Today's Outline

2. Resistive Circuits

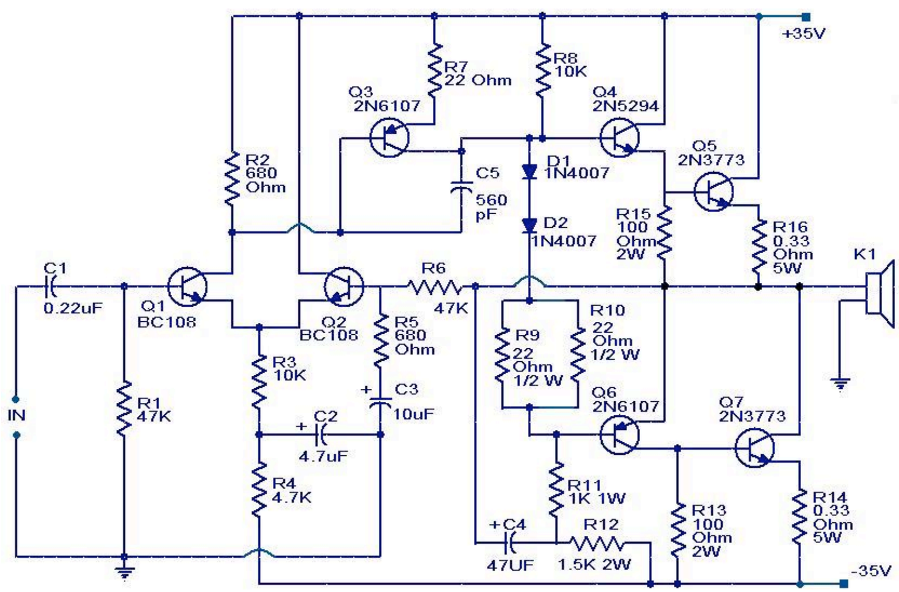
- Equivalent Circuits
- Series Resistors
- Parallel Resistors
- Voltage Divider
- Current Divider

Equivalent Circuits

In many situations, we only care about the relation between $i(t)$ and $v(t)$ at two terminals of a circuit branch (sometimes referred to as a **sub-circuit**).

Example:

When connecting a pair of speakers to an audio amplifier (as below), we might only want to know the voltage and current at the amplifier terminals. We need not know every voltage and current within the audio amplifier.

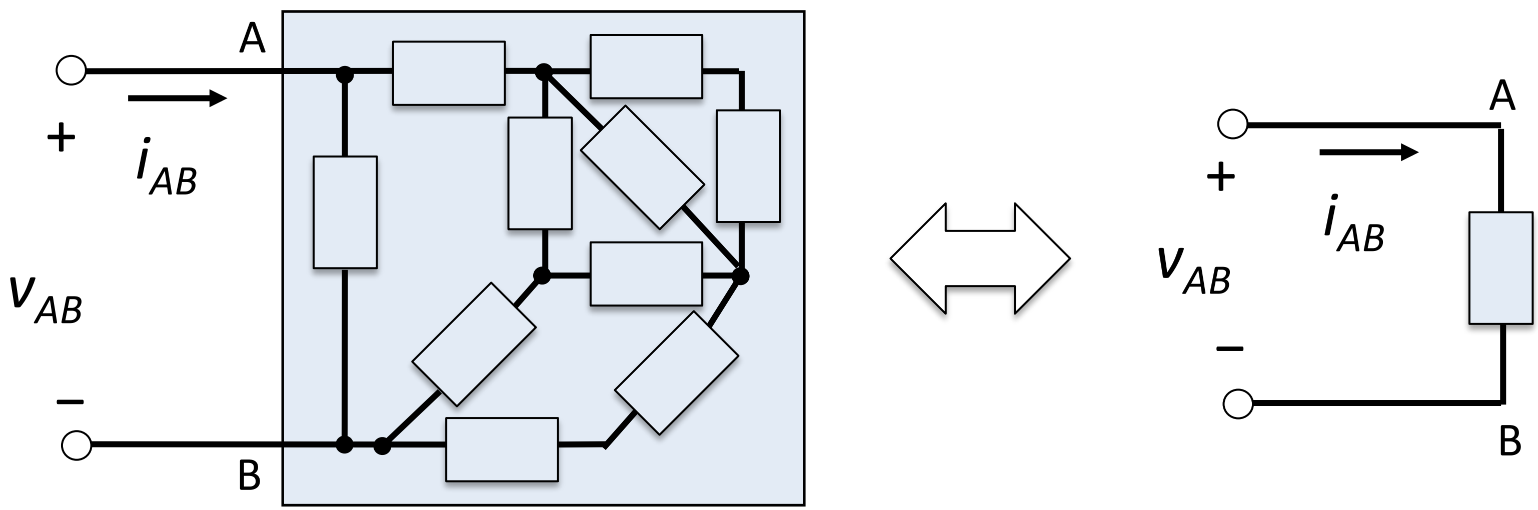


Equivalent Circuits

Equivalent Circuits: Two two-terminal circuits that have the same **terminal law**, that is the same mathematical relationship between i_{AB} and v_{AB}

– equivalent circuits produce identical terminal voltages and currents when connected to other circuits

– replacing circuits with simpler equivalent circuits is a very powerful circuit analysis technique

