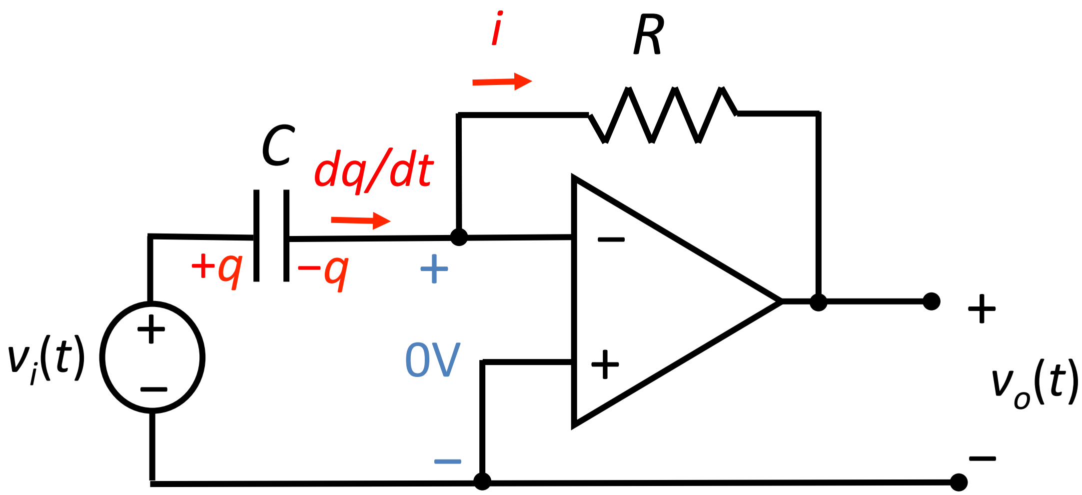
This circuit acts as an ***integrator***, with RC setting the time scale.

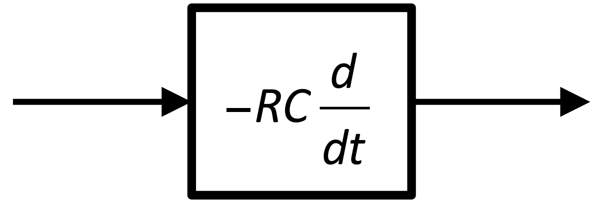




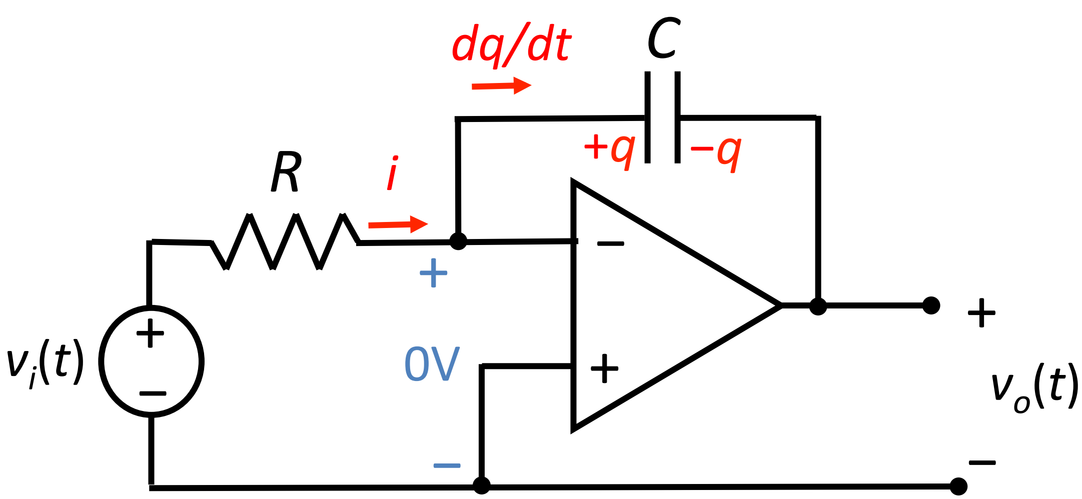
Question: How can the circuit be "reset", so that the result of a past integration is "erased"?

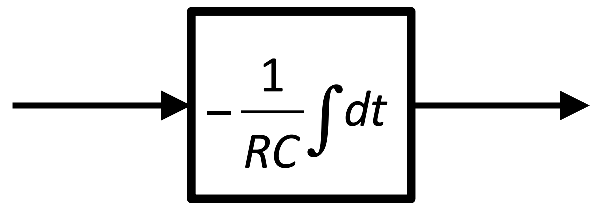
physical interpretation





C passes a current *i* in proportion to the time rate of change of input voltage, and *R* produces a voltage proportional to *i*.



