A general graph based method for geographical resource optimization

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Contents

Abstract

In this paper, our team presents a global framework for resource optimization in the context of this problem and beyond. We construct a systematic formalism based on graph theory, illustrating the exchange of various resources among residences, dams, water ways, factories as a graph of interacting vertices and edges, embedded in a high dimensional vector bundle of economy, geography, agriculture, hydraulics, and electricity.

We pay thoughtful attention in quantifying all the important resources and their interactions, taking into account the GDP growth, residential happiness, environmental friendliness, etc. Our goal is to maximize the total utility function, which is a functional of the function that characterizes the flow of resources and its integral function over time. We connect this special structure of optimization to the Euler-Lagrangian method in classical theoretical mechanics. By specifying concrete sets of assumptions, we explicitly write out the most general Lagrangian for our dynamical system, and then solve for Euler-Lagrange equations to obtain the optimal solution. Although analytical solutions do not exist for the full Lagrangian, we are able to derive a generic Runge–Kutta method to compute the numerical solutions. The estimated complexity of N nodes is $O(n^4)$.

1 Introduction

The problem of optimal distribution of resources is a strongly interdisciplinary field of research that concerns mathematics, economics, science, environment, and public policy, etc. Due to climate change, the drainage area of the Colorado river basin has caused shortage in water supply upon the five states of US: CO, CA, WY, NM and AZ. The needs of all parties cannot be satisfied simultaneously and thus it is a vital task to propose a flexible and sustainable plan that regulates the distribution of resources and balances supply and demand of this complex system. The need for electricity and the concern for environmental conservation are also at odds with each other.

2 Methodology

The **Lagrangian** is a physical variable that captures the instantaneous action of a dynamical system, $L(\mathbf{q}, \dot{\mathbf{q}}, t)$, where the $\mathbf{q}$ are generalized coordinates and $\dot{\mathbf{q}}$ are the instantaneous rate of change of these coordinates. This formalism can be applied to any complex systems in general, and the **Euler-Lagrange equations**, or the Equations of Motion, yield the complete solution of the system by computing the paths $\mathbf{q}(t)$ that minimizes the total action defined as

$$S = \int_{t_0}^{t_1} L dt.$$

To formalize our problem, we will replace $\mathbf{q}$ with a set of variables involved in our complex system. We will represent residences, factories, dams, water ways, and transportation of resources as a graph of interacting vertices and edges, embedded in a high dimensional vector bundle of economy, geography, agriculture, hydraulics, and electricity. Then we will solve for the Euler-Lagrange equations to find the stationary "path" within our complex system by optimizing the global reward function S. We can guarantee the maximization of this path by designing the utility rate L to be negative definite. The "path", $\mathbf{q}(t)$, in its numerical form, will inform directly the quotas of electricity assigned to the participants in the system, as well as the important metrics such as the

GDP, climate, residential satisfaction, over a time-dependent basis. Policy makers have the freedom to vary the coefficients of the specific metrics that they wish to focus on optimizing. They will also be able to manage the immediacy of their scope on these metrics by multiplying a time-dependent weight function onto the GDP growth of a region over the next decade, for instance.

3 Notations & Terminologies

3.1 Conventions

$\dot{x}$	The time derivative of a function $x(t)$
$G = (\mathbb{V}, \mathbb{E})$	The graph representation of the general settings of our model consisting of vertices and edges
M	The resource manifold, the underlying field that flows between vertices
TM	Vector bundle of the resource manifold, representing flow rates of resources
$(q_{ij}, e_{ij}) \in TM$	Any element of TM correspond to a total flow amount, and the flow rate at that flow amount
$W_{ij} \in C^\infty(TM \times TM, \mathbb{R})$	The flow flow constraint at a particular edge $E_{ij} \in \mathbb{E}$
$m = \{C^\infty(TM, \mathbb{R})\}$	A set of reward density functions contributed from different perspectives, say food supply or GDP
$\rho \in C^\infty(TM, \mathbb{R})$	Total reward density function of a single vertex
$L \in C^\infty\left(TM^{\lvert\mathbb{V}\rvert}, \mathbb{R}\right)$	Global total reward density function
$S \in \mathbb{R}$	Global total reward, also know as the **action**

3.2 Resource Manifold

M	
Water (H)	This is the amount of water being transported, unit in kL
Electricity (Q)	This is the amount of electricity being transmitted, unit in kWh
Products (A)	This is te amount of products produced from the electricities, unit in dollars, can be agricultural, industrial, and so on.

3.3 Notations

- When referring to objects from the resource manifold $q \in M$ or objects from its vector bundle $\chi \in TM$, we use superscripts to specify the type of the entry we want to isolate. In general, we write χ^k or q^k to represent the resource of type k.

- The flows at entries of vertices are represented by $e_{ij} \in TM$, they are the time derivatives of our generalized coordinates q_{ij}. That is $e_{ij} = \dot{q}_{ij}$
- The accumulation rate of flows at vertices are given by $\chi_i = \sum_j e_{ji}$, they are the time derivatives of the total accumulation q_i. That is $q_i = \dot{\chi}_i$.
- **In general, an upper index represents a type of resource of outcome; and a lower index represents the vertex location.**

4 Global Framework

We are constructing a framework for generic resource optimization in the context of this problem and beyond. Graph theory allows us to discretize regions of each state, which is in general a continuous geographical area into a vertex where resources are exchanged and consumed. The edges are set up as flows of resources such as water and electricity between states, which are represented by vertices.

