ELEN90051 Advanced Communication Systems
2018 Semester 1 Tutorial 5
Modem with Bandlimited Channel Solutions

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Instructions:

Only look at these solutions after you have had a go at solving the questions yourself. The solutions provided below enable you to find out whether your answers are correct.

1 Determine the bit rate that can be transmitted through a 4 KHz voice-band telephone (bandpass) channel if the following modulation methods are used:

(a) binary PSK,

(b) four-phase PSK,

(c) 8-point QAM.

Assume that in each case the transmitter pulse shape has a raised cosine spectrum with $\beta = 0.5$ roll-off.

Solution:
For a bandpass channel bandwidth $W = 4000$ Hz, the raised cosine spectrum pulse shape with $\beta = \frac{1}{2}$ has baseband bandwidth

$$\frac{1+\beta}{2T} = 2000$$

and hence the symbol rate is $R_s = \frac{1}{T} = 2667$ symbols/sec.

(a) For binary PSK, each symbol represents 1 bit and the corresponding bit rate is $R_b = R_s = 2667$ bits/sec.

(b) For 4-phase PSK, each symbol represents 2 bits and the corresponding bit rate is $R_b = 2R_s = 5334$ bits/sec.

(c) For 8-point QAM, each symbol represents 3 bits and the corresponding bit rate is $R_b = 3R_s = 8001$ bits/sec.

2 An ideal voice-band telephone line channel has a (ideal) bandpass frequency response characteristic spanning the frequency range 600-3000 Hz.

(a) Design an $M = 4$ PSK system for transmitting data at a rate of 2400 bits/sec and a carrier frequency $f_c = 1800$ Hz. For spectral shaping, use a raised cosine frequency-response characteristic, splitting the desired frequency response characteristic evenly between the transmit filter $G_T(f)$ and the receive filter $G_R(f)$. Write down the expression for $G_T(f)$ and sketch a block diagram of the transmitter.

(b) Repeat part (a) if the bit rate is 4800 bits/sec.

Solution:

(a) The bandwidth of the bandpass channel is

$$W = 3000 - 600 = 2400 \text{ Hz}$$

which corresponds to a baseband bandwidth of 1200 Hz. Since each QPSK symbol represents 2 bits, the bit rate of $R_b = 2400$ bits/sec results in a symbol rate of

$$R_s = \frac{1}{T} = \frac{2400}{2} = 1200 \text{ symbols/sec}$$

Thus, the roll-off factor is given by

$$\beta = 1200 \times 2T - 1 = 1$$

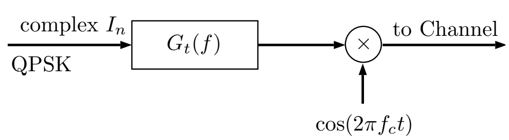
For roll-off factor $\beta = 1$, the raised cosine spectrum is (use that $1 + \cos 2\alpha = 2\cos^2 \alpha$):

$$X_{rc}(f) = \frac{T}{2}[1 + \cos(\pi T|f|)] = T\cos^2\left(\frac{\pi T|f|}{2}\right) = \frac{1}{1200}\cos^2\left(\frac{\pi|f|}{2400}\right)$$

When the channel is ideal with $C(f) = 1$, $|f| \leq W$, we have $X_{rc}(f) = G_T(f)G_R(f)$. Therefore, if the desired spectral characteristic is split evenly between the transmit and receive filters, we have

$$G_T(f) = G_R(f) = \sqrt{X_{rc}(f)} = \sqrt{\frac{1}{1200}}\cos\left(\frac{\pi|f|}{2400}\right), \qquad |f| < \frac{1}{T} = 1200$$

A block diagram of the modulated bandpass signal is shown as follows.



