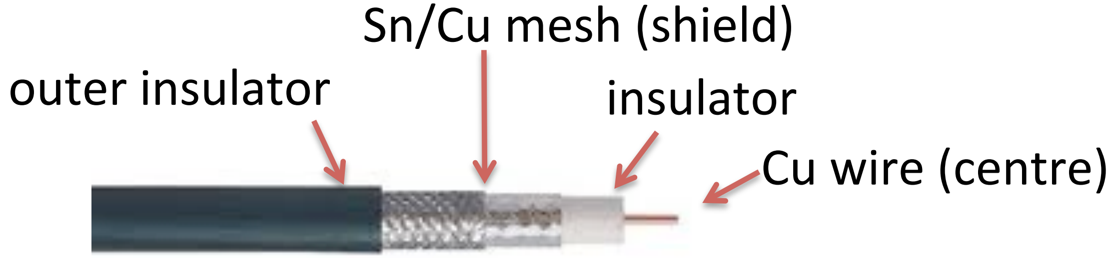
Today's Outline

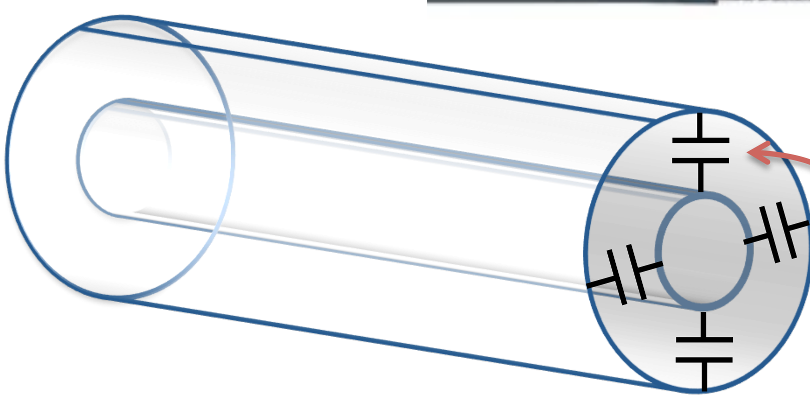
7. First Order Circuits

- response to a constant input
  - examples

example 3

You have a voltage source with 100Ω internal resistance, connected to a 1m coaxial cable (capacitance of 100pF/m), and connected to a load resistance of 1kΩ. If the voltage source switches from 0V to 5V at $t > 0$, what is the voltage on the load resistor for $t > 0$? Assume dc steady state before the voltage change.

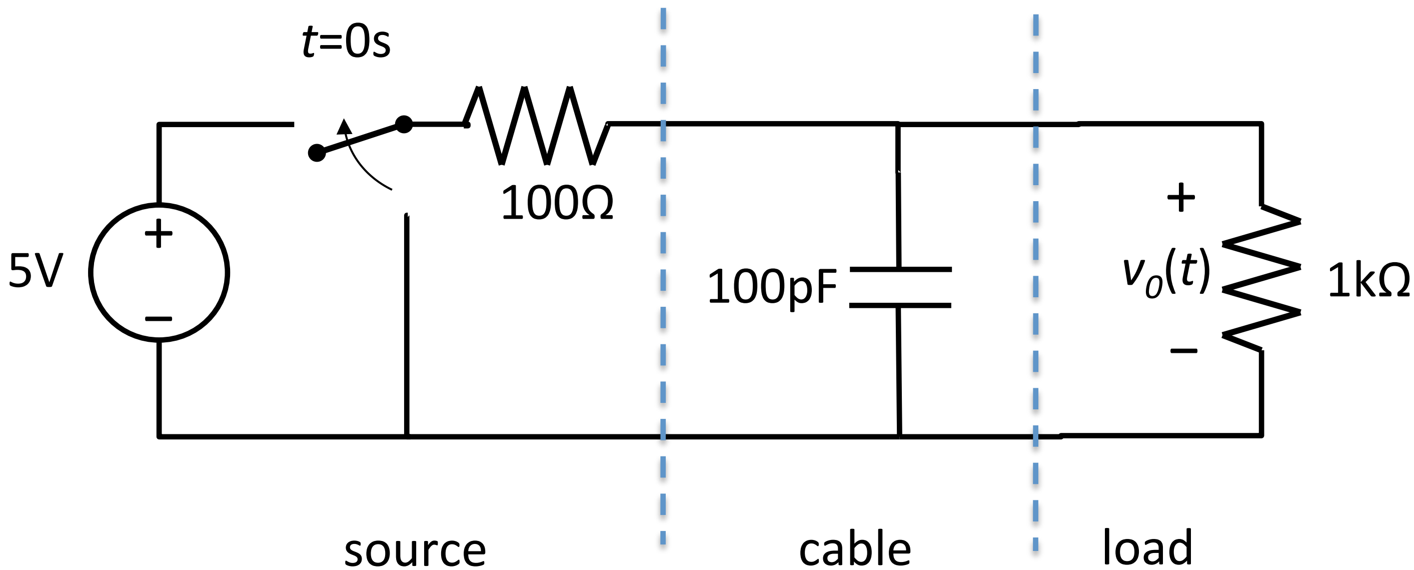




Note: A more accurate electrical model than a capacitor alone is required for "long" cable lengths for greater accuracy (as you will see in ECSE-351).

example 3

We first draw the electrical model for the circuit.

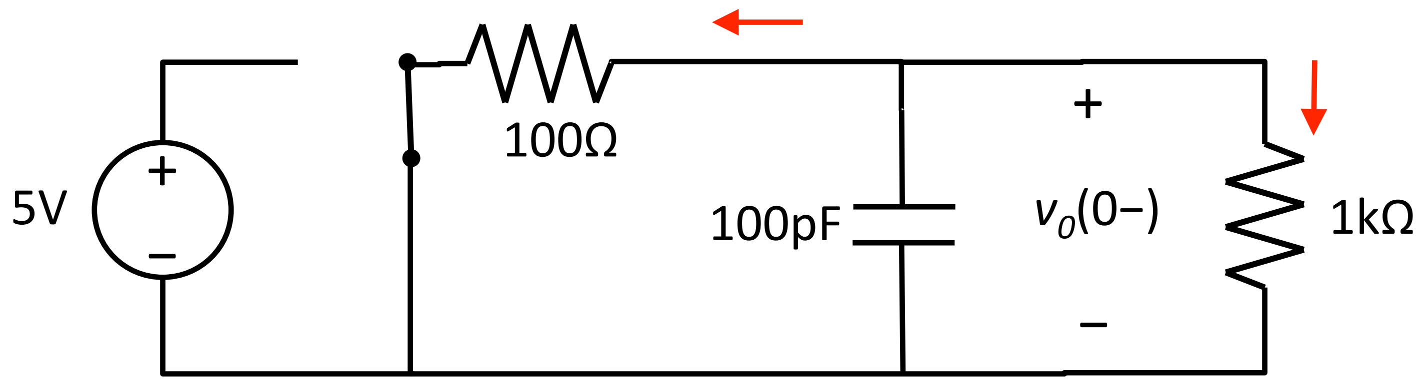


example 3

step #1: At $t < 0$, the circuit is in dc steady state, so the capacitor acts as an open. Continuity of capacitor voltage imposes continuity in the voltage v_0.

