Advanced Communication Systems

ELEN90051 (Lecturer: Margreta Kuijper)

Signal Detector Error Probability Calculations

1st Semester 2018

Material based on Chapter 4 of "Digital Communications" by Proakis & Salehi, 2008, and Chapter 7 of "Communication Systems Engineering" by Proakis & Salehi, 2002

Error Probability per Signaling Scheme

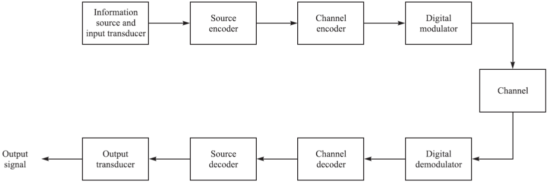


FIGURE 1.1–1
Basic elements of a digital communication system.

- Binary modulation
  - Pulse amplitude modulation
  - Orthogonal signalling
- M-ary modulation
  - Pulse amplitude modulation
  - Phase-shift keying
  - Quadrature amplitude modulation
  - Orthogonal signalling
- Spectral efficiency

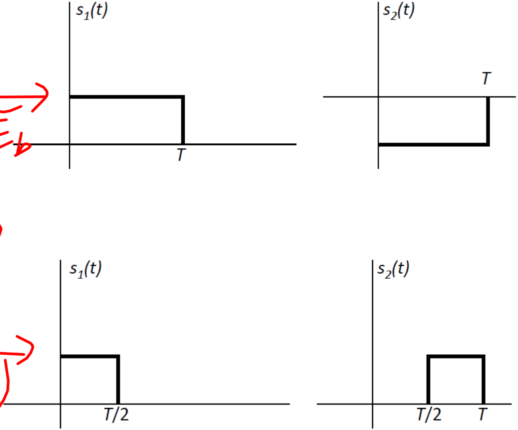
Binary Modulation

Question: Consider binary PAM and binary orthogonal signalling. An example is in the next figure. Assuming equal signal energy over the signaling interval, which one has better error performance?

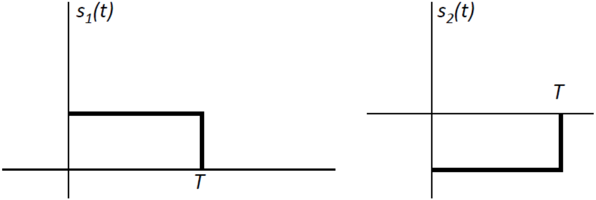
$-\sqrt{E_b}$ $\sqrt{E_b}$

$(0, \sqrt{E_b})$

$(\sqrt{E_b}, 0)$



Binary PAM



- Recall that in binary PAM the antipodal signalling waveforms (see example figure above) are defined as:

$$s_m(t) = A_m g(t) = A_m \sqrt{\mathcal{E}_b}\phi(t)$$

where $A_m = \{1, -1\}$ is the amplitude of the pulse, $\phi(t) = \frac{g(t)}{\sqrt{\mathcal{E}_b}}$ is the unit energy basis function, and the energy in the pulse $g(t)$ is equal to the energy per bit $\mathcal{E}_b$.

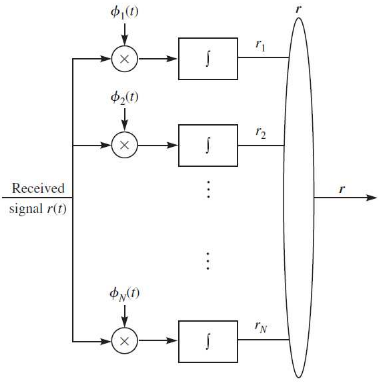
Binary PAM – Signal demodulator

Recall: assuming the signal $\mathbf{s}_1$ is transmitted, the output of the signal demodulator is a random variable

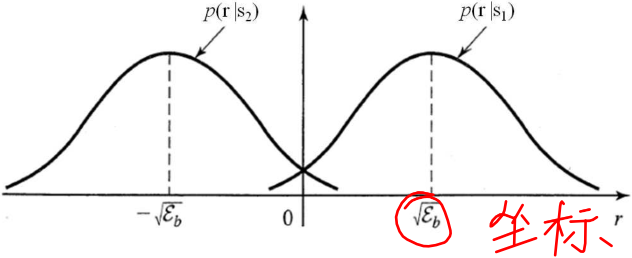
$$\begin{aligned} r &= \langle r(t), \phi(t) \rangle \\ &= \langle \sqrt{\mathcal{E}_b}\phi(t) + n(t), \phi(t) \rangle \\ &= \sqrt{\mathcal{E}_b} \int_0^T \phi^2(t)dt + \int_0^T n(t)\phi(t)dt \\ &= \sqrt{\mathcal{E}_b} + n \end{aligned}$$

where n is the AWGN projected noise rv. From $n \sim \mathcal{N}(0, N_0/2)$ it follows that $r \sim \mathcal{N}(\sqrt{\mathcal{E}_b}, N_0/2)$.

In this figure the PAM signal space dimension should be set to $N = 1$.