(b) There are two ways to answer this question. If we stick to our choice $\beta = 1$ from part a) then we need to use a 16PSK signaling scheme. Alternatively we stick to our 4PSK signaling scheme and allow ourselves to change the value of β: if the bit rate is 4800 bits/sec, then the symbol rate is

$$R_s = \frac{4800}{2} = 2400 \text{ symbols/sec}$$

In order to satisfy the Nyquist criterion, there is only one choice for the signal pulse which is the sinc spectrum given by

$$X(f) = \begin{cases} T, & |f| < 1200\text{Hz} \\ 0, & \text{otherwise} \end{cases}$$

Thus, the frequency response of the transmit filter is

$$G_T(f) = \begin{cases} \sqrt{T}, & |f| < 1200\text{Hz} \\ 0, & \text{otherwise} \end{cases}$$

3 Consider a three level PAM system with possible transmitted levels -2, 0, 2. The channel through which the data is transmitted introduces ISI over two successive symbols. The equivalent discrete-time channel model is given below

$$\begin{aligned} u_0 &= 0.8I_0 + \eta_0 \\ u_k &= 0.8I_k - 0.6I_{k-1} + \eta_k, \quad k \geq 1 \end{aligned}$$

where $\eta_k \sim \mathcal{N}(0,1)$. Suppose that the received signals are $u_0 = 0.5, u_1 = 1.2, u_2 = -0.7$. Using the Viterbi algorithm, determine the most likely transmitted sequence $\hat{I}_0, \hat{I}_1, \hat{I}_2$

Solution:
Here $M = 3$, $L = 1$. This three level PAM system has the following equivalent discrete channel model:

$$\begin{aligned} u_0 &= 0.8I_0 + \eta_0, && k = 0 \\ u_k &= 0.8I_k - 0.6I_{k-1} + \eta_k, && k \geq 1 \end{aligned}$$

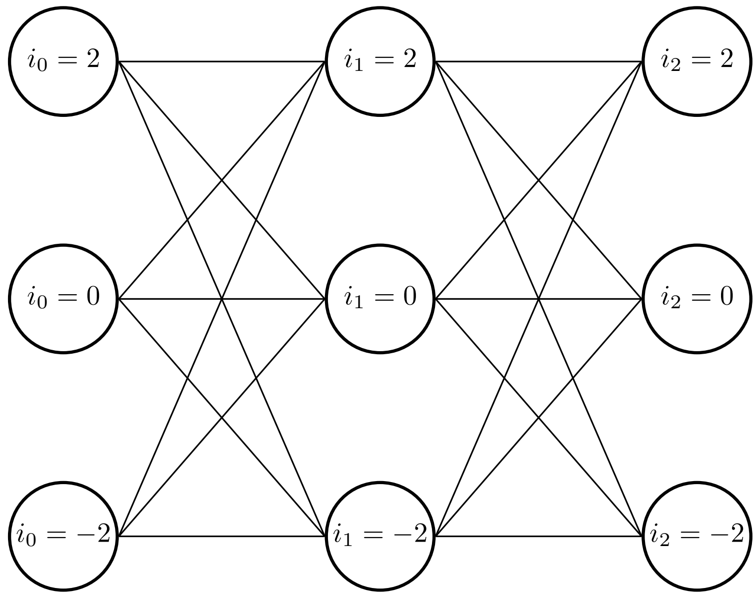
with $I_k \in \{-2, 0, 2\}$.
The noise terms are $\eta_k \sim \mathcal{N}(0,1)$, meaning that $u_k \sim \mathcal{N}(0.8I_k - 0.6I_{k-1}, 1)$ with $I_{-1} = 0$.
Assume we receive a signal with $u_0 = 0.5$ $u_1 = 1.2$ and $u_2 = -0.7$. We can find the most likely transmitted sequence by applying the Viterbi algorithm. To do this we need to construct a graph with a group of nodes for each received symbol. Each group of nodes needs a node for each possible value of the symbol. Finally, each node is connected to every node in the previous group by a path.
These normally are weighted by either the respective log likelihood functions or $f(u_k|I_{k-1} = a, I_k = b)$, but we need a way of accounting for the likelihoods for I_0 since there are no paths into it. This can be done by including an additional node for an artificial "$I_{-1} = 0$" but the way it is done in lectures is by just adding the appropriate path weights to the paths between the I_0 and I_1 nodes. Hence the path weights are given by

$$L_k(a,b) = \begin{cases} f(u_1|I_0 = a, I_1 = b) + f(u_0|I_0 = a), & k = 1 \\ f(u_k|I_{k-1} = a, I_k = b), & k > 1 \end{cases}$$

