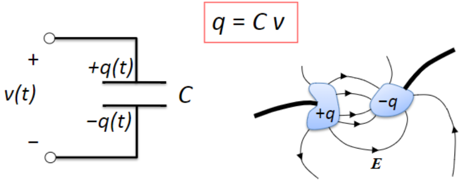
ECSE 200 - Electric Circuits 1
Problem set 9

ECE Dept., McGill University

Part 1: Capacitors

Capacitor Recap

- Charge separation of a capacitor: $q(t) = Cv(t)$ (Coulomb)
- Current: $i(t) = \frac{\partial q(t)}{\partial t} = C\frac{\partial v(t)}{\partial t}$ (A)
- Voltage difference: $v(t_1) - v(t_0) = \frac{1}{C}\int_{t_0}^{t_1} i(t)\partial t$ (V)
- Power: $p(t) = i(t)v(t) = C\frac{v(t)\partial v(t)}{\partial t}$ (W)
- Energy: $E(t) = \int_{t_0}^{t_1} p(t)\partial t = C\int_{v(t_0)}^{v(t_1)} v(t)\partial v(t) = \frac{Cv^2(t)}{2}\Big|_{t_0}^{t_1}$ (J)



Problem P 7.2-1

A 15-μF capacitor has a voltage of 5 V across it at t $=$ 0. If a constant current of 25 mA flows through the capacitor, how long will it take for the capacitor to charge up to 150 μC?

Problem P 7.2-1

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Solution: $$v(t) = v(0) + \frac{1}{C}\int_0^t i(\tau)\,d\tau \quad and \quad q = Cv$$

In our case, the current is constant so $i\,t = \int_0^t i(\tau)\,d\tau$.

$$\therefore\ Cv(t) = Cv(0) + i\,t$$

$$\therefore\ t = \frac{q - Cv(0)}{i} = \frac{150\times10^{-6} - \left(15\times10^{-6}\right)(5)}{25\times10^{-3}} \underline{= 3\text{ ms}}$$

Problem P 7.2-4

Determine $v(t)$ for the circuit shown in Figure P 7.2-4a when the $i_s(t)$ is as shown in Figure P 7.2-4b and $v_0(0^-) = -1$ mV.

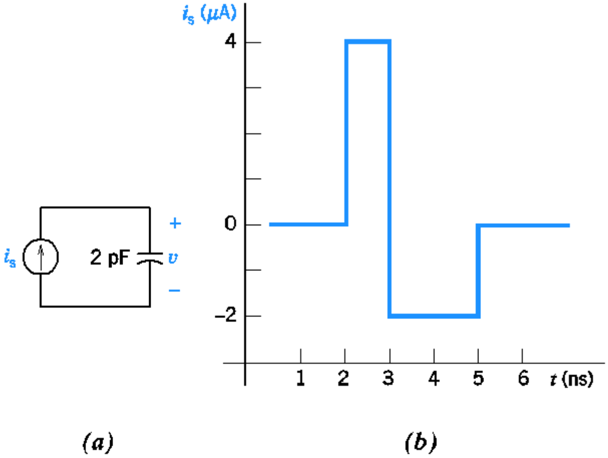


Figure P 7.2-4

Problem P 7.2-4

Solution:

$$v(t)=\frac{1}{C}\int_0^t i(\tau)\,d\tau+v(0)=\frac{1}{2\times10^{-12}}\int_0^t i(\tau)\,d\tau-10^{-3}$$

$0<t<2\times10^{-9}$

$$i_s(t)=0 \quad\Rightarrow\quad v(t)=\frac{1}{2\times10^{-12}}\int_0^t 0\,d\tau-10^{-3}=-10^{-3}$$

$2\times10^{-9}<t<3\times10^{-9}$

$$i_s(t)=4\times10^{-6}\text{ A}$$

$$\Rightarrow\quad v(t)=\frac{1}{2\times10^{-12}}\int_{2\text{ns}}^t \left(4\times10^{-6}\right)d\tau-10^{-3}=-5\times10^{-3}+\left(2\times10^{6}\right)t$$

In particular, $v\left(3\times10^{-9}\right)=-5\times10^{-3}+\left(2\times10^{6}\right)\left(3\times10^{-9}\right)=10^{-3}$

