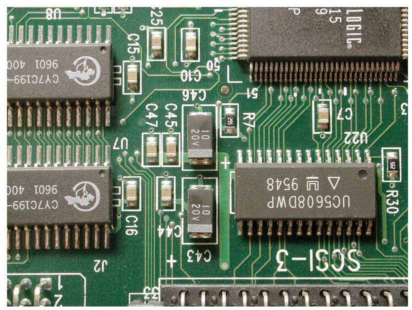
6. Energy Storage Elements

- the Capacitor
- the Inductor
- Coupled Inductors
- dc steady state
- op-amp circuits with energy storage elements

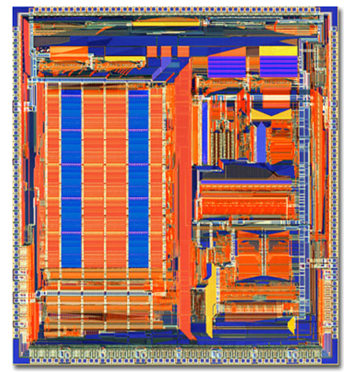
Motivation

Capacitors and inductors are circuit elements that allow one to store and release electric energy, and they appear in most circuits, here are just a few examples:

power sub-stations



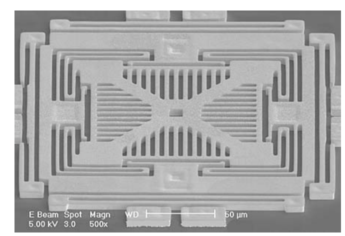
audio + video circuits



digital integrated circuits

power supplies

turbines



accelerometers, manometers and other sensors

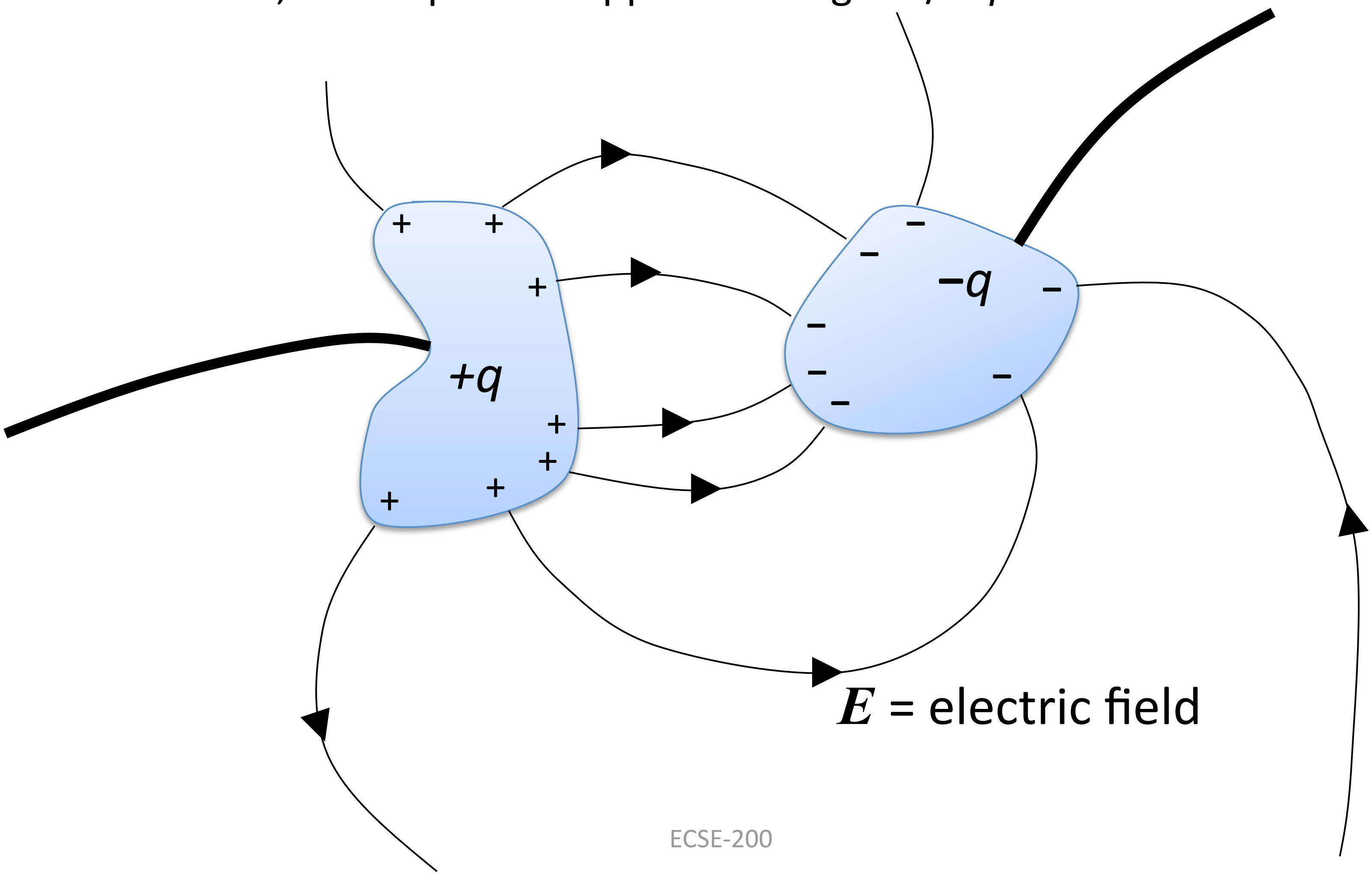
Today's Outline

6. Energy Storage Elements

- the Capacitor

the capacitor

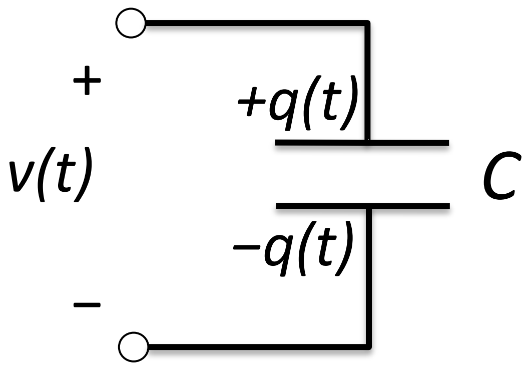