Note that the trellis diagram actually visualizes the discrete-time state representation (with $x_0 = 0$ and $k \in \mathbb{Z}_+$):

$$\begin{cases} x_{k+1} = 0 \cdot x_k + i_k \\ y_k = -0.6x_k + 0.8i_k. \end{cases}$$

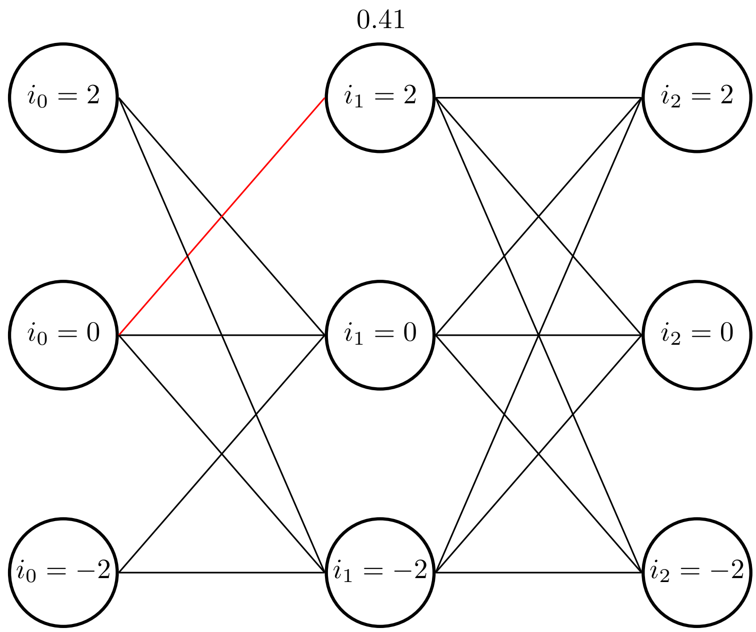
Here $i_k \in \{-2, 0, 2\}$ is the input sequence, $x_k \in \{-2, 0, 2\}$ is the state sequence and y_k is the real-valued noiseless output sequence. We have $u_k = y_k + \eta_k$, where $\eta_k \sim \mathcal{N}(0,1)$.



Now that we have the graph, the first step in the algorithm is to find the paths with the minimum weight that leads to each node of I_1. For $I_1 = 2$ we have

$$\begin{aligned}
L_1(2,2) &= (u_1 - 0.8I_1 + 0.6I_0)^2 + (u_0 - 0.8I_0)^2 \\
&= (1.2 - 0.8(2) + 0.6(2))^2 + (0.5 - 0.8(2))^2 \\
&= 0.64 + 1.21 \\
&= 1.85 \\
L_1(0,2) &= (u_1 - 0.8I_1 + 0.6I_0)^2 + (u_0 - 0.8I_0)^2 \\
&= (1.2 - 0.8(2) + 0.6(0))^2 + (0.5 - 0.8(0))^2 \\
&= 0.16 + 0.25 \\
&= 0.41 \\
L_1(-2,2) &= (u_1 - 0.8I_1 + 0.6I_0)^2 + (u_0 - 0.8I_0)^2 \\
&= (1.2 - 0.8(2) + 0.6(-2))^2 + (0.5 - 0.8(-2))^2 \\
&= 2.56 + 4.41 \\
&= 6.97
\end{aligned}$$

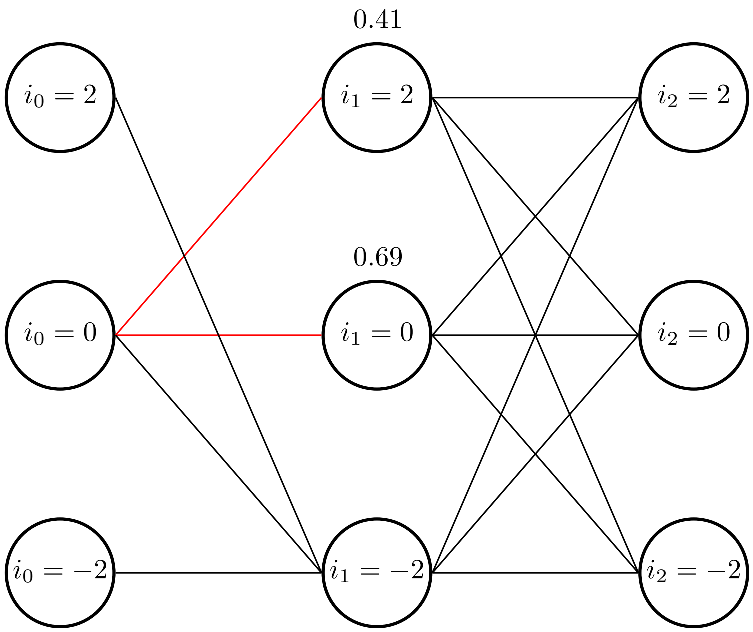
We can see that $L_1(0,2)$ has the smallest value, hence we remove every path to the node $i_1 = 2$ except the one from $i_0 = 0$.



