Vv186 Honors Mathematics II

Functions of a Single Variable

Assignment 4

Date Due: 8:00 AM, Thursday, the 22nd of October 2020

Mathematics is the part of physics where experiments are cheap.

V.I. Arnol'd, *On teaching mathematics*, March 7th, 1997[1]

This assignment has a total of **(53 Marks)**.

Exercise 4.1

Let (X, ρ) be a complete metric space and let $T\colon X \to X$ be a function on X. Let $r < 1$ be a real number such that

$$\rho(Tx, Ty) \le r \cdot \rho(x, y) \qquad \text{for all } x, y \in X.$$

(We say that T is a *contraction*.) Our goal is to prove that

a) T has a unique fixed point $x_0 \in X$, i.e., there exists a *unique* point x_0 such that $Tx_0 = x_0$ and

b) for any $y_0 \in X$, the sequence $(y_n) = (T^n y_0)$ converges to x as $n \to \infty$.

This theorem is called the *contraction mapping principle* or *Banach's fixed point theorem*[2]. It is an extremely useful tool in the proofs of many important theorems.

i) Show that if $c \neq 1$, then
$$\sum_{i=0}^{k} c^{m+i} = \frac{c^m - c^{m+k+1}}{1-c}$$
for all $m, k \in \mathbb{N}$.
(2 Marks)

ii) Show that $\rho(y_{n+k}, y_n) \le r^n (1-r)^{-1} \rho(y_1, y_0)$. Deduce that (y_n) is a Cauchy sequence. Why does this imply that (y_n) converges?
(2 Marks)

iii) Deduce that $x_0 := \lim\limits_{n\to\infty} y_n$ is a fixed point, i.e., $Tx_0 = x_0$.
(2 Marks)

iv) Prove that there can not be two fixed points $x_0, \widetilde{x}_0 \in X$.
(2 Marks)

Exercise 4.2

Consider the sequence $(a_n)_{n\in\mathbb{N}}$ given by

$$\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}},\ \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}, \ldots.$$

i) Find a recursive representation of the sequence, i.e., a value a_0 and a function f (defined on what domain?) such that $a_{n+1} = f(a_n)$ for all $n \in \mathbb{N}$.
(1 Mark)

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[1] Arnol'd (sometimes written Arnold) was a famous Russian mathematician who was very critical of "abstract nonsense" and insisted on the importance of intuitive ideas in mathematical theory. The full text of his address, which I highly recommend, can be found at http://pauli.uni-muenster.de/~munsteg/arnold.html. Since Arnol'd spent a lot of time in France, the article addresses differences in the French and Russian culture of mathematics.

[2] see also *Spivak*, Ch. 22, Ex. 22, 23.

ii) Finden an explicit representation of the sequence and use induction to show that this representation is correct (i.e., it follows from the recursive representation).
(2 Marks)

iii) Use induction to show that (a_n) is bounded and increasing, so that the limit $a := \lim_{n\to\infty} a_n$ exists. Then find the limit.
(2 Marks)

Exercise 4.3
For fixed $a, b, c \in \mathbb{R}$, $a \geq 0$, find $\alpha, \beta \in \mathbb{R}$, such that

$$\lim_{x\to\infty} \left(\sqrt{ax^2 + bx + c} - \alpha x - \beta\right) = 0.$$

Having found such $\alpha, \beta \in \mathbb{R}$, can there exist different numbers $\alpha', \beta' \in \mathbb{R}$ such that $\lim_{x\to\infty} (\sqrt{ax^2 + bx + c} - \alpha' x - \beta') = 0$? Explain!
(4 Marks)

Exercise 4.4

i) For which α is $\sqrt{1 + x^4} = O(x^\alpha)$ as $x \to 0$?
(1 Mark)

ii) For which α is $\sqrt{1 + x^4} = o(x^\alpha)$ as $x \to 0$?
(1 Mark)

iii) Show that $f(x) = o(\phi(x))$ as $x \to 0$ if and only if $\lim_{x\to 0} \frac{|f(x)|}{|\phi(x)|} = 0$.
(2 Marks)

iv) Find functions f and ϕ such that $f(x) = O(\phi(x))$ as $x \to 0$ but $\lim_{x\to 0} \frac{|f(x)|}{|\phi(x)|}$ does not exist.
(1 Mark)

Exercise 4.5
Interpret and prove the following relations as $x \to x_0 \in \mathbb{R}$:

$$\begin{aligned}
O(f(x)) + O(g(x)) &= O(|f(x)| + |g(x)|), \\
O(f(x))O(g(x)) &= O(f(x)g(x)), \\
O(f(x))o(g(x)) &= o(f(x)g(x)), \\
O(O(f(x))) &= O(f(x)), \\
o(O(f(x)) &= o(f(x)).
\end{aligned}$$

(7 Marks)

Exercise 4.6
Let $\lceil x \rceil := \min\{n \in \mathbb{Z} \colon n \geq x\}$ denote the smallest integer greater than $x \in \mathbb{R}$. Sketch the following real functions f and state (without proof) at which points of their domain they are continuous.

i) $f(x) = \lceil x \rceil$, ii) $f(x) = \lceil x \rceil - x$, iii) $f(x) = \sqrt{\lceil x \rceil - x}$,

iv) $f(x) = \lceil x \rceil + \sqrt{\lceil x \rceil - x}$, v) $f(x) = \left\lceil \frac{1}{x} \right\rceil$, vi) $f(x) = \frac{1}{\lceil \frac{1}{x} \rceil}$,

(12 Marks)

Exercise 4.7
A number $p/q \in \mathbb{Q}$, $p, q \in \mathbb{Z}$, $q \neq 0$, is in *lowest terms* if p and q have no common factor and $q > 0$. We assume (without proof) that every $x \in \mathbb{Q} \setminus \{0\}$ has a unique representation in lowest terms. Define

$$f \colon [0,1] \to \mathbb{R}, \qquad f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q} \text{ or } x = 0, \\ \frac{1}{q} & \text{if } x \in \mathbb{Q} \text{ and } x = p/q \text{ in lowest terms.} \end{cases}$$

Plot f as well as you can and state (without proof) at which points f is continuous
(2 Marks)

Exercise 4.8
Suppose that $f\colon \mathbb{R} \to (0, \infty)$ is continuous and satisfies $\lim\limits_{x\to-\infty} f(x) = \lim\limits_{x\to\infty} f(x) = 0$.

i) Show that f attains its maximum, i.e., there exists some $x_0 \in \mathbb{R}$ such that $f(x_0) \geq f(x)$ for all $x \in \mathbb{R}$.
(2 Marks)

ii) Let x_0 be given as in i) above and let $y_0 := f(x_0)$. Show that $\operatorname{ran} f = (0, y_0]$, i.e., for every $\eta \in (0, y_0]$ there exists some $\xi \in \mathbb{R}$ such that $f(\xi) = \eta$.
(2 Marks)

Exercise 4.9

i) Prove that $f(x) = 1/x$ is not uniformly continuous on $(0, 1]$.
(2 Marks)

ii) Prove that $f(x) = 1/x$ is uniformly continuous on $[1, \infty)$.
(2 Marks)

iii) Prove that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.
(2 Marks)