$$\frac{v_0(0-)}{1\text{k}\Omega} + \frac{v_0(0-)}{100\Omega} = 0$$

$$v_0(0-) = 0\text{V} \quad \rightarrow \quad v_0(0+) = v_0(0-) = 0\text{V}$$

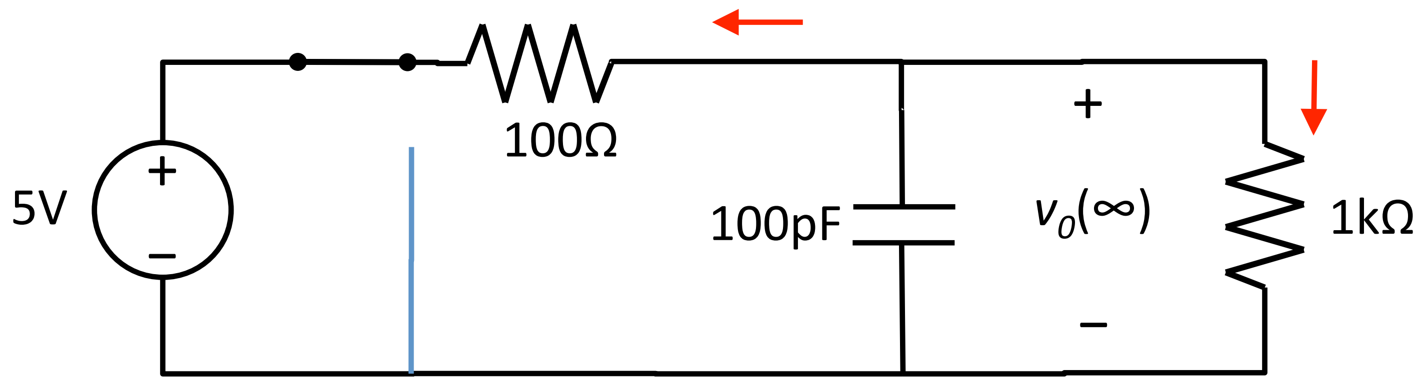


example 3

step #2: As *t* -> ∞, the circuit again reaches dc steady state, and the capacitor acts as an open.

$$\frac{v_0(\infty)}{1\text{k}\Omega} + \frac{v_0(\infty) - 5\text{V}}{100\Omega} = 0$$

$$v_0(\infty) = 5\text{V}\,\frac{1\text{k}\Omega}{1\text{k}\Omega + 100\Omega} = 4.545\text{V}$$

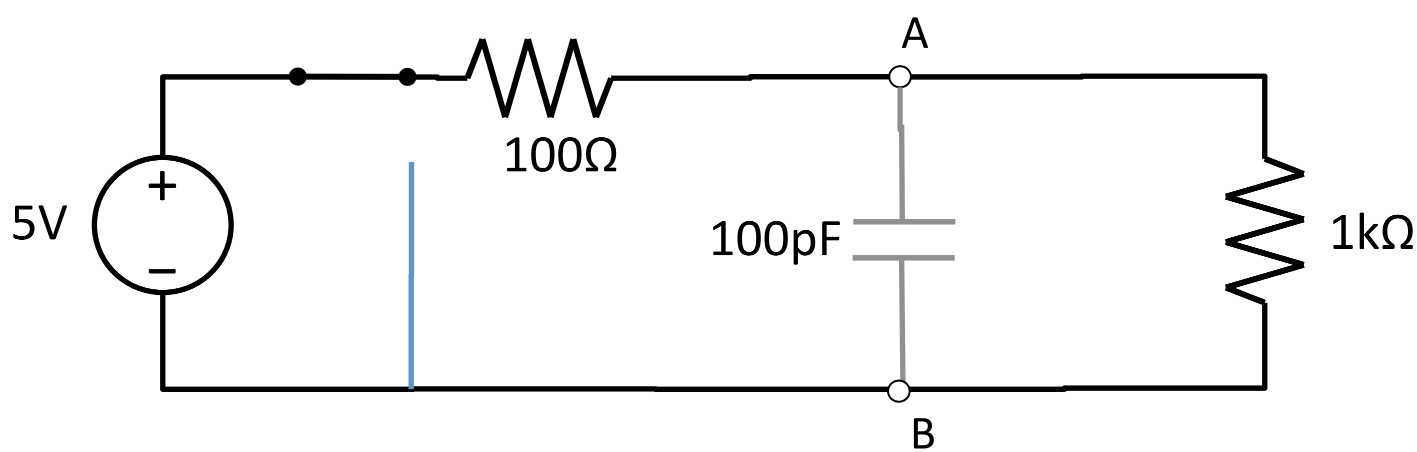


example 3

step #3: For $t > 0$, find R_T as seen from the capacitor terminals (identified A and B). Find the time constant $\tau = R_T C$.

$$R_T = 1\text{k}\Omega \,||\, 100\Omega = 90.91\Omega$$

$$\tau = R_T C = 90.91\Omega \cdot 100\text{pF} = 9.09\text{ns}$$



example 3

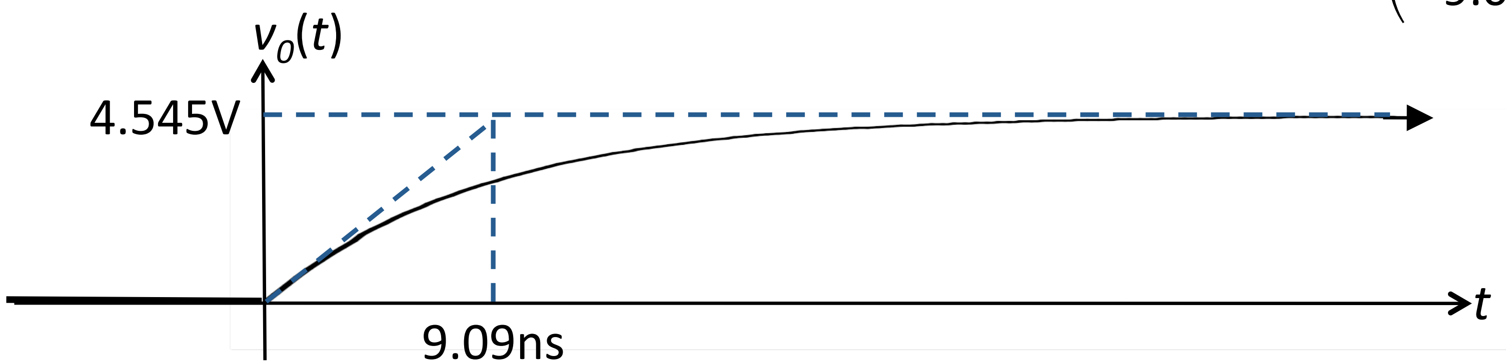
step #4: Assemble the solution.

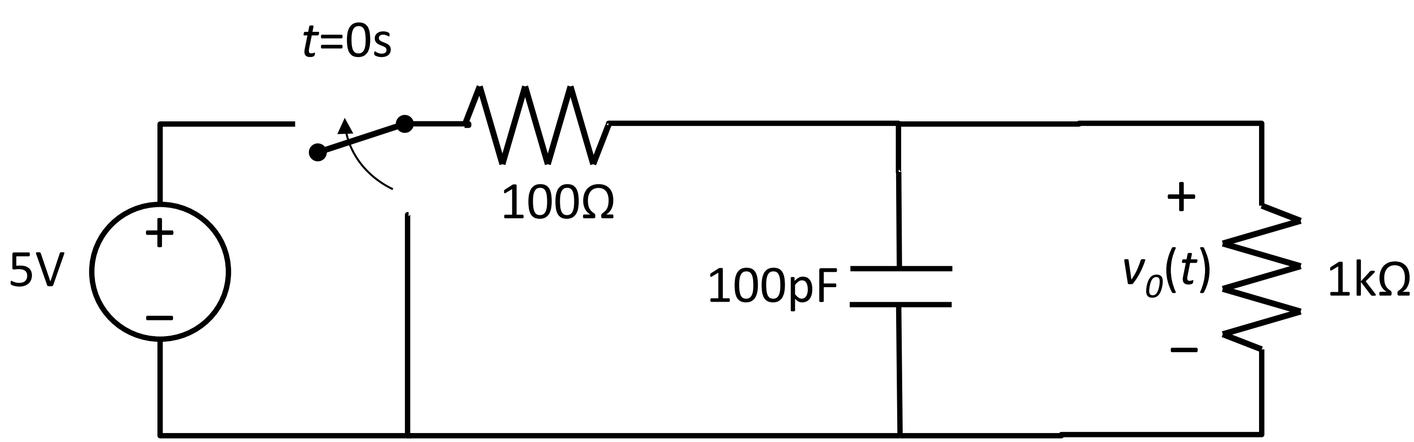
$$v_0(\infty) = 4.545\text{V} \quad v_0(0+) = 0\text{V}$$

$$\tau = 9.09\text{ns}$$

$$v_0(t) = v_0(\infty) + \left[v_0(0+) - v_0(\infty)\right]\exp\left(-\frac{t}{\tau}\right)$$