We need to repeat this with the node $i_1 = 0$

$$
\begin{aligned}
L_1(2,0) &= (u_1 - 0.8I_1 + 0.6I_0)^2 + (u_0 - 0.8I_0)^2 \\
&= (1.2 - 0.8(0) + 0.6(2))^2 + (0.5 - 0.8(2))^2 \\
&= 5.76 + 1.21 \\
&= 6.97 \\
L_1(0,0) &= (u_1 - 0.8I_1 + 0.6I_0)^2 + (u_0 - 0.8I_0)^2 \\
&= (1.2 - 0.8(0) + 0.6(0))^2 + (0.5 - 0.8(0))^2 \\
&= 1.44 + 0.25 \\
&= 1.69 \\
L_1(-2,0) &= (u_1 - 0.8I_1 + 0.6I_0)^2 + (u_0 - 0.8I_0)^2 \\
&= (1.2 - 0.8(0) + 0.6(-2))^2 + (0.5 - 0.8(-2))^2 \\
&= 0 + 4.41 \\
&= 4.41
\end{aligned}
$$

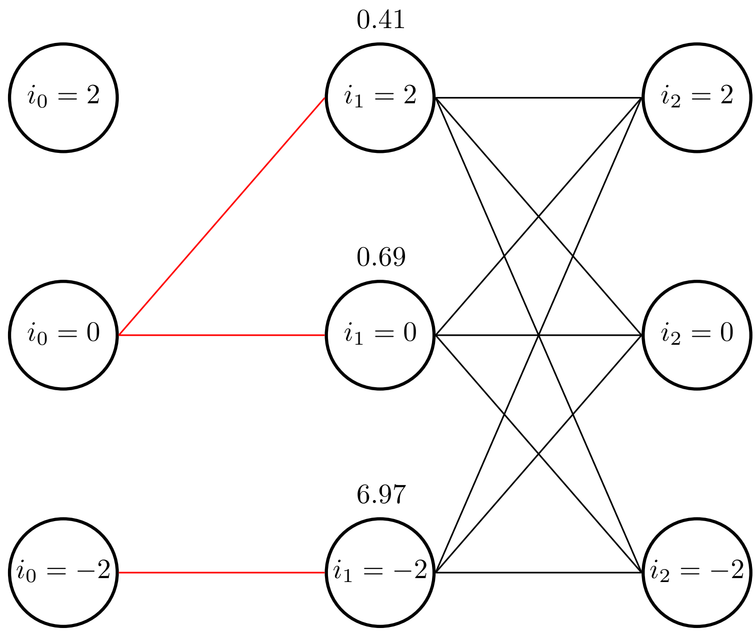
The path with the lowest weight is $L_1(0,0)$



Repeating for the node $i_1 = -2$ gives

$$\begin{aligned}
L_1(2,-2) &= (u_1 - 0.8I_1 + 0.6I_0)^2 + (u_0 - 0.8I_0)^2 \\
&= (1.2 - 0.8(-2) + 0.6(2))^2 + (0.5 - 0.8(2))^2 \\
&= 16 + 1.21 \\
&= 17.21 \\
L_1(0,-2) &= (u_1 - 0.8I_1 + 0.6I_0)^2 + (u_0 - 0.8I_0)^2 \\
&= (1.2 - 0.8(-2) + 0.6(0))^2 + (0.5 - 0.8(0))^2 \\
&= 7.84 + 0.25 \\
&= 8.09 \\
L_1(-2,-2) &= (u_1 - 0.8I_1 + 0.6I_0)^2 + (u_0 - 0.8I_0)^2 \\
&= (1.2 - 0.8(-2) + 0.6(-2))^2 + (0.5 - 0.8(-2))^2 \\
&= 2.56 + 4.41 \\
&= 6.97
\end{aligned}$$

We can see that $L_1(-2,-2)$ has the smallest weight.