Binary PAM – ML detector

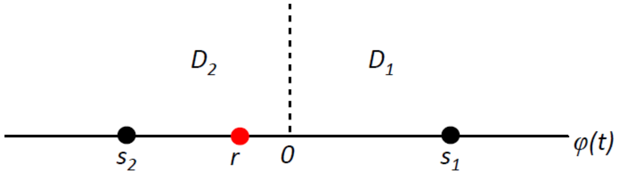


- Recall: the conditional PDF's of the random variable r are

$$p(r|\mathbf{s}_1) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{(r-\sqrt{\mathcal{E}_b})^2}{N_0}\right),$$

and

$$p(r|\mathbf{s}_2) = \frac{1}{\sqrt{\pi N_0}} \exp\left(-\frac{(r+\sqrt{\mathcal{E}_b})^2}{N_0}\right).$$



- Recall: the ML detection criterion is

$$\widehat{m} = \arg\max_{m} p(r|\mathbf{s}_m) = \arg\min_{m} (r - s_m)^2,$$

where $s_1 = \sqrt{\mathcal{E}_b}$, and $s_2 = -\sqrt{\mathcal{E}_b}$ and r is the value of the demodulator output (which is a real number).

- ML detector: If $r > 0$, decide $\mathbf{s}_1$ was transmitted. If $r < 0$, decide $\mathbf{s}_2$ was transmitted

Binary PAM – Probability of Error

$P(\text{error}) = P(s_1) \cdot P(\text{error}|s_1) + P(s_2) \cdot P(e|s_2)$

- Assuming $\mathbf{s}_1$ is transmitted, the probability of error is

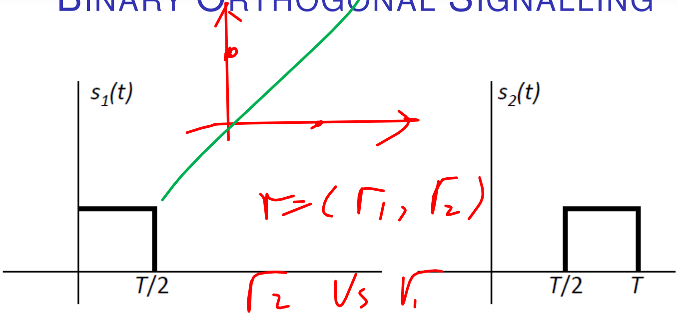
$$
\begin{aligned}
P(e|\mathbf{s}_1) &= P(\text{decide for } \mathbf{s}_2|\mathbf{s}_1 \text{ transmitted}) = \int_{D_2} p(r|\mathbf{s}_1)dr \\
&= P(r < 0|\mathbf{s}_1) \qquad (1)\\
&= P\left(\frac{r-\sqrt{\mathcal{E}_b}}{\sqrt{N_0/2}} < \frac{0-\sqrt{\mathcal{E}_b}}{\sqrt{N_0/2}}\right) = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right),
\end{aligned}
$$

here we used that $P(X < x) = 1 - Q(x) = Q(-x)$ for $X \sim \mathcal{N}(0,1)$.

- Due to symmetry of decision regions, we have $P(e|\mathbf{s}_1) = P(e|\mathbf{s}_2)$.
- Thus, for equiprobable signals, the average bit error probability is

$$
P_b = \frac{1}{2}P(e|\mathbf{s}_1) + \frac{1}{2}P(e|\mathbf{s}_2) = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right).
$$

BINARY ORTHOGONAL SIGNALLING



- Recall that binary orthogonal signalling waveforms (see example figure above) are defined as:

$$s_m(t) = \sqrt{\mathcal{E}_b}\phi_m(t)$$

where $\phi_1(t)$ and $\phi_2(t)$ are orthonormal.

Binary Orthogonal Signalling – Signal Demodulator

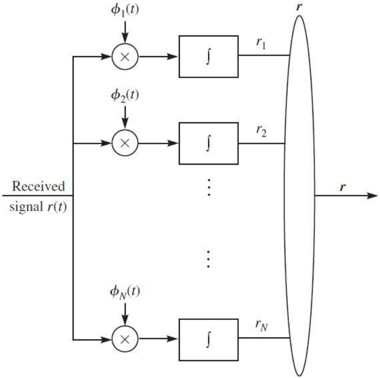
Recall: assuming $\mathbf{s}_2$ is transmitted, the output of the demodulator is the random vector $\mathbf{r}$ with component rv's

$$r_1 = \langle \sqrt{\mathcal{E}_b}\phi_2(t) + n(t), \phi_1(t) \rangle = n_1 \text{ and}$$

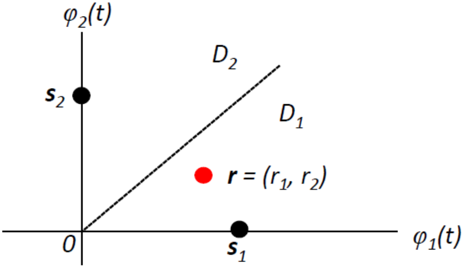
$$\begin{aligned} r_2 &= \langle \sqrt{\mathcal{E}_b}\phi_2(t) + n(t), \phi_2(t) \rangle \\ &= \sqrt{\mathcal{E}_b} + n_2 \end{aligned}$$

where $n_1 \sim \mathcal{N}(0, N_0/2)$ and $n_2 \sim \mathcal{N}(0, N_0/2)$.

In this figure the signal space dimension should be set to $N = 2$.