$3\times10^{-9}<t<5\times10^{-9}$

$$i_s(t)=-2\times10^{-6}\text{ A}$$

$$\Rightarrow\quad v(t)=\frac{1}{2\times10^{-12}}\int_{3\text{ns}}^t \left(-2\times10^{-6}\right)d\tau+10^{-3}=4\times10^{-3}-\left(10^{6}\right)t$$

In particular, $v\left(5\times10^{-9}\right)=4\times10^{-3}-\left(10^{6}\right)\left(5\times10^{-9}\right)=-10^{-3}\text{ V}$

$5\times10^{-9}<t$

$$i_s(t)=0 \quad\Rightarrow\quad v(t)=\frac{1}{2\times10^{-12}}\int_{5\text{ns}}^t 0\,d\tau-10^{-3}=-10^{-3}\text{ V}$$

Problem P 7.2-8

Find i for the circuit of Figure P 7.2-8 if $v = 5(1 - 2e^{-2t})$ V.

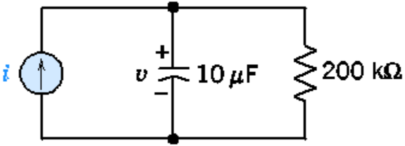
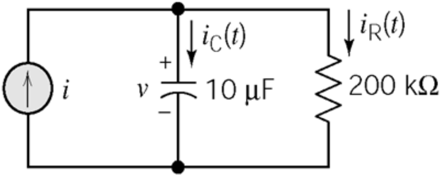


Figure P 7.2-8

Problem P 7.2-8

Solution:



$$i_R = \frac{v}{200\times10^3} = \frac{1}{40}\left(1-2e^{-2t}\right)\times10^{-3} = 25\left(1-2e^{-2t}\right)\ \mu\text{A}$$

$$i_C = C\,\frac{dv}{dt} = \left(10\times10^{-6}\right)(-2)\left(-10\,e^{-2t}\right) = 200\,e^{-2t}\ \mu\text{A}$$

$$i = i_R+i_C = 200\,e^{-2t}+25-50\,e^{-2t}$$

$$= \underline{25+150e^{-2t}\quad \mu\text{A}}$$

Problem P 7.2-14

The capacitor voltage in the circuit shown in Figure P 7.2-14 is given by

$$v(t) = 10 - 8e^{-5t}\ \text{V} \quad \text{for} \quad t \geq 0$$

Determine $i(t)$ for $t > 0$.

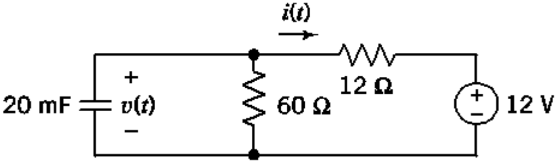


Figure P 7.2-14

Problem P 7.2-14

The capacitor voltage in the circuit shown in Figure P 7.2-14 is given by

$$v(t) = 10 - 8e^{-5t}\ \text{V} \quad \text{for} \quad t \ge 0$$

Determine $i(t)$ for $t > 0$.

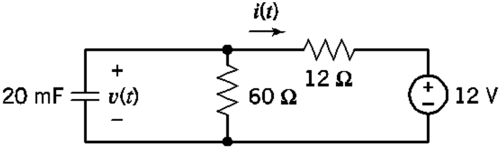


Figure P 7.2-14

Solution: Apply KVL to the outside loop to get

$$12\,i(t) + 12 - v(t) = 0 \quad \Rightarrow \quad i(t) = \frac{12 - 10 - 8e^{-5t}}{12} = -\frac{1}{6} - \frac{2}{3}e^{-5t}\ \text{A} \quad \text{for} \quad \text{t} > 0$$

Part 2: Energy Storage in a Capacitor

Problem P 7.3-2

In a pulse power circuit the voltage of a 10-μF capacitor is zero for $t < 0$ and

$$v = 5(1 - e^{-4000t})\text{V} \ \ t \geq 0.$$

Determine the capacitor current and the energy stored in the capacitor at $t = 0$ ms and $t = 10$ ms.

Problem P 7.3-2

In a pulse power circuit the voltage of a 10-μF capacitor is zero for $t < 0$ and

$$v = 5(1 - e^{-4000t})\,\mathrm{V} \quad t \geq 0$$

Determine the capacitor current and the energy stored in the capacitor at $t = 0$ ms and $t = 10$ ms.