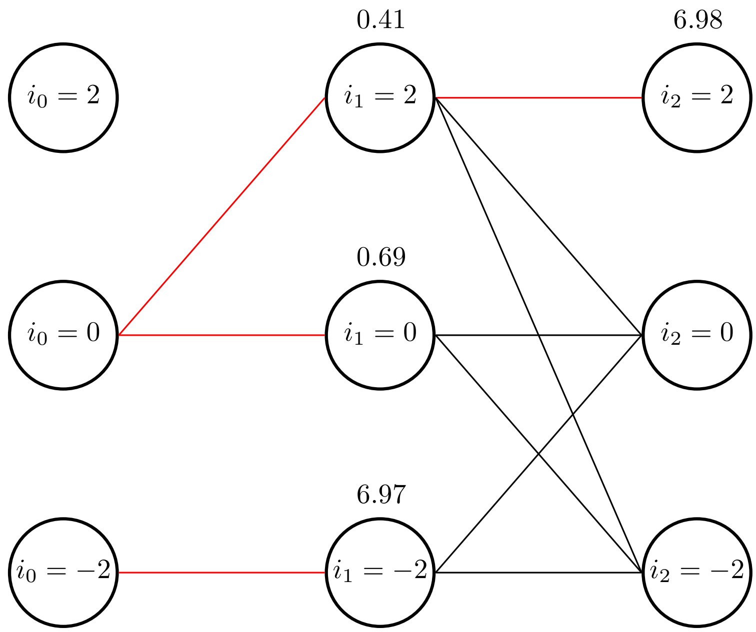
The next stage of the algorithm is very similar to the previous one, but we need to **add the path weights** selected by the previous stage to get the total path weight from the respective i_0 node.

$$L_2'(a,b) = L_2(a,b) + \min_c L_1(c,a)$$

For $i_2 = 2$ we have

$$\begin{aligned}
L_2'(2,2) &= (u_2 - 0.8I_2 + 0.6I_1)^2 + 0.41 \\
&= (-0.7 - 0.8(2) + 0.6(2))^2 + 0.41 \\
&= 1.21 + 0.41 \\
&= 1.62 \\
L_2'(0,2) &= (u_2 - 0.8I_2 + 0.6I_1)^2 + 1.69 \\
&= (-0.7 - 0.8(2) + 0.6(0))^2 + 1.69 \\
&= 5.29 + 1.69 \\
&= 6.98 \\
L_2'(-2,2) &= (u_2 - 0.8I_2 + 0.6I_1)^2 + 6.97 \\
&= (-0.7 - 0.8(2) + 0.6(-2))^2 + 6.97 \\
&= 12.25 + 6.97 \\
&= 19.22
\end{aligned}$$

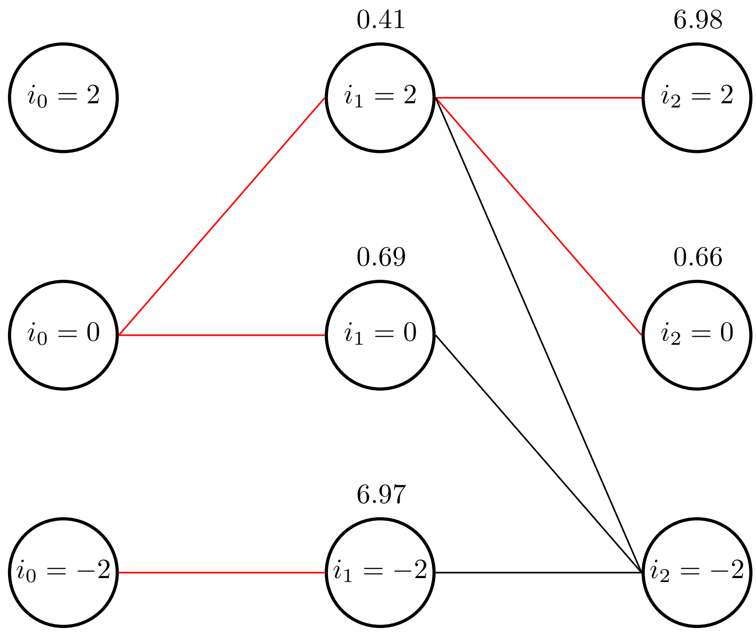
Hence $L_2'(2,2)$ has the shortest path length