4.1 The Graph Representation

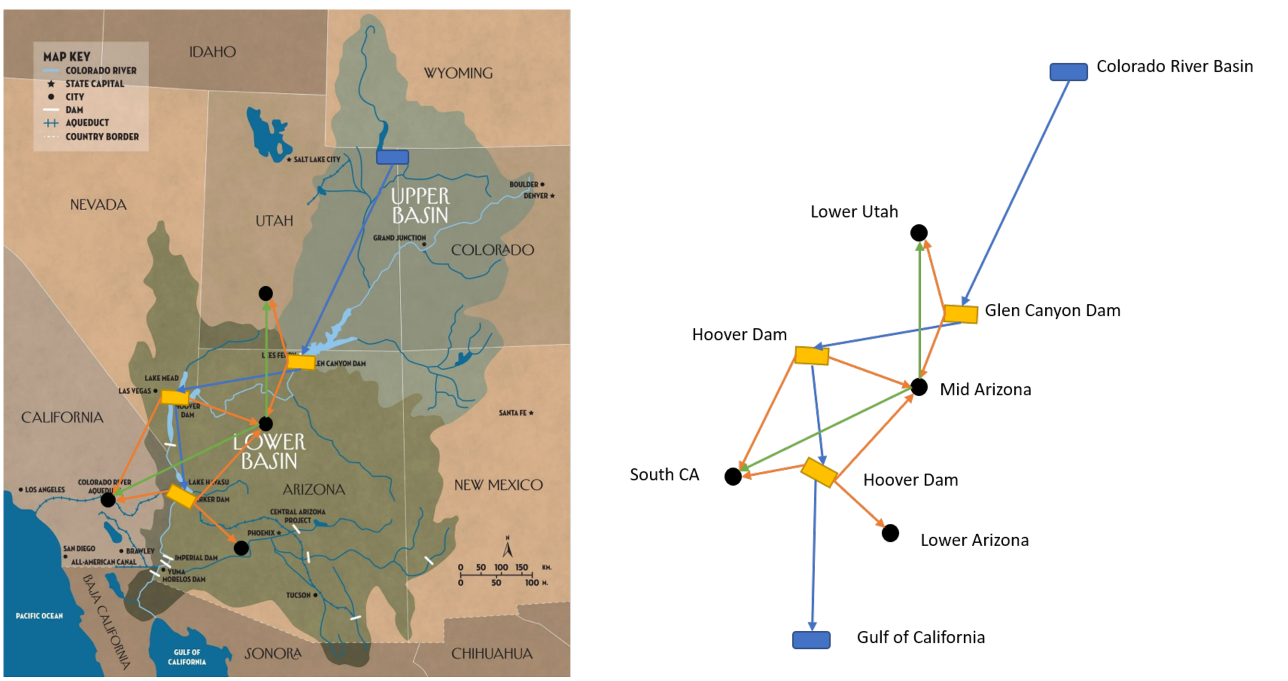

Figure 1: This is the graph[1] representation of the Colorado River system. The black dots represent production sites that consume electricity and water. Blue lines in the example represent waterways. Green lines represent the flow of agricultural products. Orange lines represent the electric transmission lines connecting the dam and the surrounding sites that demand electricity.

As exemplified in figure 1, the Colorado River ecosystem can be captured by vertices representing the participating sites, and edges between them that describe the continuous flow of various resources. This model can be easily generalized to include more residential areas and dams connected by complex electric transmission lines, water bodies interlinked by various waterways, cities linked to agricultural production sites through various transportation routes and so forth.

We will now make another **important abstraction** before getting to the mathematical formulations. Since the essential things that the sites (i.e. the dams, the factories, the residences...) carry out are **taking resources** from other sites, **converting** them into other resources, and **redistributing** them, dams for example take the kinetic energy from the water that goes in and redistribute them as electricity, factories take electricity and turn them into products (measured in $), we will treat all the sites as the same type of object: vertices, and denote them with $V_i \in \mathbb{V}$.

On the other hand, the routes, waterways and transmission lines all serve the same function of **transporting** resources, so we will simply call them edges and denote them by $E_{ij} \in \mathbb{E}$. Since the edges themselves don't carry a sense of direction, $E_{ij} = E_{ji}$.

The collection of **resources**: money, water, potential energy, kinetic energy, electricity, etc, that the vertices take and the edges transport can be represented by a single vector $\mathbf{v}$ in the resource manifold M.

4.2 Interactions at Vertices

The various resources flowing from site j through an edge **into** a site i are denoted by the time dependent function $e_{ji} : \mathbb{R} \to TM$. The amount of resources flowing **out of** the site j into the site i is denoted by e_{ij}. When the edge E_{ij} fully conserves what flows through it, we have the relation that $e_{ij} = -e_{ji}$, more on this later.

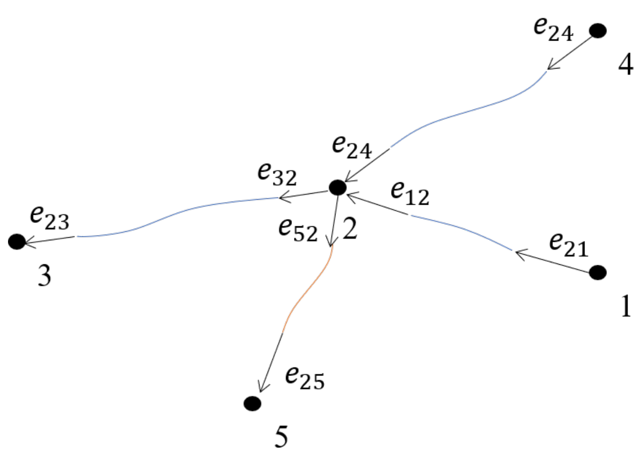

Figure 2: Hydro Plant Example, V_2 represents the dam, V_1 and V_4 are the two upstreams that water flow out from, v_3 is the down stream, and V_5 is a consumption site