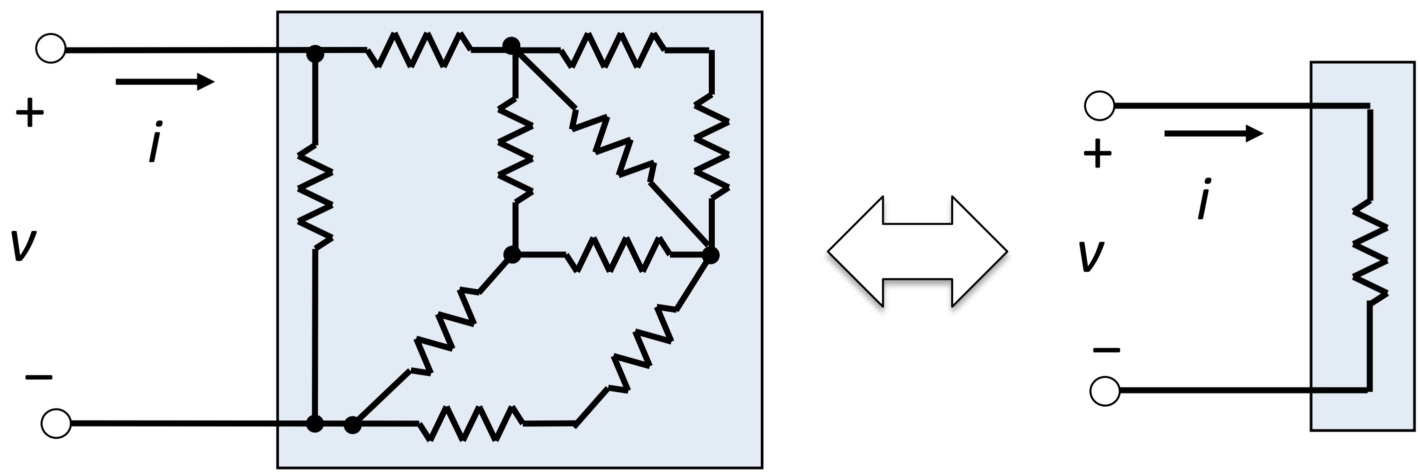


$i_{AB} = F\ (\ v_{AB}\ )$ $\qquad$ $i_{AB} = F\ (\ v_{AB}\ )$

Equivalent Resistance

Equivalent Resistance: a single ideal resistor that is equivalent to a two-terminal circuit composed *only* of ideal resistors.

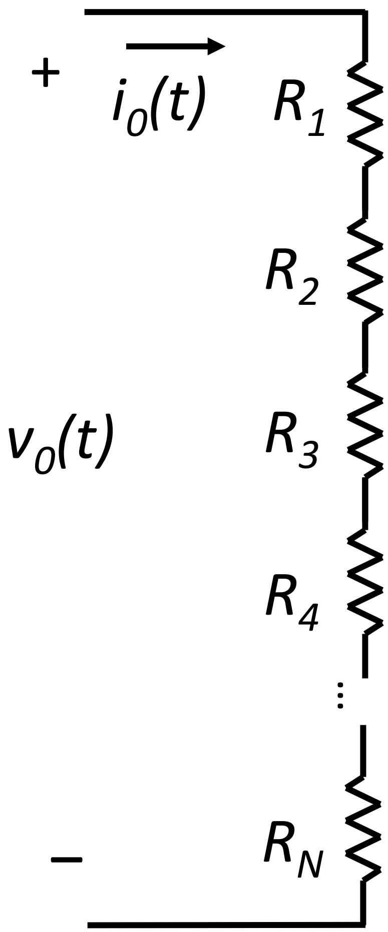
– *any* two-terminal circuit that is composed of ideal resistors alone is equivalent to a single ideal resistor



$R_{eq} = v / i$

Question: why is it that a *single* resistance is sufficient?

Series Equivalent Resistance

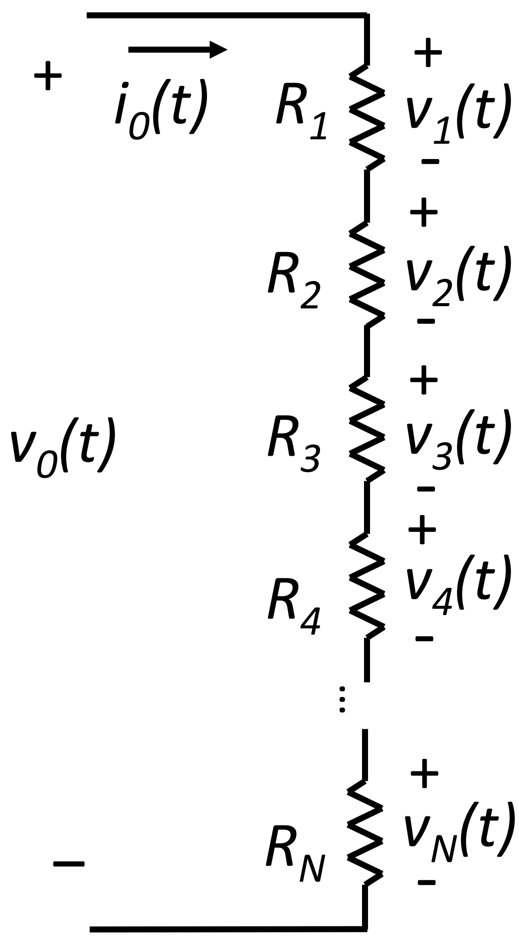


In a **series** connection, each element conducts the same current (by KCL).

