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CS188: Introduction to Robotics

Worksheet 1 Solutions

Problem 1: DH Parameters

Note that on exams, the axes will be given to you, so the more important part of these types of problems is translating from the given diagram and the axes to the DH parameter table. For this class, we will use the definition in *Modern Robotics*, which has been reproduced below:

- a_{i-1} (link length): distance along the common normal from z_{i-1} to z_i.
- α_{i-1} (link twist): twist angle from z_{i-1} to z_i, measured about x_{i-1}.
- d_i (link offset): displacement along z_i; distance from the intersection of x_{i-1} and z_i to the origin of the link-i frame.
- θ_i (joint angle): the angle from x_{i-1} to x_i, measured about the z_i-axis.

We have assigned frames for the robot manipulator from question 5C) on problem set 1. Furthermore, in this diagram, the slider bar is parallel to the ground and $\theta_1 = 0, \theta_2 = 90$. There are many other ways to assign frames which will affect the final table. Using the following diagram, please fill in the DH table.

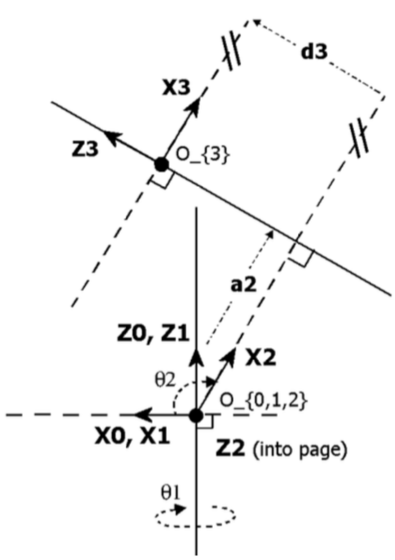

Figure 1: RRP manipulator with frames

i	a_{i-1}	α_{i-1}	d_i	θ_i
1				
2				
3				

Figure 2: DH parameters (fill in!)

Problem 1 Solution

We first look at row 1 of the DH table, meaning we are moving from frames $\{0\} \rightarrow \{1\}$.

- For a_{i-1}, we want the distance from z_0 to z_1 about x_0. Since z_0 and z_1 lie on the same line, there is no displacement along the x-axis, giving us $\mathbf{a_0} = \mathbf{0}$.

- For α_{i-1}, we want the angle between z_0 and z_1 about x_0: said another way, the angle z_0 has to rotate to align with z_1 about the x_0 axis. As they are both in-line with each other, we have $\alpha_\mathbf{0} = \mathbf{0}°$.

- For d_i, we want the distance from x_0 to x_1 measured about z_1. Since the origins of the frames $\{0\}$ and $\{1\}$ coincide along z_1, there is no displacement along z_1 and $\mathbf{d_1} = \mathbf{0}$.

- For θ_i, the angle from x_0 to x_1 about z_1 is θ_1. We do not need to rotate x_0 to align with x_1 about z_1, which is why $\boldsymbol{\theta_1} = \mathbf{0}°$: no rotation is necessary.

Next we look at row 2 of the DH table, meaning we are moving from frames $\{1\} \rightarrow \{2\}$.

- For a_{i-1}, we want the distance from z_1 to z_2 about x_1. Notice that z_2 is pointing into the page at the origin of its frame $O_{\{2\}}$. The common normal between z_1 and z_2 (measured along x_1) is zero length if both axes intersect, so $\mathbf{a_1} = \mathbf{0}$ (see pg. 587 of *Modern Robotics*).

- For α_{i-1}, we want the angle between z_1 and z_2 about x_0. Notice that for the choice of z_2, it is perpendicular to z_1 (z_2 points directly into the page, z_1 is pointing vertically, both share the same origin). Because of that $90°$ change in axis direction, we get $\alpha_\mathbf{1} = \mathbf{90}°$. Note that since a mutually perpendicular line between the joint axes fails to exist, $\alpha_1 = -90°$ is also acceptable (see pg. 587 of *Modern Robotics*).

- There is no distance from the intersection of x_1 and z_2 to the origin of the link-2 frame, since their intersection is the origin. Then, $\mathbf{d_2} = \mathbf{0}$.

- In order to rotate x_1 to align with x_2 about z_2, we would need to rotate by $\boldsymbol{\theta_2} = \mathbf{90}°$.

Finally, we look at row 3 of the DH table, meaning we are moving from frames $\{2\} \rightarrow \{3\}$.

- For a_{i-1}, we want the distance from z_2 to z_3 in the direction of x_2. Since we have z_2 pointing into the origin, and z_3 parallel to the ground pointing left, our mutually perpendicular line is a_2 which is pointing in the direction of x_2. Then, $\mathbf{a_2} = \mathbf{a_2}$.