Ideal capacitor: physically consists of two ideal conductors separated by an ideal insulator, with equal but opposite charges +/− *q* on each conductor

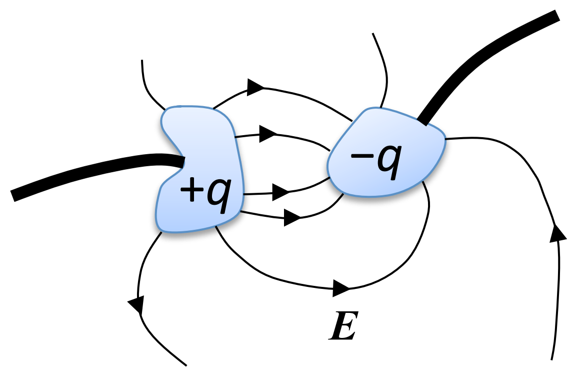


the capacitor

Ideal Capacitor: the **charge separation** *q* on an ideal capacitor is proportional to the voltage drop *v* across the capacitor

$$\boxed{q = C\,v}$$



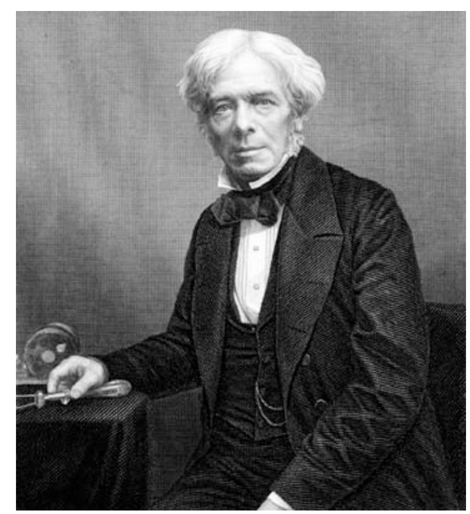


– the capacitor is a *passive* circuit element

– the constant of proportionality between charge and voltage is the **capacitance**, given the symbol *C*

– SI unit of capacitance is the Farad (abbreviated F)

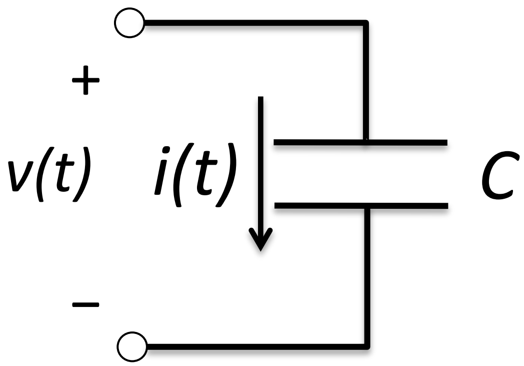
$$1\ \mathrm{F} = 1\ \mathrm{C} / \mathrm{V} = 1\ \mathrm{A\,s} / \mathrm{V}$$