Solution:

$$i_c = C\frac{dv}{dt} = \left(10\times10^{-6}\right)(-5)(-4000)e^{-4000t} = \underline{0.2e^{-4000t}\ \mathrm{A}} \Rightarrow \begin{cases} \underline{i_c(0) = 0.2\ A} \\ \underline{i_c(10ms) = 8.5\times10^{-19}\ A} \end{cases}$$

$$\mathcal{W}(t) = \frac{1}{2}Cv^2(t) \quad \text{and} \quad v(0) = 5 - 5e^0 = 0 \Rightarrow \underline{\mathcal{W}(0) = 0}$$

$$v\left(10\times10^{-3}\right) = 5 - 5e^{-40} = 5 - 21.2\times10^{-18} \cong 5 \Rightarrow \underline{\mathcal{W}(10) = 1.25\times10^{-4}\ \mathrm{J}}$$

Problem P 7.3-3

If $v_c(t)$ is given by the waveform shown in Figure P 7.3-3, sketch the capacitor current for $-1\text{s} \le t \le 2\text{s}$. Sketch the power and the energy for the capacitor over the same time interval when $C = 1$ mF.

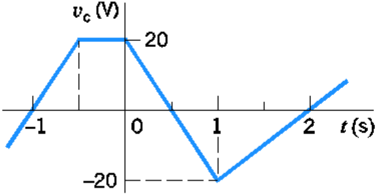


Figure P 7.3-3

Problem P 7.3-3 (Solution)

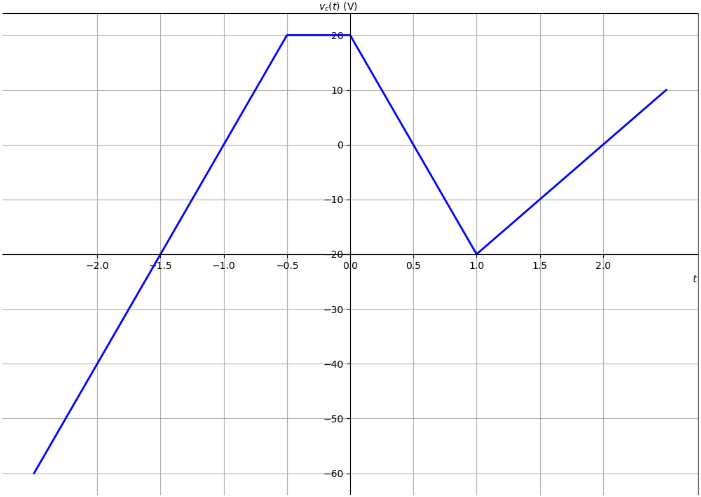


Figure: $v(t)$

Problem P 7.3-3 (Solution-cnt.)

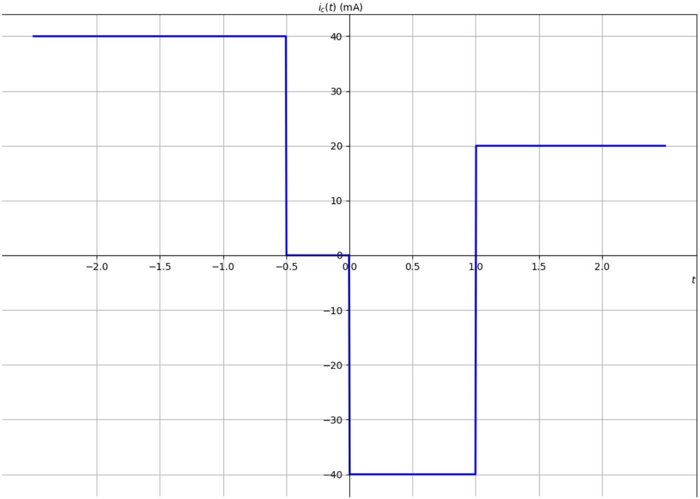


Figure: $i(t)$

Problem P 7.3-3 (Solution-cnt.)

