# Vv186 Honors Mathematics II
# Functions of a Single Variable
# Assignment 7

**Date Due: 8:00 AM, Thursday, the 12th of November 2020**

*"Logic moves in one direction, the direction of clarity, coherence and structure. Ambiguity moves in the other direction, that of fluidity, openness, and release. Mathematics moves back and forth between these two poles. [...] It is the interaction between these different aspects that gives mathematics its power."*

William Byers, *How Mathematicians Think*, Princeton University Press, 2007

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This assignment has a total of **(39 Marks)**.

**Exercise 7.1**

In this exercise, $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ denote elements of $\mathbb{R}^n$.

i) Use (1) to prove *Hölders inequality*:

$$\sum_{j=1}^{n} |x_j y_j| \le \Big(\sum_{j=1}^{n} |x_j|^p\Big)^{1/p} \Big(\sum_{j=1}^{n} |y_j|^q\Big)^{1/q} \qquad \text{for } p \in \mathbb{N} \setminus \{0\}, \qquad \frac{1}{p} + \frac{1}{q} = 1$$

*Hint:* first deduce from Jensen's inequality that $a \cdot b \le a^p/p + b^q/q$ for $a, b > 0$ and $p^{-1} + q^{-1} = 1$. Then let $\widetilde{x} = x/\|x\|_p$, $\widetilde{y} = y/\|y\|_q$ and show that $\sum |\widetilde{x}_j \widetilde{y}_j| \le 1$.
**(2 Marks)**

ii) Prove *Minkowski's Inequality*:

$$\Big(\sum_{j=1}^{n} |x_j + y_j|^p\Big)^{1/p} \le \Big(\sum_{j=1}^{n} |x_j|^p\Big)^{1/p} + \Big(\sum_{j=1}^{n} |y_j|^p\Big)^{1/p} \qquad p \in \mathbb{N} \setminus \{0\}$$

*Hint:* use $|x_j + y_j|^p \le (|x_j| + |y_j|)|x_j + y_j|^{p-1}$. Then apply Hölder's inequality. It will be useful to prove that $p^{-1} + q^{-1} = 1$ implies $(p-1)q = p$.
**(2 Marks)**

iii) Deduce that

$$\|x\|_p := \Big(\sum_{j=1}^{n} |x_j|^p\Big)^{\frac{1}{p}},$$

defines a norm on $\mathbb{R}^n$ for all $p \in \mathbb{N} \setminus \{0\}$.
**(2 Marks)**

iv) Show that $\|x\|_p \ge \|x\|_q$ for $p < q$, $p, q \in \mathbb{N} \setminus \{0\}$ and $x \ne 0$. *Hint*: Set $\xi := \|x\|_p$ so $|x_j|/\xi \le 1$ for $j = 1, \dots, n$. Deduce $(|x_j|/\xi)^q \le (|x_j|/\xi)^p$ for $p < q$ and hence

$$\sum_{j=1}^{n} \left(\frac{|x_j|}{\xi}\right)^q \le \sum_{j=1}^{n} \left(\frac{|x_j|}{\xi}\right)^p = 1.$$

**(2 Marks)**

v) Verify that $\|x\|_\infty := \max\limits_{1 \le j \le n} |x_j|$ defines a norm on $\mathbb{R}^n$ and prove that $\|x\|_\infty = \lim\limits_{p \to \infty} \|x\|_p$ for any $x \in \mathbb{R}^n$ .
*Hint:* Use that $\lim\limits_{m \to \infty} \sqrt[m]{y} = 1$ for fixed $y > 0$. (See Exercise 6.7.)
**(2 Marks)**

**Exercise 7.2**
Which of the following objects define real vector spaces? The addition and scalar multiplication of vectors is taken to be pointwise.

i) $(\{x = (x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 \leq 0\}, +, \cdot)$,

ii) $(\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 x_n = 0\}, +, \cdot)$,

iii) $(\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 + 5x_2 = 0\}, +, \cdot)$,

**(3 Marks)**

**Exercise 7.3**
Calculate[1] the pointwise limit, if it exists, for each of the following function sequences $\{f_n\}_{n\in\mathbb{N}}$ on the given domain and decide whether the convergence is uniform.

i) $f_n(x) = \sqrt[n]{x}, \quad \operatorname{dom} f_n = [0, 1]$,

ii) $f_n(x) = \dfrac{nx}{1 + n + x}, \quad \operatorname{dom} f_n = [0, \infty)$,

iii) $f_n(x) = \begin{cases} 0 & x \leq n \\ 1/x & x > n \end{cases}, \quad \operatorname{dom} f_n = \mathbb{R}$,

iv) $f_n(x) = \sqrt{1/n + x} - \sqrt{x}, \quad \operatorname{dom} f_n = (0, \infty)$,

v) $f_n(x) = n\big(\sqrt{1/n + x} - \sqrt{x}\big), \quad \operatorname{dom} f_n = (0, \infty)$,

**(10 Marks)**

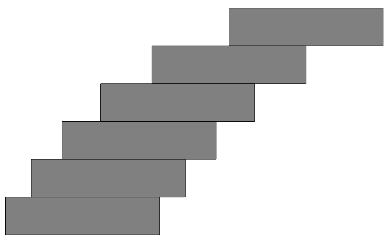
**Exercise 7.4**
A ball is dropped from height $h$ on an even surface and bounces. At each bounce, the ball attains $r$ times $(0 < r < 1)$ the height of the last bounce. What is the total distance travelled by the ball?
**(2 Marks)**

**Exercise 7.5**
It is well-known that the harmonic series $\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$ diverges. But what happens if all summands that contain the number 9 are eliminated? Denote by $X$ the set of all positive natural numbers not containing the number 9,

$$X = \{1, 2, 3, 4, 5, 6, 7, 8, 10, 11, \dots, 18, 20, 21, \dots, 87, 88, 100, 101, \dots\}.$$

Investigate the convergence of $\sum_{n\in X} \frac{1}{n}$ and $\sum_{n\in\mathbb{N}^*\setminus X} \frac{1}{n}$.
**(4 Marks)**

**Exercise 7.6**
We want to build a tower using an infinite number of bricks by stacking them onto (or *under*) each other. At each level of the tower we may use only a single brick. How far can the tower extend horizontally without toppling?
**(4 Marks)**



**Exercise 7.7**
Analyze the convergence of the following series:

$$\sum_{n=1}^{\infty} \frac{2^{2n}3^{3n}}{5^{3n}}, \qquad \sum_{n=1}^{\infty} \frac{n+4}{n^2 - 3n + 1}, \qquad \sum_{n=1}^{\infty} \frac{n^4}{3^n}, \qquad \sum_{n=1}^{\infty} \frac{2^n}{n!}, \qquad \sum_{n=1}^{\infty} \left(\frac{2}{n}\right)^n, \qquad \sum_{n=1}^{\infty} \frac{n}{10n^3 - 100}$$

**(6 Marks)**

[1]See *Spivak*, Ch. 23, Ex. 2