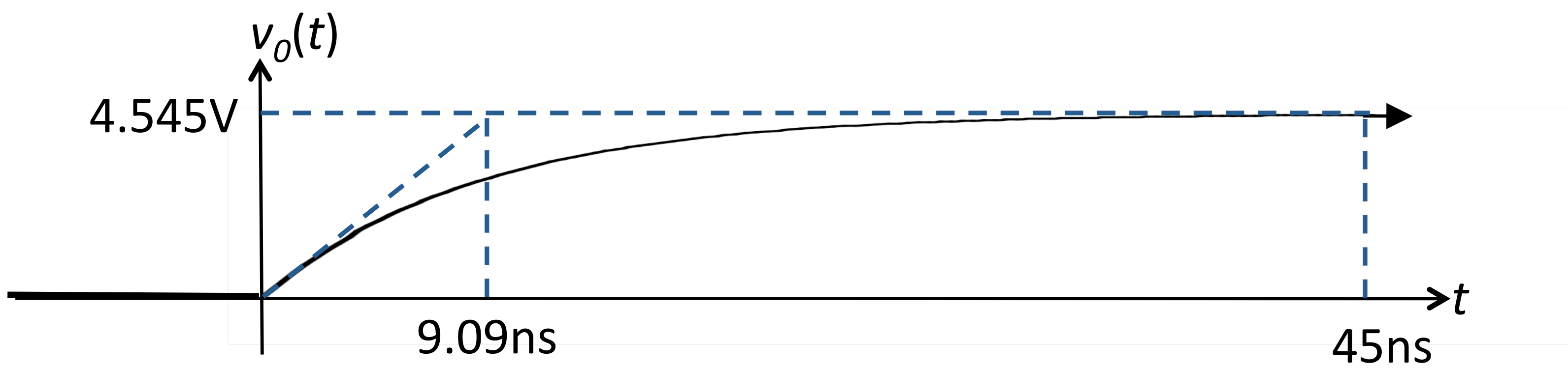
$$v_0(t) = 4.545\text{V} - 4.545\text{V}\exp\left(-\frac{t}{9.09\text{ns}}\right)$$





example 3

In this example, the time constant (proportional to the cable capacitance) limits the rate at which the voltage develops at the load. This limits the rate at which a signal (ie. ***information***) can be sent from source to load.



If we require the transient to decay to <1% of its value to consider the signal received at the source, then 5τ = 45ns is required to send the signal. This corresponds to a signalling rate of $f = 1/5\tau$ = 22MHz.

Telecommunications is the detailed study of the information transmission problem, quantifying how many bits per second can transmitted through a communication channel.

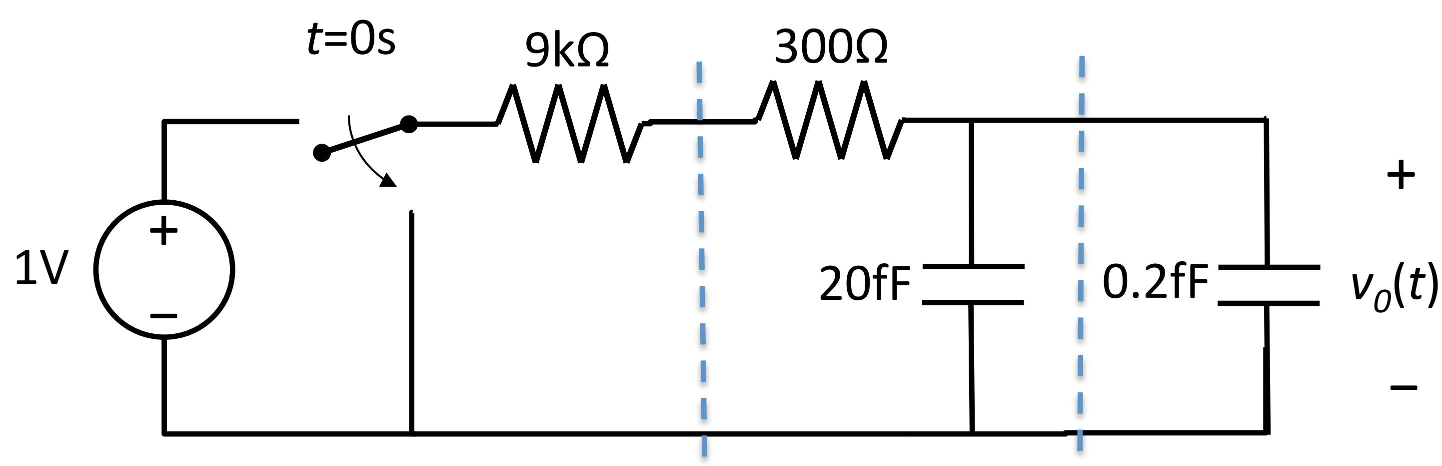
example 4

A voltage source with 9kΩ internal resistance and a 0.2fF load capacitor are connected by wiring that is modeled by a capacitance of 20fF and a resistance of 300Ω (see circuit below). The source voltage switches from 1V to 0V at t=0. Assume dc steady state conditions for t<0.

a) How much energy is stored in the load capacitor and wire capacitor at t = 0?

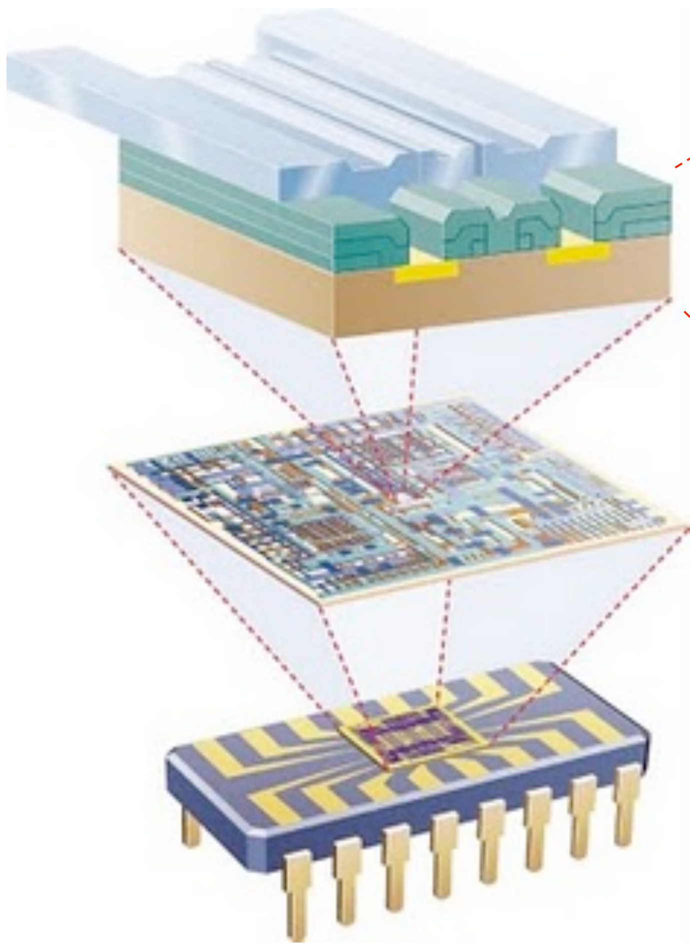
b) What is the voltage on the load capacitor versus time t?