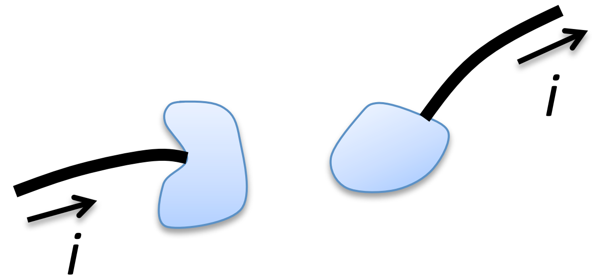
Michael Faraday
(1791-1867)

the capacitor

Although an insulator separates the conductors, a current *i* equal to the time rate of change of charge separation *q* can flow "through" the capacitor

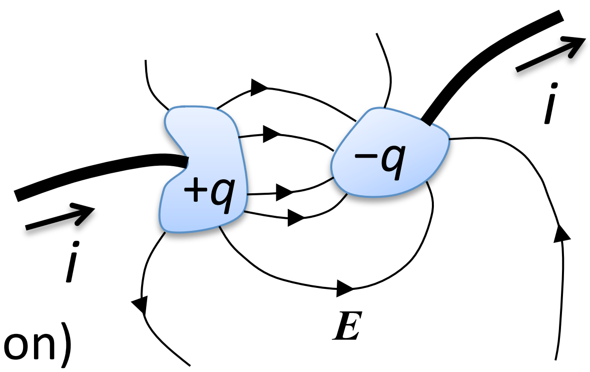


$$\boxed{i = \frac{dq}{dt} = C\frac{dv}{dt}}$$



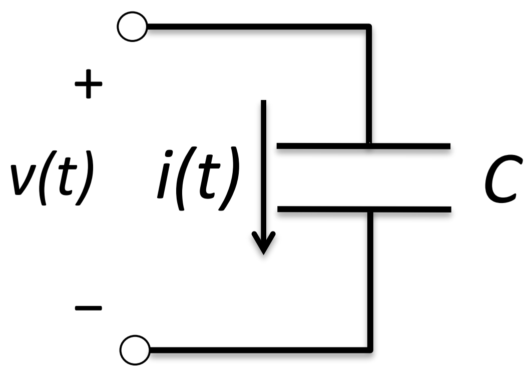
– the voltage *v* and current *i* are defined above to satisfy ***passive sign convention***

– the voltage *v* and current *i* are related to each other by a linear operator (differentiation / integration)



the capacitor

There are alternative but equivalent forms of the equations describing terminal behaviour of ideal capacitors.



differential form:

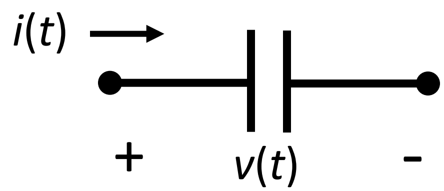
$$i = C\frac{dv}{dt} = \frac{dq}{dt}$$

integral form:

$$v(t) - v(t_0) = \frac{1}{C}\int_{t_0}^{t} i(\tau)\,d\tau = \frac{q(t) - q(t_0)}{C}$$

energy storage in the capacitor

Consider the energy *stored* in a capacitor. The instantaneous power *absorbed* (note the *passive sign convention*) by a capacitor is:



$$p(t) = i(t) \cdot v(t) = C\frac{dv(t)}{dt}v(t)$$

The energy absorbed by the capacitor from time t_0 to time t is:

$$W_{t_0 \to t} = \int_{t_0}^{t} p(t')dt' = \int_{t_0}^{t} Cv(t')\frac{dv(t')}{dt'}dt' = \int_{v(t_0)}^{v(t)} Cv(t')dv(t') = \frac{1}{2}Cv^2(t) - \frac{1}{2}Cv^2(t_0)$$

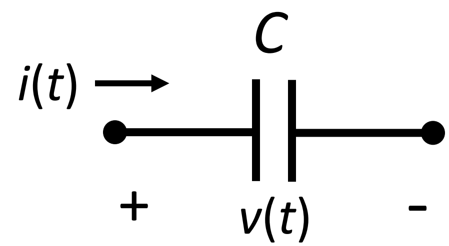
The energy absorbed is *stored* as electric potential energy *U(t)*:

$$\boxed{U(t) = \frac{1}{2}Cv^2(t) = \frac{1}{2}\frac{q^2(t)}{C}} \qquad W_{t_0 \to t} = U(t) - U(t_0)$$