To find the equivalent resistance of several resistors in series:

1. we assume unknown terminal voltage $v_0(t)$ and terminal current $i_0(t)$
2. we find the relationship between $v_0(t)$ and $i_0(t)$, by applying KCL, KVL and Ohm's Law to the network of resistors
3. we apply the definition of equivalent resistance, $R_{eq} = v_0(t) / i_0(t)$

Series Equivalent Resistance



By KVL:

$$0 = -v_0(t) + v_1(t) + v_2(t) + \ldots + v_N(t)$$

By KCL and Ohm's Law: $v_m(t) = i_0(t)\, R_m$

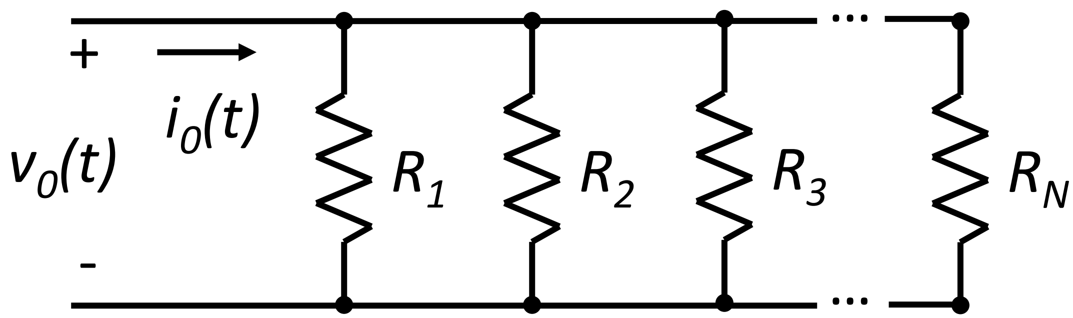
Combining the above:

$$0 = -v_0(t) + i_0(t)\,(R_1 + R_2 + \ldots + R_N)$$

By definition $R_{eq} = v_0(t)/i_0(t)$, thus:

$$\boxed{R_{eq} = R_1 + R_2 + \ldots R_N}$$

Parallel Equivalent Resistance

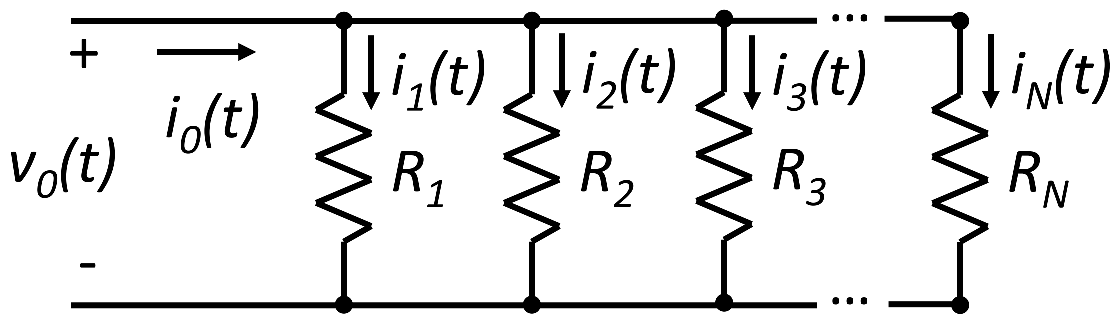


In a **parallel** connection, each element carries the same voltage (by KVL).

To find the equivalent resistance of several resistors in parallel:

1. we assume unknown terminal voltage $v_0(t)$ and terminal current $i_0(t)$
2. we find the relationship between $v_0(t)$ and $i_0(t)$, by applying KCL, KVL and Ohm's Law to the network of resistors
3. we apply the definition of equivalent resistance, $R_{eq} = v_0(t) / i_0(t)$

Parallel Equivalent Resistance



By KCL: $0 = -\, i_0(t) + i_1(t) + i_2(t) + \ldots\, i_N(t)$

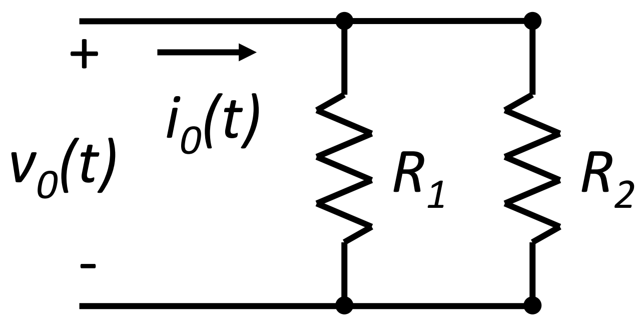
By KVL and Ohm's Law: $i_m(t) = v_0(t)/R_m$

Combining the above: $0 = -i_0(t) + v_0(t)\,(1/R_1 + 1/R_2 + \ldots + 1/R_N)$

By definition $R_{eq} = v_0(t)/i_0(t)$, thus:

$\boxed{1/R_{eq} = 1/R_1 + 1/R_2 + \ldots\, 1/R_N}$ *or* $\boxed{G_{eq} = G_1 + G_2 + \ldots\, G_N}$

Two Parallel Resistors



The parallel combination of two resistors appears frequently enough that it is worth remembering this specific case:

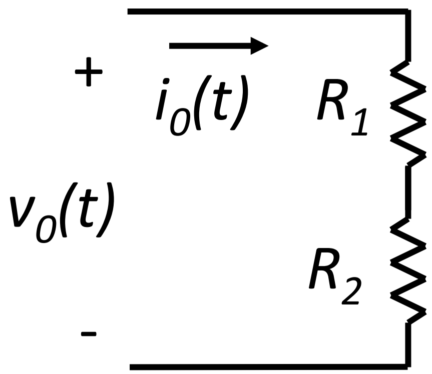
$$1/R_{eq} = 1/R_1 + 1/R_2$$

$$R_{eq} = 1 / (1/R_1 + 1/R_2\,)$$

$$\boxed{R_{eq} = R_1\, R_2 / (\, R_1 + R_2\,)}$$

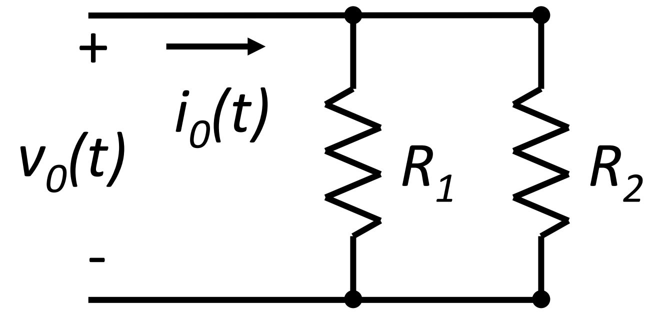
A short hand notation is often used, $R_{eq} = R_1 || R_2$, which is read as "R_1 in parallel with R_2".

Series Resistors



$$\frac{v_0(t)}{i_0(t)} = R_{eq} = R_1 + R_2$$

Parallel Resistors

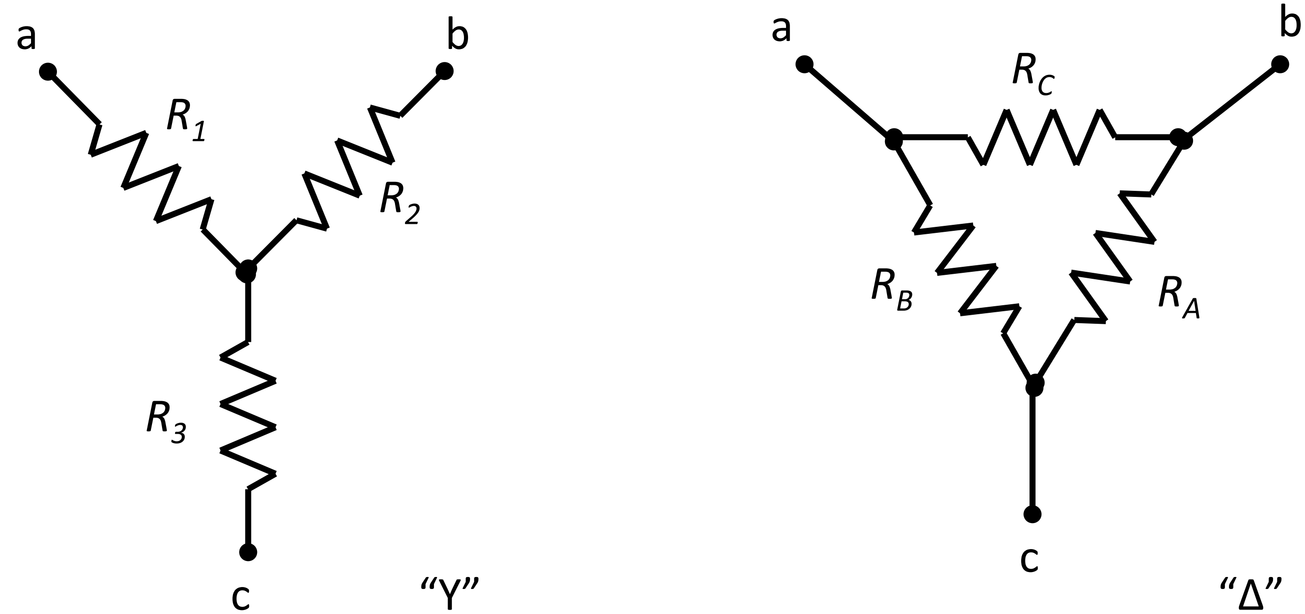


$$\frac{v_0(t)}{i_0(t)} = R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

$$\frac{i_0(t)}{v_0(t)} = G_{eq} = G_1 + G_2$$

Example

What value should R_1, R_2 and R_3 have so that the Y network has the same equivalent resistances between node pairs as the Δ network?

