WINCHESTER THURSTON ADVANCED TOPICS IN MATHEMATICS 2019

An Occurrence of π in a Quasi-Elastic Collisions Scenario

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ABSTRACT

In a classical scenario of particles colliding elastically as a system, with the singular exception that the collision with a wall reverses the velocities of the colliding parties and therefore their momenta, hence quasi-elastic, it has been observed that for a two-block system (see **Introduction**)–a block of some mass static and closer to the wall, and another moving from afar toward this block and the wall, with the mass of the some integer n power of 100 to the latter–the number of inter-block and block-wall collisions will be equivalent to the integer listing of the first n digits of π. To examine in greater detail this scenario, this paper purports to illustrate the proof of this observation, provide a general formula for the number of collisions of a two-block scenario with any masses, and produce an emulator for a block system of an arbitrary number of blocks.

1. Introduction

Physics often relates to mathematics in surprising ways, in part due to their history of mutual development. In an idealized scenario, we setup the problem with a wall on the left and two blocks that are free to slide on a frictionless surface(1). The larger block on the right has a mass of some integer power of 100, denoted henceforth as n, to that of its lighter counterpart (the size and mass of blocks in figures and animations are represented as somehow proportional). The larger block has an initial velocity greater than zero toward the wall. The smaller block toward the left has no initial velocity. If we allow the scenario to proceed, collisions will occur between both blocks and with the wall in a methodical fashion, until both blocks shall have their velocities toward the right with the smaller one never able to reach its departing counterpart.

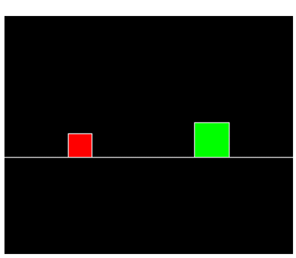


FIG. 1. Initial Scenario

Remarkably, if one were to enumerate the number of collisions occurred in this process described, it would be an integer list of the numbers from the unit to the sequential decimals of π. 3 to 31 to 314, etc.

For example, if the larger block has a mass that is as heavy as the smaller one, the number of collisions will be 3, which is the unit digit of π. Then if the larger block has a mass that is 100 times the smaller one, the total number of collisions will be 31, as for the coefficients of the unit and the one-tenth places.

2. Observations

Before we exert ourselves to the analysis of this scenario, let us enumerate our assumptions, or principles.

1. That all inter-block collisions are elastic, that is, both the total momentum and kinetic energy of the system are conserved, or

$$p_0 = \sum_{i\in c} p_i = \sum_{i\in c} {p_i}'$$

$$K_0 = \sum_i \frac{1}{2} m_i {v_i}^2 = \sum_i \frac{1}{2} m_i {v_i}'^2$$

2. That upon contact with the wall with relatively negative velocity (colliding unto), the momentum of the block will be reversed, or

$${p_w}' = -p_w$$

3. That the masses of blocks remain unchanged.
4. That as elasticity assumes the instantaneous nature of collisions, collisions are always considered to be binary–either between the wall and a block, or between a block and a block. The case where two collisions advent coincidentally is ignored.

It then follows from the first two tenets that the kinetic energy K is always conserved, whereas the momentum of the two blocks are conserved when they collide with each other independent of the wall. The third assumption essentially states that the alterations in momenta are by shifts in velocities of the blocks. We shall have more to say about the fourth assumption later in this paper.

Now, prior to plumbing the essence of the problem, a few observations prove to be useful.

1. That the relative order of blocks, that is, their sequential positions from left to right, can never be altered, as the one block can never transcend another.

2. That upon any collision succeeding which the more massive block to the right still owns a leftward velocity, the post-collision velocity of the smaller will perforce be one leftward with greater magnitude.

3. That the only occasion where the two blocks collide and the less massive block does not have a left-bound velocity is when the impulse it imparts to the other block renders that block proceed to the right with a higher speed, thus essentially entailing this as the last collision.

The first observation is plain as blocks can never physically pass over one another. The second and third arise from the fact that the result of elastic collisions must be the disengagement of the blocks. Regarding the second, for the blocks to disengage, the block on the left must move left with a higher speed. Of the third, were the block on the left to have an equal or greater mass, the blocks would, again, not disengage, which contradicts the assumption of elasticity.