4.2.1 Introduction of χ by an example

To formulate the interactions that take place at sites (vertices), we will consider the simplified **example** of a hydro-power station. We will coordinatize $\mathbf{v}$ by: the potential energy contained in the **water** flow u (J), the kinetic energy of the **water** flow k (J), the amount of water in the **water** flow h, measured in tons, and the amount of energy in the **electricity** flow Q (J), so $\mathbf{v} = (u, k, h, Q)$. Now suppose what happens at the site is that kinetic energy in the water are converted into electricity according to thermal efficiency η:

$$\dot{Q}_5 = \eta(\dot{k}_1 + \dot{k}_4 - \dot{k}_3).$$

We account for the negative sign in the equation by taking directions into consideration, so k_{32} represents amount of kinetic energy flowing into V_2, and takes on a negative value when kinetic energies are flowing out, so:

$$-\dot{Q}_{52} = \eta(\dot{k}_{12} + \dot{k}_{42} + \dot{k}_{32}). \tag{1}$$

Next we put the equation in vector form, by introducing an active variable $\dot{k}_2(t)$ that describes the conversion rate of kinetic energy in water into electricity as controlled by the dam:

$$\begin{bmatrix} \dot{k}_2 \\ -\eta\dot{k}_2 \end{bmatrix} = \begin{bmatrix} \dot{k}_{12} + \dot{k}_{42} + \dot{k}_{32} \\ \dot{Q}_{52} \end{bmatrix}. \tag{2}$$

Accounting for conservation of water and assuming that a slight drop in the dam accounts for the conversion of potential energy into the kinetic energy, will get:

$$\begin{bmatrix} -\delta\dot{u} \\ \dot{k}_2 \\ 0 \\ -\eta(\dot{k}_2 + \delta\dot{u}) \end{bmatrix} = \begin{bmatrix} \dot{u}_{12} + \dot{u}_{42} + \dot{u}_{32} \\ \dot{k}_{12} + \dot{k}_{42} + \dot{k}_{32} \\ \dot{h}_{12} + \dot{h}_{42} + \dot{h}_{32} \\ \dot{Q}_{52} \end{bmatrix}. \tag{3}$$

If we have some way of **constraining** the edges, such that electricity cannot flow through waterways E_{12}, E_{42} and E_{32}, and that water cannot flow through electric transmission lines E_{52}, we can guarantee that our optimization result will yield $\dot{u}_{52} = \dot{k}_{52} = \dot{h}_{52} = 0$, and that $\dot{Q}_{12} = \dot{Q}_{42} = \dot{Q}_{32} = 0$. In that case we can safely write:

$$\begin{bmatrix} -\delta\dot{u} \\ \dot{k}_2 \\ 0 \\ -\eta(\dot{k}_2 + \delta\dot{u}) \end{bmatrix} = \sum_{j\neq 2} \begin{bmatrix} \dot{u}_{j2} \\ \dot{k}_{j2} \\ \dot{h}_{j2} \\ \dot{Q}_{j2} \end{bmatrix}. \tag{4}$$

The vector on the left is will be denoted χ_2, and the vectors in the sum on the right are the e_{j2}'s characterizing the flow of resources into vertex 2. So the compact formula is:

$$\chi_2 = \sum_{j\neq 2} e_{j2}. \tag{5}$$

In general, to characterize the interactions at any vertex V_i, we have:

$$\chi_i = \sum_{j\neq i} e_{ji}. \tag{6}$$

We will use superscript $k \in [n]$ to specify the type of flows by isolating an entry:

$$\chi_i^k = \sum_{j} e_{ji}^k. \tag{7}$$

4.3 Flow Constraints on Edges

As have been mentioned, if the amount of resources flowing into and out of an edge E_{ij} are fully conserved, we have the relation that $e_{ij} = -e_{ji}$, or:

$$e_{ij} + e_{ji} = 0. \tag{8}$$

But to account for the possibility of, for instance, loss of energy due to electricity resistance in electric transmission lines, or loss of water volume due to permeation, we may want to introduce a vector of factors $\alpha \in [0, 1]^n$, such that the constraints shall satisfy:

$$e_{ij} + \alpha \cdot e_{ji} = 0, \tag{9}$$

where i is upstream and j downstream, and the resources that flow out an edge into the down stream j now have a slightly reduced quantity given by α^k.

In other cases we might want to specify that an edge is a waterway or an electric transmission line that permits the flow of only related resources, then we may have the constraint for example:

$$e_{ij}^{\text{electricity}} = e_{ji}^{\text{electricity}} = 0.$$

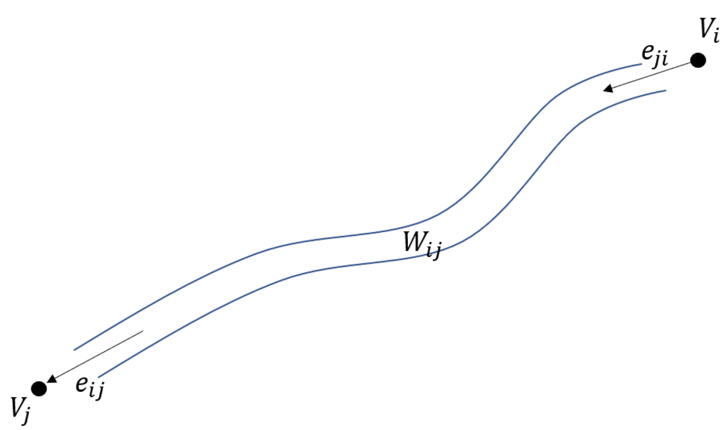

Figure 3: Edge constraint function W_{ij} on E_{ij}

In general, the method of including these conditions into our optimization is by introducing a **negative definite constraint function** $W_{ij} : TM^2 \to \mathbb{R}$ that corresponds to each edge E_{ij} and acts on its inflow and outflows:

$$W_{ij}(e_{ij}, e_{ji}) \leq 0, \tag{10}$$

that only evaluates to $W(e_{ij}, e_{ji}) = 0$ when all the constraint relations stated above are satisfied. For instances, for the free conserved flow, we will have:

$$W_{ij} = -(e_{ij} + e_{ji}) \cdot (e_{ij} + e_{ji}), \tag{11}$$

to prevent the set of unsupported resources $K_{ij} \subset \mathbb{N}$ from going through, we have:

$$W_{ij} = -(e_{ij} + e_{ji})^2 - \sum_{k \in K_{ij}} (e_{ij}^k)^2. \tag{12}$$

Noting the symmetry in these quadratic forms, we have the relation that $W_{ij} = W_{ji}$, and we shall let $W_{ii} = 0$ as they don't post any valid constraints. To capture the constraints on all $E \in \mathbb{E}$ of the graph, we can introduce a single negative-definite term $W : TM^{|E|} \to \mathbb{R}$ by:

$$W = \sum_{i<j} W_{ij}. \tag{13}$$

When $W = 0$, the constraints of the whole graph are simultaneously satisfied.