continuity of capacitor voltage

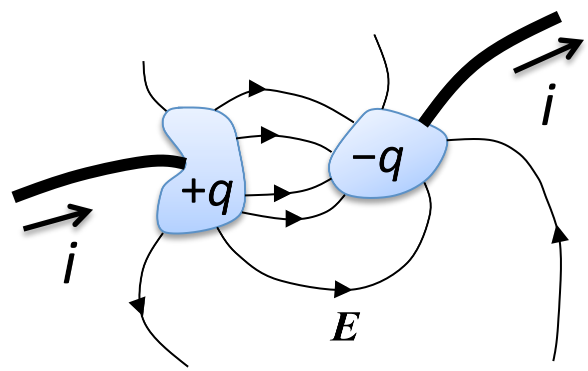
The current flow through a capacitor is:

$$i = C\frac{dv}{dt} = C\lim_{\Delta t \to 0}\frac{v(t+\Delta t) - v(t)}{\Delta t}$$



where we restate the definition of the derivative.

An instantaneous change in capacitor voltage (and charge separation) requires an infinite (unphysical) current. *For a finite current to flow, we require that the capacitor voltage v(t) is* ***continuous***.



continuity of capacitor voltage

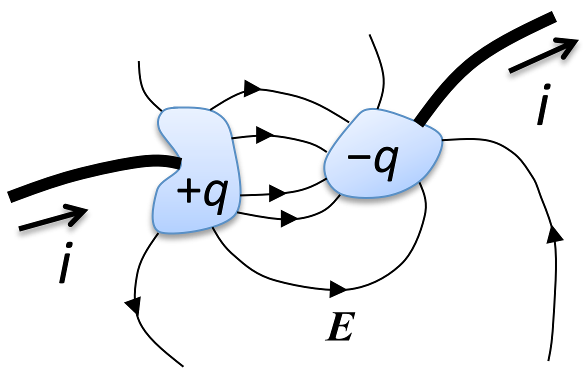
Continuity of capacitor voltage ensures that:

– the current i is finite

– the power absorbed $p = iv$ by the capacitor is finite

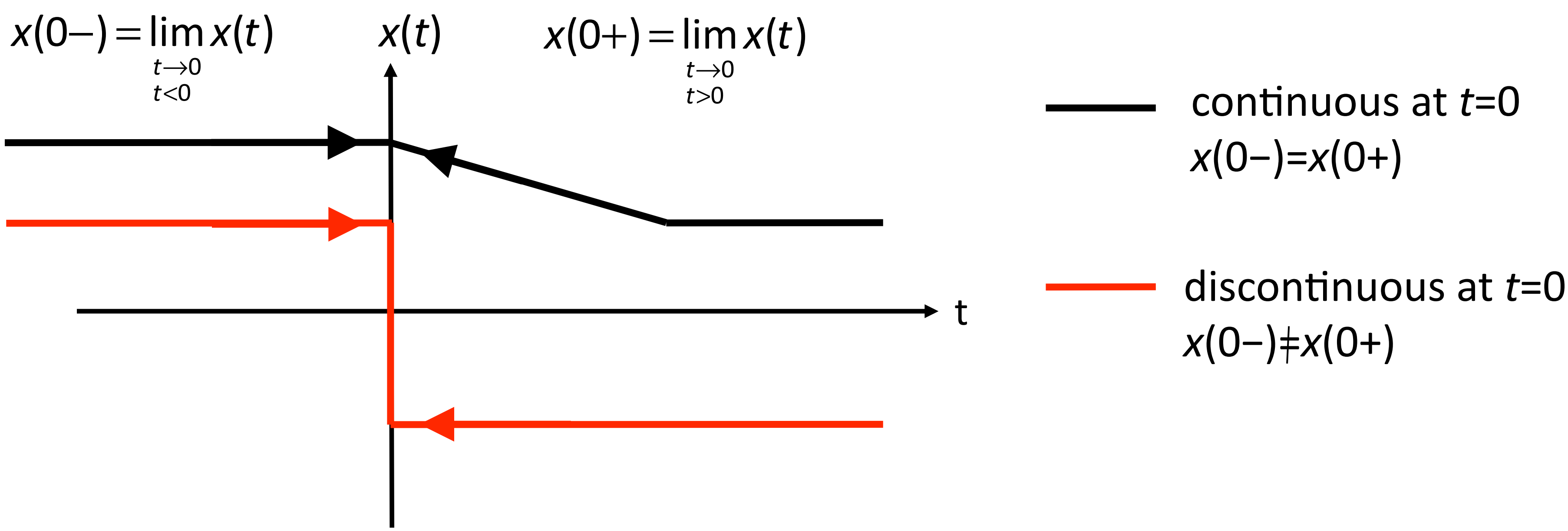
– the charge separation q is continuous, satisfying the **conservation of charge**

