Vv186 Honors Mathematics II

Functions of a Single Variable

Assignment 11

This assignment will not be graded. Marks assigned to the exercises are for reference only.

"Il est facile de voir que..." [1]

Pierre-Simon Laplace (1749–1827)

"Whenever I meet in La Place with the words "Thus it plainly appears," I am sure that hours, and perhaps days, of hard study will alone enable me to discover how *it plainly appears."*

Nathaniel Bowditch, *Memoir of Nathaniel Bowditch* (1839) p. 62.

Exercise 11.1

Calculate the following integrals:

$$\text{i)} \int e^x \sin x\,dx, \quad \text{ii)} \int \tan x\,dx, \quad \text{iii)} \int \frac{1}{x^2(1+x)^2}\,dx, \quad \text{iv)} \int e^{x^2} x(1+x^2)\,dx, \quad \text{v)} \int \sqrt{\frac{1-x}{1+x}}\,dx.$$

(5 Marks)

Exercise 11.2

Calculate the indefinite integral

$$\int \frac{dx}{ax^2+bx+c}, \qquad a,b,c \in \mathbb{R}.$$

Take care to distinguish various cases according to the sign of $\Delta := 4ac - b^2$.

(4 Marks)

Exercise 11.3

The *Weierstrass substitution* allows the transformation of any rational integrand containing sine and cosine functions into a purely rational integrand.

i) Show that the substitution $t = \tan\frac{x}{2}$, $-\pi < x < \pi$, gives the following identities

$$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2} \qquad dx = \frac{2}{1+t^2}\,dt. \tag{$*$}$$

(2 Marks)

ii) Use ($*$), to calculate the integrals a), b) and to derive formula c),

$$\text{a)} \int \frac{1}{\sin x}\,dx \qquad \text{b)} \int \frac{1}{\cos x}\,dx \qquad \text{c)} \int \frac{1}{\sin x + \cos x}\,dx = \frac{1}{\sqrt{2}} \ln\left|\tan\left(\frac{x}{2} + \frac{\pi}{8}\right)\right|$$

Note: $\tan\frac{\pi}{4} = 1$, $\tan\frac{\pi}{8} = \sqrt{2} - 1 = (1+\sqrt{2})^{-1}$, $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$.

(1 Mark)+**(2 Marks)**+**(3 Marks)**

Exercise 11.4

Through *trigonometric substitution* integrands of the form $\sqrt{\pm x^2 \pm \beta^2}$ may be treated. The substitution rules are listed below:

Expression	Substitution		Identity
$\sqrt{\beta^2 - x^2}$	$x = \beta \sin\theta,$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{\beta^2 + x^2}$	$x = \beta \tan\theta,$	$-\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = 1/\cos^2\theta$
$\sqrt{x^2 - \beta^2}$	$x = \beta/\cos\theta,$	$0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$	$1/\cos^2\theta - 1 = \tan^2\theta$

[1] "It is therefore obvious that..." Frequently used in the Traité de mécanique céleste when Laplace had proved something and mislaid the proof, or found it clumsy. Notorious as a signal for something true, but hard to prove.

Use trigonometric substitution to calculate the integrals

$$J(a) = \int_0^1 \sqrt{1+a^2x^2}\,dx, \qquad I(a) = \int \frac{dx}{\sqrt{a^2-x^2}}, \qquad a > 1.$$

(6 Marks)

Exercise 11.5

i) Decide whether or not the following improper integrals converge:

$$\int_0^\infty \frac{1}{\sqrt{1+x^3}}\,dx, \qquad \int_0^\infty \frac{x}{1+x^{3/2}}\,dx, \qquad \int_0^\infty \frac{1}{x\sqrt{1+x}}\,dx.$$

(3 Marks)

ii) Find a continuous function $f\colon [0,\infty) \to \mathbb{R}$ such that $\sup\limits_{x\in[0,\infty)} f(x)$ does not exist but $\int_0^\infty f(x)\,dx$ converges. (This example shows that the analogy between improper integrals and series is limited; if $\sum_{n=0}^\infty a_n$ converges, then $a_n \to 0$ as $n \to \infty$.)
(1 Mark)

Exercise 11.6
Calculate the following integrals:

$$\text{i)} \int_0^1 \ln x\,dx, \qquad \text{ii)} \int \frac{\ln\ln x}{x}\,dx, \qquad \text{iii)} \int \tan^2 x\,dx, \qquad \text{iv)} \int_1^{e^{\pi/2}} \sin(\ln x)\,dx, \qquad \text{v)} \int_0^{\frac{\pi}{2}} \ln(\sin x)\,dx.$$