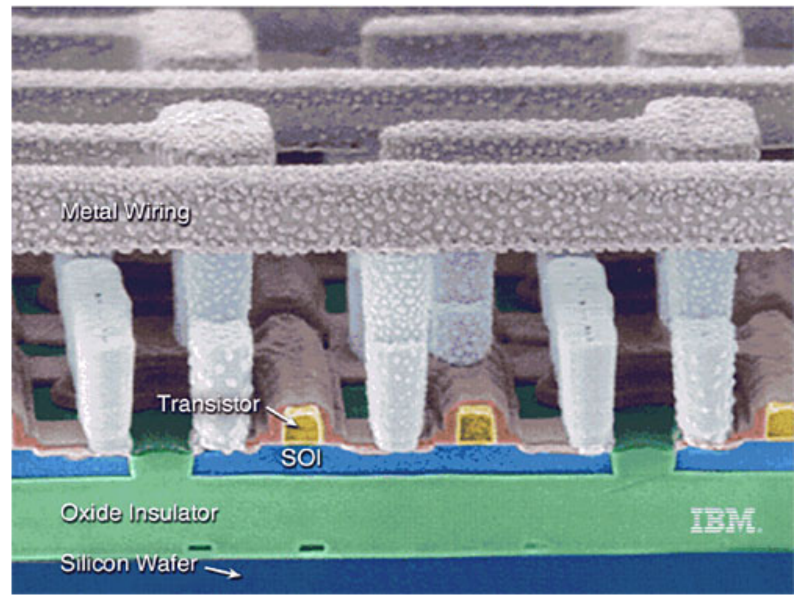
c) What is the total energy dissipated by the resistance in the circuit for t > 0?



example 4

This is a very simple model for the connection of two transistors in an integrated circuit.





example 4

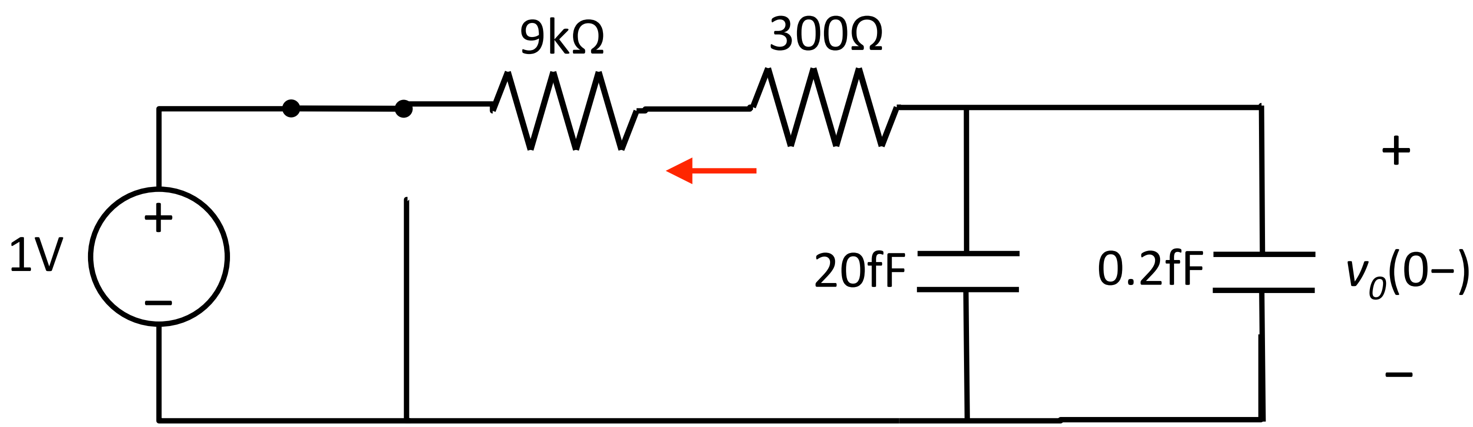
a) For $t < 0$, the circuit is in dc steady state and the capacitors are equivalent to open circuits. The voltage and energy stored is easily found.

$$\frac{v_0(0-) - 1\text{V}}{9\text{k}\Omega + 300\Omega} = 0$$

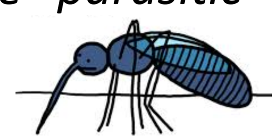
$$v_0(0-) = 1\text{V}$$

$$U_{load} = \tfrac{1}{2} \cdot 0.2\text{fF} \cdot \left(1\text{V}\right)^2 = 0.1\text{fJ}$$

$$U_{wire} = \tfrac{1}{2} \cdot 20\text{fF} \cdot \left(1\text{V}\right)^2 = 10\text{fJ}$$