– the electric energy stored $U = ½\, Cv^2$ is continuous, satisfying the **conservation of energy**



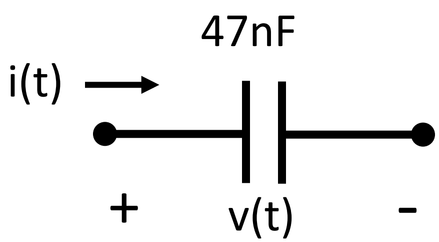
a note on notation

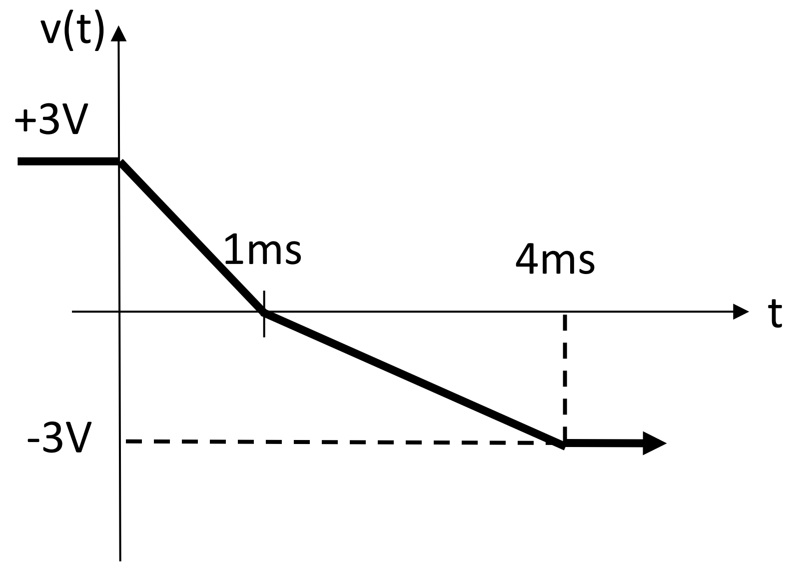
The notations $t = 0+$ and $t = 0-$ are often used in circuit analysis. The value of a circuit variable $x(t)$ as t approaches 0 from the past (left) or from the future (right) are identified separately:



Example 1

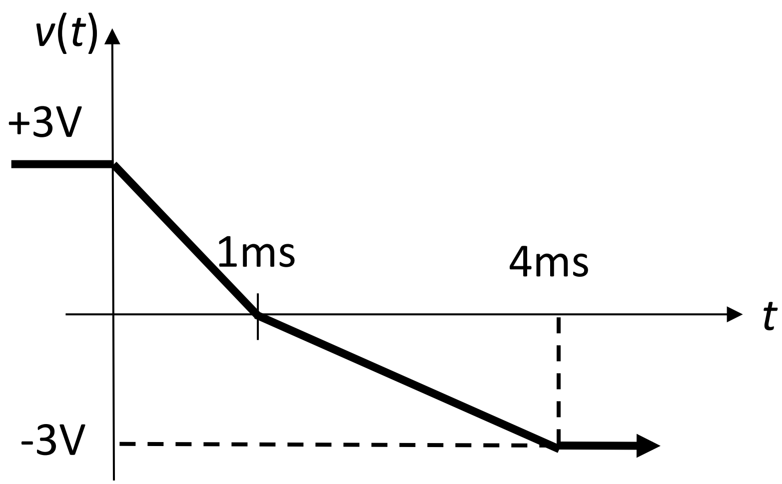
A 47nF capacitor has a voltage across its terminals given by the following diagram. Plot the charge separation and current as a function of time.