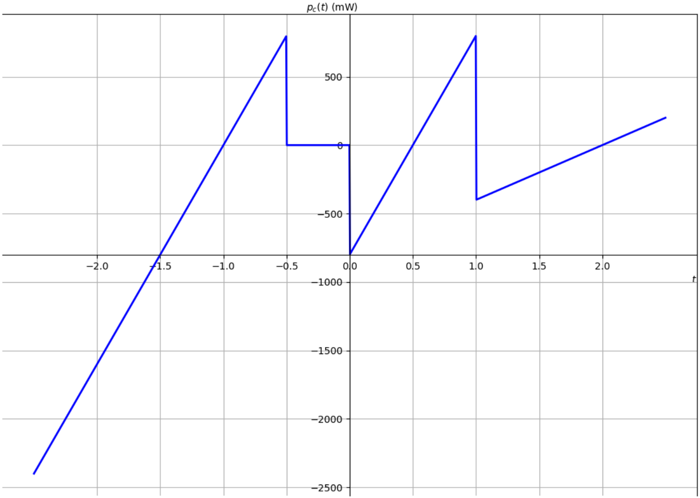


Figure: $p(t)$

Problem P 7.3-3 (Solution-cnt.)

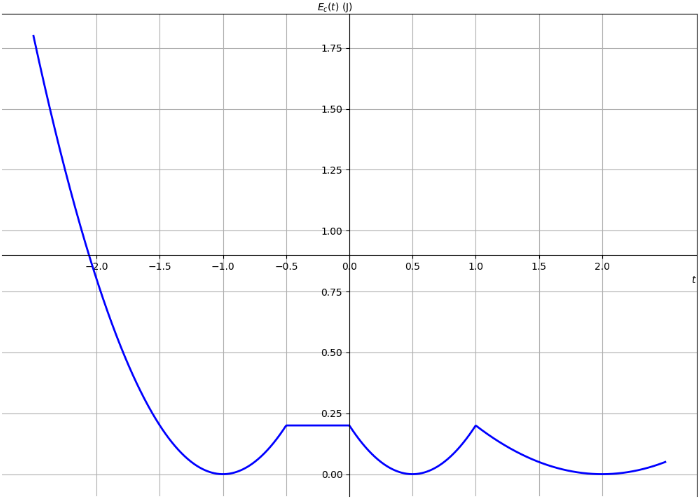


Figure: $E(t)$

Part 3: Series and Parallel Capacitors

Problem P 7.4-1

Find the current $i(t)$ for the circuit of Figure P 7.4-1.

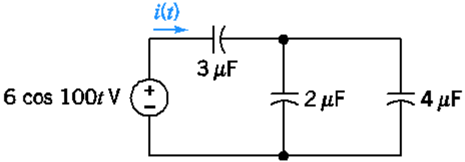


Figure P 7.4-1

Problem P 7.4-1

Find the current $i(t)$ for the circuit of Figure P 7.4-1.

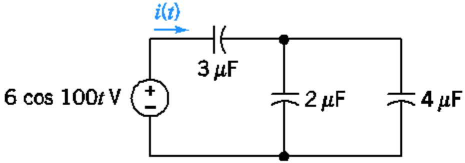


Figure P 7.4-1

Solution:

$$2\mu\text{F} \parallel 4\mu\text{F} = 6\mu\text{F}$$

$$6\mu\text{F in series with } 3\mu\text{F} = \frac{6\mu\text{F}\cdot 3\mu\text{F}}{6\mu\text{F}+3\mu\text{F}} = 2\mu\text{F}$$

$$i(t) = 2\ \mu\text{F}\ \frac{d}{dt}\,(6\cos 100t) = (2\times 10^{-6})\ (6)\ (100)\ \ (-\sin 100t)\ \text{A} = \underline{-1.2\ \sin 100t\ \ \text{mA}}$$

Problem P 7.4-7

The circuit shown in Figure P 7.4-7 consists of nine capacitors having equal capacitance, C. Determine the value of the capacitance C, given that $C_{eq} = 50$ mF.