Binary Orthogonal Signalling – ML Detector



- Recall: the ML detection criterion uses the demodulator output value **r** (which is a 2-dimensional real-valued vector):

$$\widehat{m} = \arg\min_{m} ||\mathbf{r} - \mathbf{s}_m||$$

- Denote the two components of the demodulator output value **r** by r_1 and r_2. Then ML detector rule: if $r_1 > r_2$, decide that $\mathbf{s}_1$ was transmitted. If $r_1 < r_2$, decide that $\mathbf{s}_2$ was transmitted

Binary Orthogonal Signalling – Probability of Error

- Assuming $\mathbf{s}_2$ is transmitted, the probability of error is

$$P(e|\mathbf{s}_2) = P(\text{decide for } \mathbf{s}_1|\mathbf{s}_2 \text{ transmitted}) = \int_{D_1} p(\mathbf{r}|\mathbf{s}_2)dr_1dr_2$$

$$= P(r_1 > r_2|\mathbf{s}_2) = P\left(r_1 - r_2 > 0|\mathbf{s}_2\right) = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right),$$

越大，值越小

Vari 相加

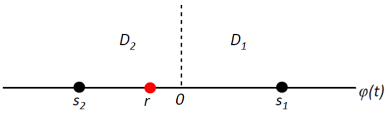
here we used that $r_1 - r_2 \sim \mathcal{N}(-\sqrt{\mathcal{E}_b}, N_0)$. (Why? because the sum of two independent Gaussians $Z = X + Y$ has distribution $Z \sim \mathcal{N}(\mu_X + \mu_Y, \sigma_X^2 + \sigma_Y^2)$ for $X \sim \mathcal{N}(\mu_X, \sigma_X^2)$ and $Y \sim \mathcal{N}(\mu_Y, \sigma_Y^2)$.)

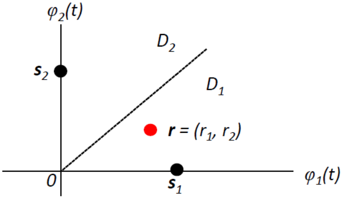
- Thus, for equiprobable signals and due to symmetry, $P_b = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right)$.

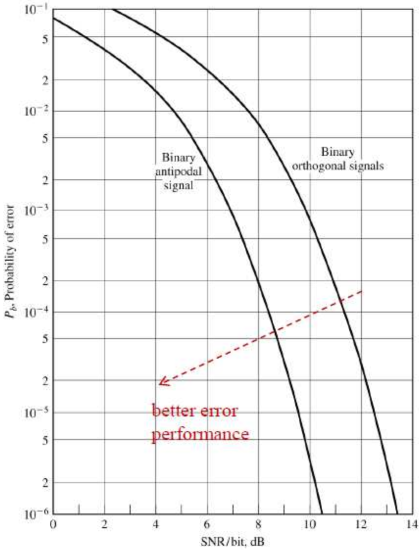
Comparison 2-PAM $\leftrightarrow$ 2-Orthogonal Signaling

Answer to the question on page 3: Binary PAM, $P_b = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$, has better error performance than binary orthogonal signalling, $P_b = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right)$.

- Binary PAM can achieve the same error probability as binary orthogonal signalling with half as much energy.
- Distance between signals in binary PAM is $d_{\min} = 2\sqrt{\mathcal{E}_b}$, whereas in binary orthogonal signalling we have $d_{\min} = \sqrt{2\mathcal{E}_b}$.






10^{-1}
5
2
10^{-2}
5
2
10^{-3}
5
2
10^{-4}
5
2
10^{-5}
5
2
10^{-6}
P_b, Probability of error
0
2
4
6
8
10
12
14
SNR/bit, dB
Binary antipodal signal
Binary orthogonal signals
better error performance

M-ARY PULSE AMPLITUDE MODULATION (M-PAM)

- Recall: **Baseband** M-PAM signal waveforms are represented by
$$s_m(t) = A_m g(t) = A_m \sqrt{\mathcal{E}_g}\phi(t), \quad 1 \le m \le M, \quad 0 \le t \le T$$
where $A_m = (2m - 1 - M)$ is the set of possible amplitudes, $\phi(t) = \frac{g(t)}{\sqrt{\mathcal{E}_g}}$, and the energy in the pulse $g(t)$ is $\mathcal{E}_g$. *Note that here we cannot easily express the signal amplitudes in terms of $\mathcal{E}_b$, so for now we express them in terms of $\mathcal{E}_g$.*
- Minimum distance between constellation points is $d_{\min} = 2\sqrt{\mathcal{E}_g}$

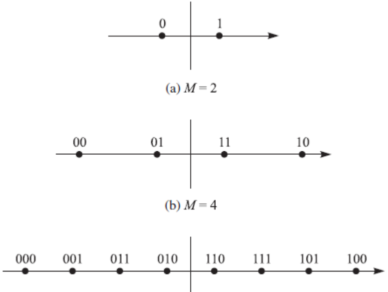


FIGURE 3.2–1
Constellation for PAM signaling.

M-PAM – Demodulation

- Recall that the signal demodulator's output is the random variable

$$
\begin{aligned}
r &= \langle r(t), \phi(t) \rangle \\
&= \langle s_m(t), \phi(t) \rangle + \langle n(t), \phi(t) \rangle \\
&= A_m \sqrt{\mathcal{E}_g} + n \quad \text{therefore } r \sim \mathcal{N}(A_m \sqrt{\mathcal{E}_g}, N_0/2)
\end{aligned}
$$

- as usual, the ML detection criterion uses the value r of the demodulator output: $\widehat{m} = \arg\max_m p(r|\mathbf{s}_m)$ where

$$
p(r|\mathbf{s}_m) = \frac{1}{\sqrt{\pi N_0}} \exp\left(- \frac{(r - A_m \sqrt{\mathcal{E}_g})^2}{N_0} \right).
$$

Important to note: for $M > 2$ some decision regions are larger than others; we need to make a distinction between outer points ($m = 1 - M$ and $m = M - 1$) and inner points.