4.4 General Formulation of the Model

Let the utility set be
m = {Productivity,Residential Happiness, Agricultural Output, Environment Conservation}. The resources is a point on the resource manifold $\mathbf{v} \in M$.

Our model is **completely** and **uniquely** represented by the set of coordinates $\{e_{ij} \in TM | i, j \in [n]\}$ and their time integrals, $\{q_{ij} \in M | i, j \in [n]\}$, where e_{ij} is the set of resources (water, electricity) flowing from site j *into* site i.

$$q_{ij}(t) = \int_{t_0}^{t} e_{ij}(t)dt \tag{14}$$

$$e_{ij}(t) = \dot{q}_{ij}(t) \tag{15}$$

We let $e_{ij} = \dot{q}_{ij}$ because e_{ij} represents the flow rate of all the resources, so that q_{ij} represents the total "volume" that have flowed through a port from an inception t_0. Note that e_{ij} might be negative, representing the case where water is leaving from j with destination i. To better illustrate this relation, see figure below.

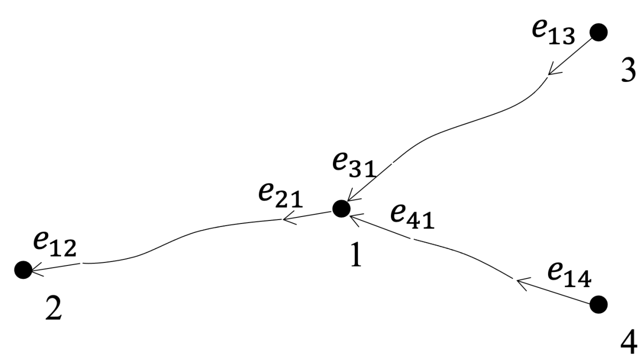

Figure 4: This is the flow relations between these four vertices. Here we may look at vertex 1 when relating the above equation.

The smallest unit we have is a vertex representing a site that needs resources and produces rewards.For a given vertex V_i, the k-th resources flowing into it is calculated simply by

$$\chi_i^k = \sum_j e_{ji}^k. \tag{16}$$

What these resources bring us are the the satisfaction of these resources, reflecting whether they are adequate. If not, the satisfaction goes low. Now we should have the satisfaction factor of the

j-th utility at site i to be m_i^j, which is a function $TM \to [0,1]$ and is dependent on the adequacy of water and electricity. This should be subjective to **the law of diminishing marginal utility**:

$$\begin{cases} \dfrac{\partial^2}{\partial {\chi_i^k}^2} m_i^j(\{\chi_i^k\}) \leq 0; \\ \lim\limits_{\chi \to 0} m_i^j = 0; \\ \left.\dfrac{\partial m_i^j}{\partial \chi_i^{k_0}}\right|_{\chi_i^{k_0} \geq N_i^{k_0}} \approx 0. \end{cases} \tag{17}$$

here N_i^k is the need of the k-th resource at site i, is a parameter in the expression of m_i^j.

Different resources will bring rewards at different efficiencies, and therefore a weight a must be added to each resources, yielding a total reward density function ρ_i at site V_i representing a state (NM, CA, etc.) or a reservoir:

$$\rho_i := \sum_{j \in M} \alpha^j m_i^j(\{\chi_i^k\}), \tag{18}$$

where a^j is the weight of the j-th resource. To acquire the global reward density function, we simply need to sum up the reward density function L at all sites.

$$L_0 = \sum_{V_i \in G} \rho_i(\chi_i, N_i). \tag{19}$$

The meaning of L is the global total reward produced per second, and hence the final total reward S should be simply a time integral:

$$S = \int_{t_0}^{t_1} L_0(\{q_{ij}^k\}, \{e_{ij}^k\}, t) dt. \tag{20}$$

While our aim is to maximize this quantity S, we have not imposed any restriction on this model. For example, the conservation of water and the lost in electricity transportation have not been a restriction in this model. These restrictions defines a parameter manifold on which the quantity S is to be maximized. This manifold can be described as an equation

$$W(\{e_{ij}\}_{i \neq j}) = 0. \tag{21}$$

Then this integral that we are maximizing will become[2]

$$S = \int_{t_0}^{t_1} L_0 + \lambda W(\{e_{ij}\}) dt, \quad \lambda \to \infty. \tag{22}$$

Now we define the **Lagrangian** L as

$$L(\{q_{ij}^k\}, \{e_{ij}^k\}, t) = L_0 + \lambda W(\{e_{ij}^k\}), \quad \lambda \to \infty, \tag{23}$$

and the **action** S as

$$S = \int_{t_0}^{t_1} L(\{q_{ij}^k\}, \{e_{ij}^k\}, t) dt. \tag{24}$$

Maximizing S, the maximizing solutions $e_{ij}^k \in \{e_{ij}^k\}$ will satisfy the **Euler-Lagrange Equations**.

$$\frac{\partial L}{\partial q_{ij}^k} = \frac{d}{dt} \frac{\partial L}{\partial e_{ij}^k}. \tag{25}$$

The solution, after checking the convexity of the critical points, will maximize our action.(see supplementary material)

Note that in reality, there are numerous sites located densely, and so the sum should be treated continuously:

$$L = \int_{\Sigma} \rho d^3\mathbf{r} + \lambda \int_{\Sigma\times\Sigma} W d^3\mathbf{r}_1 d^3\mathbf{r}_2. \tag{26}$$

Here we condensing each region containing to a point, by assigning a delta function $\prod_i \delta(\mathbf{r}-\mathbf{r}(V_i))$ to the position vector $\mathbf{r}$, restoring the last equation.

