Vv186 Honors Mathematics II

Functions of a Single Variable

Assignment 3

Date Due: 8:00 AM, Saturday, the 10$^{\text{th}}$ of October 2020

"Obvious" is the most dangerous word in mathematics.

Eric Temple Bell

This assignment has a total of **(35 Marks)**.

Exercise 3.1

In this exercise, you are supposed to argue directly from the definition of convergence. Your answers should start with the words "Let $\varepsilon > 0$ be fixed" and should include an expression for finding N depending on the given ε.

i) Show that $\lim_{n\to\infty} \frac{1}{n^2} = 0$ and $\lim_{n\to\infty} \frac{2n-5}{n} = 2$.
(2 Marks)

ii) Let $(a_n), (b_n)$ be complex sequences with limits $a, b \in \mathbb{C}$, respectively. Show that $\lim_{n\to\infty} a_n b_n = ab$.
(2 Marks)

Exercise 3.2

Find the limits of the sequences given by the expressions below. You may use any limit laws that we have proven or discussed in the lecture (such as the squeeze theorem). However, you are required to indicate which law you are using and justify its use at every step.

$$a_n = \frac{n^2 - 3n + 2}{2n^2 + 5n + 10} \qquad b_n = (1 - n^{-2})^n \qquad c_n = \frac{n^n}{(2n)!} \qquad d_n = \frac{n!}{n^n} \qquad e_n = \sqrt{n+1} - \sqrt{n}$$

(5 Marks)

Exercise 3.3

Let (a_n) be a sequence and (a_{2n}), (a_{2n+1}) the subsequences of odd- and even-numbered values of (a_n). Prove that if $a_{2n} \to a$ and $a_{2n+1} \to a$ as $n \to \infty$ then (a_n) converges to a.
(2 Marks)

Exercise 3.4

In Exercise 2.1 we saw that any rational approximation m/n to $\sqrt{2}$ can be replaced by a better approximation $(m + 2n)/(m + n)$. In particular, starting with $m = n = 1$, we obtain a sequence of improving approximations

$$1, \ \frac{3}{2}, \ \frac{7}{5}, \ldots$$

i) Prove[1] that this sequence is given recursively by

$$a_1 = 1, \qquad\qquad a_{n+1} = 1 + \frac{1}{1 + a_n}.$$

(1 Mark)

ii) Consider the subsequence (a_{2n}). Show that this subsequence is monotonic and bounded (you may use results from Exercise 2.1) and therefore converges, i.e., there exists a number $a \in \mathbb{R}$ such that $a_{2n} \to a$. Then prove that $a^2 = 2$ and argue that $a = +\sqrt{2}$.
(2 Marks)

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[1] See *Spivak*, Ch. 22, Ex. 7

iii) Do the same for the subsequence (a_{2n+1}). Conclude from Exercise 3.3 that $a_n \to \sqrt{2}$. The sequence (a_n) can be written as a *continued fraction*,

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \dots}}.$$

(1 Mark)

iv) Prove that for any natural numbers $a \geq 1$ and b

$$\sqrt{a^2 + b} = a + \cfrac{b}{2a + \cfrac{b}{2a + \dots}}.$$

(2 Marks)

Exercise 3.5

Let (x_n) be a bounded real sequence.[2] Let

$$y_n = \sup\{x_m : m \geq n\} = \sup\{x_n, x_{n+1}, x_{n+2}, \dots\}.$$

i) Prove that the sequence (y_n) converges. The limit $\lim\limits_{n\to\infty} y_n$ is denoted by $\varlimsup\limits_{n\to\infty} x_n$ or $\limsup\limits_{n\to\infty} x_n$ and called the *limes superior* or *upper limit*, of the sequence (x_n).
(2 Marks)

ii) Define the *limes inferior* $\varliminf\limits_{n\to\infty} x_n$ for a bounded real sequence (x_n). Then state (without proof) $\varlimsup\limits_{n\to\infty} x_n$ and $\varliminf\limits_{n\to\infty} x_n$ for each of the following sequences:

a) $x_n = \dfrac{1}{n}$, b) $x_n = (-1)^n \dfrac{1}{n}$ c) $x_n = (-1)^n \left(1 + \dfrac{1}{n}\right)$.

(3 Marks)

iii) Prove that $\varliminf\limits_{n\to\infty} x_n \leq \varlimsup\limits_{n\to\infty} x_n$ if (x_n) is a bounded sequence.
(1 Mark)

iv) Let (x_n) be a bounded real sequence and A be the set of all accumulation points of (x_n). Prove that $\min A$, $\max A$ exist and

$$\varliminf_{n\to\infty} x_n = \min A \qquad \text{and} \qquad \varlimsup_{n\to\infty} x_n = \max A.$$

(2 Marks)

v) Prove that a bounded real sequence (x_n) that only has a single accumulation point a converges, and that $\lim\limits_{n\to\infty} x_n = a$. Deduce that $\lim\limits_{n\to\infty} x_n$ exists if and only if $\varliminf\limits_{n\to\infty} x_n = \varlimsup\limits_{n\to\infty} x_n$ and that in this case

$$\lim_{n\to\infty} x_n = \varliminf_{n\to\infty} x_n = \varlimsup_{n\to\infty} x_n.$$

(2 Marks)

vi) Recall the definition of $\varliminf A$ for a bounded set A (see Exercise 2.4). Let (x_n) be a bounded sequence such that $x_n \neq x_m$ for $n \neq m$. Prove that

$$\varlimsup_{n\to\infty} x_n = \varlimsup \operatorname{ran}(x_n).$$

(2 Marks)

[2] See *Spivak*, Ch. 22, Ex. 27

Exercise 3.6

i) Let $a > b \geq 0$ be real numbers and let

$$a_1 = \frac{a+b}{2}, \qquad b_1 = \sqrt{ab}$$

be their *arithmetic* and *geometric mean*, respectively. Show that the recursively defined sequences

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad b_{n+1} = \sqrt{a_n b_n}, \qquad n = 1, 2, \ldots$$

converge, and that their limits are equal. This limit is called the *arithmetic-geometric mean* of a and b.

You may proceed as follows (but other methods are also acceptable): Show first that (a_n), (b_n) are monotonic (increasing or decreasing?) and hence deduce the existence of $\lim a_n$ and $\lim b_n$. Then estimate $|a_{n+1} - b_{n+1}|$ by $|a_n - b_n|$, and deduce the equality of the limits.
(3 Marks)

ii) The *harmonic mean* b_1 of $a > b > 0$ is defined by

$$\frac{1}{b_1} = \frac{1}{2}\left(\frac{1}{a} + \frac{1}{b}\right).$$

Let a_1 be the arithmetic mean of a, b, $a > b$ as above and set

$$a_{n+1} = \frac{a_n + b_n}{2}, \qquad \frac{1}{b_{n+1}} = \frac{1}{2}\left(\frac{1}{a_n} + \frac{1}{b_n}\right) \qquad n = 1, 2, \ldots.$$

Show that the sequences $(a_n), (b_n)$ converge to the same limit and calculate this limit.
(3 Marks)