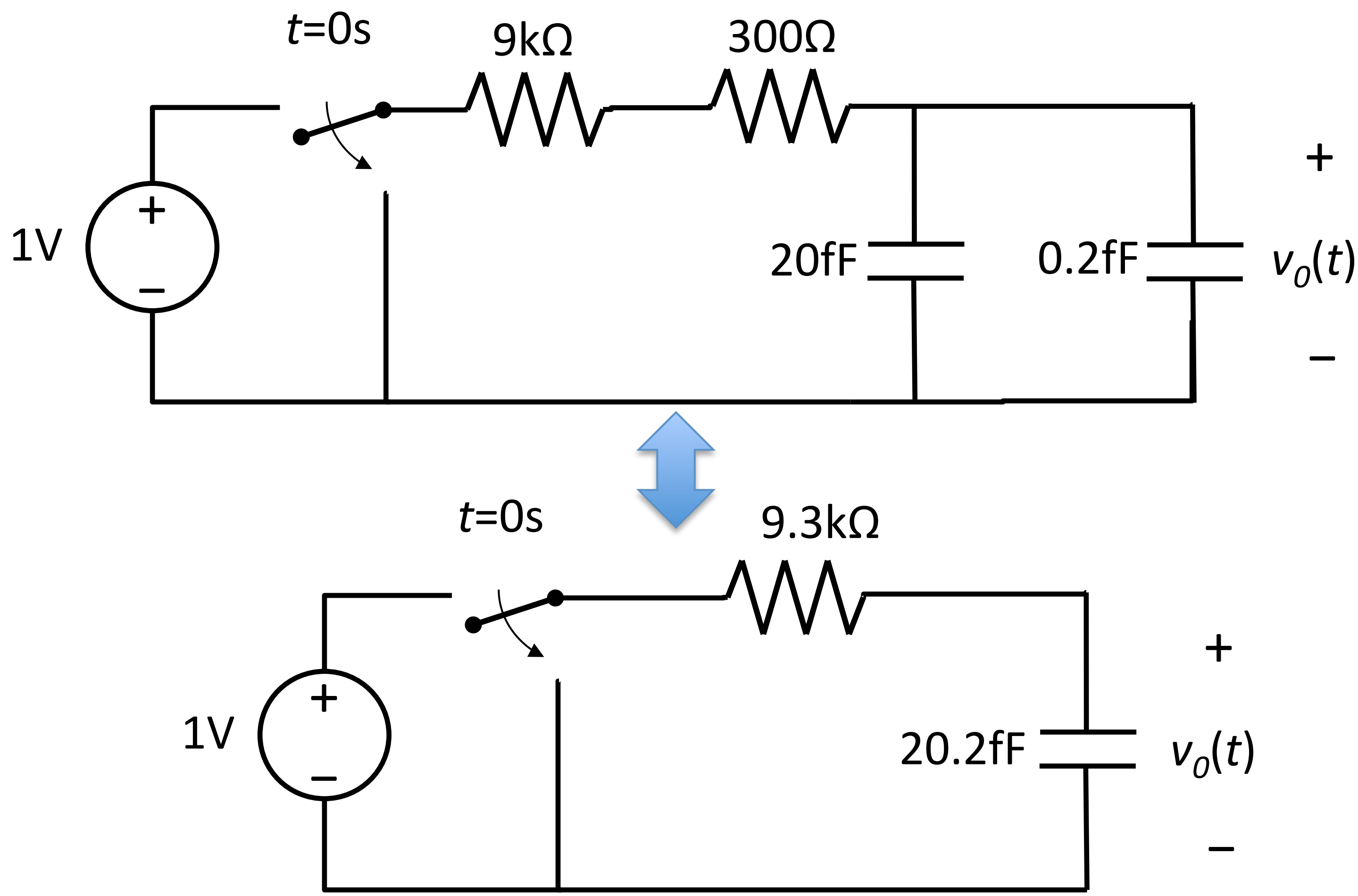


Most of the energy is stored in the "parasitic" wire capacitor (20fF) and not in the load capacitor (0.2fF).



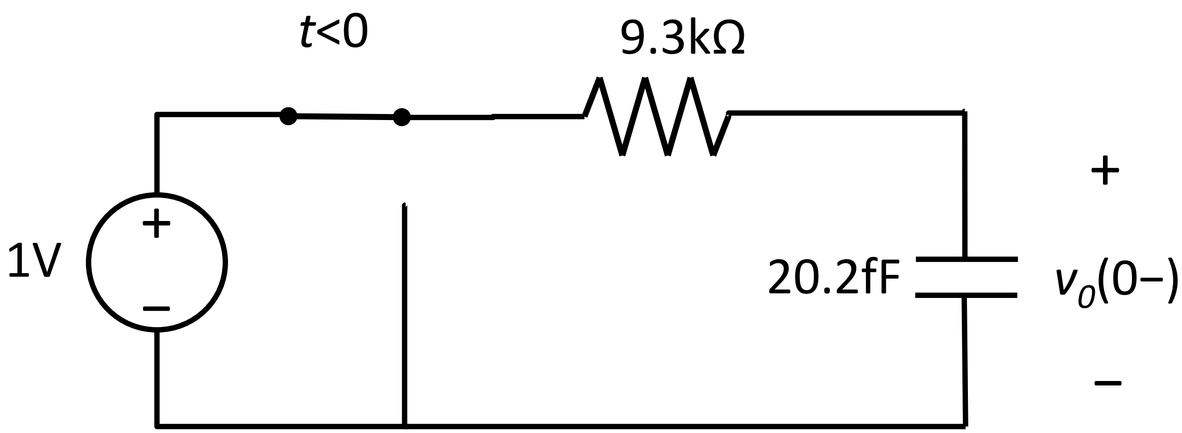
example 4

b) To find the voltage on the load capacitor versus t, we first find an equivalent first order circuit (it is possible here, but generally this is not the case if there is more than one energy storage element in the circuit).

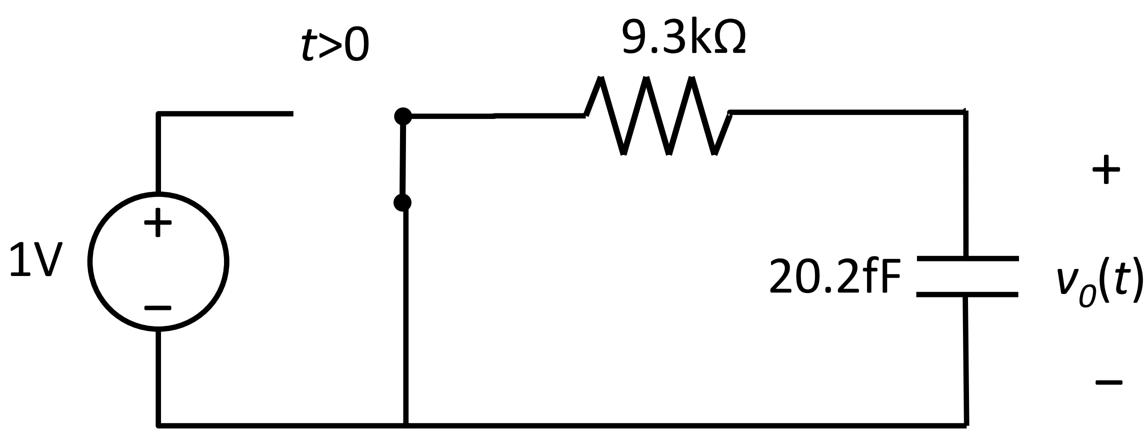


example 4

b) We analyze the familiar looking circuit.



