Vv186 Honors Mathematics II

Functions of a Single Variable

Assignment 9

Date Due: 8:00 AM, Thursday, the 26$^{\text{th}}$ of November 2020

"Trigonometry, so far as this, is most valuable to every man; there is scarcely a day in which he will not resort to it for some of the purposes of common life. The science of calculation also is indispensable as far as the extraction of the square and cube roots; algebra as far as the quadratic equation and the use of logarithms is often of value in ordinary cases. But all beyond these is but a luxury; a delicious luxury, indeed, but not to be indulged in by one who is to have a profession to follow for his subsistence."

Thomas Jefferson, in a letter to W.G. Mumford of June 18, 1799[1]

This assignment has a total of **(29 Marks)**.

Exercise 9.1

This exercise shows some pitfalls associated with l'Hopital's rule:

i) Attempt to evaluate

$$\lim_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}}$$

by using l'Hopital's rule directly. Then find the limit using a different approach.
(2 Marks)

ii) Attempt to evaluate

$$\lim_{x\to 0}\frac{x^2\cos(1/x)}{\sin x}$$

by using l'Hopital's rule directly. Then find the limit using a different approach.
(2 Marks)

iii) Let $f, g\colon \mathbb{R}\to\mathbb{R}$ be given by $f(x)=x+\sin x\cos x$, $g(x)=f(x)e^{\sin x}$. Show that the limit

$$\lim_{x\to\infty}\frac{f'(x)}{g'(x)} \quad \text{exists} \qquad\qquad \text{but} \qquad\qquad \lim_{x\to\infty}\frac{f(x)}{g(x)} \quad \text{does not exist.}$$

Does this contradict l'Hopital's rule?
(3 Marks)

Exercise 9.2

i) Find a differentiable function $f\colon\mathbb{R}\to\mathbb{R}$ such that $\sup\limits_{x\in\mathbb{R}}|f(x)|=1$ but $\sup\limits_{x\in\mathbb{R}}|f'(x)|=\infty$.
(2 Marks)

ii) Find a differentiable function $f\colon\mathbb{R}\to\mathbb{R}$ such that $\lim\limits_{x\to\infty}|f(x)|=\infty$ but $\lim\limits_{x\to\infty}|f'(x)|=0$.
(2 Marks)

Exercise 9.3

Let $\lambda_1,\dots,\lambda_n>0$ such that $\sum\limits_{i=1}^{n}\lambda_i=1$ and $x_1,\dots,x_n\in\mathbb{R}$.[2]

i) Show that

$$\min_{1\le i\le n} x_i \le \sum_{i=1}^{n}\lambda_i x_i \le \max_{1\le i\le n} x_i.$$

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[1] See http://www.britannica.com/presidents/article-9116908 for the full text.
[2] See *Spivak*, App. to Ch. 11, Ex. 9.

ii) Let $t = \sum_{i=1}^{n-1} \lambda_i$. Show that

$$\min_{1\le i\le n} x_i \le \frac{1}{t}\sum_{i=1}^{n-1} \lambda_i x_i \le \max_{1\le i\le n} x_i.$$

iii) Let $I \subset \mathbb{R}$ be an interval with $x_1, \ldots, x_n \in I$. Suppose that $f\colon I \to \mathbb{R}$ is a convex function. Use Exercise 6.4 i) with $1 - t = p_n$ to show *Jensen's inequality*,

$$f\Big(\sum_{i=1}^{n} \lambda_i x_i\Big) \le \sum_{i=1}^{n} \lambda_i f(x_i).$$

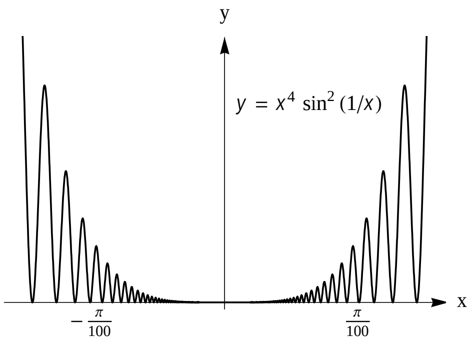
(You will need the inequalities in ii) above to verify that $\frac{1}{t}\sum_{i=1}^{n-1} \lambda_i x_i \in I$.) What form does Jensen's inequality take if f is concave?

iv) Show that the logarithm $\ln\colon \mathbb{R}_+ \to \mathbb{R}$ is concave. Apply Jensen's inequality to establish

$$\prod_{i=1}^{n} x_i^{\lambda_i} \le \sum_{i=1}^{n} \lambda_i x_i$$

for $x_1, \ldots, x_n > 0$. (This is formula (1) of Exercise 6.7.)

(1 + 1 + 3 + 2 Marks)



The following exercise shows that a function does not need to be increasing in an interval to the right of a local minium or decreasing in an interval to the left.

Exercise 9.4

Let $f(x) = x^4 \sin^2(1/x)$ for $x \neq 0$, $f(0) = 0$. Prove[a] that 0 is a local miminum point for f, and that $f'(0) = f''(0) = 0$. **(3 Marks)**

[a]See *Spivak*, Ch. 11, Ex. 62

Exercise 9.5

Find all solutions[3] $z, w \in \mathbb{C}$ to the following equations.

$$z^7 = 3 + 4i, \qquad z^2 - iz + 1 = 0, \qquad z^4 + z^2 + 1 = 0, \qquad \begin{cases} iz - (1+i)w = 3 \\ (2+i)z + iw = 4 \end{cases}$$

(4 Marks)

Exercise 9.6

i) Use the addition theorems[4] for sine and cosine to prove that

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

for suitable x, y. What are the restrictions on x and y?
(2 Marks)

ii) Prove that $\arctan x + \arctan y = \arctan\left(\dfrac{x+y}{1-xy}\right)$ for $xy \neq 1$.
(1 Mark)

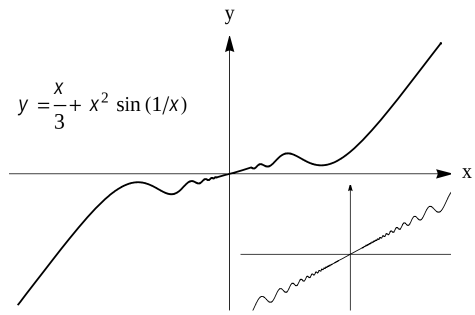
iii) Prove that $\arctan x + \arctan \dfrac{1}{x} = \dfrac{\pi}{2}$ for $x > 0$.
(2 Marks)

[3]See *Spivak*, Ch. 25, Ex. 1,2
[4]See *Spivak*, Ch. 15, Ex. 9

Exercise 9.7

i) Prove[a] that if $f'(a) > 0$ and f' is continuous at $a \in \mathbb{R}$, then f is increasing in some interval containing a. **(3 Marks)**

ii) Let $\alpha \in (0, 1)$ and $f(x) = \alpha x + x^2 \sin(1/x)$ for $x \neq 0$, $f(0) = 0$. Show that $f'(0) > 0$ but f is not increasing in any interval containing 0 by showing that in any interval containing 0 there are points x, y with $f'(x) > 0$ and $f'(y) < 0$. **(3 Marks)**



[a]See *Spivak*, Ch. 11, Ex. 63; see also Ex. 64