M-PAM – Probability of Error

- For outer points $m = 1$ and $m = M$, the probability of error is

$$P(e|\mathbf{s}_m) = Q\left(\sqrt{\frac{2E_g}{N_0}}\right)$$

- For inner points $m = 2, \ldots, M-1$, the probability of error is

$$P(e|\mathbf{s}_m) = P(|r - s_m| > \sqrt{\mathcal{E}_g}) = P(|n| > \sqrt{\mathcal{E}_g}) = 2Q\left(\sqrt{\frac{2E_g}{N_0}}\right)$$

- The average symbol error probability is

$$P_e = \frac{1}{M}\left(2Q\left(\sqrt{\frac{2E_g}{N_0}}\right) + (M-2)2Q\left(\sqrt{\frac{2E_g}{N_0}}\right)\right)$$
$$= \frac{2(M-1)}{M} Q\left(\sqrt{\frac{2E_g}{N_0}}\right)$$

← energy of g(t),

M-PAM – Symbol energy vs bit energy

- The average symbol energy for M-PAM is (assuming equiprobable symbols)

$$\begin{aligned}\mathcal{E}_{av} &= \frac{1}{M}\sum_{m=1}^{M} A_m^2 \mathcal{E}_g = \frac{\mathcal{E}_g}{M}\sum_{m=1}^{M}(2m-1-M)^2 \\ &= \frac{\mathcal{E}_g}{M}\frac{M(M^2-1)}{3} = \frac{\mathcal{E}_g(M^2-1)}{3}\end{aligned}$$

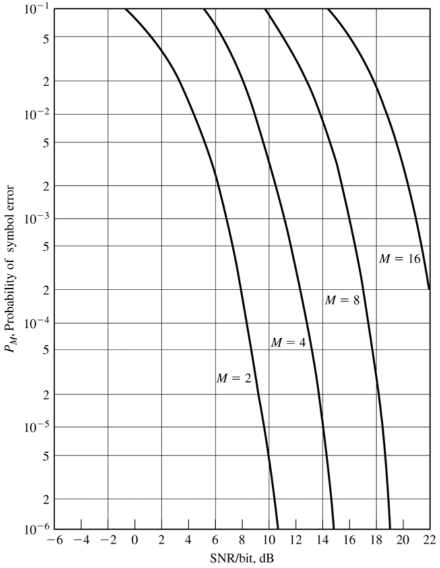
- For comparison with other modulation schemes, it is more useful to consider the average bit energy

$$\mathcal{E}_b := \frac{\mathcal{E}_{av}}{\log_2 M} = \frac{\mathcal{E}_g(M^2-1)}{3\log_2 M}$$

- After substitution, the average symbol error probability is

$$P_e = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6\log_2 M \mathcal{E}_b}{(M^2-1)N_0}}\right)$$

M-PAM ERROR PLOT



Larger M has worse symbol error performance but less bandwidth required per bit.

M-PAM – Bandpass

- Recall: M**-PAM bandpass** signal waveforms are

$$s_m(t) = A_m g(t)\cos(2\pi f_c t) = A_m\sqrt{\frac{\mathcal{E}_g}{2}}\phi(t), \quad 1 \le m \le M, \quad 0 \le t \le T$$

where $A_m = (2m - 1 - M)$ and $\phi(t) = \sqrt{\frac{2}{\mathcal{E}_g}}g(t)\cos(2\pi f_c t)$.

- Minimum distance between constellation points is $d_{\min} = \sqrt{2\mathcal{E}_g}$

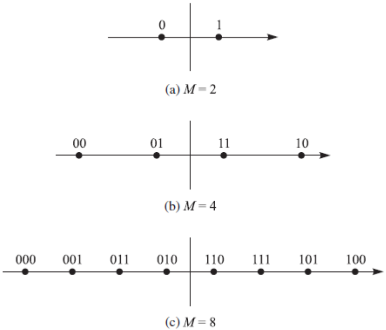


FIGURE 3.2–1
Constellation for PAM signaling.

- Average symbol error probability for M-PAM bandpass is

$$P_e = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{E_g}{N_0}}\right).$$

- Average symbol energy is

$$\mathcal{E}_{av} = \frac{1}{M}\sum_{m=1}^{M} A_m^2 \frac{\mathcal{E}_g}{2} = \frac{\mathcal{E}_g(M^2-1)}{6}$$

and the average bit energy is now $\mathcal{E}_b = \frac{\mathcal{E}_g(M^2-1)}{6\log_2 M}$.

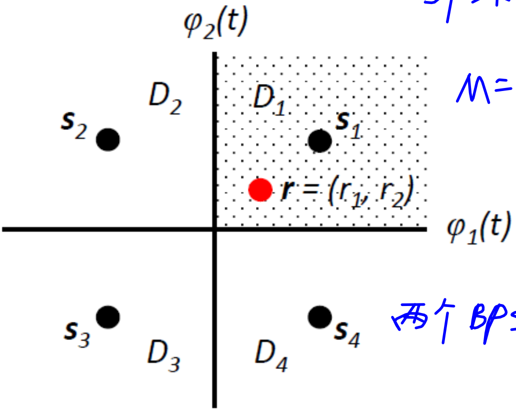
- After substitution, the average symbol error probability is

$$P_e = \frac{2(M-1)}{M} Q\left(\sqrt{\frac{6\log_2 M\mathcal{E}_b}{(M^2-1)N_0}}\right)$$

$\Rightarrow$ Same performance as M-ary PAM baseband!