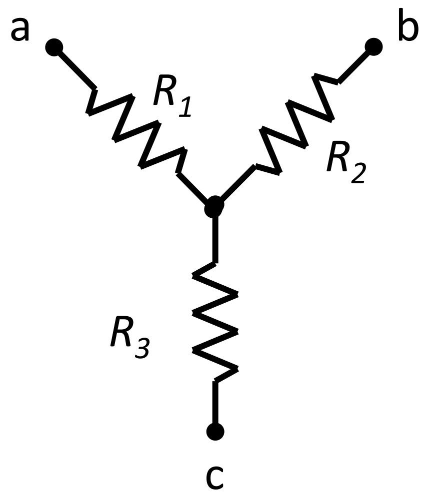


Strategy:

1. *Find equivalent resistance for all terminal pairs in Y and Δ.*
2. *Equate terminal resistances.*
3. *Solve for R_1,R_2 and R_3.*

Example

Y



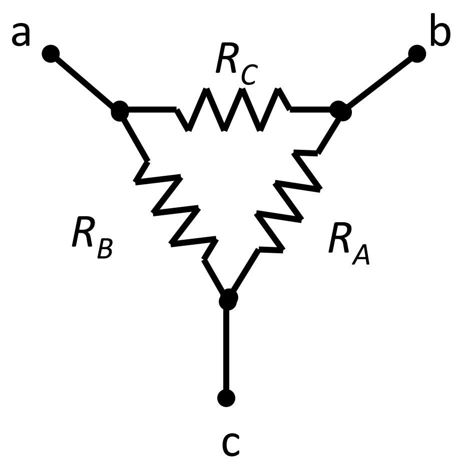
Equivalent resistances for Y network:

$R_{ab}=R_1+R_2$

$R_{bc}=R_2+R_3$

$R_{ca}=R_3+R_1$

Equivalent resistances for Δ network:



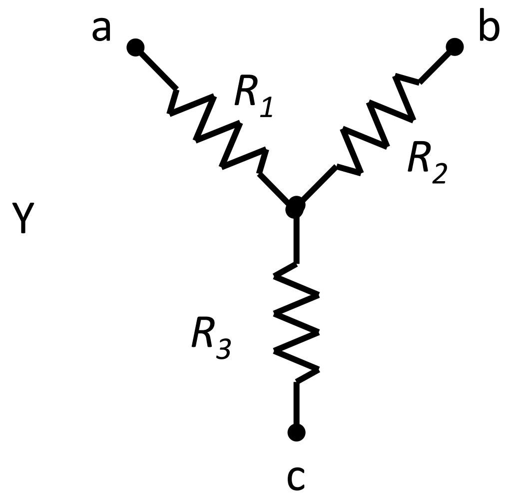
Δ

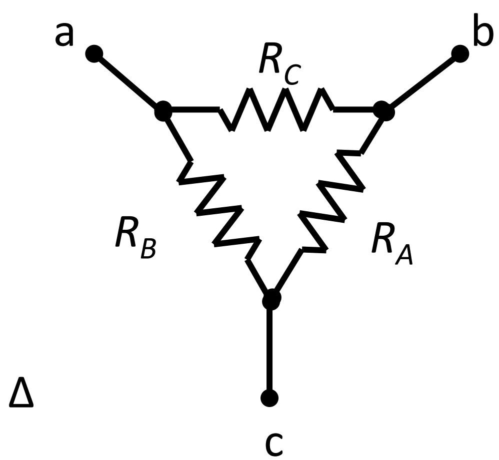
$R_{ab}=R_C||(R_B+R_A)=R_C(R_B+R_A)/(R_A+R_B+R_C)$

$R_{bc}=R_A||(R_C+R_B)=R_A(R_C+R_B)/(R_A+R_B+R_C)$

$R_{ca}=R_B||(R_A+R_C)=R_B(R_A+R_C)/(R_A+R_B+R_C)$

Example





Equate R_{ab}:

$R_{ab}=R_1+R_2=R_C(R_B+R_A)/(R_A+R_B+R_C)$

$R_{bc}=R_2+R_3=R_A(R_C+R_B)/(R_A+R_B+R_C)$

$R_{ca}=R_3+R_1=R_B(R_A+R_C)/(R_A+R_B+R_C)$

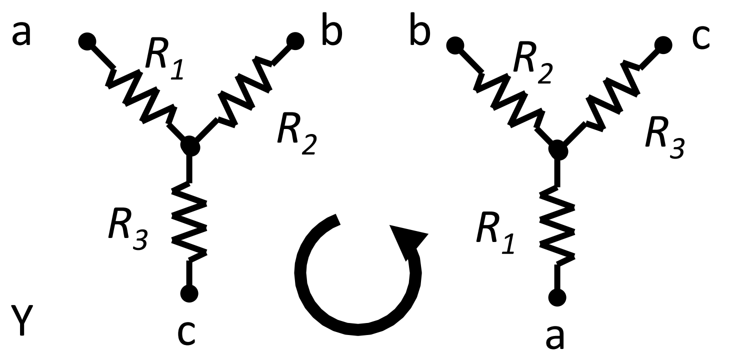
Solve for R_1:

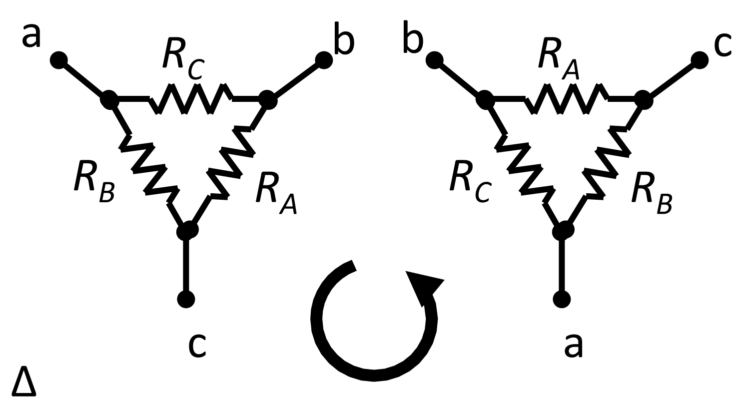
$R_1=(R_{ab}+R_{ca}-R_{bc})/2$ (by inspection, or substitution)