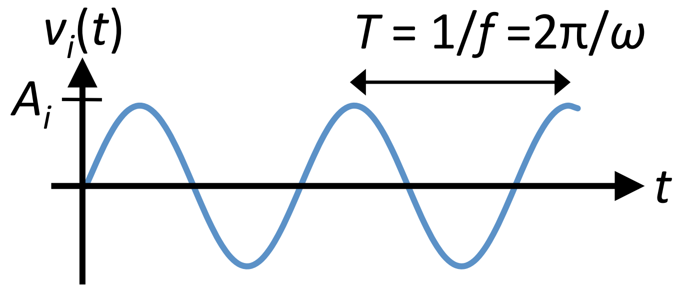


R passes a current *i* in proportion to the input voltage, and *C* produces a voltage proportional to the collected charge.

special case: sinusoidal inputs

Voltage signals are often of an ***ac*** form, being a sinusoidal function of amplitude A_i and radial frequency $\omega = 2\pi f$:

$$v_i(t) = A_i \sin(\omega t)$$



The corresponding output signals for the differentiator and integrator are:

$$v_i(t) = A_i \sin(\omega t) \longrightarrow \boxed{-RC\frac{d}{dt}} \longrightarrow v_0(t) = \underbrace{-RC\omega A_i}_{A_0} \cos(\omega t)$$

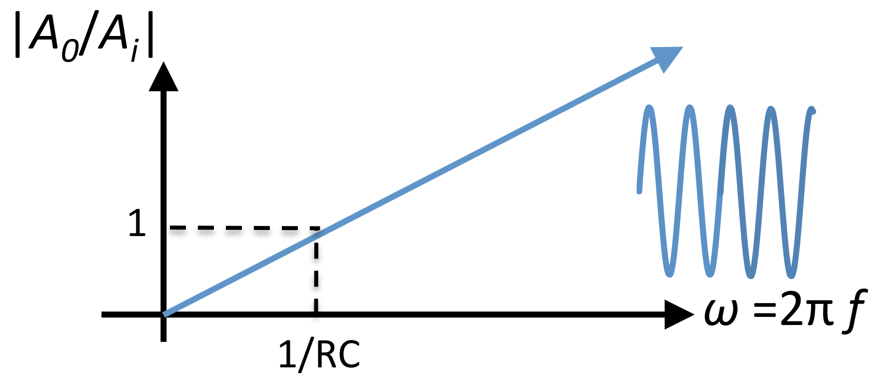
$$v_i(t) = A_i \sin(\omega t) \longrightarrow \boxed{-\frac{1}{RC}\int dt} \longrightarrow v_0(t) = \underbrace{\frac{1}{RC\omega} A_i}_{A_0} \cos(\omega t)$$

special case: sinusoidal inputs

The voltage gain of the sinusoidal amplitude is highly dependent on the frequency of the input sinusoid.

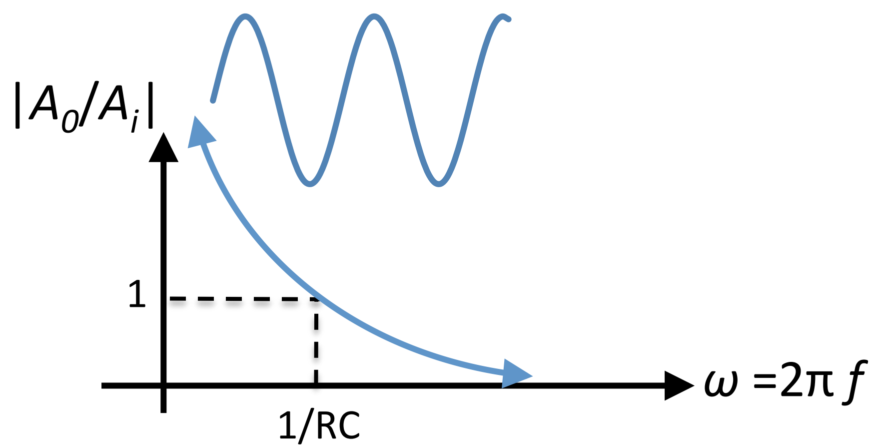
$$-RC\frac{d}{dt} \qquad \left|\frac{A_0}{A_i}\right| = RC\omega$$

High frequencies are amplified more than low frequencies.