$$v_0(0-) = v_0(0+) = 1\text{V}$$

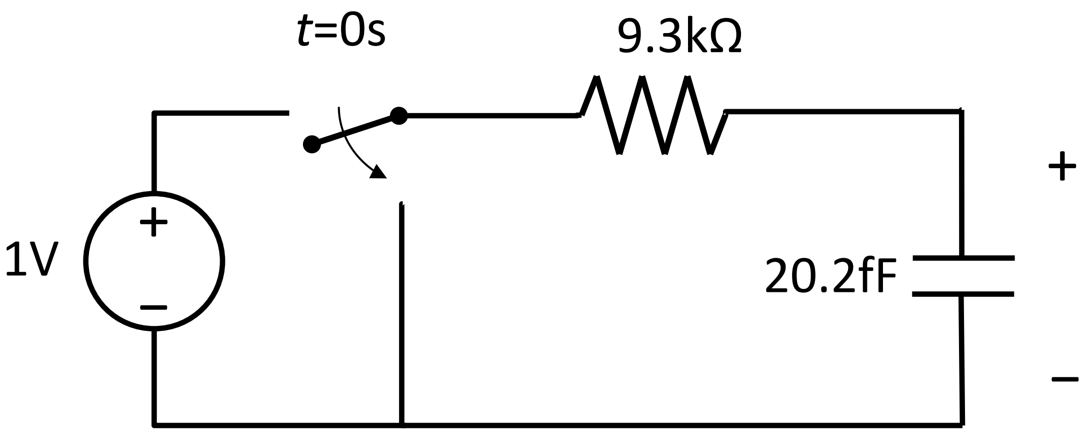


$$v_0(\infty) = 0\text{V}$$

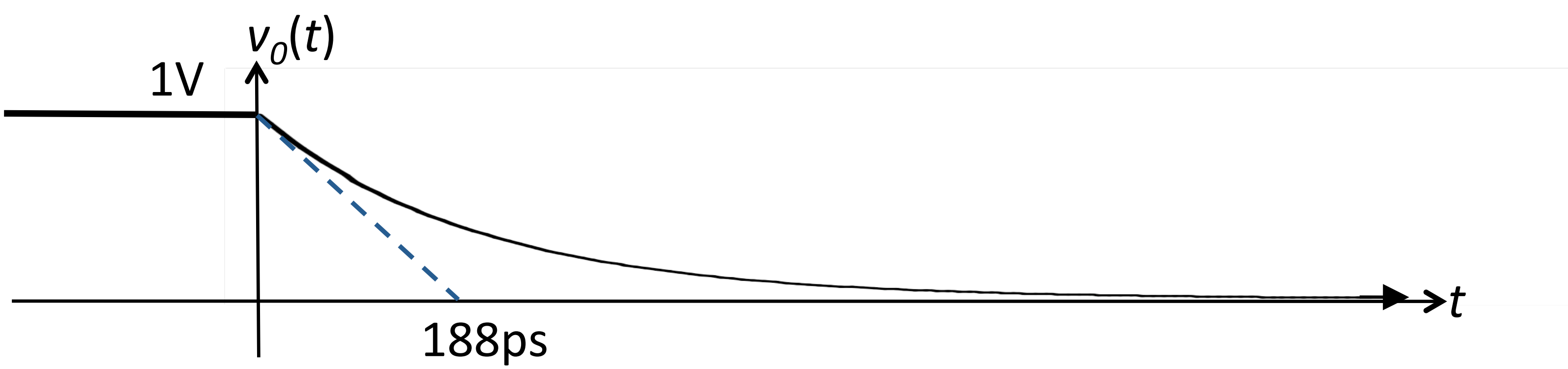
$$\tau = R_T C = 9.3\text{k}\Omega \cdot 20.2\text{fF}$$
$$= 188\text{ps}$$

example 4

b) The solution corresponds to the loss of charge separation on the capacitors.

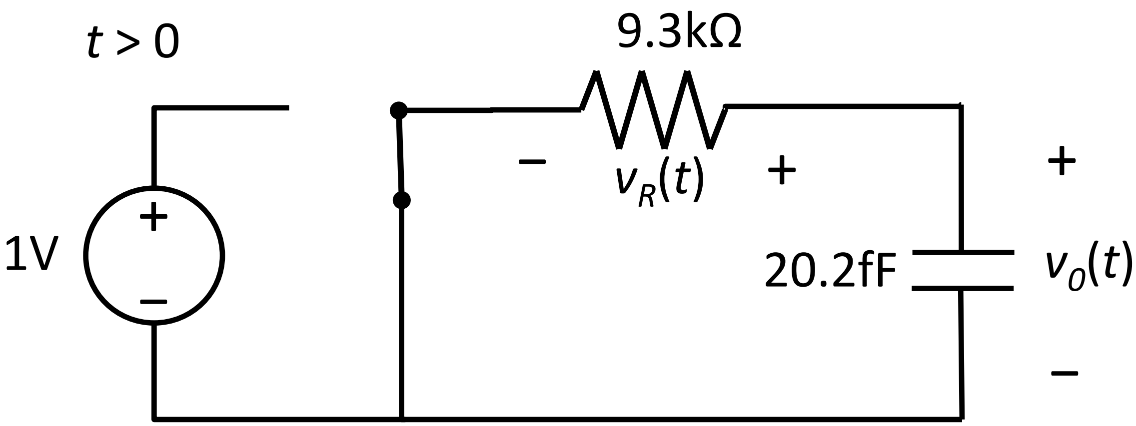


$$v_0(t) = v_0(\infty) + \left[v_0(0+) - v_0(\infty)\right]\exp\left(-\frac{t}{\tau}\right)$$
$$= 1\text{V}\exp\left(-\frac{t}{188\text{ps}}\right)$$



example 4

c) The energy dissipated in the resistor can be found by explicit calculation.



KVL:

$$0 = -v_R(t) + v_0(t)$$

$$v_R(t) = v_0(t) = 1\text{V}\exp\left(-\frac{t}{188\text{ps}}\right)$$

Power dissipated by the resistor:

$$P_{abs}(t) = \frac{v_R^2(t)}{9.3\text{k}\Omega} = \frac{\left(1\text{V}\exp(-t/188\text{ps})\right)^2}{9.3\text{k}\Omega} = 107.5\mu\text{W}\exp(-t/94\text{ps})$$

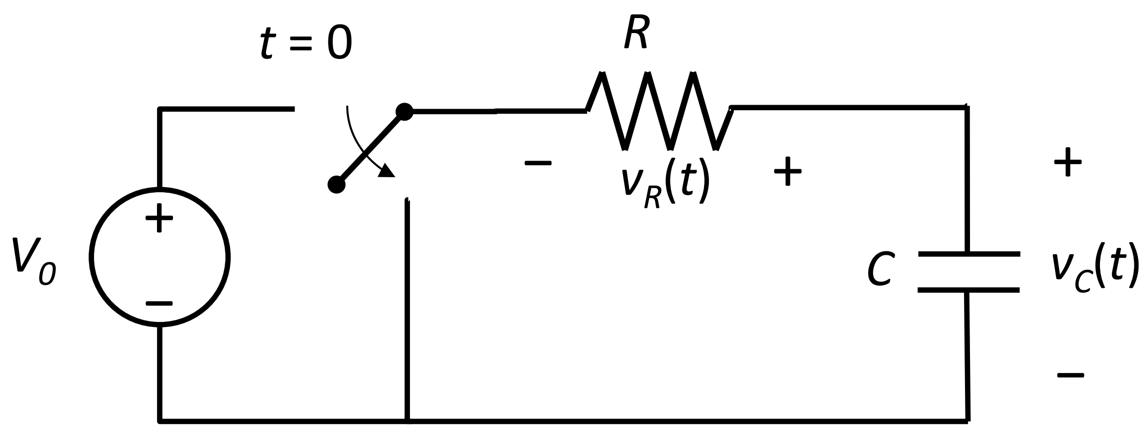
The total energy dissipated by the resistor over $0 < t < \infty$:

$$U_{abs} = \int_0^\infty 107.5\mu\text{W}\exp\left(-\frac{t}{94\text{ps}}\right)dt = \frac{107.5\mu\text{W}}{\left(-1/94\text{ps}\right)}\left[\exp\left(-\infty\right) - \exp\left(0\right)\right]$$

$$= 107.5\mu\text{W}\cdot 94\text{ps} = 10.1\text{fJ}$$

a note about energy

We see that the total electric energy stored in the capacitor is dissipated as heat in the resistor, as required by ***conservation of energy***. A calculation for arbitrary *R* and *C* shows how this comes about:



$$v_R(t) = v_C(t) = V_0 \exp\left(-\frac{t}{\tau}\right)$$

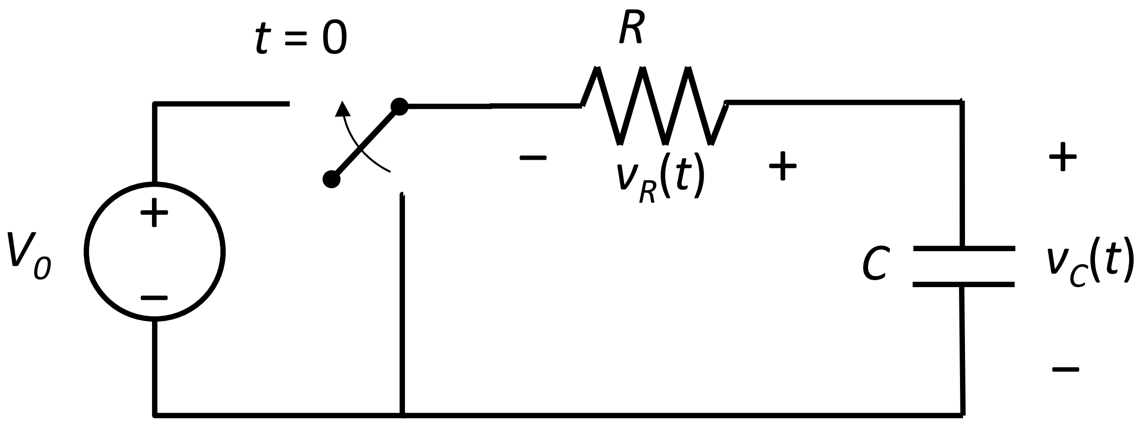
Power dissipated by *R*: $P_{abs}(t) = \frac{v_R^2(t)}{R} = \frac{V_0^2}{R}\exp\left(-\frac{2t}{\tau}\right)$