$R_1= 1/2 \cdot(R_CR_B+R_CR_A+R_BR_A+R_BR_C-R_AR_C- R_AR_B) /(R_A+R_B+R_C)$

$\boxed{R_1=R_CR_B/(R_A+R_B+R_C)}$

Example





For R_2 and R_3, use symmetry:

Rotate networks counter-clockwise:

$(R_1 \rightarrow R_2;\ R_C \rightarrow R_A;\ R_A \rightarrow R_B;\ R_B \rightarrow R_C)$

$R_1 = R_C R_B / (R_A + R_B + R_C) \rightarrow \boxed{R_2 = R_A R_C / (R_A + R_B + R_C)}$

Rotate networks counter-clockwise again:

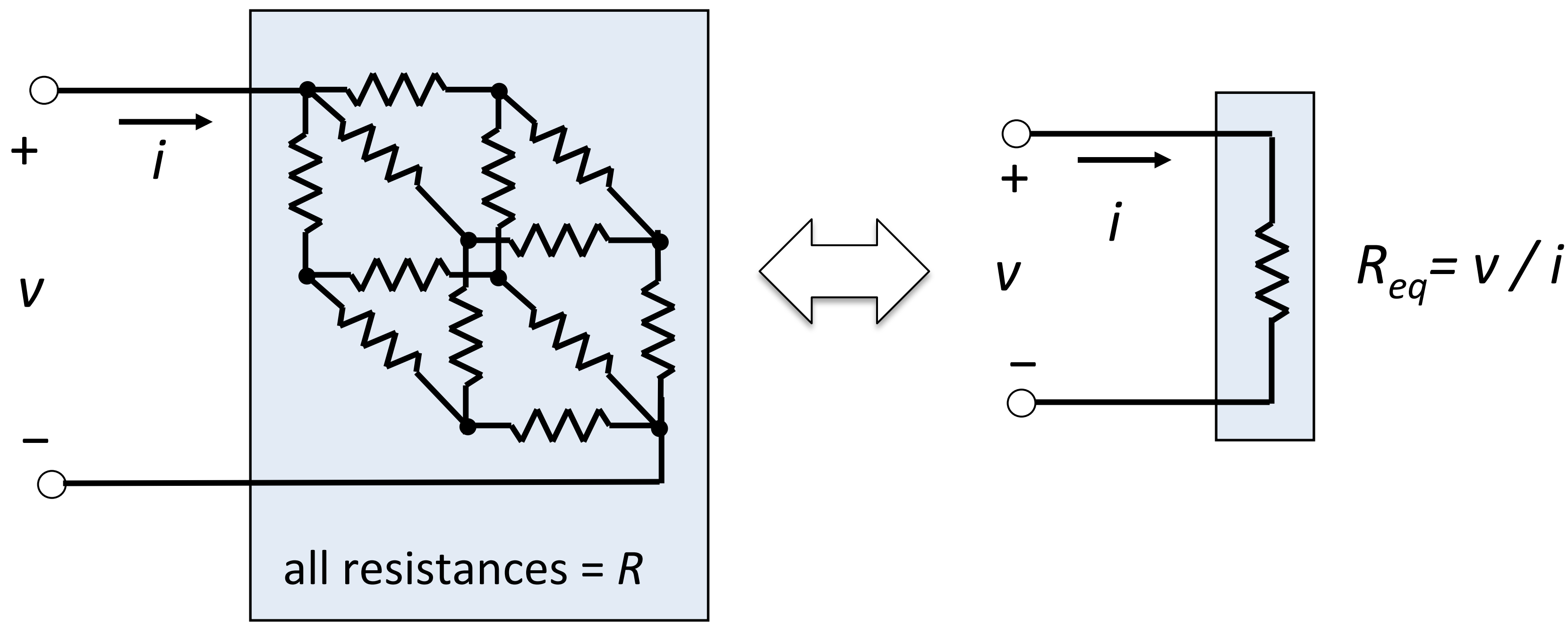
$(R_2 \rightarrow R_3;\ R_C \rightarrow R_A;\ R_A \rightarrow R_B;\ R_B \rightarrow R_C)$

$R_2 = R_A R_C / (R_A + R_B + R_C) \rightarrow \boxed{R_3 = R_B R_A / (R_A + R_B + R_C)}$