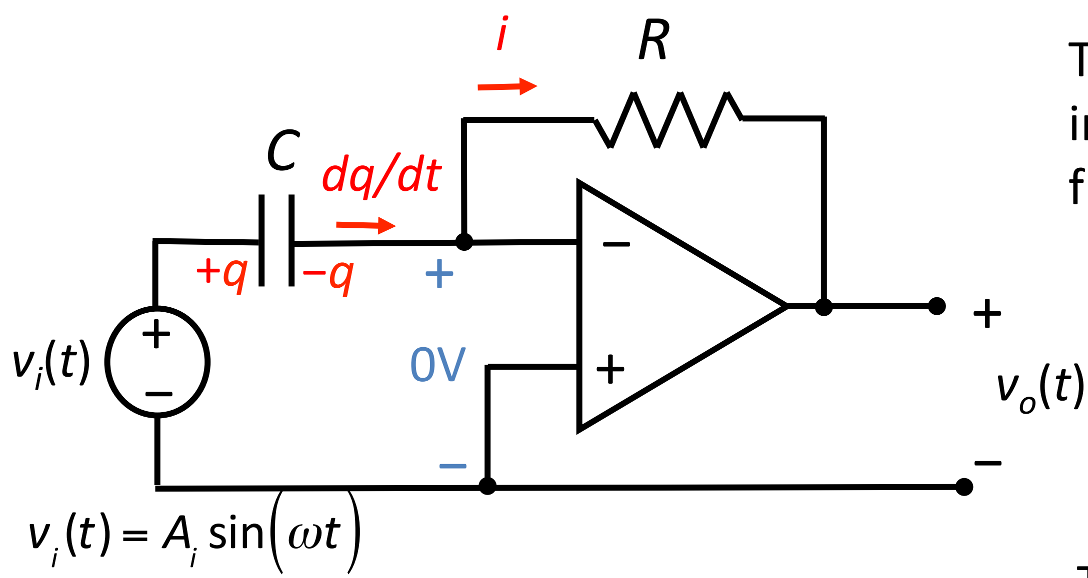


$$-\frac{1}{RC}\int dt \qquad \left|\frac{A_0}{A_i}\right| = \frac{1}{RC\omega}$$

Low frequencies are amplified more than high frequencies.

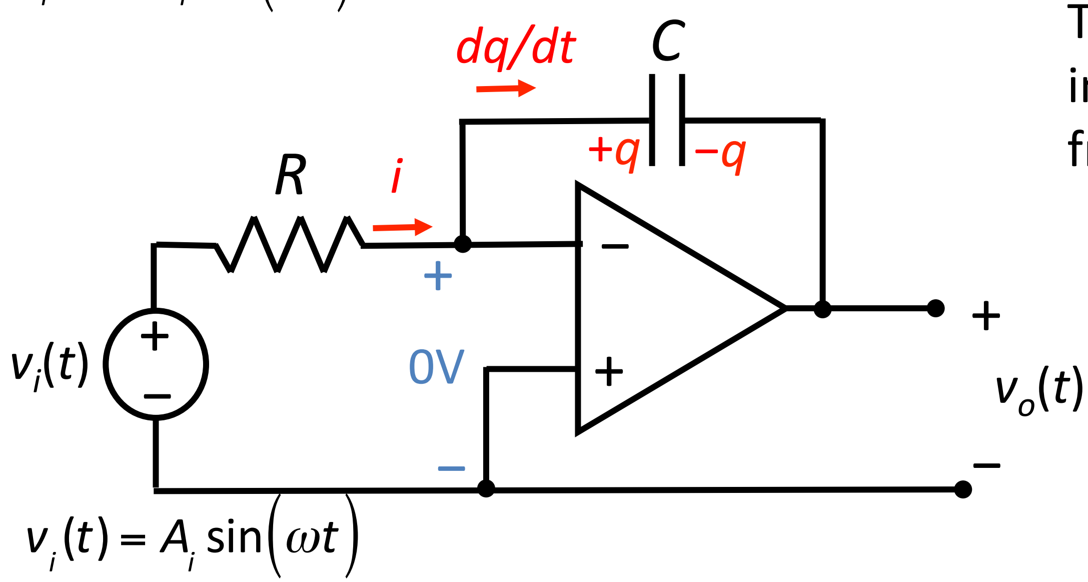


physical interpretation



The current on the capacitor C increases with increasing frequency:

$$i = \frac{dq}{dt} = C\frac{dv_i}{dt} \propto \omega$$



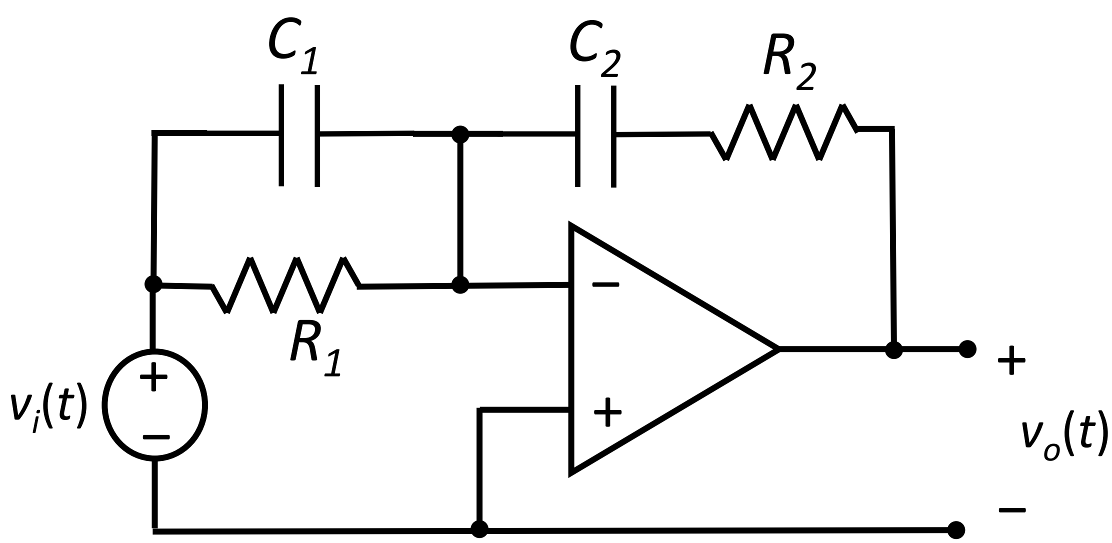
The voltage on the capacitor C increases with decreasing frequency:

$$v_o = -\frac{q}{C} = -\frac{\int i dt}{C} \propto \frac{1}{\omega}$$

Question: What is the output voltage if a triangular pulse is applied to the input of the differentiator and the integrator? Is the circuit linear?

example 5

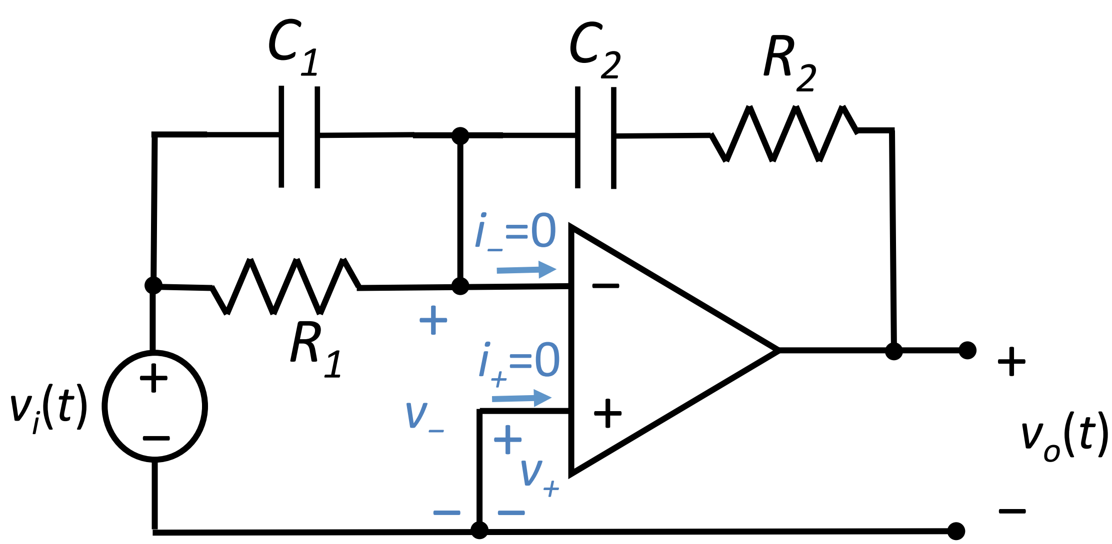
Assuming ideal op-amp behaviour, what is $v_o(t)$ as a function of $v_i(t)$? Assume that at time t=0, the capacitors C_1 and C_2 have zero charge separation.



example 5

Ideal op-amp: $v_+ = v_-$

$i_+ = i_- = 0$



example 5

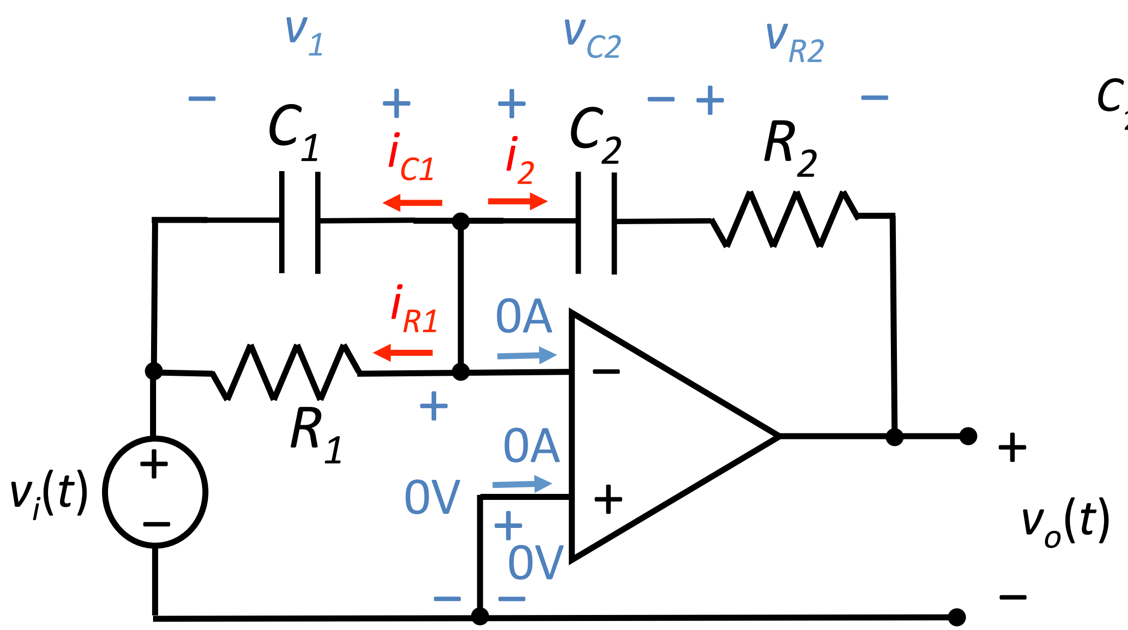
node-voltage equation at inverting input node:

$$i_{C1} + i_{R1} + i_2 = 0$$

$$C_1 \frac{dv_1(t)}{dt} + \frac{v_1(t)}{R_1} + C_2 \frac{dv_{C2}(t)}{dt} = 0$$

$$C_1 \frac{d}{dt}\left(0 - v_i(t)\right) + \frac{0 - v_i(t)}{R_1} + C_2 \frac{dv_{C2}(t)}{dt} = 0$$

$$C_2 \frac{dv_{C2}(t)}{dt} = C_1 \frac{d}{dt} v_i(t) + \frac{v_i(t)}{R_1}$$



example 5

node equation between R_2 and C_2:

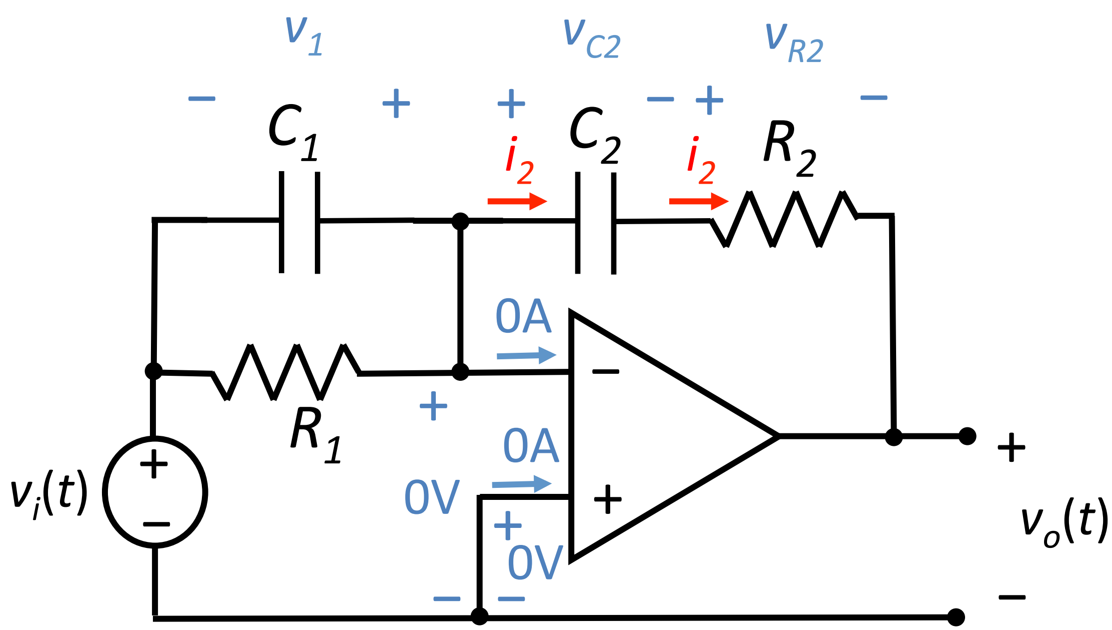
$$i_2 = i_2$$

$$\frac{v_{R2}(t)}{R_2} = C_2 \frac{dv_{C2}(t)}{dt}$$

KVL:

$$0 = 0 + v_{C2}(t) + v_{R2}(t) + v_o(t)$$

$$v_o(t) = -v_{C2}(t) - v_{R2}(t)$$



example 5

putting the results together: $$C_2 \frac{dv_{C2}(t)}{dt} = C_1 \frac{d}{dt} v_i(t) + \frac{v_i(t)}{R_1}$$

$$v_{C2}(t) - v_{C2}(0) = \frac{C_1}{C_2}\Big(v_i(t) - v_i(0)\Big) + \frac{1}{R_1 C_2}\int_0^t v_i(t')dt'$$

$$v_{C2}(t) = \frac{C_1}{C_2} v_i(t) + \frac{1}{R_1 C_2}\int_0^t v_i(t')dt'$$ given that zero charge sep. at t=0