Taken together, the second and third observations amount to the statement that a inter-block collision, if preceded or succeeded, must be by a block-wall collision, the aforementioned "collision cycle." The general strategy to prove this interesting phenomenon would be to represent collisions numbers in this problems in a way that relates to geometry, and more specifically that of a circle for π is involved in the result. It is important to note that all collisions in our problem setup is elastic, which means that all energies are conserved. Since in this scenario there are only kinetic energies that contribute to the total energy, we have $\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = E$, where E is constant. This expression is actually quite similar to he analytical expression of a circle, which is $x^2 + y^2 = r^2$. In fact with some transformations, the law of conservation of energy will be the key to proving this seemingly chaotic scenario.

3. Derivation

An important transformation, enabled by our previous observations, that provides a sterlingly succinct management of the scenario is, as suggested in the video that prompted this paper [1], *a phase diagram.*

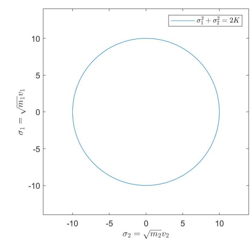
From the form of the kinetic energy-conservation equation, one observes a resemblance to a circle,

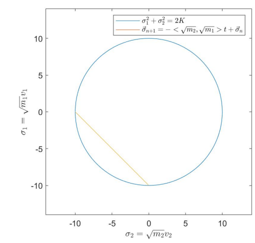
$$2K_0 = m_1{v_1}^2 + m_2{v_2}^2$$

This can, in turn, be transformed to a circle in defining

$$\sigma_1 = \sqrt{m_1}v_1,\ \sigma_2 = \sqrt{m_2}v_2$$

$$\sigma_1^2 + \sigma_2^2 = 2K_0 \tag{1}$$





Thus, σ_1, σ_2 become weighted velocities of the blocks. Accordingly, the momentum equality can be manipulated so

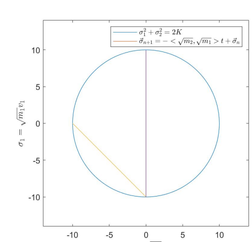
$$\sqrt{m_1}\sqrt{m_1}v_1 + \sqrt{m_2}\sqrt{m_2}v_2 = \sqrt{m_1}\sqrt{m_1}{v_1}' + \sqrt{m_2}\sqrt{m_2}{v_2}'$$

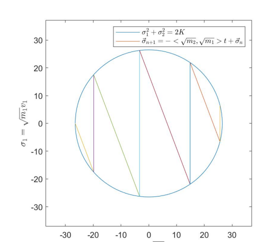
$$\sqrt{m_1}\sigma_1 + \sqrt{m_2}\sigma_2 = p_0 \tag{2}$$

Which assumes a linear form, with the slope of the line $-\sqrt{\frac{m_1}{m_2}}$. Notice that σ_1, which concerns the leftward block of lesser mass, is marked as the vertical axis, while the other, horizontal. Also, the particular scales of the plot may be ignored. The specific example used in this is two blocks of identical mass, with $v_2 = 10$.

The image on the right illustrates the method we manage or account for collisions. Mark that the intersection to the left has $\sigma_1 = 0$, which corresponds to the block on the left being stationary, while σ_2 rests on the left-most of the kinetic energy-conservation circle, representing that the block to the right is moving left-bound and contain all the kinetic energy of the system.

The behavior of the leftward block bouncing succeeding collision with the wall is represented as the diagram on the left, with its velocity reverted, thus its σ.





Thus, we can conclude that every inter-block collision is represented as a gradually descending line, with the leftward intersection with the circle representing the pre-collision conditions and the rightward intersection the post-collision. And a block-wall collision a vertical line, signifying the reversal of velocity.

By our previous observations, we demonstrated that inter-block and block-wall collisions precede or succeed each other, or the "collision cycle," which is shown in the construction of the diagram on the right.