Equivalent Resistance

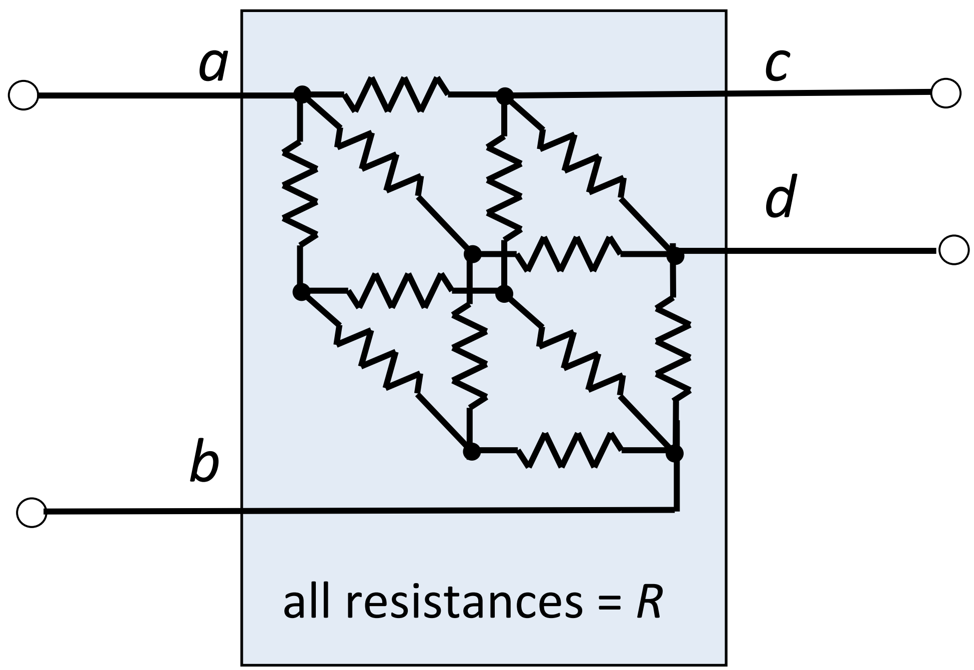
There are resistive circuits that *cannot* be reduced to R_{eq} by a sequence of series and parallel resistance reductions.

In such cases, the circuit must be solved from basic principles. Find the linear relation between i and v using KCL, KVL and Ohm's law, as we did in deriving the series equivalent resistance rule.



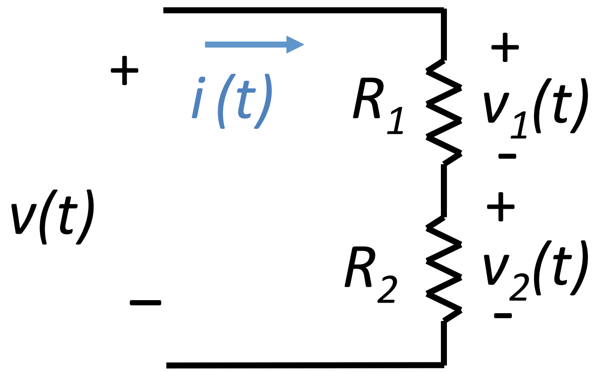
Equivalent Resistance

Equivalent resistance is always defined with respect to two terminals. A network with multiple terminal pairs can have more than one equivalent resistance between those pairs.



$$R_{ab} \neq R_{cd}$$

Voltage Divider



$$R_{eq} = R_1 + R_2$$

Consider a series combination of two resistors.

By KCL and Ohm's Law:

$$v_1(t) = i(t)\, R_1$$

$$v_2(t) = i(t)\, R_2$$

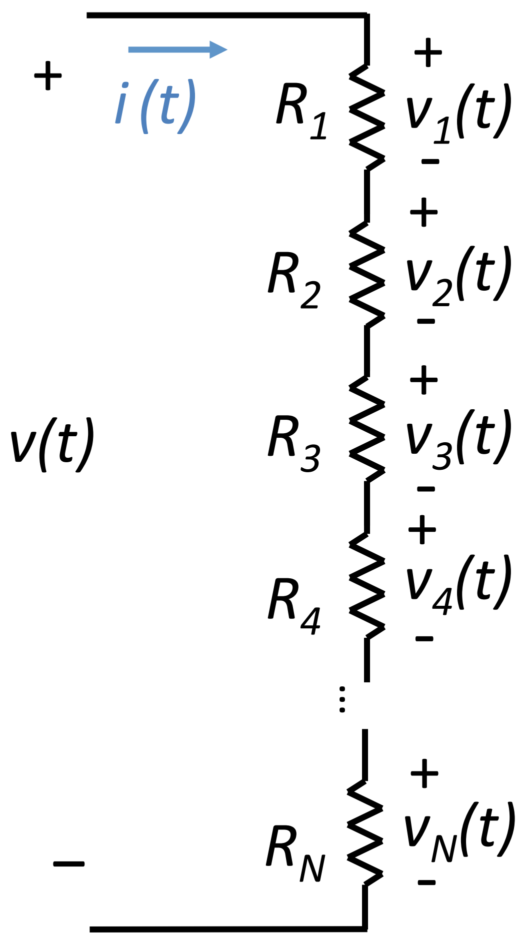
From the series equivalent circuit:

$$v(t) = i(t)\, R_{eq} = i(t)\, (\, R_1 + R_2\,)$$

This circuit is called a ***voltage divider***:

$$\frac{v_1(t)}{v(t)} = \frac{R_1}{R_1 + R_2} \qquad \frac{v_2(t)}{v(t)} = \frac{R_2}{R_1 + R_2}$$

Voltage Divider with *N* Resistors



A series combination of *N* resistors also acts as a voltage divider.

