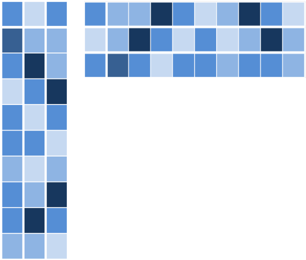
Tuning-Free Convex Methods for Noisy Matrix Completion

Yuepeng Yang

joint work with Cong Ma

Noisy low-rank matrix completion

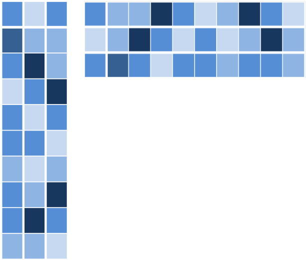


unknown rank-r matrix $\boldsymbol{L}^\star \in \mathbb{R}^{n\times n}$

$$\begin{bmatrix} \checkmark & ? & ? & ? & \checkmark & ? \\ ? & ? & \checkmark & \checkmark & ? & ? \\ \checkmark & ? & ? & \checkmark & ? & ? \\ ? & ? & \checkmark & ? & ? & \checkmark \\ \checkmark & ? & ? & ? & ? & ? \\ ? & \checkmark & ? & ? & \checkmark & ? \\ ? & ? & \checkmark & \checkmark & ? & ? \end{bmatrix}$$

sampling set Ω

Noisy low-rank matrix completion



unknown rank-r matrix $\boldsymbol{L}^\star \in \mathbb{R}^{n\times n}$

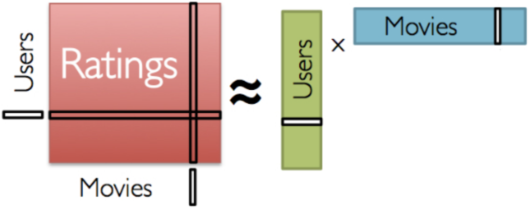
$$\begin{bmatrix} \checkmark & ? & ? & ? & \checkmark & ? \\ ? & ? & \checkmark & \checkmark & ? & ? \\ \checkmark & ? & ? & \checkmark & ? & ? \\ ? & ? & \checkmark & ? & ? & \checkmark \\ \checkmark & ? & ? & ? & ? & ? \\ ? & \checkmark & ? & ? & \checkmark & ? \\ ? & ? & \checkmark & \checkmark & ? & ? \end{bmatrix}$$

sampling set Ω

observations: $M_{i,j} = L^\star_{i,j} + \text{noise}, \quad (i,j) \in \Omega$

goal: estimate $\boldsymbol{L}^\star$

One application: Netflix challenge



- Netflix challenge: Netflix provides highly incomplete ratings from nearly 0.5 million users & 20k movies
- How to predict unseen user ratings for movies?

Convex relaxation for matrix completion

observations: $\quad M_{i,j} = L^{\star}_{i,j} + E_{i,j}, \quad (i,j) \in \Omega$

goal: $\quad$ estimate $\boldsymbol{L}^{\star}$

Convex relaxation for matrix completion

observations: $M_{i,j} = L^{\star}_{i,j} + E_{i,j}, \quad (i,j) \in \Omega$

goal: estimate $\boldsymbol{L}^{\star}$

convex relaxation:

$$\boldsymbol{L}_{\mathsf{cvx}} \quad := \quad \underset{\boldsymbol{L} \in \mathbb{R}^{n \times n}}{\text{argmin}} \quad \underbrace{\sum_{(i,j)\in\Omega} \left(L_{i,j} - M^{\star}_{i,j}\right)^2}_{\text{squared loss}} + \lambda \|\boldsymbol{L}\|_*$$

— $\|\boldsymbol{L}\|_* = \sum_{i=1}^{n} \sigma_i(\boldsymbol{L})$

Known statistical guarantees

- **random sampling:** each $(i,j) \in \Omega$ indep. with prob. p
- **random noise:** i.i.d. sub-Gaussian noise with variance proxy σ^2
- true matrix $\boldsymbol{L}^\star \in \mathbb{R}^{n\times n}$: $r = O(1)$, well-conditioned, incoherent

Setting $\lambda \asymp \sigma\sqrt{np}$ yields minimax optimal estimation rate [CM: Add references]

Issue: tuning parameter λ requires knowledge of both σ and p

A solution: square root MC

— borrowing ideas from square root Lasso

$$\boldsymbol{L}_{\mathsf{cvx}} \quad := \quad \underset{\boldsymbol{L}\in\mathbb{R}^{n\times n}}{\text{argmin}} \quad \underbrace{\sum_{(i,j)\in\Omega} \left(L_{i,j} - M_{i,j}\right)^2}_{\text{squared loss}} + \lambda \|\boldsymbol{L}\|_*$$

$$\downarrow$$

$$\boldsymbol{L}_{\mathsf{cvx}} \quad := \quad \underset{\boldsymbol{L}\in\mathbb{R}^{n\times n}}{\text{argmin}} \quad \underbrace{\sqrt{\sum_{(i,j)\in\Omega} \left(L_{i,j} - M_{i,j}\right)^2}}_{\text{sqaure root squared loss}} + \lambda \|\boldsymbol{L}\|_*$$

The intuition

λ is often chosen based on size of sub-gradient:

$$\frac{\partial}{\partial \boldsymbol{L}} \left(\sum_{(i,j)\in\Omega} \left(L_{i,j} - M_{i,j}\right)^2 \right) = 2\mathcal{P}_\Omega(\boldsymbol{L} - \boldsymbol{M}) \tag{1}$$

$$\frac{\partial}{\partial \boldsymbol{L}} \sqrt{\sum_{(i,j)\in\Omega} \left(L_{i,j} - M_{i,j}\right)^2} = \frac{\mathcal{P}_\Omega(\boldsymbol{L} - \boldsymbol{M})}{\|\mathcal{P}_\Omega(\boldsymbol{L} - \boldsymbol{M})\|_{\mathrm{F}}} \tag{2}$$

When $\boldsymbol{L} = \boldsymbol{L}^\star$, (1) is σ-dependent, (2) is not!