5 Formulation of the Components in the General Model

To build up this model, we will first introduce several basic presumptions—they will be treated as axioms henceforth.

Assumption I

Electricity is only generated at and distributed from the dam ensembles.

Assumption II

Recall that N_i^j denotes the need of the j-th resource at site i. We have

$$N_\mu^j : N_\nu^j = \text{Size}(\mu) : \text{Size}(\nu). \tag{27}$$

Assumption III

Potential energy due to geographic is the primary source of kinetic motion in water bodies.

Corollaries from Physics: **Water-Energy transformation**

$$\Delta E = \eta(mg\Delta h + \Delta k), \quad \eta \text{ is the efficiency factor.} \tag{28}$$

This factor is approximately 0.9[3]

5.1 Utility Metrics m

The first function with parameters to be determined is the m_i^j, utility function at a state vertex. In section 6.1.3, the utility function at the reservoir vertices. The simplest way to construct the function is to harness the dimensionless variable $\xi^k := \chi^k/N^k \in [0,1]$. (This k is an upper index, instead of power.) Then we may define (see supplementary material)

$$m^j = \left(\sqrt{\xi^1} + \frac{1-\sqrt{\xi^1}}{1+e^{\beta(1-\xi^1)}}\right)\left(\sqrt{\xi^2} + \frac{1-\sqrt{\xi^2}}{1+e^{\beta(1-\xi^2)}}\right), \quad \beta \text{ is large, say} = 100, \tag{29}$$

and here index 1 represents water while 2 hydropower electricity.

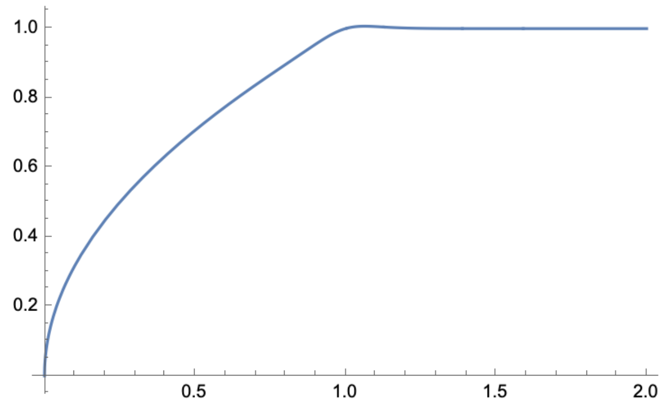

Figure 5: This is the graph of the function $m^j(\xi^k)$, here β is chosen to be 20.

For different types of utility(GDP, Residential satisfaction,etc.), we have different needs and so the utility function m will differ. However, their basic structures are in the form given in Eq.29.

Using the unit of kL and kWh, the domestic needs of the states are shown in the tables below.

5.1.1 The Residential Satisfaction Metric m^{Resident}

To acquire the needs of different utilities, we have found the following information:

1.Hydropower represents about 17% (International Energy Agency) of total electricity production.[4]

2.According to the U.S. Energy Information Administration, the average U.S. home(3.15 person) uses 893 kilowatt-hours (kWh) of electricity per month.[5]

3.Every 1 million US people uses 325 Billion gals of water per year[6]

Now, we may come up with the following chart, representing the residential needs of the states.

States	Water Need (kL/sec)	Electricity Need (kWh/sec)
AZ	293	145
NM	83	41
WY	23	11
CO	231	114
CA	1538	760

Table 1: The residential needs of the states is shown in this table.

5.1.2 The GDP Metric m^{GDP}

In the forms provided by the Bureau of Economic Analysis, we may find the GDP portion of each state.[7] Then, combining with the total use of electricity and water [8-9], we may come up with the following table representing the GDP needs of the states.

States	Water Need (kL/sec)	Electricity Need (kWh/sec)
AZ	11	4
NM	3	1
WY	1	0.5
CO	12	4
CA	95	33

Table 2: The GDP needs of the states is shown in this table.

5.1.3 The Climate Conservation Metric m^{Climate}

The water level in the reservoirs constitutes an absolutely crucial aspect of environmental conservation of the Colorado River region[10]. Observing from the maps (figure 6), large reservoirs are formed in proximity of the dams since the amount of water entering the reservoir exceeds amount that are permitted to exit by the damn. If the water levels in these reservoirs grow to be too large, rapid evaporation of water will take place and cause environmental detriments. On the other hand, if water levels fall below a certain height, residents upstream might experience shortage of water and electricity. When water level of a reservoir or lake goes to 0, that means dysfunction of the natural environment.

Figure 6: Ths figure shows the Hoover dam (circled in gold) and the reservoir formed behind it. (From Google Map.)

To fully model the situation, we will suppose that the **ideal** water level q_i of a vertex V_i is q_a, measured in volumes (kL). We will let the metric reach its maximum 1 when its water level is at exactly q_a. When water level drops to 0, we will let the metric to also drop to 0. It's natural to propose here that the metric is a parabolic function of the water level q_i. Then

$$m_i^{\text{Climate}} = \frac{q_i}{q_a}\left(2 - \frac{q_i}{q_a}\right), \tag{30}$$

which also goes to 0 when q_i go beyond $2q_a$.

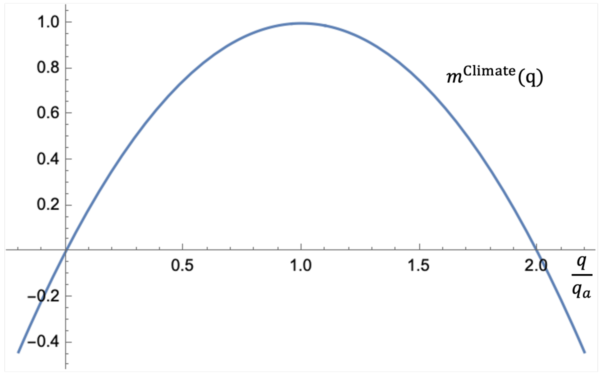

Figure 7: This is the graph of the climate utility function: $m(q_i) = \xi(2-\xi)$, for $\xi = q/q_a$.