For $i_2 = 0$ we have

$$\begin{aligned}
L_2'(2,0) &= (u_2 - 0.8I_2 + 0.6I_1)^2 + 0.41 \\
&= (-0.7 - 0.8(0) + 0.6(2))^2 + 0.41 \\
&= 0.25 + 0.41 \\
&= 0.66 \\
L_2'(0,0) &= (u_2 - 0.8I_2 + 0.6I_1)^2 + 1.69 \\
&= (-0.7 - 0.8(0) + 0.6(0))^2 + 1.69 \\
&= 0.49 + 1.69 \\
&= 2.18 \\
L_2'(-2,0) &= (u_2 - 0.8I_2 + 0.6I_1)^2 + 6.97 \\
&= (-0.7 - 0.8(0) + 0.6(-2))^2 + 6.97 \\
&= 3.61 + 6.97 \\
&= 10.58
\end{aligned}$$

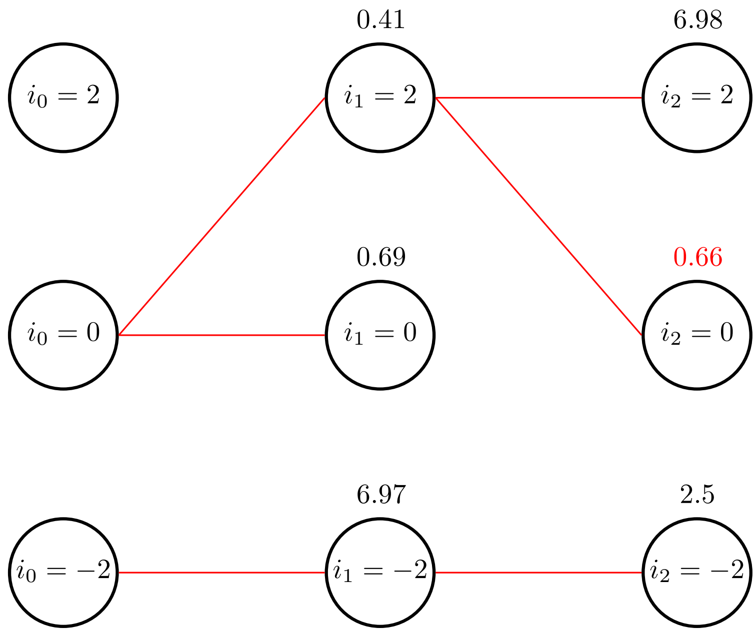
Hence $L_2'(2,0)$ has the shortest path length



Finally, for $i_2 = -2$ we have

$$\begin{aligned}
L_2'(2,-2) &= (u_2 - 0.8I_2 + 0.6I_1)^2 + 0.41 \\
&= (-0.7 - 0.8(-2) + 0.6(2))^2 + 0.41 \\
&= 4.41 + 0.41 \\
&= 4.82 \\
L_2'(0,-2) &= (u_2 - 0.8I_2 + 0.6I_1)^2 + 1.69 \\
&= (-0.7 - 0.8(-2) + 0.6(0))^2 + 1.69 \\
&= 0.81 + 1.69 \\
&= 2.5 \\
L_2'(-2,-2) &= (u_2 - 0.8I_2 + 0.6I_1)^2 + 6.97 \\
&= (-0.7 - 0.8(-2) + 0.6(-2))^2 + 6.97 \\
&= 0.09 + 6.97 \\
&= 7.06
\end{aligned}$$

Hence $L_2'(0,-2)$ has the shortest path length



From this we can see that the path from i_0 to i_2 with the lowest weight corresponds to $\hat{I}_0 = 0$, $\hat{I}_1 = 2$ and $\hat{I}_2 = 0$. Hence this is the most likely transmitted sequence.