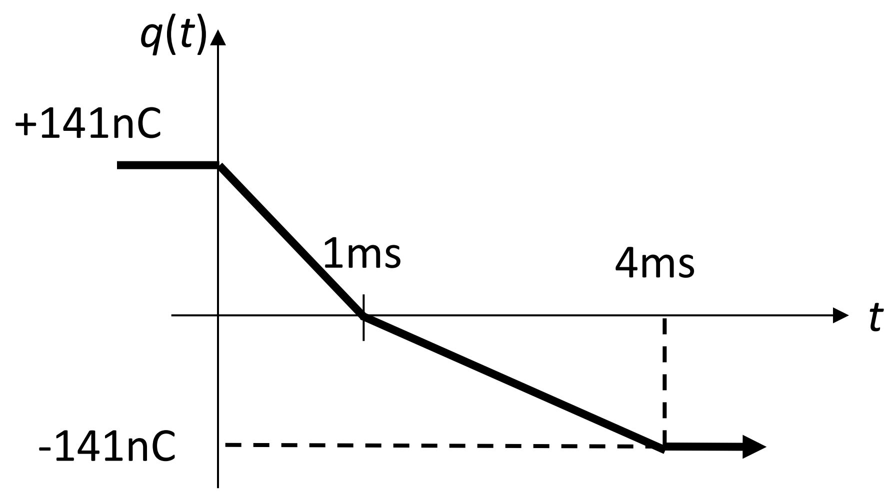




Example 1

Charge separation is given by $q(t) = C\,v(t)$: 47nF×3V= 141nC





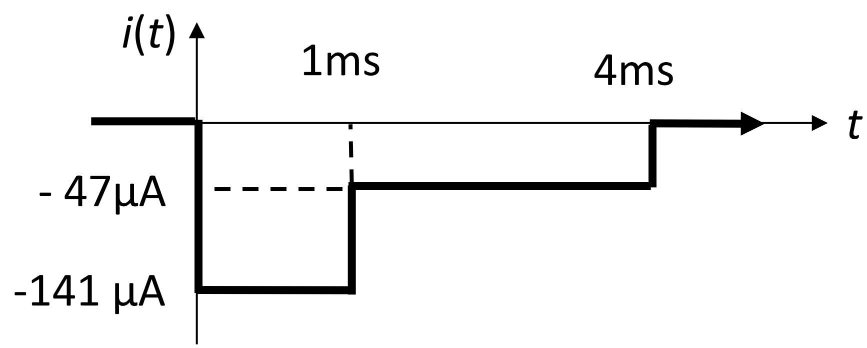
Current is given by $i(t) = dq/dt$, the slope of the charge-time plot:

t<0ms : dq/dt = 0A

0<t<1ms : dq/dt = -141µA

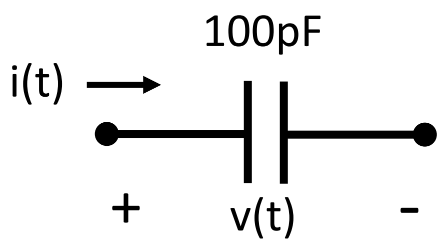
1ms<t<4ms : dq/dt = - 47µA

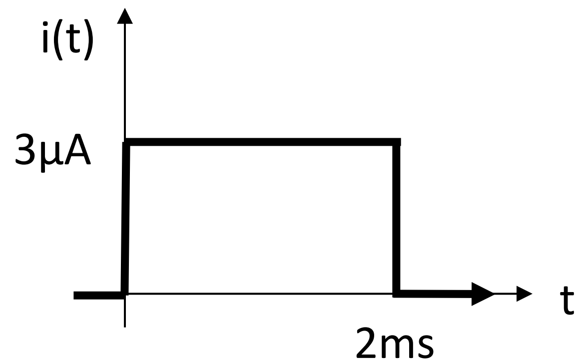
4ms<t : dq/dt = 0A



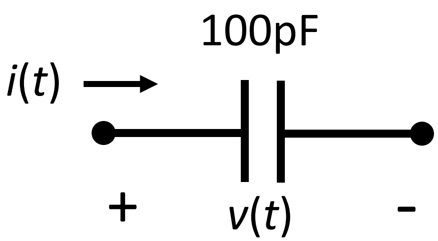
Example 2

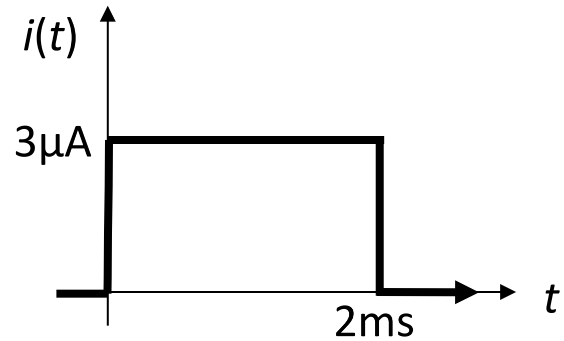
A 100pF capacitor, initially uncharged, passes a current given in the diagram below. Plot the voltage as a function of time.