4. Proof

The first step to approach this problem would be to find the generalized equation for velocity change during elastic collisions. The fundamental characteristic of an elastic collision is that both momentum and energy are conserved. Now consider two blocks, 1 and 2, with masses m_1, m_2 and velocities v_1, v_2. Their total energy is $E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$ and total momentum would be $p = m_1 v_1 + m_2 v_2$. Because the momentum is conserved during the collision, we can setup the equation with an impulse J that was transferred from block 1 to block 2, resulting in new values of momenta:

$$p_1' = m_1 v_1 - J, \ p_2' = m_2 v_2 + J;$$

and that translates to their new velocities:

$$v_1' = \frac{m_1 v_1 - J}{m_1}, \ v_2' = \frac{m_2 v_2 + J}{m_2}.$$

Then with the conservation of energy equation, we have:

$$\frac{1}{2}m_1 \left(\frac{m_1 v_1 - J}{m_1}\right)^2 + \frac{1}{2}m_2 \left(\frac{m_2 v_2 + J}{m_2}\right)^2 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$$

$$\Rightarrow m_1 v_1^2 + \frac{J^2 - 2m_1 v_1 J}{m_1} + m_2 v_2^2 + \frac{J^2 + 2m_2 v_2 J}{m_2} = m_1 v_1^2 + m_2 v_2^2$$

That reduces to:

$$\frac{J^2 - 2m_1 v_1 J}{m_1} + \frac{J^2 + 2m_2 v_2 J}{m_2} = 0$$

$$\Rightarrow \frac{m_1 + m_2}{m_1 m_2} J^2 + 2(v_2 - v_1)J = 0$$

Solving for I we have that $J = 0$ or $J = 2(v_1 - v_2)\frac{m_1 m_2}{m_1 + m_2}$. The solution 0 occurs because if the net momentum and energy doesn't change, it could be the case that neither of the block has any velocity change. But in our case the impulse cannot be equal to zero because an collision has occurred, and that leaves us with the second value of impulse. Plugging it back to solve for the new velocities yields:

$$v_1' = v_1 - \frac{J}{m_1}, \ v_2' = v_2 + \frac{J}{m_2}.$$

$$\Rightarrow v_1' = v_1 - \frac{2m_2(v_1 - v_2)}{m_1 + m_2} \text{ and } v_2' = v_2 + \frac{2m_1(v_1 - v_2)}{m_1 + m_2}$$

As mentioned in our observations, the collision cycle also requires a collision with the wall. So to complete one cycle on the phase diagram, v_1' needs to be reversed upon its collision with the wall, making the updated set of velocities:

$$v_1'' = -v_1 + \frac{2m_2(v_1 - v_2)}{m_1 + m_2} \text{ and } v_2' = v_2 + \frac{2m_1(v_1 - v_2)}{m_1 + m_2}.$$

These will correspond to neighboring points a and c on our phase diagram as shown in figure 2, and finding the angle of each procession is reduced to a geometry problem.

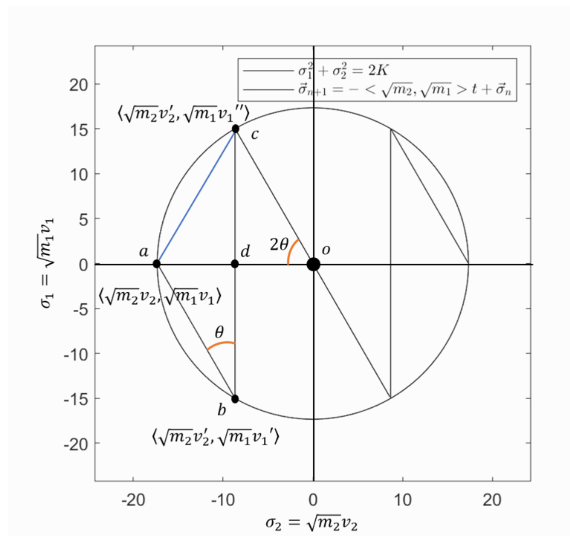


FIG. 2. Phase Diagram Representation

From the diagram above, it's easy to see that $\angle aoc$ which the arc ac corresponds to is twice the angle of $\angle abc$, according to inscribed angle theorem. So by finding the angle of $\angle abc$, we can easily infer the size of $\angle aoc$. The easiest way to compute the size of $\angle abc$, as shown in the diagram, is to first find the lengths of ad and db, and use their inverse tangent to acquire the degree, in that way, the starting velocities v_1 and v_2 will be cancelled out and hopefully we will be left with some kind of constant.

$$ad = \sqrt{m_2}v_2' - \sqrt{m_2}v_2 = \sqrt{m_2}\frac{2m_1(v_1 - v_2)}{m_1 + m_2} \tag{3}$$

$$db = \sqrt{m_1}v_1 - \sqrt{m_1}v_1' = \sqrt{m_1}\frac{2m_2(v_1 - v_2)}{m_1 + m_2} \tag{4}$$

Then we have

$$\tan(\theta) = \frac{ad}{db} = \frac{\sqrt{m_2}m_1}{\sqrt{m_1}m_2} = \frac{\sqrt{m_1}}{\sqrt{m_2}}$$

and hence

$$\theta = \tan^{-1}(\frac{\sqrt{m_1}}{\sqrt{m_2}}).$$

With this result, we know that the arc ac always corresponds to a constant angle $\phi = 2\theta = 2\tan^{-1}(\frac{\sqrt{m_1}}{\sqrt{m_2}})$ regardless of the starting velocities of each collision. We call this measurement ϕ the angular procession of each collision cycle.