M-ARY PHASE-SHIFT KEYING (M-PSK)

BPSK → QPSK



M=4

两个 BPSK ⇒ QPSK

M-PSK – Probability of Error

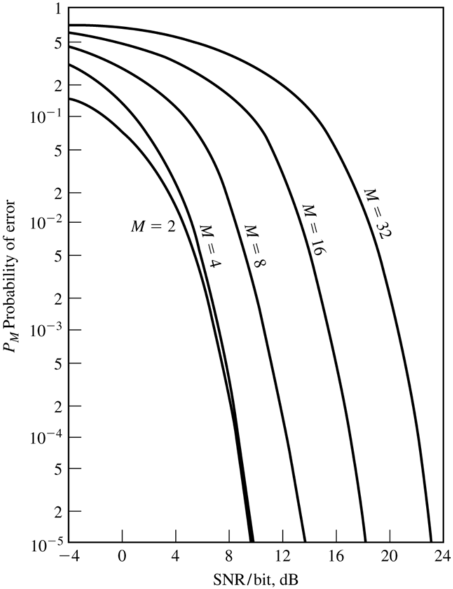
- For binary PSK ($M = 2$), average bit error probability is the same as for binary PAM, i.e., $P_b = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$
- For QPSK ($M = 4$),

$$\begin{aligned} P(\text{correct}|\mathbf{s}_1) &= P(\mathbf{r} \in D_1|\mathbf{s}_1) = P(r_1 > 0, r_2 > 0|\mathbf{s}_1) \\ &= P(r_1 > 0|\mathbf{s}_1)P(r_2 > 0|\mathbf{s}_1) \quad \text{due to independence} \\ &= (1 - P_{\text{BPSK}})^2 \end{aligned}$$

$\Rightarrow$ average symbol error probability is
$P_e = 1 - (1 - P_{\text{BPSK}})^2 = 2Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right) - (Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right))^2.$

- For $M \geq 8$, the symbol error probability calculations need a transformation to polar coordinates and can only be approximated.

M-PSK ERROR PLOT

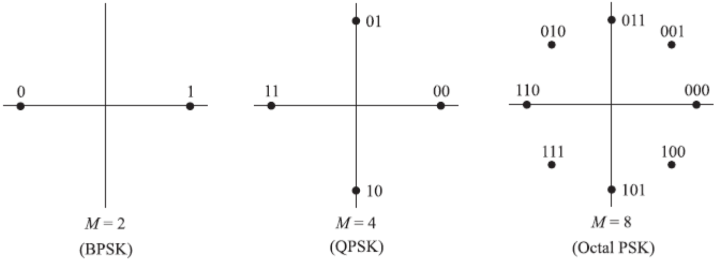


M-PSK – Bit Error Probability

- The average bit error probability is the average proportion of erroneous bits per symbol
- Of course this depends on the mapping from k-bit codeword to symbol m
- For example, for $k = 3$, if the detector mistakes the 5th symbol (100) for the 4th symbol (011), then 3 bits are in error, but if it mistakes it for the 1st symbol (000), then only 1 bit is in error.

Symbol	Bits
1	000
2	001
3	010
4	011
5	100
6	101
7	110
8	111

- At high SNR, most symbol errors involve erroneous detection of the transmitted symbol as nearest neighbour symbol.
- Therefore fewer bit errors occur by ensuring that neighbouring symbols differ by *only 1 bit* ⇒ Gray coding



Let P_b be the average bit error probability, how does it relate to the average symbol error probability P_e?

- At high SNR, P_e can be approximated as the probability of picking a neighbouring signal, therefore, assuming Gray coding,

$$P_b \approx \frac{1}{k} P_e.$$

$\Rightarrow P_b \approx \frac{P_e}{\log_2 M}$ where $k = \log_2 M$.

Differential Phase-Shift Keying (DPSK)

For completeness, we include a figure that compares the theoretical performance of BPSK and DBPSK. As you can see from the figure, DBPSK performs far worse. Why would anyone use it???

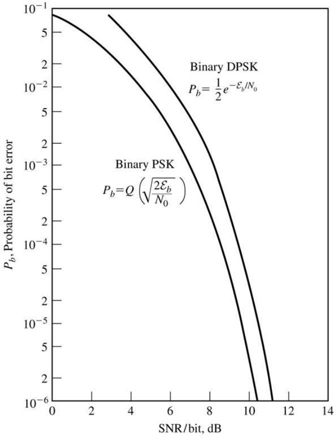


Figure 7.58

Refer to textbook pp.223-224 for further details.

M-ARY QUADRATURE AMPLITUDE MODULATION (M-QAM)

- Recall that there are many different signal constellations possible for QAM
- Example: $M = 4$

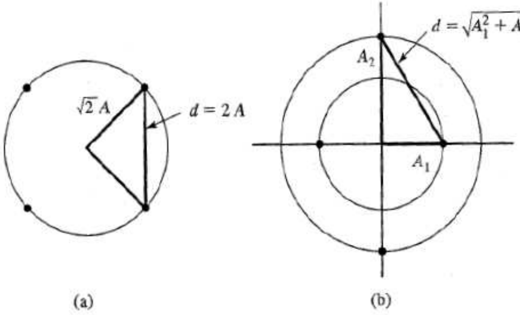
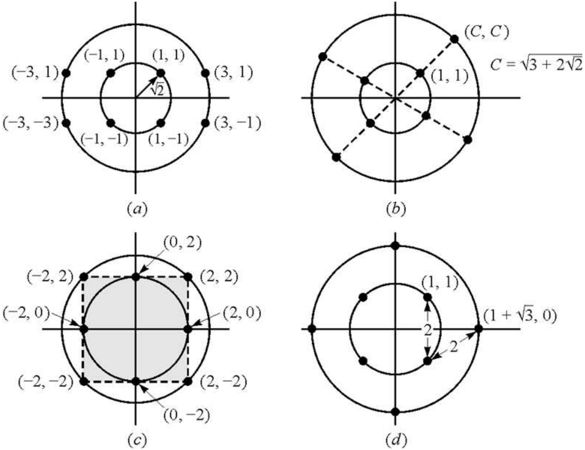


Figure 7.59 Two 4-point signal constellations.