Example 2





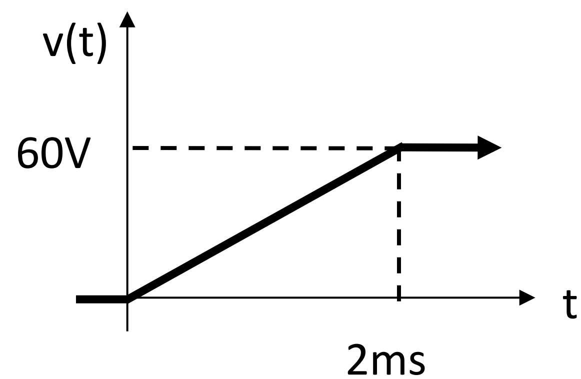
Use the integral form of the capacitor i-v relationship:

$$v(t) - v(0) = \frac{1}{C}\int_0^t i(t')\,dt'$$

$$v(t) - 0\text{V} = \frac{1}{100\text{pF}}\int_0^t 3\mu\text{A}\,dt' = 30\tfrac{\text{kV}}{\text{s}}\cdot t \quad 0 < t < 2\text{ms}$$

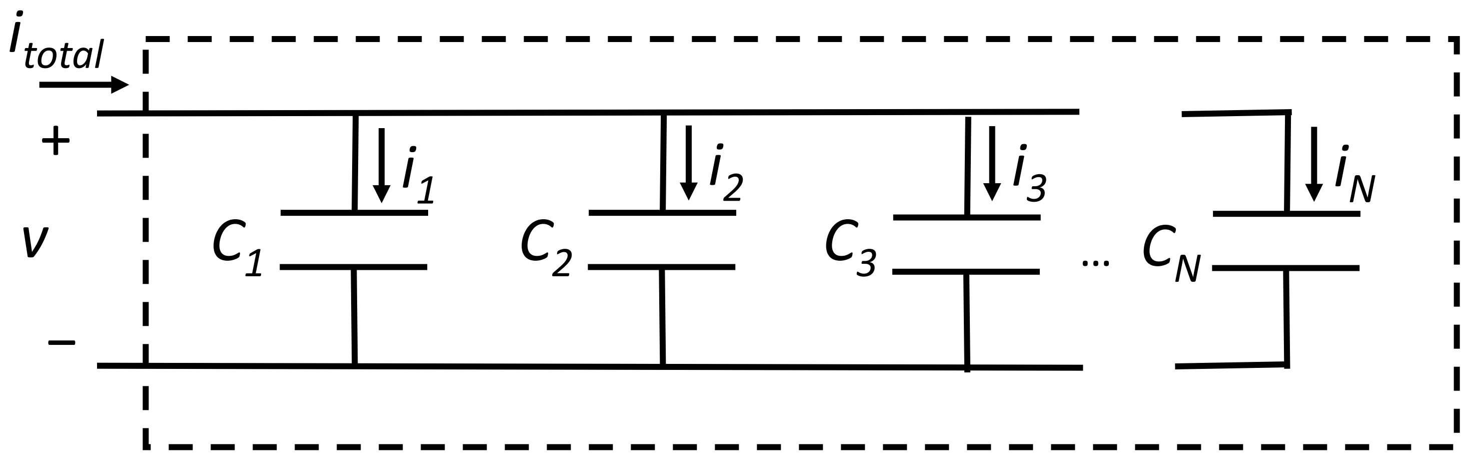
$$\therefore$$

$$v(t) - 0\text{V} = 60\text{V} + \frac{1}{100\text{pF}}\int_{2\text{ms}}^t 0\;dt' = 60V \qquad 2\text{ms} < t$$



capacitors in parallel

A parallel combination of capacitors has an equivalent capacitance C_{eq}.



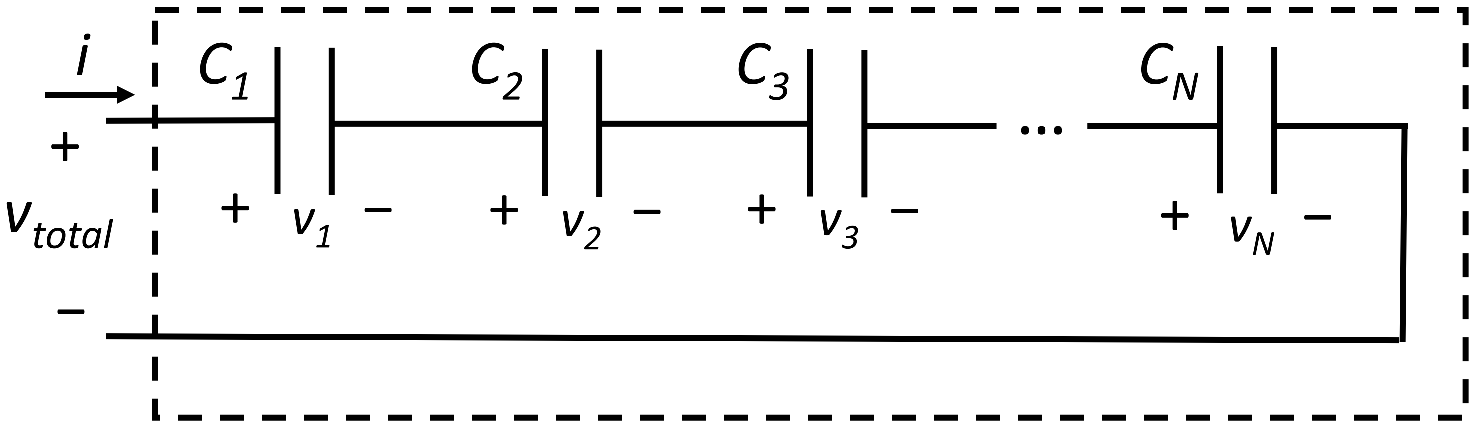
Current through each capacitor: $i_m = C_m\, dv/dt$