Starting from the initial state, the collisions will come to a termination when the following two conditions are met: 1. Both blocks have a positive velocity (travelling to the positive infinity) 2. v_2 is larger than v_1 With these conditions, we can determine the region where the block collisions are going to terminate on the phase diagram:

$$v_1, v_2 \geq 0 \Rightarrow \sigma_1, \sigma_2 \geq 0$$

$$v_2 > v_1 \Rightarrow \frac{\sigma_2}{\sqrt{m_2}} > \frac{\sigma_1}{\sqrt{m_1}} \Rightarrow \sigma_1 > \frac{\sqrt{m_1}}{\sqrt{m_2}}\sigma_2,$$

with the combination of which we can locate the orange region in diagram 3, knowing that the last collision has to occur in this region.

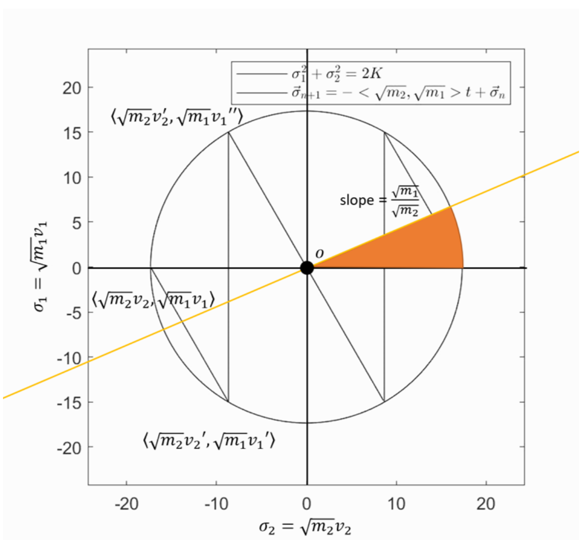


FIG. 3. Termination Region

In other words, when the collisions terminates, the total angular measurement Φ of our collision cycle has to has to land itself between the upper bound of the termination region and its lower bound, which happens to be the σ_2 axis. To find the angular measurement of the upper bound, we can once again take the inverse tangent of its slope and subtract it from π (since the procession occurs in a clockwise direction), giving us:

$$\pi - \tan^{-1}(\frac{\sqrt{m_1}}{\sqrt{m_2}}) \leq \Phi \leq \pi.$$

The Φ here, which is the sum of all the individual angular processions, can be expressed by the total number of collision cycles n_{cycle} times the angular procession of each collision cycle ϕ, that is:

$$\pi - \tan^{-1}(\frac{\sqrt{m_1}}{\sqrt{m_2}}) \leq n_{cycle} \cdot \phi \leq \pi.$$

And because there are two collisions during each collision cycle, the total collision count n_{col} can be obtained by multiplying the total collision cycle count by two, hence giving: $n_{col} = 2n_{cycle}$. Substituting n_{col} into the current inequality yields:

$$\pi - \tan^{-1}(\frac{\sqrt{m_1}}{\sqrt{m_2}}) \leq \frac{n_{col}}{2} \cdot \phi \leq \pi$$

$$\Rightarrow \pi - \tan^{-1}(\frac{\sqrt{m_1}}{\sqrt{m_2}}) \leq n_{col} \cdot \tan^{-1}(\frac{\sqrt{m_1}}{\sqrt{m_2}}) \leq \pi,$$

yielding the range of the collision count:

$$\frac{\pi}{\tan^{-1}(\frac{\sqrt{m_1}}{\sqrt{m_2}})} - 1 \leq n_{col} \leq \frac{\pi}{\tan^{-1}(\frac{\sqrt{m_1}}{\sqrt{m_2}})}$$