(6 Marks)

Exercise 11.7
Find the Taylor polynomial of degree 2 at $x_0 = 0$ for

$$g\colon \mathbb{R} \to \mathbb{R}, \qquad g(x) = \frac{1}{(1+\sin x)^2}$$

(2 Marks)

Exercise 11.8
Prove that the function $f\colon \mathbb{R} \to \mathbb{R}$,

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is differentiable any number of times. Calculate the Taylor series $T_{f,0}$ of f at $x_0 = 0$. Deduce that for any $n \in \mathbb{N}$ there exists a $c_n > 0$ such that $f(x) \le c_n \cdot |x|^n$ for x in a neighborhood of 0.
(3 Marks)

Exercise 11.9
Let $a_n := \int_0^{\pi/2} \sin^n x\,dx$.

i) Show that $\{a_n\}_{n\in\mathbb{N}}$ is a convergent sequence by establishing that it is bounded below and decreasing.
(1 Mark)

ii) Prove the recursion formula

$$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx \qquad (*)$$

and calculate $\int_0^{\pi/2} \sin^2 x\,dx$, $\int_0^{\pi/2} \sin x\,dx$.
(2 Marks)

iii) Using ii) together with mathematical induction, find expressions for a_{2k} and a_{2k+1}, $k \in \mathbb{N}$. Deduce *Wallis's Product formula*

$$\frac{\pi}{2} = \frac{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8}{1\cdot3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9}\cdots$$

(2 Marks)

Exercise 11.10

We consider the so-called *complete elliptic integral of the first kind*,

$$T(a,b) := \frac{2}{\pi}\int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}, \qquad a,b \in \mathbb{R}\setminus\{0\}.$$

(This integral plays an important role when computing the exact period of a pendulum.)

i) Substitute $t := b\tan\theta$ to obtain

$$T(a,b) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{dt}{\sqrt{(a^2+t^2)(b^2+t^2)}}.$$

ii) Substitute $u := (t - ab/t)/2$ to obtain

$$T(a,b) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{du}{\sqrt{\left(\left(\frac{a+b}{2}\right)^2 + u^2\right)\left(ab+u^2\right)}}.$$

iii) Let $M(a,b)$ denote the arithmetic-geometric mean of $a,b\in\mathbb{R}$ (see question 3.6). Show that

$$T(a,b) = \frac{1}{M(a,b)}.$$

(2 + 2 + 3 Marks)

Exercise 11.11

i) Integrate the binomial series to show that for $|x| \le 1$

$$\begin{aligned}\arcsin x &= x + \frac{1}{2}\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7} + \dots \\ &= \sum_{n=0}^{\infty} \frac{(2n+1)!!}{2^n n!}\frac{x^{2n+1}}{(2n+1)^2}\end{aligned} \tag{1}$$

where $(2n+1)!! = 1\cdot 3\cdot 5\cdot\dots\cdot(2n-1)\cdot(2n+1)$.
(3 Marks)

ii) Set $x = \sin t$ and integrate both sides of (1) from 0 to $\pi/2$, using $\int_0^{\pi/2}\sin^n x\,dx = \frac{n-1}{n}\int_0^{\pi/2}\sin^{n-2}x\,dx$. Deduce

$$\frac{\pi^2}{8} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}.$$

(2 Marks)

iii) Show that

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{3}{4}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

and deduce the *Euler series*

$$\frac{\pi^2}{6} = \sum_{n=1}^{\infty}\frac{1}{n^2}.$$

(2 Marks)

Remark. Finding the sum of $\sum\frac{1}{n^2}$ is a famous problem in mathematics. The following excerpt is quoted from Wikipedia:[2]

[2] Wikipedia contributors, "Basel problem," Wikipedia, The Free Encyclopedia, http://en.wikipedia.org/wiki/Basel_problem (accessed December 9, 2010).

> The Basel problem is a famous problem in number theory, first posed by Pietro Mengoli in 1644, and solved by Leonhard Euler in 1735. Since the problem had withstood the attacks of the leading mathematicians of the day, Euler's solution brought him immediate fame when he was twenty-eight. Euler generalised the problem considerably, and his ideas were taken up years later by Bernhard Riemann in his seminal 1859 paper "On the Number of Primes Less Than a Given Magnitude", in which he defined his zeta function and proved its basic properties. The problem is named after Basel, hometown of Euler as well as of the Bernoulli family, who unsuccessfully attacked the problem.

It is also interesting that $\sum \frac{1}{n^4} = \frac{\pi^4}{90}$ (we will prove this in the future) but that until today there exists no closed form expression for $\sum \frac{1}{n^3}$.