Prior works

— for $\|\boldsymbol{L} - \boldsymbol{L}^{\star}\|_{\mathrm{F}}$, ignoring log factors

[CM: add line between rows]

Minimax limit	$O(\sigma\sqrt{n/p})$
Gaïffas and Klopp '17	$O(\max\{\sigma, \|\boldsymbol{L}\|_{\infty}\}\sqrt{n/p})$
Zhang, Yan, and Wright '21	$O(\sigma n^2)$

Main results for $r, \kappa = O(1)$

- **Random sampling**: Each (i,j) observed with prob $p \gtrsim \frac{\log^3 n}{n}$
- **Random noise**: Sub-Gaussian noise with sd $\sigma \lesssim \sqrt{\frac{np}{\log n}} \|\boldsymbol{L}^\star\|_\infty$
- **Regularity condition**: $\boldsymbol{L}^\star$ is incoherent and well-conditioned

$$\boldsymbol{L}_{\mathsf{cvx}} := \underset{\boldsymbol{L} \in \mathbb{R}^{n\times n}}{\mathrm{argmin}} \quad \sqrt{\sum_{(i,j)\in\Omega} \left(L_{i,j} - M_{i,j}\right)^2} + \lambda \|\boldsymbol{L}\|_*,$$

Main results for $r, \kappa = O(1)$

- **Random sampling**: Each (i, j) observed with prob $p \gtrsim \frac{\log^3 n}{n}$
- **Random noise**: Sub-Gaussian noise with sd $\sigma \lesssim \sqrt{\frac{np}{\log n}} \|\boldsymbol{L}^\star\|_\infty$
- **Regularity condition**: $\boldsymbol{L}^\star$ is incoherent and well-conditioned

$$\boldsymbol{L}_{\mathsf{cvx}} := \underset{\boldsymbol{L} \in \mathbb{R}^{n \times n}}{\mathsf{argmin}} \quad \sqrt{\sum_{(i,j) \in \Omega} \left(L_{i,j} - M_{i,j}\right)^2} + \lambda \|\boldsymbol{L}\|_*,$$

Theorem 1 (Yang and Ma, 2022)

Set $\lambda = 32/\sqrt{n}$. With high probability, $\boldsymbol{L}_{\mathrm{cvx}}$ achieves

$$\|\boldsymbol{L}_{\mathrm{cvx}} - \boldsymbol{L}^\star\|_{\mathrm{F}} \lesssim \sigma \sqrt{\frac{n}{p}}$$

Implications

- Minimax optimal for a wide range of noise sizes $\sigma \lesssim \sqrt{\frac{np}{\log n}} \|\boldsymbol{L}^{\star}\|_{\infty}$
- Improves the error bound in XXX from $O(\sigma n^2)$ to the $O(\sigma\sqrt{n/p})$
- A byproduct of our analysis: $\boldsymbol{L}_{\mathrm{cvx}}$ is nearly rank-r

A peek at analysis

[CM: An ugly page...Needs to be improved] —follows general roadmap of bridging convex and nonconvex analysis [CM: Add some citations]

- Sqrt-MC $\boldsymbol{L}_{\mathsf{cvx}} := \underset{\boldsymbol{L}\in\mathbb{R}^{n\times n}}{\operatorname{argmin}} \sqrt{\sum_{(i,j)\in\Omega}\left(L_{i,j}-M_{i,j}\right)^2} + \lambda\|\boldsymbol{L}\|_*$ is hard to analyze due to non-smoothness
- "Translate" Sqrt-MC into smooth but nonconvex counterpart

$$\underset{\boldsymbol{X},\boldsymbol{Y}\in\mathbb{R}^{n\times r}}{\operatorname{argmin}} \frac{\sum_{(i,j)\in\Omega}\left([\boldsymbol{X}\boldsymbol{Y}^\top]_{i,j}-M_{i,j}\right)^2}{\theta} + \theta + \lambda(\|\boldsymbol{X}\|_{\mathrm{F}}^2 + \|\boldsymbol{Y}\|_{\mathrm{F}}^2)$$

- Suffices to show (1) solution $(\boldsymbol{X}_{\mathsf{ncvx}}, \boldsymbol{Y}_{\mathsf{ncvx}})$ to nonconvex is close to $\boldsymbol{L}_{\mathsf{cvx}}$, and (2) $(\boldsymbol{X}_{\mathsf{ncvx}}, \boldsymbol{Y}_{\mathsf{ncvx}})$ is optimal

Conclusion and discussion

- We sharpen analysis of Sqrt-MC, a tuning-free convex scheme for noisy matrix completion
- Analysis is based on a nonconvex proxy that is close to both convex solution and ground truth
- Future directions: inference on $\boldsymbol{L}^\star$

Reference

Yuxin Chen, Yuejie Chi, Jianqing Fan, Cong Ma, and Yuling Yan.
Noisy matrix completion: Understanding statistical guarantees for convex relaxation via nonconvex optimization.
SIAM journal on optimization, 30(4):3098–3121, 2020.

Stéphane Gaïffas and Olga Klopp.
High dimensional matrix estimation with unknown variance of the noise.
Statistica Sinica, pages 115–145, 2017.

Junhui Zhang, Jingkai Yan, and John Wright.
Square root principal component pursuit: Tuning-free noisy robust matrix recovery.
Advances in Neural Information Processing Systems, 34, 2021.