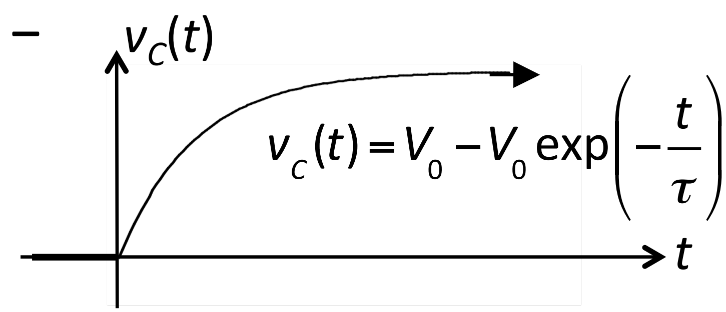
The total energy dissipated by the resistance over $0 < t < \infty$:

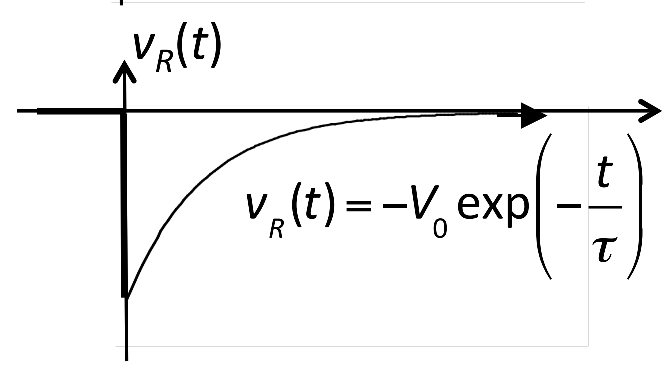
$$U_{abs} = \int_0^\infty \frac{V_0^2}{R}\exp\left(-\frac{2t}{RC}\right)dt = \frac{V_0^2}{R}\cdot\frac{RC}{-2}\cdot\left[\exp\left(-\infty\right) - \exp\left(0\right)\right] = \tfrac{1}{2}CV_0^2$$

a note about energy

Surprisingly, the same amount of energy is dissipated through the resistor when building up a non-zero charge separation on the capacitor!







Power dissipated by R:

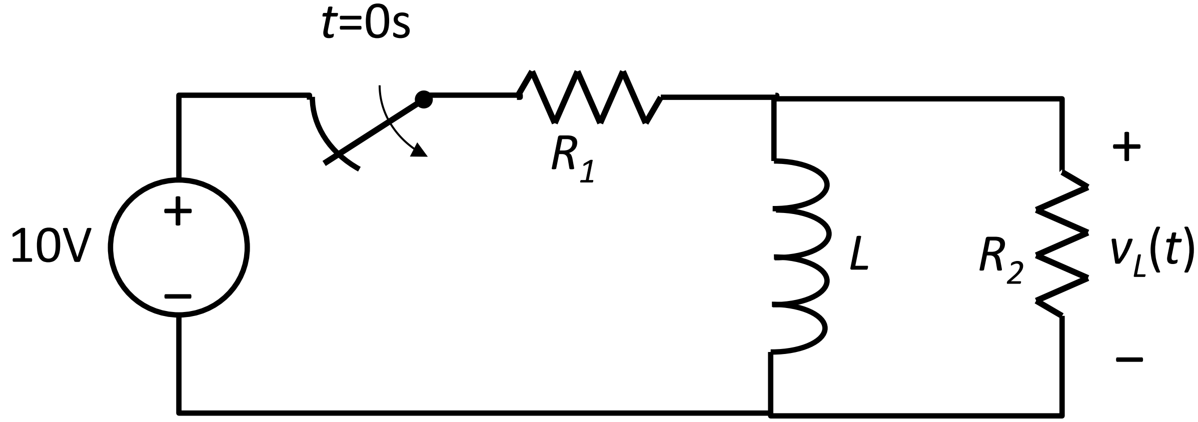
$$P_{abs}(t) = \frac{v_R^2(t)}{R} = \frac{V_0^2}{R}\exp\left(-\frac{2t}{\tau}\right)$$

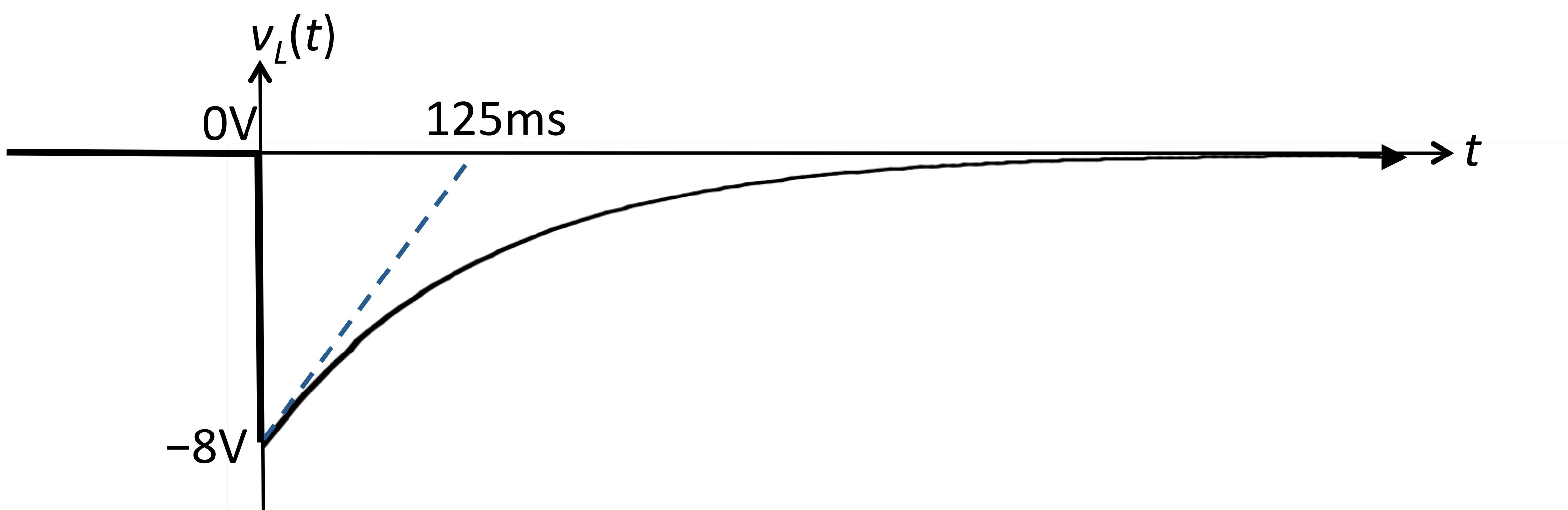
The total energy dissipated by the resistance over $0 < t < \infty$:

$$U_{abs} = \int_0^\infty \frac{V_0^2}{R}\exp\left(-\frac{2t}{RC}\right)dt = \tfrac{1}{2}CV_0^2$$