Note that this metric only applies to vertices that correspond to dams or natural lakes and reservoirs. To write the water level q_i explicitly, suppose the current water level is q_0 measured in kL, then since the accumulation rate of water at vertex V_i is given by

$$\chi_i^{\text{water}} = \sum_j e_{ji}^{\text{water}}. \tag{31}$$

The total water level is then given by:

$$q_i = q_0 + \int_{t_0}^{t} \chi_i^{\text{water}} dt = q_0 + \int_{t_0}^{t} \sum_j e_{ji}^{\text{water}} dt = q_0 + \sum_j q_{ji}^{\text{water}}. \tag{32}$$

5.2 The constraints

5.2.1 Time-dependent Inflow of Water

The inflow of the Colorado river is almost completely provided by rains. The precipitation graph is shown in Fig.8. Then at each site i, we can have the inflow of water into the river from the chart above, denote that number q_i^{rain}. Then we have

$$\oint_{t\in\text{year}} \frac{d\chi_i^{\text{water}}}{dt} dt = q_i^{\text{rain}}. \tag{33}$$

However, we need to find $d\chi_i^{\text{water}}/dt$ contributed by the rain. To achieve this, we have to find the precipitation function as a function of time, as shown in Fig.9. We denote this function $f(t)$. Then

$$\frac{d\chi_i^{\text{water}}}{dt} = q_i^{\text{rain}} \frac{f(t)}{\oint_{t'\in\text{year}} f(t')dt'}. \tag{34}$$

Basin or Sub-basin (gage)	Natural Streamflow (maf)	Proportion of Colorado River at Imperial Runoff (%)	Precipitation (maf)	Runoff Efficiency (%)
Green River (nr Green River, UT)	5.4	34%	92 maf Upper Basin Total	16%
Colorado River (nr Cisco, UT)	6.8	42%		
San Juan River (nr Bluff, UT)	2.1	13%		
Total Upper Basin (Colorado River at Lees Ferry)	**14.8**	**92%**		
Inflows between Powell and Mead	0.8	5%	78 maf Lower Basin Total (includes Gila River Basin)	3%
Inflows between Mead and Imperial Dam	0.4	3%		
Total inflows between Powell and Imperial Dam	1.3	8%		
Total Colorado River above Imperial Dam	**16.1**	**100%**		
Gila River (nr Dome, AZ at mouth)	1.1			
Total Colorado River at Yuma, AZ	**17.2**		**170 maf**	**10%**

Figure 8: This is the yearly precipitation graph of the Colorado river.[12]

Figure 9: This is the precipitation function of time t, denoted as $f(t)$. [13]

5.2.2 Electric transmission line constraint function W_Q

According to NRDC, "the U.S grid loses about 5 percent of electricity generated in transmission and distribution" [11]. Thus we let the transmission factor of electric energy be $\alpha = 0.95$. Since unlike water, we cannot guarantee the flow direction of electricity in power lines, the design of our relations must be direction dependent. That is when $e_{ij} < 0$, $e_{ji} > 0$, such that energy is flowing from V_j into V_i through electricity,

$$\alpha e_{ij}^{\text{electricity}} + e_{ji}^{\text{electricity}} = 0.$$

Then when $e_{ij} > 0$ and $e_{ji} < 0$, such that the electric energy is flowing out of V_i,

$$e_{ij}^{\text{electricity}} + \alpha e_{ji}^{\text{electricity}} = 0,$$

where $b \in [0, \infty]$ determines how fast the transition of slope takes place. We will construct the relation $f : e_{ij}^{\text{electricity}} = f(e_{ji}^{\text{electricity}})$ by linearly combinations two functions $g(x)$ that has slope 0 when $x < 0$ and quickly change to slope 1 when $x > 0$. A good candidate for g will be

$$g(x) = \frac{\sqrt{x^2 + \frac{1}{b}} + x}{2}.$$

Then we have equation of f:

$$f(x) = \alpha g(-x) - \frac{g(x)}{\alpha},$$

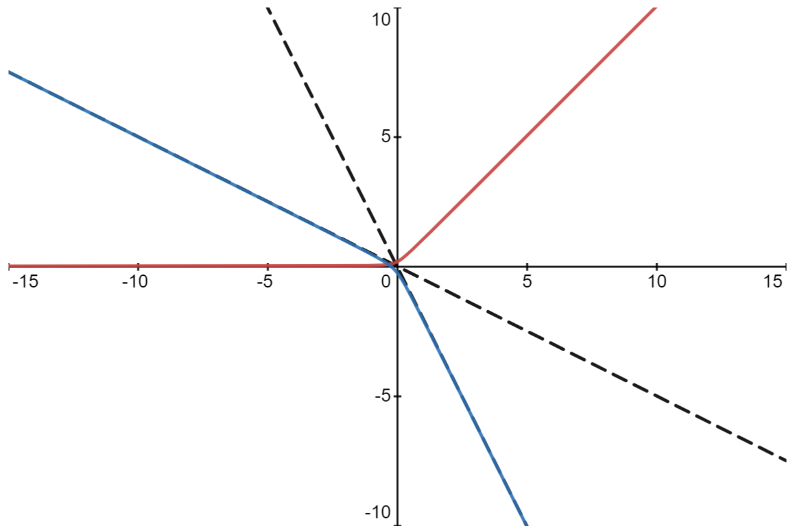

Figure 10: $f(x)$ in blue and $g(x)$ in red with $\alpha = 0.5$ and $b = 10$

and the edge constraint is defined by:

$$e_{ij}^{\text{electricity}} = f(e_{ji}^{\text{electricity}}) = \alpha g(-e_{ji}^{\text{electricity}}) - \frac{g(e_{ji}^{\text{electricity}})}{\alpha} \tag{35}$$

$$\Rightarrow e_{ij}^{\text{electricity}} + \frac{g(e_{ji}^{\text{electricity}})}{\alpha} - \alpha g(-e_{ji}^{\text{electricity}}) = 0 \tag{36}$$

$$\Rightarrow W_Q = -(e_{ij}^{\text{electricity}} + \frac{g(e_{ji}^{\text{electricity}})}{\alpha} - \alpha g(-e_{ji}^{\text{electricity}}))^2 - (e_{ji}^{\text{water}})^2 - (e_{ji}^{\text{product}})^2. \tag{37}$$

W_Q is indeed concave (negative definite) upon examination.

