Vv186 Honors Mathematics II

Sample Exercises for the First Midterm Exam

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The following exercises have been compiled from past first midterm exams of Ve401. A first midterm will usually consist of 4-7 such exercises to be completed in 100 minutes. In the actual exam, necessary tables of values of distributions will be provided. You may use all tables in Appendix A of the textbook to solve the sample exercises.

General Information for the exam:

- The exam will be 100 minutes long and include 4-5 exercises.
- The contents of the exam include the material indicated in the lecture slides **as well as all material introduced in the assignments**.
- You may not use a calculator or any other aid. You may bring only pens and writing material (paper will be provided), food and drink or other necessities for your well-being.
- Calculators, laptops, cell phones, electronic dictionaries and other electronic devices are **strictly forbidden**. Bring a watch if you need to keep track of time, although regular announcements of the time will be provided.
- The TAs and I will not offer any translations into Chinese. You may bring a paper (book) monolingual english dictionary, if you wish.
- **Read through all exercises before starting the exam.** The first exercises will not necessarily be the easiest exercise (for you). Start with an easy exercise, then move to the enxt easiest and so on. It is not necessary to complete the exercises in order.
- **Look at how many marks each exercise is worth and plan your time.** If the exam has 20 marks in total, you should not spend more than four minutes per mark. For example, if an exercise is worth 3 marks, you should not spend more than 12 minutes on it. After that, move to the next exercise, regardless of whether you have finished or not. If you do not follow this advice, you may end up missing "easy" marks from the following exercises for lack of time. You can always come back to a difficult exercise when you have finished the others.

Exercise 1.
In the following exercises, mark the boxes corresponding to true statements with a cross (⊠). In each case, it is possible that none of the statements are true or that more than one statement is true.

i) Let $A \subset \mathbb{R}$ be a non-empty set.

- ☐ If $\inf A$ exists, then $\underline{\lim}\, A$ exists.
- ☐ If $\underline{\lim}\, A$ exists, then $\inf A$ exists.
- ☐ $\underline{\lim}\, A$ exists if and only if A is bounded below.
- ☐ $\inf A$ exists if and only if A is bounded below.

ii) Let (a_n) be a sequence of real numbers.

- ☐ If (a_n) is convergent, then (a_n) is bounded.
- ☐ If (a_n) is bounded, then (a_n) is convergent.
- ☐ If $a_n \leq a$ for some $a \in \mathbb{R}$, then $a_n \to a$.
- ☐ If $a_n \leq a$ for some $a \in \mathbb{R}$ and (a_n) is strictly increasing, then $a_n \to a$.

(1/4 Mark) for each correctly checked or unchecked box.

Exercise 2.
Describe in words the difference between **an accumulation point of a sequence** and **a limit of a sequence**. How are these two concepts different? If you can, state some properties that they have in common or that serve to differentiate them from each other. Is one of them also always an example of the other? Give examples.
(5 Marks)

Exercise 3.

i) Consider the set $U \subset \mathbb{R}$, where $U = A \cup B \cup C$ with

$$\begin{aligned} A &= \{x \in \mathbb{R} \colon 0 < x \leq 1\}, \\ B &= \{x \in \mathbb{R} \colon x = 2 - 1/n,\ n \in \mathbb{N} \setminus \{0\}\}, \\ C &= \{x \in \mathbb{R} \colon x = -1/n,\ n \in \mathbb{N} \setminus \{0\}\}. \end{aligned}$$

State (without proof) $\min U$, $\max U$, $\inf U$, $\sup U$, $\underline{\lim}\, U$ and $\overline{\lim}\, U$ (if one or more of these do not exist, simply state this).

ii) Consider the sequence (a_n) given by

$$a_n = \frac{1}{2} + (-1)^n \frac{2+n}{2n}.$$

Calculate $\overline{\lim}\, a_n$ and $\underline{\lim}\, a_n$.

($6 \times \frac{1}{2} + 2$ Marks)

Exercise 4.
Suppose that f is a function such that $f \circ g = g \circ f$ for *all* functions g. Prove that $f(x) = x$, i.e., f is the identity function.
(2 Marks)

Exercise 5.
Prove the following statement using induction in n:

$$\sum_{j=1}^{n} x^{n-j} y^{j-1} = \frac{x^n - y^n}{x - y}, \qquad x, y \in \mathbb{R},\ x \neq y,\ n \geq 1.$$

(3 Marks)

Exercise 6.
Prove (e.g., with mathematical induction), that the sequence

$$\sqrt{2},\ \sqrt{2+\sqrt{2}},\ \sqrt{2+\sqrt{2+\sqrt{2}}},\ \ldots$$

is increasing and bounded. (Be sure to argue carefully and precisely.) Then calculate the limit of the sequence.
(6 Marks)

Exercise 7.
Show that the sequence defined by

$$a_1 = 2, \qquad a_{n+1} = \frac{1}{3-a_n}, \qquad (n \geq 1)$$

satisfies $0 < a_n \leq 2$ and is decreasing. Deduce that the sequence is convergent and find its limit.
(4 Marks)

Exercise 8.
Using the ε-δ definition of continuity, show that the function f given by $f(x) = 1/\sqrt{x}$ is continuous at $x = 1$.
(3 Marks)

Exercise 9.
If $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\lim_{x\to 0} \frac{f(x)}{x} = \alpha$ for some $\alpha \in \mathbb{R}$, calculate

i) $\lim\limits_{x\to 0} \dfrac{f(2x)}{x}$,

ii) $\lim\limits_{x\to 0} \dfrac{[f(2x)]^2}{x^2}$,

iii) $\lim\limits_{x\to 0} \dfrac{[f(2x)]^2}{x}$.

(3 × 1 Mark)

Exercise 10.
Prove that the equation

$$x^2 + |x|^{5/2} - 4x + 1 = 0, \qquad x \in \mathbb{R},$$

has a solution in the interval $(1, 2)$.
(2 Marks)

Exercise 11.
Let $a > 0$ and $f\colon [0, 2a] \to \mathbb{R}$ be a continuous function with $f(0) = f(2a)$. Prove that there exists some $c \in [0, a]$ such that $f(c) = f(c + a)$.
(4 Marks)

Exercise 12.
Let $f\colon (0, 1) \to \mathbb{R}$ be uniformly continuous on $(0, 1)$. Show that $\lim_{x\searrow 0} f(x)$ exists.
(3 Marks)

Exercise 13.
Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that $A \subset \operatorname{ran} f$ is an open set. Prove that then the *pre-image*

$$f^{-1}(A) := \{x \in \mathbb{R}\colon f(x) \in A\}$$

is also an open set.
(4 Marks)

Exercise 14.
Suppose the function $f\colon \mathbb{R} \to \mathbb{R}$ is such that $xf(x) + f(2-x) = x^2$.

i) Find an expression for $f(x)$.

ii) Is f continuous everywhere? Explain your answer.

(2 + 2 Marks)

Exercise 15.
Let $\Omega \subset \mathbb{R}$, $I \subset \Omega$ an interval and $f\colon \Omega \to \mathbb{R}$ a real function. Then f is called *Lipschitz-continuous on I* if there exists a constant $L > 0$ (called a *Lipschitz constant*) such that

$$|f(x) - f(y)| \leq L|x - y| \qquad \text{for all } x, y \in I.$$

i) Show that if f is Lipschitz-continuous on I, f is also uniformly continuous on I.

ii) Why is the function $f\colon [0,\infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$ uniformly continuous on $[0,1]$?

iii) Show that $f\colon [0,\infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$ is not Lipschitz-continuous on $[0,1]$.

(3+1+2 Marks)