$$\frac{v_{R2}(t)}{R_2} = C_2 \frac{dv_{C2}(t)}{dt}$$

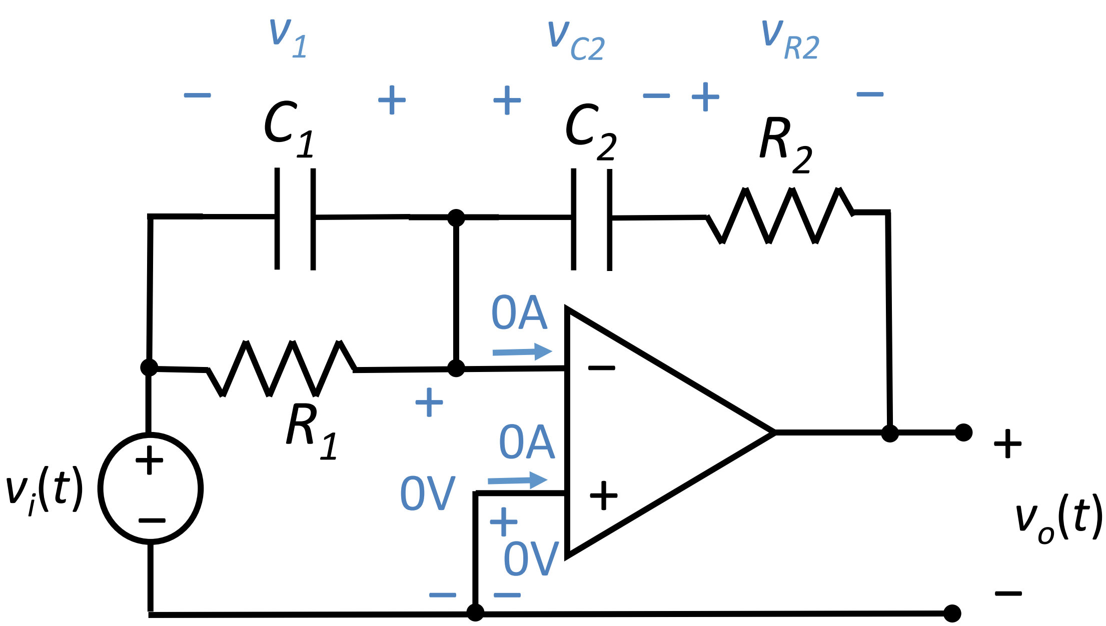
$$v_{R2}(t) = R_2 C_1 \frac{dv_i(t)}{dt} + \frac{R_2}{R_1} v_i(t)$$

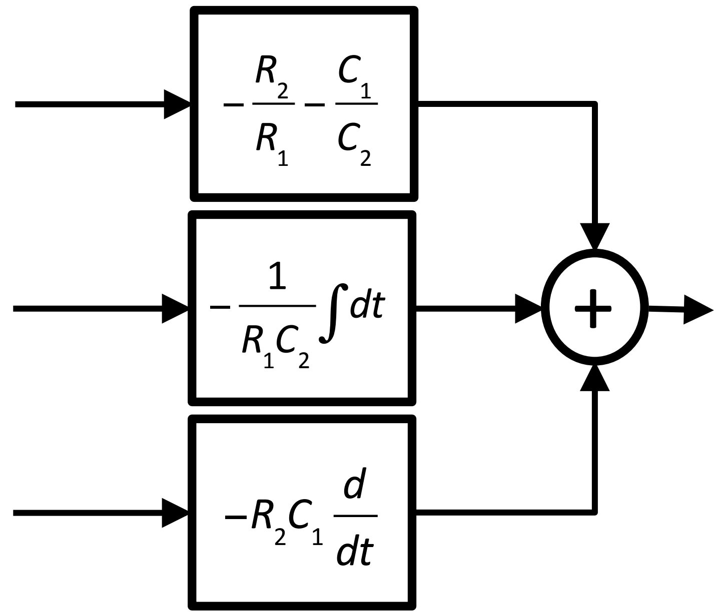
$$v_o(t) = -v_{C2}(t) - v_{R2}(t)$$

example 5

This single circuit has a ***proportional***, ***integral*** and ***differential*** output.

$$v_o(t) = -\left(\frac{R_2}{R_1} + \frac{C_1}{C_2}\right)v_i(t) - \frac{1}{R_1 C_2}\int_0^t v_i(t')dt' - R_2 C_1 \frac{dv_i(t)}{dt}$$

