ELEN90051 Advanced Communication Systems
2018 Semester 1 Tutorial 6
Channel Capacity Solutions

Department of Electrical and Electronic Engineering
University of Melbourne

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Instructions:

Only look at these solutions after you have had a go at solving the questions yourself. The solutions provided below enable you to find out whether your answers are correct.

1 Let X and Y be two binary random variables, distributed according to the joint distributions

$$P(X = Y = 0) = P(X = 0, Y = 1) = P(X = Y = 1) = \frac{1}{3}.$$

Compute $H(X), H(Y), H(X|Y), H(Y|X), H(XY)$ and $I(X, Y)$.

Solution:
Clearly $H(XY) = \log_2 3 \approx 1.585$. The marginal probabilities are given by

$$\begin{aligned}
P(X = 0) &= P(X = 0, Y = 0) + P(X = 0, Y = 1) = \frac{2}{3} \\
P(X = 1) &= P(X = 1, Y = 1) = \frac{1}{3} \\
P(Y = 0) &= P(X = 0, Y = 0) = \frac{1}{3} \\
P(Y = 1) &= P(X = 0, Y = 1) + P(X = 1, Y = 1) = \frac{2}{3}
\end{aligned}$$

Hence,

$$\begin{aligned}
H(X) &= -\left(\frac{1}{3}\log_2\frac{1}{3} + \frac{2}{3}\log_2\frac{2}{3}\right) = -\left(\log_2\frac{1}{3} + \frac{2}{3}\right) \approx 0.9183 \\
H(Y) &= -\left(\frac{2}{3}\log_2\frac{2}{3} + \frac{1}{3}\log_2\frac{1}{3}\right) = H(X) \approx 0.9183 \\
H(X|Y) &= H(XY) - H(Y) = \log_2 3 - \left(\log_2\frac{1}{3} + \frac{2}{3}\right) = \frac{2}{3} \approx 0.6667 \\
H(Y|X) &= H(X, Y) - H(X) = \frac{2}{3} \approx 0.6667 \\
I(X, Y) &= H(X) - H(X|Y) = H(Y) - H(Y|X) \approx 0.252
\end{aligned}$$

2 Compute the capacity of the BSC.

Solution:
Denote $P(X=0)$ by ϵ.

x	y	$P(y\|x)$	$P(x,y)$
0	0	$1-p$	$(1-p)\epsilon$
0	1	p	$p\epsilon$
1	0	p	$p(1-\epsilon)$
1	1	$1-p$	$(1-p)(1-\epsilon)$

$$I(X,Y) = H(Y) - H(Y|X)$$
$$H(Y) = -\alpha\log\alpha - (1-\alpha)\log(1-\alpha) = H_b(\alpha),$$

where $\alpha = P(Y=0)$ is given by

$$\alpha = P(Y=0|X=0)P(X=0) + P(Y=0|X=1)P(X=1) = (1-p)\epsilon + p(1-\epsilon).$$

Note that $\alpha = \frac{1}{2}$ for $\epsilon = \frac{1}{2}$.
Because of $H(Y|X=0) = H(Y|X=1) = H_b(p)$, we have $H(Y|X) = \epsilon H_b(p) + (1-\epsilon)H_b(p) = H_b(p)$.
Therefore

$$I(X,Y) = H_b((1-p)\epsilon + p(1-\epsilon)) - H_b(p)$$
$$C = \max_{\epsilon} I(X,Y) = 1 - H_b(p)$$

Note: an alternative less straightforward solution method is via $I(X,Y) = H(X) - H(X|Y)$.

3 Compute the capacity of the BEC.

Solution:
Denote $P(X=0)$ by ϵ.

x	y	$P(x\|y)$	$P(y\|x)$	$P(x,y)$
0	0	1	$1-p$	$(1-p)\epsilon$
0	1	0	0	0
0	e	ϵ	p	$p\epsilon$
1	0	0	0	0
1	1	1	$1-p$	$(1-p)(1-\epsilon)$
1	e	$1-\epsilon$	p	$p(1-\epsilon)$

$$I(X,Y) = H(X) - H(X|Y) = H_b(\epsilon) - H(X|Y)$$

There are two ways to obtain $H(X|Y)$:

(a)

$$\begin{aligned} H(X|Y) =& -\sum_{x,y} P(x,y)\log P(x|y) \\ =& -p\epsilon\log\epsilon - p(1-\epsilon)\log(1-\epsilon) \\ =& pH_b(p) \end{aligned}$$

(b)
- $H(X|Y=0) = H(X|Y=1) = 0$
- $H(X|Y=e) = H_b(\epsilon)$

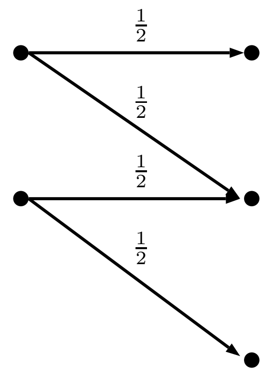
Therefore $H(X|Y) = P(Y=e)H(X|Y=e) = pH_b(\epsilon)$.