- Example: $M = 8$



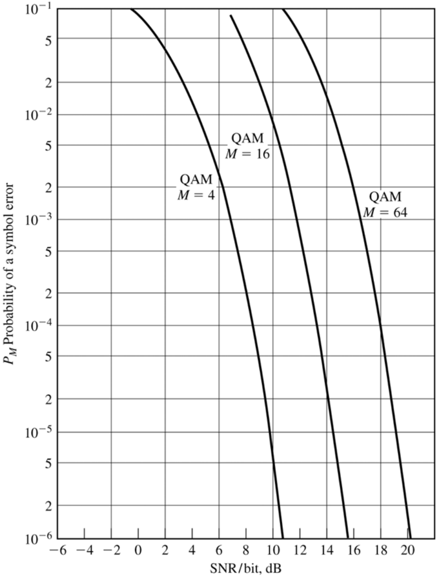
- Symbol error probability is dominated by: 1) minimum distance in signal constellation, and 2) average transmitter power

- Rectangular QAM is most frequently used in practice.
- When $M = 2^k$ with k even, M-ary QAM is equivalent to two $\sqrt{M}$-ary PAM signals on quadrature carriers, with each having half the equivalent QAM power
- If $P_{\sqrt{M}}$ is the probability of error of $\sqrt{M}$-ary PAM, then similar to the case of QPSK, the probability of a correct decision is given by

$$P(\text{no error}) = (1 - P_{\sqrt{M}})^2$$

$\Rightarrow$ average probability of error is $P_e = 1 - (1 - P_{\sqrt{M}})^2$.

M-QAM ERROR PLOT



M-ARY ORTHOGONAL SIGNALLING

- Recall that orthogonal signaling employs the waveforms:

$$s_m(t) = \sqrt{\mathcal{E}_s}\phi_m(t), \quad 1 \leq m \leq M,$$

- Recall the signal demodulator:

$$\begin{aligned} r_k &= \langle r(t), \phi_k(t) \rangle = \langle s_m(t), \phi_k(t) \rangle + \langle n(t), \phi_k(t) \rangle \\ &= \sqrt{\mathcal{E}_s}\delta_{km} + n_k \end{aligned}$$

where

$$\delta_{km} = \begin{cases} 1 & k = m \\ 0 & k \neq m \end{cases}$$

- ML detector rule uses components $r_1, r_2, \ldots, r_M$ of the demodulator output vector value: if $r_m > r_k$ for all $k \neq m$, then decide $\mathbf{s}_m$ was transmitted

P_e for M-ary Orthogonal Signaling

- If $\mathbf{s}_m$ is transmitted, then $r_m \sim \mathcal{N}(\sqrt{\mathcal{E}_s}, N_0/2)$ and $r_k \sim \mathcal{N}(0, N_0/2)$ for all $k \neq m$
- Probability of correct detection conditioned on $\mathbf{s}_m$ and r_m is

$$\begin{aligned} P(\text{no error}|\mathbf{s}_m, r_m) &= P(\max_{k\neq m}\{r_k\} < r_m|\mathbf{s}_m, r_m) \\ &= \prod_{k\neq m} P(r_k < r_m|\mathbf{s}_m, r_m) \quad \text{(independence)} \\ &= \prod_{k\neq m} (1 - P(r_k \geq r_m|\mathbf{s}_m, r_m)) \\ &= \left[1 - Q\left(\frac{r_m}{\sqrt{N_0/2}}\right)\right]^{M-1} \quad \text{(identically distributed).} \end{aligned}$$

Note that this expression involves the value of r_m.

- Probability of correct detection averaged over $r_m \sim \mathcal{N}(\sqrt{\mathcal{E}_s}, N_0/2)$ is

$$P(\text{no error}|\mathbf{s}_m) = \int_{-\infty}^{\infty} P(\text{no error}|\mathbf{s}_m, r_m)p(r_m|\mathbf{s}_m)dr_m$$

$$= \int_{-\infty}^{\infty} \left[1 - Q\left(\frac{r_m}{\sqrt{N_0/2}}\right)\right]^{M-1} \frac{\exp\left(-\frac{(r_m-\sqrt{\mathcal{E}_s})^2}{N_0}\right)}{\sqrt{\pi N_0}} dr_m$$

- Now, we have $P(\text{error}|\mathbf{s}_m) = 1 - P(\text{no error}|\mathbf{s}_m)$. For equiprobable signals, the average symbol error probability is

$$P_e = 1 - \int_{-\infty}^{\infty} \left[1 - Q\left(\frac{r_m}{\sqrt{N_0/2}}\right)\right]^{M-1} \frac{\exp\left(-\frac{(r_m-\sqrt{\mathcal{E}_s})^2}{N_0}\right)}{\sqrt{\pi N_0}} dr_m$$

where the integral can be computed using numerical integration. Later we will derive a bound for this expression—the *Union Bound*.