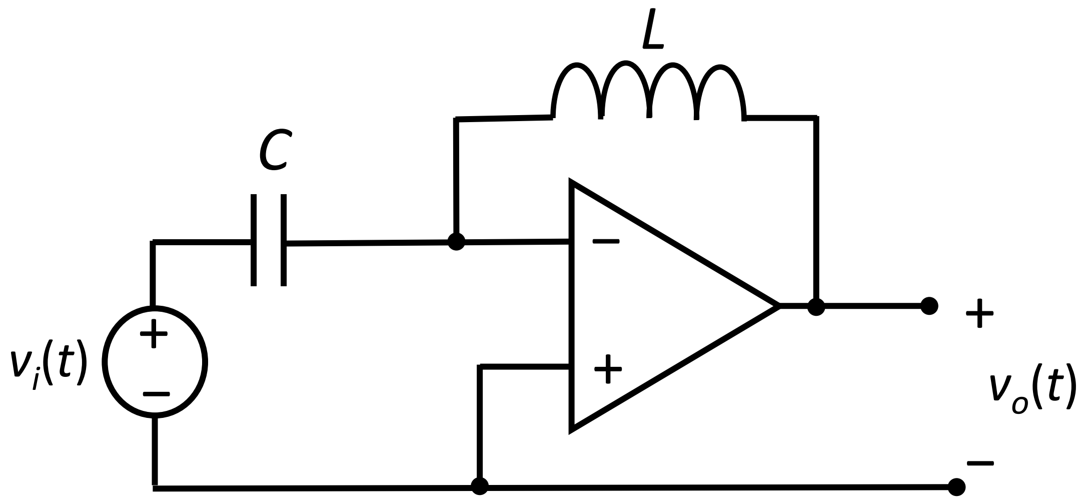




Question: One could also sum the response from an inverting amp., a differentiator and an integrator. What are the advantages and disadvantages of the two approaches?

example 6

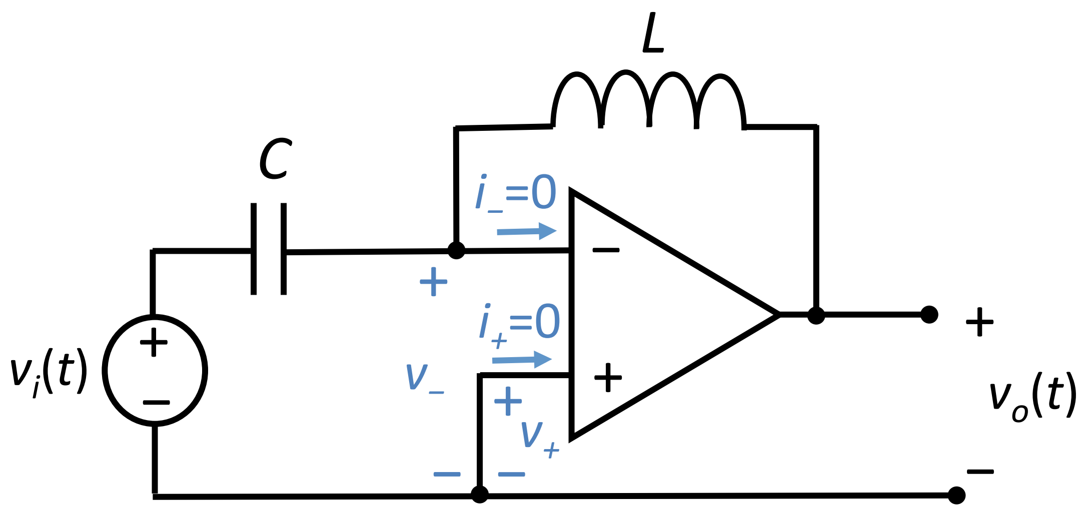
Assuming ideal op-amp behaviour, what is the $v_o(t)$ as a function of $v_i(t)$?



example 6

Ideal op-amp: $v_+ = v_-$

$i_+ = i_- = 0$



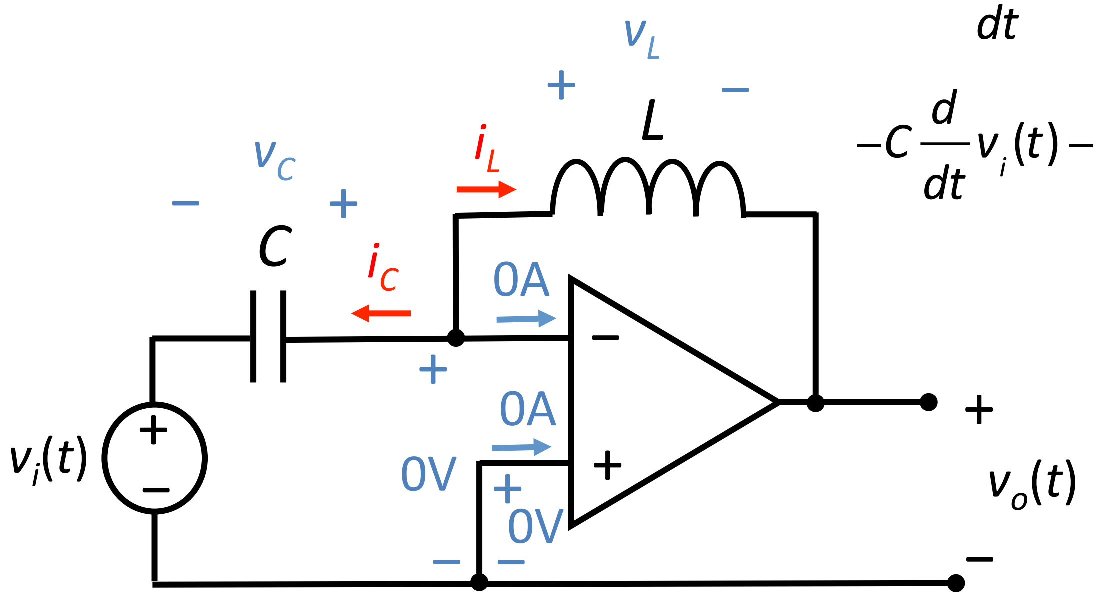
example 6

node-voltage equation at inverting input node:

$$i_C + i_L = 0$$

$$C\frac{dv_C(t)}{dt} + \frac{1}{L}\int_0^t v_L(t')dt' = 0$$

$$-C\frac{d}{dt}v_i(t) - \frac{1}{L}\int_0^t v_0(t')dt' = 0$$

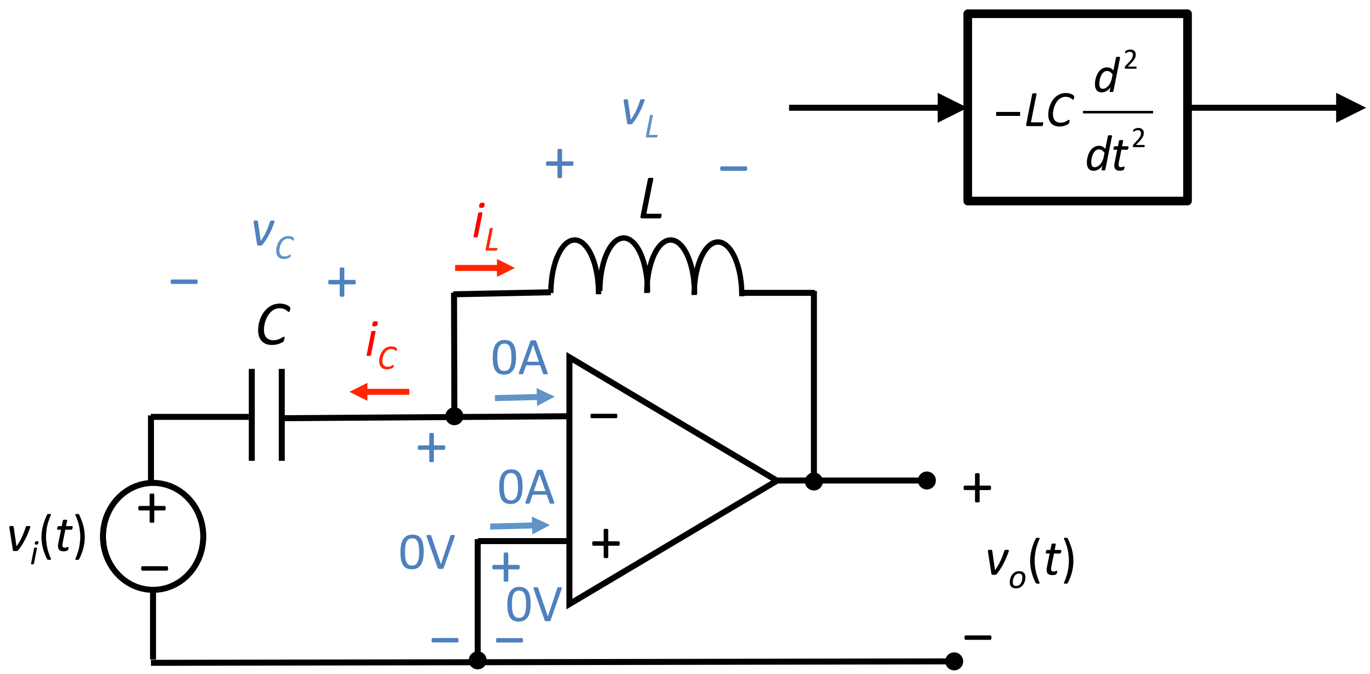


example 6

simplifying the equation: $-C\frac{d}{dt}v_i(t)-\frac{1}{L}\int_0^t v_0(t')dt'=0$

$$\boxed{v_0(t)=-LC\frac{d^2}{dt^2}v_i(t)}$$

This circuit acts as a ***second order differentiator***.



McGill

Section 6 Summary

Ideal capacitors: Store electric energy $\frac{1}{2}Cv^2$, charge separation $q=Cv$ and has terminal equation $i = C\, dv/dt$. Voltage is continuous. Replaced by open in dc steady state.

Ideal inductors: Store magnetic energy $\frac{1}{2}Li^2$, flux linkage $\lambda=Li$ and has terminal equation $v = L\, di/dt$. Current is continuous. Replaced by short in dc steady state.

Coupled inductors: Self-inductance and mutual inductance describe flux linkage between coupled ideal inductors.

Op-amps + energy storage elements: Op-amps can be programmed to perform many calculus operations by including capacitors or inductors.