- For α_{i-1}, we want the angle between z_2 and z_3 about x_2. z_3 is perpendicular to z_2 about x_2, which is a $90°$ twist. This gives $\alpha_\mathbf{2} = \mathbf{90}°$.

- The sliding distance along z_3 is the prismatic variable d_3, so we have $\mathbf{d_3} = \mathbf{d_3}$.

- x_2 and x_3 are parallel to each other, so the angle from x_2 to x_3 about z_3 is $0°$ and $\boldsymbol{\theta_3} = \mathbf{0}°$.

A short note on prismatic and revolute joints:

- d_i values typically exist only for prismatic joints.

- θ_i values typically exist only for revolute joints.

Problem 2: DoF

Three identical SRS open-chain arms are grasping a common object. Find the number of degrees of freedom of this system. You may find the following table useful.

Joint type	dof f	Constraints c between two spatial rigid bodies
Revolute (R)	1	5
Spherical (S)	3	3

Additionally provided to you is *Grübler's formula*.

$$\begin{aligned}\mathrm{DoF} &= m(N-1) - \sum_{i=1}^{J} c_i \\ &= m(N-1) - \sum_{i=1}^{J} (m - f_i) \\ &= m(N-1-J) + \sum_{i=1}^{J} f_i\end{aligned}$$

This exercise is taken from *Modern Robotics*, exercise 2.7. There are more examples and exercises involving calculating degrees of freedom in the textbook.

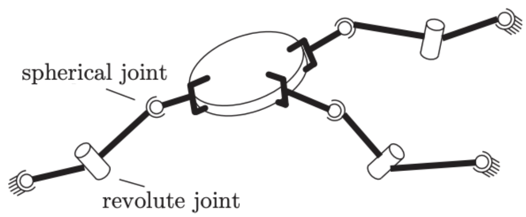

Figure 2.17: Three cooperating SRS arms grasping a common object.

DoF of the system =

Problem 2 Solution

We apply the following version of *Grübler's formula*:

$$m(N-1)-\sum_{i=1}^{J}(m-f_i)$$

Since we have a 3D spatial mechanism, $m = 6$. Each of the three SRS arms has two links, which gives us $2 \times 3 = 6$ links. Add the common grasped object (1 body) and the ground (1 body), giving

$$N = 6 + 1 + 1 = 8$$

Each arm has three joints:

- A spherical joint at the base (f = 3, c = 3).
- A revolute joint in the middle (f = 1, c = 5).
- Another spherical joint at the end-effector (f = 3, c = 3).

Each arm has $3 + 1 + 3 = 7$ internal degrees of freedom. There are 3 joints per arm, and there are 3 arms, giving us a total of 9 joints in the system. Based on the categorization in the bullet points, we have 6 spherical joints whom each impose 3 contraints, and 3 revolute joints, each imposing 5 constraints. This gives us a total of $(6 \times 3) + (3 \times 5) = 33$ total constraints. We can then calculate the degrees of freedom of the system:

$$\boxed{\text{DoF of the system} = 6(8-1) - 33 = 9}$$

Problem 3: Cartesian and Homogeneous Coordinates

You have a point $\mathbf{p} = (8, -2, 5)$ in 3D **Cartesian** coordinates.

1. Convert $\mathbf{p}$ into **homogeneous** coordinates.
2. Suppose instead you are given the same point in homogeneous form $\tilde{p} = (8, -2, 5, w)$. If $w \neq 1$, derive the corresponding Cartesian coordinates in terms of w.

Problem 4: Rotation about an axis

You have a rotation of 90° about the Y-axis in 3D space. Write the standard rotation matrix $\mathbf{R}_y(90°)$ that implements this transformation. Then show what happens to the point $\mathbf{p} = (1, 0, 0)$ under this rotation.

Problem 3 Solution

1. **Homogeneous representation:**

   For a 3D Cartesian point $\mathbf{p} = (x, y, z)$, the homogeneous version is $\tilde{\mathbf{p}} = (x, y, z, 1)$. Hence, $\tilde{\mathbf{p}} = (8, -2, 5, 1)$.