5.2.3 Waterway constraint function W_H

The loss of water volume is due to evaporation, erosion, permeation and other turbulent environmental factors during the transmission of water in between various sites. Constraining the unrelated resources, we arrive at:

$$W_H = -(e_{ij}^{\text{water}} - e_{ji}^{\text{water}})^2 - e_{ji}^{\text{electricity}2} - e_{ji}^{\text{product}2}. \tag{38}$$

5.2.4 Interactions at Damsχ

At a dam, the water flowing out creates electricity, and so a relation is formed. The amount of water flowing out from a dam with vertex representation j is (note that here ρ_{water} is the density of water, instead of the reward function.)

$$-\sum_{i:e_{ij}^{\text{water}}<0} e_{ij}^{\text{water}} \Rightarrow \text{Mass of the water is } \rho_{\text{water}}\left(\sum_{i:e_{ij}^{\text{water}}<0} e_{ij}^{\text{water}}\right). \tag{39}$$

Define a sign function $S(x)$:

$$S(x) := \begin{cases} 1, & \text{if } x \geq 0. \\ 0, & \text{if } x < 0. \end{cases} \tag{40}$$

Then the total electricity produced is

$$-\sum_i e_{ij}^{\text{electricity}} = -\eta\rho_{\text{water}} g \Delta h \sum_i e_{ij}^{\text{water}} S(-e_{ij}^{\text{water}}). \tag{41}$$

However, the height Δh is not just a constant. Instead, it depends on the current amount of water stored in the reservoir: the greater the amount is, the greater the height difference will be. This height is simply

$$\Delta h = q_i^{\text{water}}/A, \tag{42}$$

where A is the mean area of the reservoir.

6 Solving the Model

The time dependent variables that are subject to variational principles are $q_{ij}(t)$, the total flows at each ports of the vertices, their time derivatives are $e_{ij}(t)$. The general form of our lagrangian is:

$$L(\{q_{ij}^k\}, \{e_{ij}^k\}, t) = L_0 + \lambda W(\{e_{ij}^k\}), \quad \lambda \to \infty, \tag{43}$$

With

$$\begin{aligned} L_0 &= \sum_{V_i \in G} \rho_i(q_i, \chi_i, N_i) && (44) \\ &= \sum_{V_i \in G} \rho_i\left(\sum_j q_{ji}, \sum_j e_{ji}, N_i\right). && (45) \end{aligned}$$

The Euler-Lagrange of each entry of the resources in our generalized coordinates yield:

$$\frac{d}{dt}\frac{\partial L}{\partial e_{ij}^k} = \frac{\partial L}{\partial q_{ij}^k}. \tag{46}$$

For the sake of computation, we use the following *simplified* model of interstate water, dam, electricity interactions:

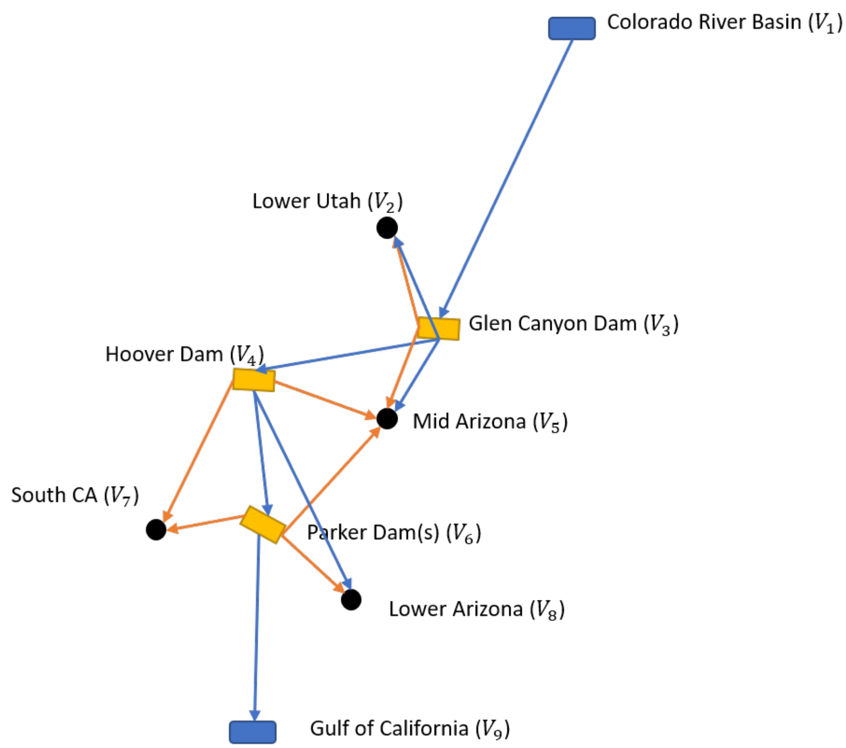

Figure 11: This shows the structure of our graph $G = (\mathbb{V}, \mathbb{E})$.

6.1 The Simulation

We use Java to recreate object representations of the graph and the simulation.

- Each ones of the vertices contain a double function that describe uniquely its set of metric given by m_i^j. Further, the vertices also store the set of parameters that the m_i^j's depend upon, as given in table 1 and 2 of section 6.1. The vertices belong to the class Vertex. They also store the set of edges that they are connected to.
- The edges belong to the class Edge, and each of them store their corresponding edge constraint function W_{ij}. The also store the two vertices that they are connected to.
- The generalized coordinates are stored independently in a class called Simulation, along with all the edges and vertices. Lambda λ is set to be a particularly large number.
- To evaluate the Lagrangian, the coordinates q_{ij} and e_{ij} are passed into the m_i^j of each of the vertices and each of the constraint functions W_{ij}.