Next, we use $H(X|Y)$ in calculating $I(X,Y)$:

$$I(X,Y) = H_b(\epsilon) - pH_b(\epsilon) = (1-p)H_b(\epsilon)$$
$$C = \max_{\epsilon} I(X,Y) = 1 - p$$

Note: an alternative solution method is via $I(X,Y) = H(Y) - H(Y|X)$.

4 Consider a discrete memoryless channel given by the figure below. Determine the channel's capacity.



Solution:
Denote $P(X=0)$ by ϵ.

x	y	$P(y\|x)$	$P(x,y)$	$P(y)$
0	A	$\frac{1}{2}$	$\frac{1}{2}\epsilon$	$\frac{1}{2}\epsilon$
0	B	$\frac{1}{2}$	$\frac{1}{2}\epsilon$	$\frac{1}{2}$
0	C	0	0	$\frac{1}{2}(1-\epsilon)$
1	A	0	0	
1	B	$\frac{1}{2}$	$\frac{1}{2}(1-\epsilon)$	
1	C	$\frac{1}{2}$	$\frac{1}{2}(1-\epsilon)$	

$$H(Y|X=0) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = 1$$

Similarly,

$$H(Y|X=1) = -\frac{1}{2}\log\frac{1}{2} - \frac{1}{2}\log\frac{1}{2} = 1$$

Therefore

$$H(Y|X) = 1$$
$$I(X,Y) = H(Y) - H(Y|X) = H(Y) - 1$$

where

$$H(Y) = -\frac{1}{2}\epsilon \log \frac{1}{2}\epsilon - \frac{1}{2}\log\frac{1}{2} - \frac{1}{2}(1-\epsilon)\log\frac{1}{2}(1-\epsilon)$$

which is maximal for $\epsilon = \frac{1}{2}$. Then

$$H(Y) = -\frac{1}{4}\log\frac{1}{4} - \frac{1}{4}\log\frac{1}{4} - \frac{1}{2}\log\frac{1}{2} = \frac{3}{2}$$

Therefore,

$$C = \max_{\epsilon} I(X,Y) = \frac{3}{2} - 1 = \frac{1}{2}$$

5 Let X and Y be random variables. Show that

$$H(XY) \leq H(X) + H(Y).$$

Also show that equality holds if and only if X and Y are independent random variables.
Hint 1: First show that

$$H(X) = -\sum_{x,y} P(x,y)\log_2 P(x).$$

Hint 2: Then use the inequality $\ln w \leq w - 1$.

Solution:
To prove the first Hint, observe that

$$P(x) = \sum_{y} P(x,y) \qquad P(y) = \sum_{x} P(x,y),$$

so that

$$\begin{aligned} H(X) &= -\sum_{x}\sum_{y} P(x,y)\log_2 P(x) = -\sum_{x,y} P(x,y)\log_2 P(x) \\ H(Y) &= -\sum_{y}\sum_{x} P(x,y)\log_2 P(y) = -\sum_{x,y} P(x,y)\log_2 P(y) \end{aligned}$$

Next, it is clearly sufficient to prove that

$$H(XY) - H(X) - H(Y) \leq 0.$$

The proof is as follows:

$$\begin{aligned} H(XY) - H(X) - H(Y) &= -\sum_{x,y} P(x,y)\log_2 P(x,y) + \sum_{x,y} P(x,y)\log_2 P(x) + \sum_{x,y} P(x,y)\log_2 P(y) \\ &= \sum_{x,y} P(x,y)\log_2 \frac{P(x)P(y)}{P(x,y)} \\ &= \log_2 e \sum_{x,y} P(x,y)\ln\frac{P(x)P(y)}{P(x,y)} \\ &\leq \log_2 e \sum_{x,y} P(x,y)\left[\frac{P(x)P(y)}{P(x,y)} - 1\right] \\ &= 0. \end{aligned}$$

Equality holds if and only if

$$\frac{P(x)P(y)}{P(x,y)} = 1,$$

that is, if X and Y are independent.

6 Let X and Y be random variables. Show that

$$H(XY) = H(X) + H(Y|X)$$

Solution:

$$\begin{aligned}
H(Y|X) &= -\sum_{x,y} P(x,y)\log P(y|x) \\
&= -\sum_{x,y} P(x,y)\log\left(\frac{P(x,y)}{P(x)}\right) \\
&= -\sum_{x,y} P(x,y)\log P(x,y) + \sum_{x,y} P(x,y)\log P(x) \\
&= H(XY) - H(X)
\end{aligned}$$

Here we used that $P(x) = \sum_y P(x,y)$.

7 Let X and Y be random variables. Show that

$$H(Y|X) \leq H(Y)$$

When does equality hold?

Solution:
Use previous results

$$H(Y|X) = H(XY) - H(X) \leq H(X) + H(Y) - H(X) = H(Y)$$

Equality holds if and only if X and Y are independent.