4 Data is transmitted using a signal pulse with a raised cosine spectrum through a channel with the following impulse response:

$$f_k = \begin{cases} -0.5, & k = -2 \\ 0.1, & k = -1 \\ 1, & k = 0 \\ -0.2, & k = 1 \\ 0.05, & k = 2 \\ 0, & \text{otherwise} \end{cases}$$

(a) Determine the tap coefficients of a three-tap linear equalizer based on the zero-forcing criterion, i.e. we want to set $q_{-1} = 0, q_0 = 1, q_1 = 0$.

(b) Using the equalizer tap coefficients in part (a), determine the output of the equalizer for the case of the isolated pulse, i.e. determine q_k for all k.

Solution:

(a) The output of the three-tap zero-forcing equalizer is

$$q_k = \sum_{j=-1}^{1} c_j f_{k-j}$$

With $q_{-1} = 0$, $q_0 = 1$ and $q_1 = 0$, we obtain the system

$$\begin{bmatrix} 1.0 & 0.1 & -0.5 \\ -0.2 & 1.0 & 0.1 \\ 0.05 & -0.2 & 1.0 \end{bmatrix} \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Solving the previous system in terms of the equalizer's coefficients, we obtain

$$\begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.9804 \\ 0.1961 \end{bmatrix}$$

(b) The output of the equalizer for all k is

$$q_k = \begin{cases} 0, & k \leq -4 \\ c_{-1}f_{-2} = 0, & k = -3 \\ c_{-1}f_{-1} + c_0 f_{-2} = -0.4902, & k = -2 \\ 0, & k = -1 \\ 1, & k = 0 \\ 0, & k = 1 \\ c_0 f_2 + c_1 f_1 = 0.0098, & k = 2 \\ c_1 f_2 = 0.0098, & k = 3 \\ 0, & k \geq 4 \end{cases}$$

Hence, the residual ISI sequence is

$$\text{residual ISI} = \{\ldots, 0, -0.4902, 0, 0, 0, 0.0098, 0.0098, 0, \ldots\}$$

and its span is 6 symbols.

5 A nonideal bandlimited channel introduces ISI over three successive symbols. The (noise-free) impulse response of the matched filter demodulator sampled at the sampling time $t = kT$ is

$$f(kT) = \begin{cases} \mathcal{E}_g, & k = 0 \\ 0.9\mathcal{E}_g, & k = \pm 1 \\ 0.1\mathcal{E}_g, & k = \pm 2 \\ 0, & \text{otherwise} \end{cases}$$

Determine the tap coefficients of a three-tap linear equalizer that equalizes the channel (received signal) response to an equivalent partial response signal

$$q_k = \begin{cases} \mathcal{E}_g, & k = 0, 1 \\ 0, & \text{otherwise} \end{cases}$$

also known as a duobinary signal.

Solution:

If $\{c_j\}$ denotes the coefficients of the equalizer and $\{q_k\}$ is the sequence of the equalizer's output samples, then

$$q_k = \sum_{j=-1}^{1} c_j f_{k-j}$$

where $\{f_k\}$ is the noise free response of the matched filter demodulator sampled at $t = kT$. With $q_{-1} = 0$, $q_0 = q_1 = \mathcal{E}_g$, we obtain the system

$$\begin{bmatrix} \mathcal{E}_g & 0.9\mathcal{E}_g & 0.1\mathcal{E}_g \\ 0.9\mathcal{E}_g & \mathcal{E}_g & 0.9\mathcal{E}_g \\ 0.1\mathcal{E}_g & 0.9\mathcal{E}_g & \mathcal{E}_g \end{bmatrix} \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ \mathcal{E}_g \\ \mathcal{E}_g \end{bmatrix}$$

which is equivalent to solving

$$\begin{bmatrix} 1 & 0.9 & 0.1 \\ 0.9 & 1 & 0.9 \\ 0.1 & 0.9 & 1 \end{bmatrix} \begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

which results in

$$\begin{bmatrix} c_{-1} \\ c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 0.2137 \\ -0.3846 \\ 1.3248 \end{bmatrix}$$