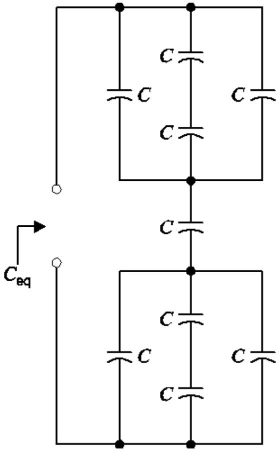
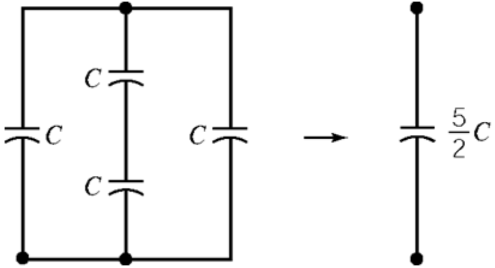


Figure P 7.4-7

Problem P 7.4-7

Solution: First



Then

$$50 = C_{\text{eq}} = \frac{1}{\frac{1}{C} + \frac{2}{5\,C} + \frac{2}{5\,C}} \quad \Rightarrow \quad C = 90 \text{ mF}$$

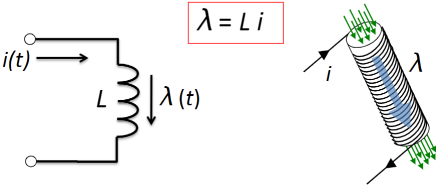
Part 4: Inductors

Inductor Recap

- Magnetic flux: $\lambda(t) = Li(t)$ (HA)

- Voltage (Faraday's law of inductor): $v(t) = \frac{\partial \lambda(t)}{\partial t} = \frac{L \partial i(t)}{\partial t}$ (V)
  In words: the voltage drop across an inductor is proportional to the rate of change of the total magnetic flux passing through the inductor.

- Current difference: $i(t_1) - i(t_0) = \frac{1}{L} \int_{t_0}^{t_1} v(t) \partial t$ (A)
- Power: $p(t) = i(t)v(t) = L \frac{i(t) \partial i(t)}{\partial t}$ (W)
- Energy: $E(t) = \int_{t_0}^{t_1} p(t) \partial t = L \int_{i(t_0)}^{i(t_1)} i(t) \partial i(t) = \frac{Li^2(t)}{2} \Big|_{t_0}^{t_1}$ (J)

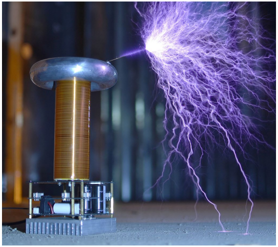


Problem P 7.5-1

Nikola Tesla (1857–1943) was an American electrical engineer who experimented with electric induction. Tesla built a large coil with a very large inductance. The coil was connected to a source current

$$i_s = 100\sin(400t)\ \text{A}$$

so that the inductor current $i_L = i_s$. Find the voltage across the inductor and explain the discharge in the air shown in the figure. Assume that $L = 200$ H and the average discharge distance is 2 m. Note that the dielectric strength of air is 3×10^6 V/m. Link



Problem P 7.5-1

Solution

Find max. voltage across coil: $v(t) = L\dfrac{di}{dt} = 200\ [100(400)\cos\ 400t]$ V

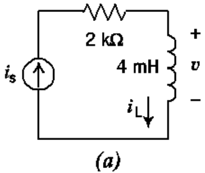
$\therefore\ v_{max} = 8\times10^6$ V thus have a field of $\dfrac{8\times10^6}{2}\ \text{V}/\text{m} = 4\times10^6\ \text{V}/\text{m}$

which exceeds dielectric strength in air of 3×10^6 V/m

$\therefore$ We get a discharge as the air is ionized.

Problem P 7.5-6

Determine $v(t)$ for $t > 0$ for the circuit of Figure P 7.5-6*a* when $i_L(0) = 0$ and i_s is as shown in Figure P 7.5-6*b*.



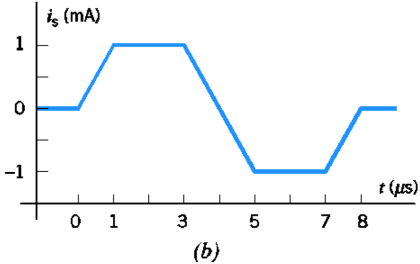


Figure P 7.5-6

Problem P 7.5-6

Solution

In general $v(t) = \left(2\times10^3\right)i_s(t) + \left(4\times10^{-3}\right)\frac{d}{dt}i_s(t)$