2. **Cartesian from general homogeneous:**

   If $\tilde{\mathbf{p}} = (x, y, z, w)$ and $w \neq 0$, then

   $$\mathbf{p}_{\text{Cartesian}} = \left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}\right).$$

   Thus, if $\tilde{\mathbf{p}} = (8, -2, 5, w)$, the corresponding Cartesian point is

   $$\left(\frac{8}{w}, \frac{-2}{w}, \frac{5}{w}\right).$$

Problem 4 Solution

We can approach this using the right-hand rule. Recall that initially, our thumb points along $+X$ (up), our index finger points along $+Y$ (forward), and our middle finger points along $+Z$ (left). Now, rotate your hand upwards so that the Y axis is pointing upwards. Our thumb still points along $+X$, and is now pointing into ourselves. Our middle finger ($+Z$) still points left, and our index finger ($+Y$) points up. When rotating $90°$ about the Y-axis, you can rotate your right hand in this position $90°$ counterclockwise (when viewed from above) around the Y-axis. Notice that now, our middle finger points into ourself, our thumb points to our right, and our index finger still points up. We can now use this track the basis vector transformations:

1. The $+X$ direction (initially forward), rotates to point in the $-Z$ direction, so $(1, 0, 0) \to (0, 0, -1)$
2. The $+Y$ direction remains unchanged, so $(0, 1, 0)$ remains.
3. The $+Z$ direction (initially left), rotates to point in the $+X$ direction, so $(0, 0, 1) \to (1, 0, 0)$

This gives us

1. Rotation matrix about Y-axis by $90°$:

$$\mathbf{R}_y(90°) = \begin{bmatrix} \cos 90° & 0 & \sin 90° \\ 0 & 1 & 0 \\ -\sin 90° & 0 & \cos 90° \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

2. Transforming $\mathbf{p} = (1, 0, 0)$:

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

So the point $(1, 0, 0)$ is rotated to $(0, 0, -1)$.

Problem 5: Extrinsic Matrix from Camera Pose

A camera in the world frame is located at $\mathbf{C}_W = (2, -3, 5)$. It's **optical axis** (the camera's Z-axis) is aligned so that it points directly **along** the negative Z-direction of the world frame. The camera's Y-axis points along the world's positive Y direction, and the camera's X-axis follows the right-hand rule (it will align with $-X_W$ if Z flips sign and Y remains the same). Construct the **extrinsic matrix** ${}^{C}T_W$ that transforms a point from the **world** frame to the camera's frame.

Hint: first write the rotation that aligns the world axes with the camera axes, then add translation.

Problem 6: Projecting a Point onto an Image Plane

A camera has the following **intrinsic** matrix:

$$K = \begin{bmatrix} 200 & 0 & 100 \\ 0 & 200 & 80 \\ 0 & 0 & 1 \end{bmatrix}$$

A point in camera coordinates is $\mathbf{p}_C = (0.5, -0.2, 1.0)$. Use K to compute the **2D** pixel coordinates (u, v) of this point on the image plane.

Problem 5 Solution

1. **Rotation from world to camera:**

   - The camera's Z_C is aligned with $-Z_W$.
   - The camera's Y_C is aligned with $+Y_W$.
   - The camera's X_C follows from the right-hand rule (it will align with $-X_W$ if Z flips sign and Y remains the same).

   So the rotation from world to camera is:

   $$\mathbf{R}_{CW} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

   (Since $X_W \to -X_C$, $Y_W \to Y_C$, $Z_W \to -Z_C$.)

2. **Translation:** We want a matrix $\mathbf{C}_{TW}$ that, when applied to a world point $\mathbf{p}_W$ in homogeneous form, gives $\mathbf{p}_C$. First, we rotate the point by $\mathbf{R}_{CW}$, then translate so that the camera is at the origin. The translation part is $-\mathbf{R}_{CW}\mathbf{C}_W$.

   $$\mathbf{t}_{CW} = -\mathbf{R}_{CW}\mathbf{C}_W = -\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ -3 \\ 5 \end{bmatrix} = -\begin{bmatrix} -2 \\ -3 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}.$$

3. Our final extrinsic matrix $\mathbf{C}_{TW}$:

   $$\mathbf{C}_{TW} = \begin{bmatrix} -1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Problem 6 Solution

1. Convert to homogeneous image coordinates:

   $$\tilde{\mathbf{u}} = K\,\mathbf{p}_C = \begin{bmatrix} 200 & 0 & 100 \\ 0 & 200 & 80 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0.5 \\ -0.2 \\ 1.0 \end{bmatrix} = \begin{bmatrix} 200 \times 0.5 + 0 + 100 \times 1.0 \\ 0 + 200 \times (-0.2) + 80 \times 1.0 \\ 0 + 0 + 1 \end{bmatrix} = \begin{bmatrix} 100 + 100 \\ -40 + 80 \\ 1 \end{bmatrix} = \begin{bmatrix} 200 \\ 40 \\ 1 \end{bmatrix}.$$

2. Divide by the third coordinate (which is 1):

   $$(u, v) = \left(\frac{200}{1}, \frac{40}{1}\right) = (200, 40).$$