Now since $\frac{x}{\tan^{-1}(\frac{x}{)}}$ approaches 1 when x approaches 0, and m_2 is several magnitude larger than m_1 in general, we can substitute $\frac{\sqrt{m_1}}{\sqrt{m_2}}$ for $\tan^{-1}(\frac{\sqrt{m_1}}{\sqrt{m_2}})$ and hence resulting in

$$\frac{\pi}{\frac{\sqrt{m_1}}{\sqrt{m_2}}} - 1 \leq n_{col} \leq \frac{\pi}{\frac{\sqrt{m_1}}{\sqrt{m_2}}}$$

Since in our conditions, it is given that $m_2 = m_1 10^{2n}$, the collision count can be reduced to:

$$\pi 10^n - 1 \leq n_{col} \leq \pi 10^n$$

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5. CollisionEmulator

To ultimately comprehend the scenario as we did above required qualitative observations that could inspire the analysis. Thus, a simulation application had been created during the study, which we present here as a companion illustration. An addition incorporated in this simulator is the allowance of an arbitrary number of blocks.

This emulator is programmed in *Processing* [2], whose basic structure ordains computation conducted by individual frames and the continuous succession of those frames. Hence, the locations of blocks alter naturally, with each frame, by the effect of their velocities. The blocks reside on an levelled surface, with their motions confined to the horizontal dimension. For purpose of visualization, the mass of a block is exponentially proportional to its side length; or the side length is logarithmically proportional to

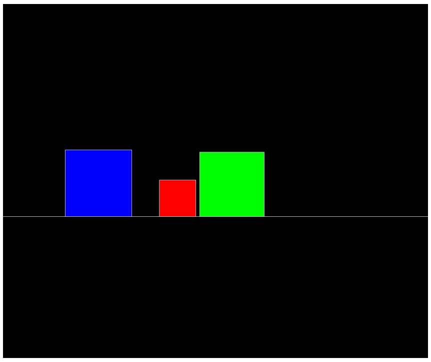

FIG. 4. Simulation Setting

the mass. Upon this setting, along with the initial proportions used in this program for determining proper conversions, great discrepancies in mass may not satisfactorily materialize in considerable difference in size.

The initial conditions of the blocks may be ordained or left to the simulation to randomize. For either case, the total number of blocks in the situation must be indicated prior to execution. Once the simulation is initiated, a collision counter maintains the number of collisions that had occurred.

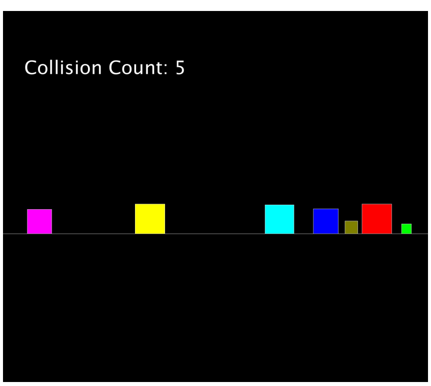

FIG. 5. Collision Counter

A peculiar feature implemented in this application is the concept of *dynamic framing*. One will notice that a simulation of our scenario constitutes two parts: the computation of the motion of the blocks and the rendition of these on the screen. To conduct the former alone, instead of tracking the motion of blocks in minute durations, the critical occurences of their collisions alone need to be considered and specially treated, while intermittence may be duly omitted. This, then, is the idea of *dynamic framing*, of framing with respect to these critical junctures.

Notable imperfections of the simulator includes the gradually offset of position of blocks from the actual flow of time, as, at times, when too many collisions occur, a frame would pause until all collisions are resolved. Furthermore, it remains to be examined when collision of more than two blocks do advent, if the binary collisions of blocks comports with observed behavior and, if so, whether the sequence of collision is irrelevant.

Do consult the source code cited in [3] for greater implementational detail.

6. Conclusion

In total, we have described a quasi-elastic scenario where kinetic energy of blocks, considered as particles, are always conserved and the system's momentum for inter-block collisions. We focused on a two-block situation, presented by the author of [1] and provided a disciplined analysis for a formula of the number of collisions of any two blocks of mass m_1 and m_2 and illustrated upon a mass ratio of 100^n, $n \in \mathbb{N}$, the number of collisions correspond to the integer listings of the first n digits of π, excluding the unit digit 3. A simulation is created for collisions based on the elasticity assumption and performs satisfactorily except for astronomical number of collisions per computation time, in computing the number of collisions advented.

Bibliography

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