For $0<t<1\ \mu s$ $i_s(t) = (1)\left(\frac{1\times10^{-3}}{1\times10^{-6}}\right)t = 10^3 t \Rightarrow \frac{d}{dt}i_s(t) = 1\times10^3$. Consequently

$$v(t) = (2\times10^3)\left(1\times10^3\right)t + 4\times10^{-3}\left(1\times10^3\right) = \left(2\times10^6 t + 4\right)\ \text{V}$$

For $1\mu s < t < 3\mu s$ $i_s(t) = 1\ \text{mA} \Rightarrow \frac{d}{dt}i_s(t) = 0$. Consequently

$$v(t) = (2\times10^3)\left(1\times10^{-3}\right) + \left(4\times10^{-3}\right)\times0 = 2\ \text{V}$$

For $3\mu s < t < 5\mu s$ $i_s(t) = 4\times10^{-3} - \left(\frac{1\times10^{-3}}{1\times10^{-6}}\right)t \Rightarrow \frac{d}{dt}i_s(t) = -\frac{1\times10^{-3}}{1\times10^{-6}} = -10^3$. Consequently

$$v(t) = \left(2\times10^3\right)\left(4\times10^{-3} - 10^3 t\right) + 4\times10^{-3}\left(-10^3\right) = 4 - \left(2\times10^6\right)t$$

Problem P 7.5-6

Solution (cnt.)

When $5\mu s < t < 7\mu s$ $\quad i_s(t) = -1\times10^{-3}$ and $\frac{d}{dt}i_s(t) = 0$. Consequently

$$v(t) = \left(2\times10^3\right)\left(10^{-3}\right) = -2 \text{ V}$$

When $7\mu s < t < 8\mu s$ $\quad i_s(t) = \left(\frac{1\times10^{-3}}{1\times10^{-6}}\right)t - 8\times10^{-3} \Rightarrow \frac{d}{dt}i_s(t) = 1\times10^3$

$$v(t) = \left(2\times10^3\right)\left(10^3\, t - 8\times10^{-3}\right) + \left(4\times10^{-3}\right)\left(10^3\right) = -12 + \left(2\times10^6\right)t$$

When $8\mu s < t$, then $i_s(t) = 0 \Rightarrow \frac{d}{dt}i_s(t) = 0$. Consequently $v(t) = 0$.

Problem P 7.5-14

The inductor current in the circuit shown in Figure P 7.5-14 is given by

$$i(t) = 3 + 2e^{-3t}\ \text{A} \quad \text{for } t \geq 0$$

Determine $v(t)$ for $t > 0$.

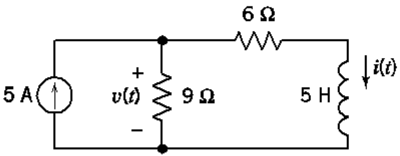


Figure P 7.5-14

Problem P 7.5-14

The inductor current in the circuit shown in Figure P 7.5-14 is given by

$$i(t) = 3 + 2e^{-3t}\ \text{A} \quad \text{for}\ t \geq 0$$

Determine $v(t)$ for $t > 0$.

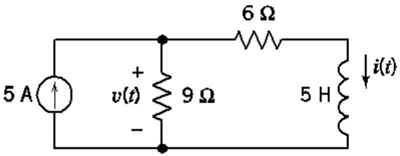


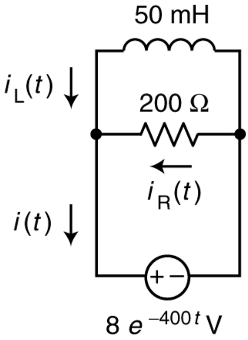
Figure P 7.5-14

Solution: Apply KVL to get

$$v(t) = 6i(t) + 5\frac{d}{dt}i(t) = 6\left(3 + 2e^{-3t}\right) + 5\frac{d}{dt}\left(3 + 2e^{-3t}\right) = 18\left(1 - e^{-3t}\right)\ \text{V} \quad \text{for}\ t > 0$$

Problem P 7.5-18

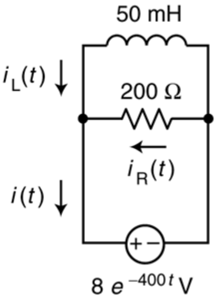
The source voltage in the circuit shown below is $v(t) = 8e^{-400t}$ after time $t = 0$. The initial inductor current is $i_L(0) = 210$ mA. Determine the source current $i(t)$ for $t > 0$.