Total current (KCL): $i_{total} = i_1 + i_2 + \ldots + i_N = (\, C_1 + C_2 + \ldots\, C_N\,)\, dv/dt$

Equivalent capacitance: $$\frac{i_{total}}{dv/dt} = \boxed{C_{eq} = C_1 + C_2 + \ldots + C_N}$$

capacitors in series

A series combination of capacitors has an equivalent capacitance C_{eq}.



Current through each capacitor: $i = C_m\, dv_m/dt$

Total voltage:

(time derivative of KVL)

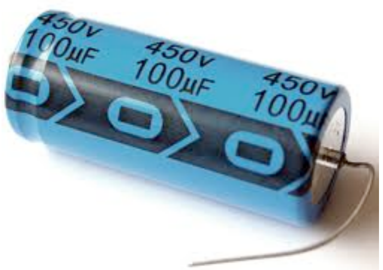
$$\frac{dv_{total}}{dt} = \frac{dv_1}{dt} + \frac{dv_2}{dt} + \ldots + \frac{dv_N}{dt} = i\left(\frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_N}\right)$$

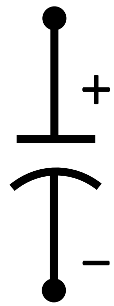
Equivalent capacitance:

$$\frac{dv_{total}/dt}{i} = \boxed{\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots + \frac{1}{C_N}}$$

practical capacitors

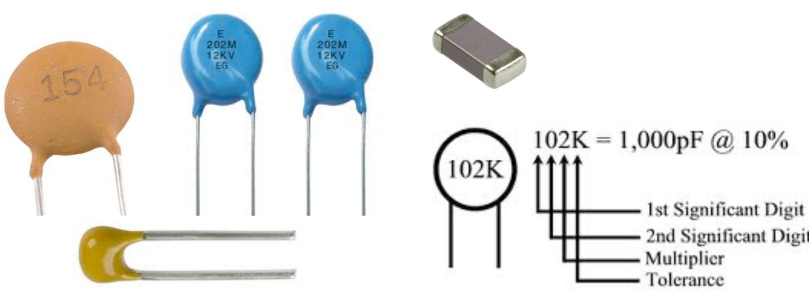
There is a very wide variety of capacitors, each with characteristics (capacitance range, breakdown voltage, polarity, dielectric leakage, price) that distinguish them for different applications.



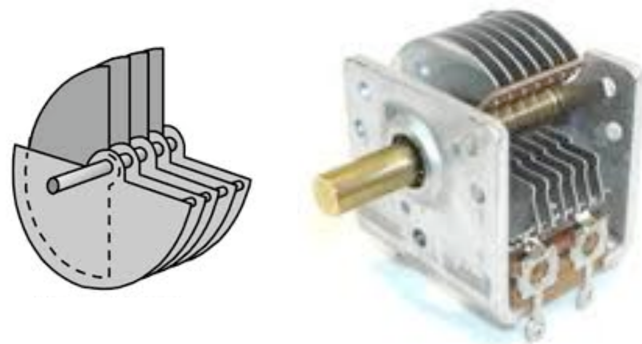


electrolytic capacitors: large C / volume, polarity due to electrolyte, low cost, high leakage, high dielectric loss

thin-film capacitors: different polymer films with different temp. and humidity stability, moderate C / volume, very low dielectric loss, low leakage



ceramic capacitors: different ceramics with different temp. stability, moderate C / volume, high dielectric loss, usually low cost



variable capacitors: low C / volume, low leakage, high cost