For vertex V_1, we will parameterize its annual flow rate (section 6.2.1) by:

$$e_{ij}^{\text{water}}(t) = 93.4454 - 30\cos(t/(2*365*24*3600))(kL). \tag{47}$$

6.2 The Numerical Integration

Without fancy Hamiltonian formulation that will require us to investigate the specific terms in our Lagrangian and re-express it in terms of the momentum coordinates, we will attempt to derive a generic Runge-Kutta method for solving the set of Euler-Lagrange equations.

Suppose that there are n vertices in our system, then the total (maximum) number of edges is equal to $|\mathbb{E}| = \frac{n(n-1)}{2}$. The number of q_{ij}^k coordinates are in turn $n(n-1)$, so are their time derivatives e_{ij}^k. The resource manifold itself is 3 dimensional. We will refer to the collection of all the q_{ij}^k's as $\mathbf{q}$, and all that of e as $\mathbf{e}$, both are $m = 3n(n-1)$ dimensional vectors. L is explicitly expressed in terms of $\mathbf{e}$ and $\mathbf{q}$. So the partial derivatives of L with its dependent coordinates are also explicitly given at $O(m)$ computational complexity.

Starting with the basic Euler-Lagrange, we have:

$$\begin{aligned}
&\frac{d}{dt}\frac{\partial L}{\partial e_{ij}^k} = \frac{\partial L}{\partial q_{ij}^k} \\
\Rightarrow\ &\frac{\partial \frac{\partial L}{\partial e_{ij}^k}}{\partial \mathbf{e}} \cdot \frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t} + \frac{\partial \frac{\partial L}{\partial e_{ij}^k}}{\partial \mathbf{q}} \cdot \frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} = \frac{\partial L}{\partial q_{ij}^k} \\
\Rightarrow\ &\frac{\partial \frac{\partial L}{\partial e_{ij}^k}}{\partial \mathbf{e}} \cdot \frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t} = \frac{\partial L}{\partial q_{ij}^k} - \frac{\partial \frac{\partial L}{\partial e_{ij}^k}}{\partial \mathbf{q}} \cdot \frac{\mathrm{d}\mathbf{q}}{\mathrm{d}t} \\
\Rightarrow\ &\frac{\partial^2 L}{\partial \mathbf{e}^2}\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t} = \frac{\partial L}{\partial \mathbf{q}} - \frac{\partial^2 L}{\partial \mathbf{q}^2}\mathbf{e}
\end{aligned}$$

Noting the $m \times m$ linear operators $\frac{\partial^2 L}{\partial \mathbf{e}^2}$ and $-\frac{\partial^2 L}{\partial \mathbf{q}^2}$ are fully explicit when $\mathbf{e}$ and $\mathbf{q}$ are given, we have:

$$\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t} = \left(\frac{\partial^2 L}{\partial \mathbf{e}^2}\right)^{-1}\left(\frac{\partial L}{\partial \mathbf{q}} - \frac{\partial^2 L}{\partial \mathbf{q}^2}\mathbf{e}\right). \tag{48}$$

Now that the change in $\mathbf{e}$ are explicitly given in terms of itself and its integral, we can use Runge-Kutta to make steps at t_0 $t_0 + \Delta t/2$, $t_0 + \Delta t/2$, and $t_0 + \Delta t$ with weights $1/6$, $1/3$, $1/3$, $1/6$ to obtain the results. The time complexity of each step is m^2 due to the need to take inverse of matrices, so the total time complexity is, if there are C time steps in the simulation, $O(Cm^2) = O(C(n-1)^2n^2)$.

7 Strength and Weakness

The Strength of our model is that it is of extremely high generality. We started off without inputting any relevant information about the Colorado river basin. After that, more detailed assumptions and constraints come into place that solidify our general model. This model applies not only to our problem, but also to all dynamical systems.

The weakness of our model is that if we want the solutions to be precise, solving such a large set of PDEs can be time costly. Therefore due to constraints of the competition, we don't have the time to implement the complete model in computer simulation. In other words, the theories are complete, but it will simply require some extra time to replicate the model and run the computation, which should by itself take about 30 minutes for a 5 year simulation.

8 Supplementary materials

If we have

$$S = \int_a^b L(q(t), q'(t), t)dt, \tag{49}$$

then the solutions that extremizes S is subjected to the Euler Lagrange equations

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial q'}. \tag{50}$$

Now we define an arbitrary non constant smooth function η such that $\eta(a) = \eta(b) = 0$. In our model, $q(t) = q_{ij}^k(t)$ and $q_{ij}^k(t)' = e_{ij}^k$, yielding

$$\frac{\partial L}{\partial q_{ij}^k} = \frac{d}{dt}\frac{\partial L}{\partial e_{ij}^k}. \tag{51}$$

If

$$\left.\frac{d^2}{d\epsilon^2}\right|_{\epsilon=0} S = \int_a^b \left.\frac{d^2}{d\epsilon^2}\right|_{\epsilon=0} L(q(t) + \epsilon\eta, q'(t) + \epsilon\eta', t)dt < 0, \tag{52}$$

the solution is maximizing. In our Lagrangian model, we have devised each function to be concave, so that checking this will not be necessary.

To derive the utility function m at a state vertex, we need this function to be concave, so

$$m(x) = \begin{cases} \sqrt{x}, & x < 1. \\ 1 & , x \geq 1. \end{cases} = \sqrt{x} + (1 - \sqrt{x})\mathbf{1}_{x \geq 1}. \tag{53}$$

One simple model for $\mathbf{1}_{x\geq 1}$ to be differentiable is given below:

$$\mathbf{1}_{x\geq 1} \approx \frac{1}{1 + e^{-\beta(1-x)}}, \quad \beta \text{ large}. \tag{54}$$

9 References

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