P_b FOR M-ARY ORTHOGONAL SIGNALING

- Recall that, in orthogonal signalling, all symbols are equidistant neighbours in M-dimensional signal space
- Therefore the transmitted $\mathbf{s}_m$ can be mistaken for another symbol with probability $P_e/(M-1)$
- Denoting the k-bit block that corresponds to the transmitted $\mathbf{s}_m$ by $\mathbf{b}_m$, there are 2^{k-1} symbols whose bit blocks differ from $\mathbf{b}_m$ in the i-th bit. Why? For example for $k = 3$ and symbols $\{000, 001, 010, 011, 100, 101, 110, 111\}$, if the 1st bit of the transmitted symbol is "0", then there are $2^{3-1} = 4$ symbol erroneous choices for the detector that cause a bit error for the 1st bit.
- Therefore the average bit error probability is

$$\begin{aligned}
P_b &= Pr(\text{error in } i\text{-th bit}) && \text{(due to symmetry)} \\
&= 2^{k-1} \times \frac{P_e}{M-1} && (k = \log_2 M) \\
&= \frac{MP_e}{2(M-1)} \approx \frac{P_e}{2} && \text{for } M >> 1
\end{aligned}$$

Orthogonal Signalling error plot

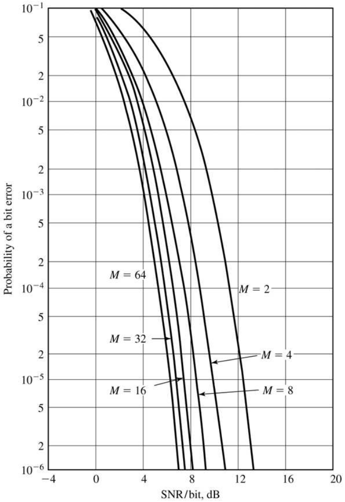
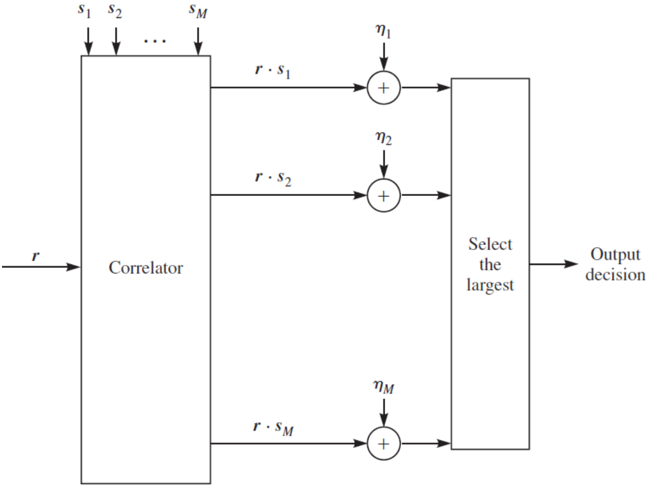


Figure 7.63

Probability of bit error for coherent detection of orthogonal signals.

Larger M has better error performance but more bandwidth required for fixed bit rate $R_b = \frac{k}{T}$

Recall: ML Detection



Bound on P_e for M-ary Orthogonal Signaling

- Recall: assuming $\mathbf{s}_1$ is transmitted, the ML detector (see figure on previous page) consists of M correlators with one output with the signal $\langle \mathbf{r}, \mathbf{s}_1 \rangle$ and $M-1$ outputs $\langle \mathbf{r}, \mathbf{s}_m \rangle$, $m = 2, 3, \ldots, M$. Therefore, an error occurs when any of the $M-1$ outputs $\langle \mathbf{r}, \mathbf{s}_m \rangle > \langle \mathbf{r}, \mathbf{s}_1 \rangle$
- Therefore the symbol error probability P_e is upper bounded by the *union bound* of the $M-1$ events:

$$P_e \leq (M-1)P_b^{bin} < MQ\left(\sqrt{\frac{\mathcal{E}_s}{N_0}}\right) < Me^{-\frac{\mathcal{E}_s}{2N_0}}$$

where $P_b^{bin} = Q\left(\sqrt{\frac{\mathcal{E}_s}{N_0}}\right)$ is the error probability of binary orthogonal signalling. In the last inequality we used the bound $Q(x) < e^{-\frac{x^2}{2}}$.

- Substituting $k = \log_2 M$ and $\mathcal{E}_s = k\mathcal{E}_b$, results in

$$P_e < e^{-k\frac{(\mathcal{E}_b/N_0 - 2\ln 2)}{2}}$$

which means that P_e can be made arbitrarily small (i.e., $P_e \to 0$) as $k \to \infty$, provided that $\mathcal{E}_b/N_0 > 2\ln 2 = 1.39 \sim 1.42$ dB

- A tighter upper bound is given by

$$P_e < 2e^{-k(\sqrt{\mathcal{E}_b/N_0} - \sqrt{\ln 2})^2}$$

which is valid when $\ln 2 \leq \mathcal{E}_b/N_0 \leq 4\ln 2$.

- The minimum SNR/bit of $\mathcal{E}_b/N_0 = \ln 2 = 0.693 \sim -1.6$ dB is called Shannon's channel coding limit for AWGN channels.

Spectral Efficiency

The **spectral efficiency** of a modulation scheme is defined as

$$\nu = \frac{R_b}{W}$$

where $R_b = \frac{k}{T}$ is the bit rate (number of bits $k = \log_2 M$ per symbol interval T) and W is the bandwidth required.

The spectral efficiency is a performance indicator for fundamental comparison of modulation schemes with respect to power and bandwidth usage.

