Mathematical Modeling of Perceptual Loss in Image Reconstruction

1 Introduction

In the process of image reconstruction, perceptual loss metrics, such as the Learned Perceptual Image Patch Similarity (LPIPS) score, are widely employed to evaluate the perceptual similarity between the original and reconstructed images. To visualize comparative performance, Figure 1 presents the relationship between the sliding average of the reconstructed area share (which correlates with the compression ratio) and the LPIPS score.

When no prompt is provided as guidance in the model input, the LPIPS curve exhibits a pattern akin to geometric Brownian motion, suggesting that the perceptual loss behaves stochastically rather than following a strictly deterministic trend.

2 Mathematical Modeling of Perceptual Loss

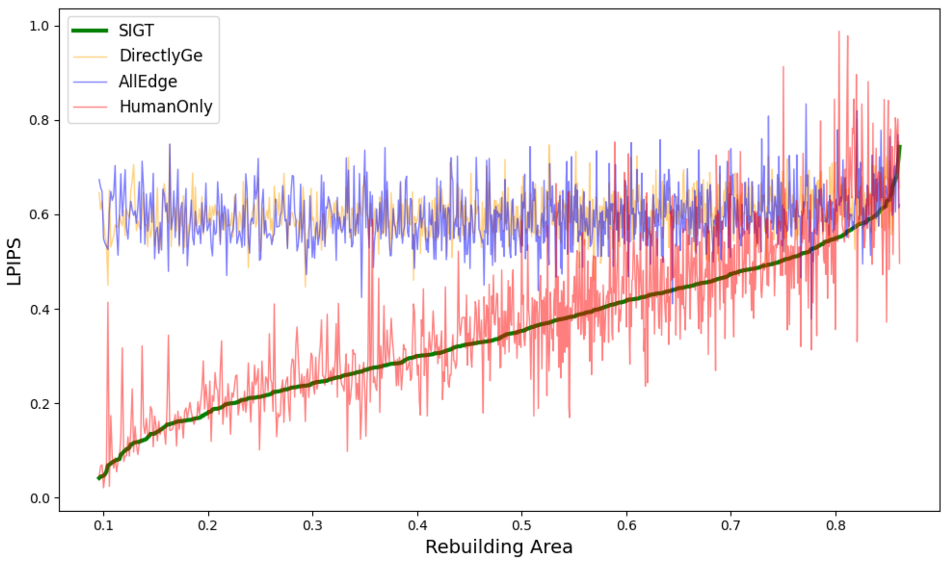

Figure 1: LPIPS Performance

We now present a stochastic differential equation (SDE) to model the evolution of the perceptual loss $D_{\bar{s}}$ as a function of the moving average of the reconstructed area $\bar{S}$. This formulation incorporates deterministic trends, intrinsic randomness, as well as adjustments due to prompts and reference graphs.

2.1 Formulation of the Stochastic Differential Equation

We propose the following SDE for the perceptual loss:

$$\begin{aligned} dD_{\bar{s}} &= \mu D_{\bar{s}}\, d\bar{S} + \sigma D_{\bar{s}}\, dW(\bar{S}) + \eta D_{\bar{s}}\, dW(\bar{S}) - \xi D_{\bar{s}}\, dW(\bar{S}) \\ &= \mu D_{\bar{s}}\, d\bar{S} + (\sigma + \eta - \xi) D_{\bar{s}}\, dW(\bar{S}), \end{aligned} \tag{1}$$

where:

- μ: The deterministic trend term reflecting the model's inpainting ability.
- σ: Intrinsic randomness of the inpainting process without guidance.
- η: Additional randomness introduced by a reference graph.
- ξ: The reduction of randomness attributed to the prompt.
- $\bar{S}$: The moving average of the reconstructed area share.
- $dW(\bar{S})$: Increment of a standard Brownian motion with respect to $\bar{S}$.

The term $\mu D_{\bar{s}}\, d\bar{S}$ captures how perceptual loss evolves deterministically as the reconstructed area increases. The remaining part $(\sigma + \eta - \xi) D_{\bar{s}}\, dW(\bar{S})$ models the stochastic component of the perceptual loss, incorporating both uncontrollable intrinsic randomness and controllable factors (prompts) that reduce volatility.

2.2 Logarithmic Transformation

To solve or analyze this SDE, it is often convenient to consider the logarithm of $D_{\bar{s}}$. Let:

$$Y_{\bar{s}} = \ln(D_{\bar{s}}).$$

Applying Itô's formula for the logarithm of a stochastic process, we start from:

$$dY_{\bar{s}} = \frac{dD_{\bar{s}}}{D_{\bar{s}}} - \frac{1}{2} \frac{(dD_{\bar{s}})^2}{D_{\bar{s}}^2}.$$

From Eq. (1):

$$dD_{\bar{s}} = \mu D_{\bar{s}}\, d\bar{S} + (\sigma + \eta - \xi) D_{\bar{s}}\, dW(\bar{S}).$$

Dividing by $D_{\bar{s}}$:

$$\frac{dD_{\bar{s}}}{D_{\bar{s}}} = \mu\, d\bar{S} + (\sigma + \eta - \xi)\, dW(\bar{S}).$$

The second-order term for Itô's lemma:

$$(dD_{\bar{s}})^2 = [(\sigma + \eta - \xi)D_{\bar{s}}]^2 (dW(\bar{S}))^2.$$

Since $(dW(\bar{S}))^2 = d\bar{S}$, we have:

$$(dD_{\bar{s}})^2 = (\sigma + \eta - \xi)^2 D_{\bar{s}}^2 d\bar{S}.$$

Thus:

$$\frac{(dD_{\bar{s}})^2}{D_{\bar{s}}^2} = (\sigma + \eta - \xi)^2 d\bar{S}.$$

Substitute these into the logarithmic differential:

$$\begin{aligned} dY_{\bar{s}} &= \left(\mu\, d\bar{S} + (\sigma + \eta - \xi)\, dW(\bar{S})\right) - \frac{1}{2}(\sigma + \eta - \xi)^2 d\bar{S} \\ &= \left(\mu - \tfrac{1}{2}(\sigma + \eta - \xi)^2\right) d\bar{S} + (\sigma + \eta - \xi)\, dW(\bar{S}). \end{aligned} \tag{2}$$

2.3 Expectation of the Logarithmic Process

Taking the expectation on both sides of Eq. (2), and noting that the expectation of the stochastic integral term $\int_0^{\bar{S}} (\sigma + \eta - \xi)\, dW(\bar{S})$ is zero, we have:

$$E[dY_{\bar{s}}] = \left(\mu - \tfrac{1}{2}(\sigma + \eta - \xi)^2\right) d\bar{S}.$$

Integrating from 0 to $\bar{S}$:

$$E[Y_{\bar{S}}] = E[Y_0] + \left(\mu - \tfrac{1}{2}(\sigma + \eta - \xi)^2\right) \bar{S}.$$

Since $Y_{\bar{S}} = \ln(D_{\bar{s}})$ and assuming an initial condition $D_0 = D_{\bar{s}=0}$, we have $Y_0 = \ln(D_0)$. Thus:

$$E[\ln(D_{\bar{s}})] = \ln(D_0) + \mu\bar{S} - \tfrac{1}{2}(\sigma + \eta - \xi)^2 \bar{S}.$$

3 Conclusion

The above derivation provides a framework for modeling and understanding the perceptual loss during image reconstruction. By treating the perceptual loss as a stochastic process influenced by deterministic trends, intrinsic and extrinsic randomness, and prompt-based corrections, this model offers insights into controlling and predicting the behavior of advanced inpainting and restoration techniques.