Problem P 7.5-18

Solution

Label the resistor current as shown. The resistor, inductor and voltage source are connected in parallel so the voltage across each is $v(t) = 2.5\ e^{-400t}$ V . Notice that the labeled voltage and current of both the resistor and inductor do not adhere to the passive convention.



The resistor current is $i_R(t) = -\dfrac{8\ e^{-400t}}{200} = -40\ e^{-400t}$ mA

The inductor current is $i_L(t) = 0.21 + \dfrac{1}{0.05}\displaystyle\int_0^t -8\ e^{-400\tau}\ d\tau$ A

$$i_L(t) = 0.21 + \frac{-8}{0.05(-400)}\int_0^t e^{-400\tau}\ d\tau \text{ A}$$

$$= 0.21 + 0.4\left(e^{-400t} - 1\right) \text{ A}$$

$$= 400 e^{-400t} - 190 \text{ mA}$$

Using KCL $\quad i(t) = 360 e^{-400t} - 190$ mA for $t > 0$

Part 5: Energy Storage in an Inductor

Problem P 7.6-4

The current in an inductor, $L = 1/4$ H, is $i = 4te^{-t}$ A for $t \geq 0$ and $i = 0$ for $t < 0$. Find the voltage, power, and energy in this inductor.

Problem P 7.6-4

The current in an inductor, $L = 1/4$ H, is $i = 4te^{-t}$ A for $t \geq 0$ and $i = 0$ for $t < 0$. Find the voltage, power, and energy in this inductor.

Solution

$$v = L\frac{di}{dt} = \left(\frac{1}{4}\right)\frac{d}{dt}\left(4\,t\,e^{-t}\right) = \underline{(1-t)\,e^{-t}\ \text{V}}$$

$$P = vi = \left[\,(1-t)e^{-t}\,\right]\left(4\,t\,e^{-t}\right) = \underline{4\,t\,(1-t)\,e^{-2t}\ \text{W}}$$

$$\mathcal{W} = \frac{1}{2}Li^2 = \frac{1}{2}\left(\frac{1}{4}\right)\left(4\,t\,e^{-t}\right)^2 = \underline{2\,t^2\,e^{-2t}\ \text{J}}$$

Part 6: Series and Parallel Inductors

Problem P 7.7-7

The circuit shown in Figure P 7.7-7 consists of 10 inductors having equal inductance, L. Determine the value of the inductance L, given that $L_{eq} = 12$ mH.

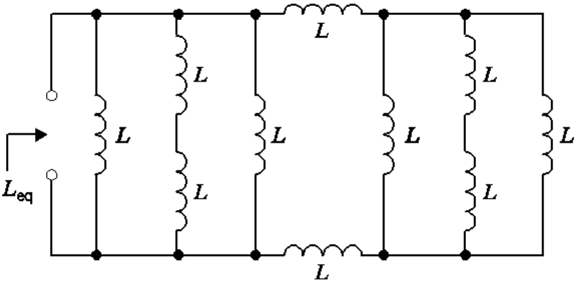


Figure P 7.7-7

Problem P 7.7-7 (Solution)

The circuit shown in Figure P 7.7-7 consists of 10 inductors having equal inductance, L. Determine the value of the inductance L, given that $L_{eq} = 12$ mH.

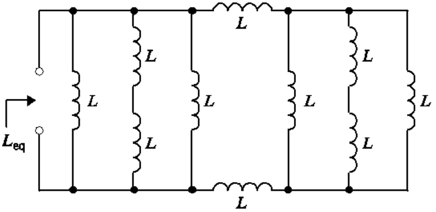
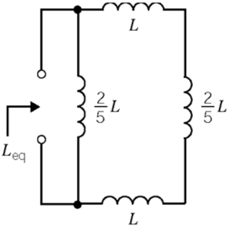


Figure P 7.7-7

Solution:

First



Then

$$12 = L_{eq} = \frac{\left(\frac{2}{5}L\right)\times\left(\frac{2}{5}L+2L\right)}{\left(\frac{2}{5}L\right)+\left(\frac{2}{5}L+2L\right)} = \frac{12}{35}L \quad \Rightarrow \quad L = 35 \text{ mH}$$

Thank you !