example 5

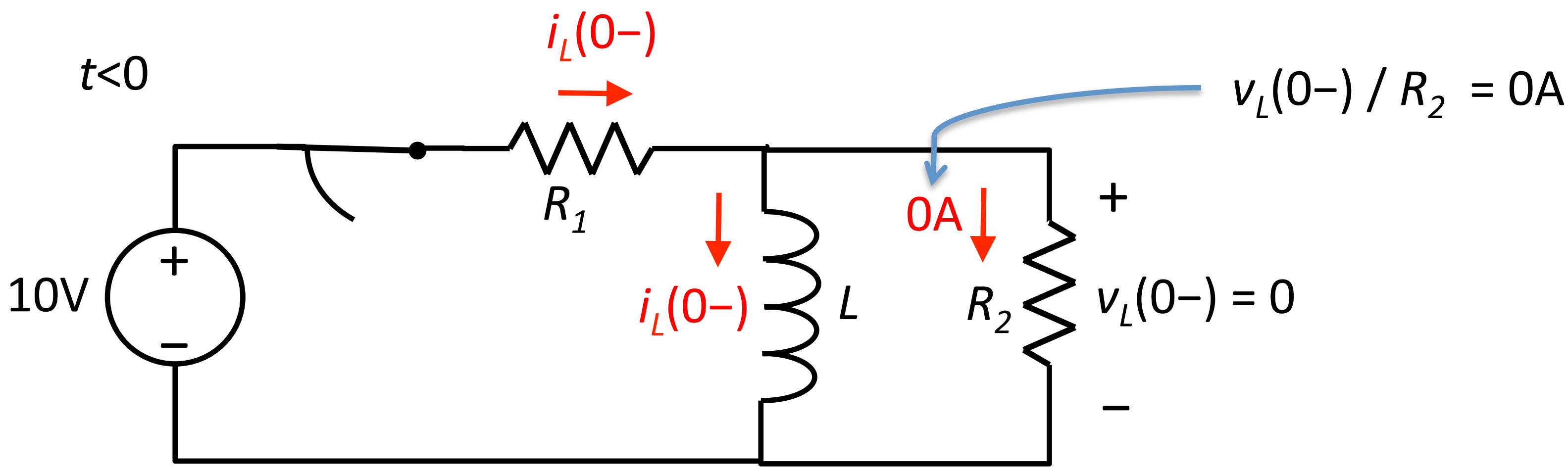
The circuit below has the voltage measured as indicated in the plot below. It is known that the voltage source delivers 10W for $t < 0$. Find the values of R_1, R_2 and L.





example 5

For $t < 0$, the circuit is in dc steady state (as indicated in plot), so the inductor acts as a short and $v_L(0-) = 0$.



Power delivered by voltage source:

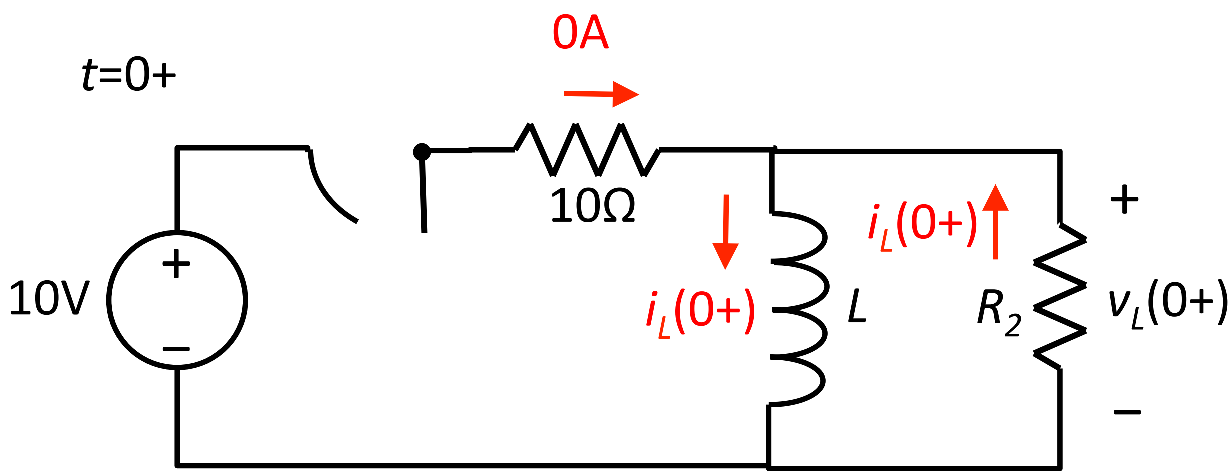
$$P_{del} = 10\text{V} \cdot i_L(0-) = 10\text{W} \rightarrow i_L(0-) = \frac{10\text{W}}{10\text{V}} = 1\text{A}$$

KVL, KCL and Ohm: $0 = -10\text{V} + i_L(0-) \cdot R_1 + 0\text{V}$

$$R_1 = \frac{10\text{V}}{1\text{A}} = 10\Omega$$

example 5

At t = 0+, inductor current continuity imposes: $i_L(0+) = i_L(0-) = 1\text{A}$



KCL + Ohm's Law:

$$v_L(0+) = -i_L(0+) \cdot R_2$$

Measured value:

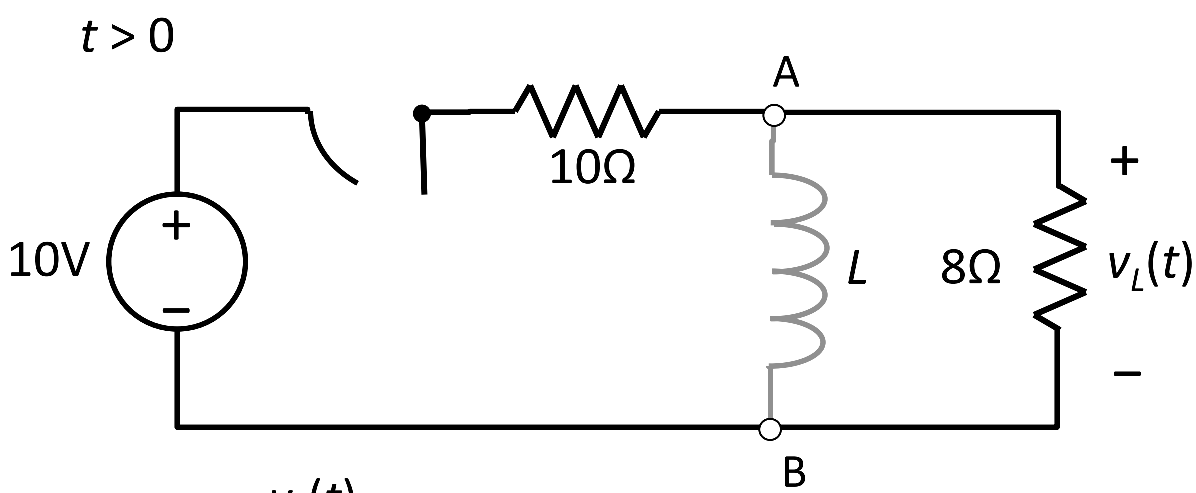
$$v_L(0+) = -8\text{V}$$

We therefore have:

$$R_2 = -\frac{v_L(0+)}{i_L(0+)} = -\frac{-8\text{V}}{1\text{A}} = 8\Omega$$

example 5

For $t > 0$, the time constant is found:



$$\tau = \frac{L}{R_T} = \frac{L}{R_2}$$

$$L = \tau \cdot R_2$$

$$= 125\text{ms} \cdot 8\Omega$$

$$= 1\text{H}$$