SPECTRAL EFFICIENCY COMPARISON

In the comparison below, it is assumed that a baseband signal pulse $g(t)$ of duration T requires a bandwidth equal to $W = \frac{1}{2T}$.

- Baseband PAM:

$$\nu = \frac{R_b}{1/2T} = 2k = 2\log_2 M \rightarrow \infty \text{ as } M \rightarrow \infty$$

- Bandpass PAM:

$$\nu = \frac{R_b}{1/T} = k = \log_2 M \rightarrow \infty \text{ as } M \rightarrow \infty$$

- M-ary PSK:

$$\nu = \frac{R_b}{1/T} = k = \log_2 M \rightarrow \infty \text{ as } M \rightarrow \infty$$

- M-ary QAM:

$$\nu = \frac{R_b}{1/T} = k = \log_2 M \rightarrow \infty \text{ as } M \rightarrow \infty$$

Spectral Efficiency – Orthogonal Signalling

- Pulse position modulation (PPM): here $g(t)$ is a pulse of duration T/M, so it requires M times as much bandwidth as baseband PAM, thus $W = \frac{M}{2T}$.

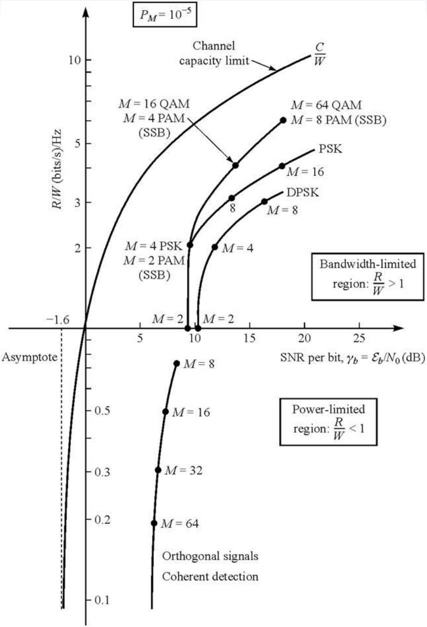
$$\nu = \frac{R_b}{\frac{M}{2T}} = \frac{2k}{M} = \frac{2\log_2 M}{M} \to 0 \text{ as } M \to \infty$$

- Frequency shift keying: $\Delta f = 1/2T$ is the minimum frequency separation between successive frequencies

$$\nu = \frac{R_b}{\frac{M}{2T}} = \frac{2k}{M} = \frac{2\log_2 M}{M} \to 0 \text{ as } M \to \infty$$

- PAM/QAM/PSK: Increasing M leads to more bandwidth efficiency and less power efficiency. Thus PAM/QAM/PSK schemes are appropriate for bandwidth-limited channels with little power constraints, e.g., telephone channel.

- PPM/FSK: Increasing M leads to less bandwidth efficiency and more power efficiency. Thus PPM/FSK schemes are appropriate for power-limited channels with little bandwidth constraints. In fact, theoretically the error probability can be made arbitrarily small as long as SNR/bit > -1.6 dB. But this would require bandwidth $W \to \infty$.



Comparison of some modulation formats for $P_e = 10^{-5}$

SUMMARY

Binary Modulation infinite Band width

- For AWGN channels and equiprobable symbols, the ML detector is the optimal detector with decision threshold equidistant from the two signal points.
- Binary PAM, $P_b = Q\left(\sqrt{\frac{2\mathcal{E}_b}{N_0}}\right)$, utilizes half the energy compared with binary orthogonal signalling, $P_b = Q\left(\sqrt{\frac{\mathcal{E}_b}{N_0}}\right)$, to achieve the same bit error probability.

M-ary Modulation

- M-PAM: the M signal points are on a line with different transmit energy, i.e., different distance from the origin. Two outer points have symbol error probability $P_e = Q\left(\sqrt{\frac{2\mathcal{E}_g}{N_0}}\right)$ whereas $M-2$ inner points have symbol error probability $P_e = 2Q\left(\sqrt{\frac{2\mathcal{E}_g}{N_0}}\right)$
- M-PSK: the M signal points are on a circle with different phases but same transmit energy. QPSK is equivalent to 2 independent BPSK signals and symbol error probability is $P_e = 1-(1-P_{\mathrm{BPSK}})^2$. At high SNRs and assuming Gray bit coding, bit error probability is $P_b \approx \frac{P_e}{k}$ where $k = \log_2 M$.

M-ary Modulation

- M-QAM: the M signal points may differ in both phase and transmit energy. For the same symbol error probability (i.e., same $d_{\min}$), there exist many possible signal constellations with different transmit energy. Rectangular M-QAM is equivalent to 2 independent PAM signals with half average energy and symbol error probability is $P_e = 1 - (1 - P_{\sqrt{M}-\mathrm{PAM}})^2$.
- M-ary orthogonal signalling: the M signal points are equidistant neighbours, therefore bit error probability is $P_b \approx \frac{P_e}{2}$. Larger M results in lower symbol error probability, i.e., can make $P_e \to 0$ as $M \to \infty$. Minimum SNR/bit of -1.6 dB is the Shannon limit for zero error probability in AWGN channels.

Spectral Efficiency

- Spectral efficiency is $\nu = \frac{R_b}{W}$ where R_b is the transmission bit rate and W is the transmission bandwidth.
- PAM/QAM/PSK: Larger M results in better spectral efficiency but worse error probability.
- PPM/FSK: Larger M results in worse spectral efficiency but better error probability.