By KCL and Ohm's Law: $v_m(t) = i(t)\, R_m$

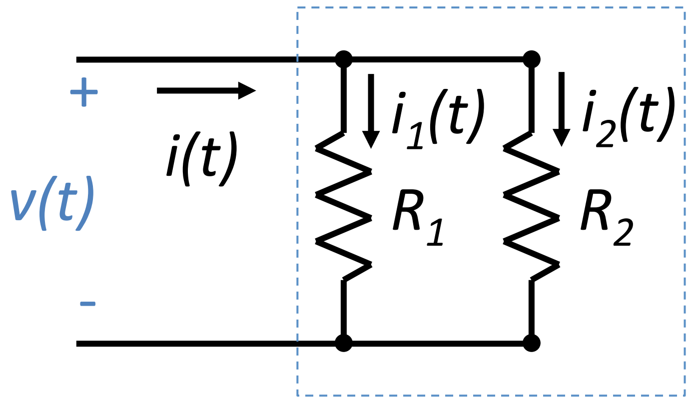
From the series equivalent circuit:

$v(t) = i(t)\, R_{eq} = i(t)\, (R_1 + R_2 + \ldots + R_N)$

We thus have:

$$\boxed{\frac{v_m(t)}{v(t)} = \frac{R_m}{R_1 + R_2 + \ldots + R_N}}$$

Current Divider



$$G_{eq} = G_1 + G_2$$

Consider a parallel combination of two resistors.

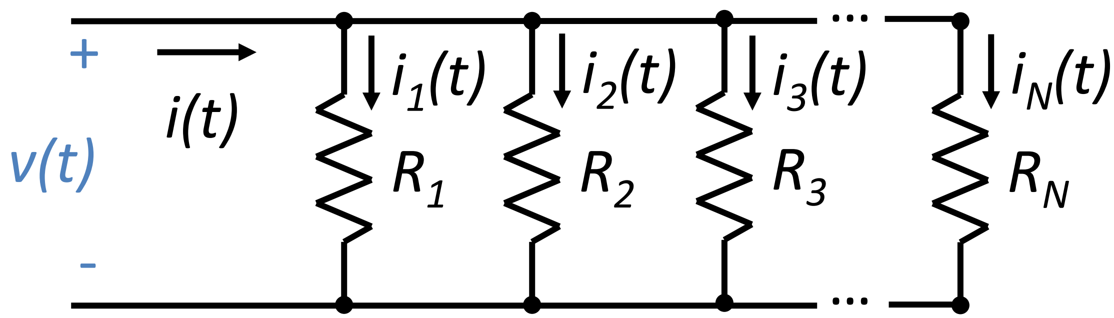
By KVL and Ohm's Law: $i_1(t) = v(t)/R_1 = v(t)G_1$ $\quad$ $i_2(t) = v(t)/R_2 = v(t)G_2$

From the parallel equivalent: $i(t) = v(t) / R_{eq} = v(t)\, G_{eq} = v(t)\,(\, G_1 + G_2\,)$

This circuit is called a ***current divider***:

$$\frac{i_1(t)}{i(t)} = \frac{G_1}{G_1 + G_2} = \frac{1/R_1}{1/R_1 + 1/R_2} = \frac{R_2}{R_1 + R_2} \qquad \frac{i_2(t)}{i(t)} = \frac{G_2}{G_1 + G_2} = \frac{1/R_2}{1/R_1 + 1/R_2} = \frac{R_1}{R_1 + R_2}$$

Current Divider with *N* Resistors



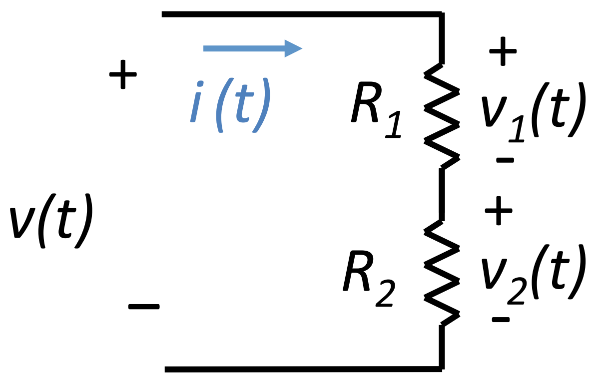
A parallel combination of *N* resistors also acts as a current divider.

By KVL and Ohm's Law: $i_m(t) = v(t)G_m$

From the parallel equivalent: $i\,(t) = v(t)\; G_{eq} = v(t)\;(\; G_1 + G_2 + \ldots + G_N\,)$

We thus have:
$$\boxed{\frac{i_m(t)}{i(t)} = \frac{G_m}{G_1 + G_2 + \ldots + G_N} = \frac{1/R_m}{1/R_1 + 1/R_2 + \ldots + 1/R_N}}$$

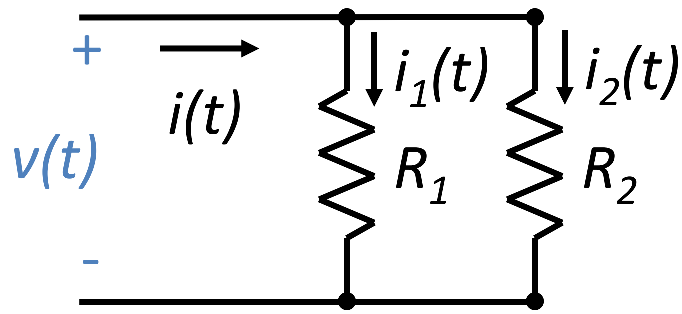
Voltage Divider



$$\boxed{\frac{v_1(t)}{v(t)} = \frac{R_1}{R_1 + R_2}}$$

$$\frac{v_2(t)}{v(t)} = \frac{R_2}{R_1 + R_2}$$

Current Divider



$$\boxed{\frac{i_1(t)}{i(t)} = \frac{R_2}{R_1 + R_2}}$$

$$\frac{i_2(t)}{i(t)} = \frac{R_1}{R_1 + R_2}$$