VV186 Big RC for Mid 1

Sequence

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Overview

Convergence

- Definition1:

$$\forall \varepsilon > 0, \exists N \in N^*, \forall n > N, |a_n - a| < \varepsilon \Leftrightarrow \lim_{n \to \infty} a_n = a$$

Convergence

- Definition2:

$$\lim_{n\to\infty} a_n = a \quad :\Leftrightarrow \quad \underset{\varepsilon>0}{\forall}\ \underset{N\in\mathbb{N}}{\exists}\ \underset{n>N}{\forall}\ a_n \in B_\varepsilon(a)$$

where ε - neighborhood is

$$B_\varepsilon(a) = \{z \in X : |a - z| < \varepsilon\}$$

$\varepsilon > 0$, $a \in X$ for $X = \mathbb{R}$ or $\mathbb{C}$.

So we can get the description in words: For sufficiently large n, a_n is arbitrarily close to a.

Convergence

- Definition3: $\forall \varepsilon > 0, \exists N \in N^*, n > N$, there are only finite number of $\{a_n\}$ in $\mathbb{R} \setminus (a-\varepsilon, a+\varepsilon)$

Divergence

- Divergent: A sequence that does not converge (to any limit).
- Divergent to infinity:

$$\underset{C>0}{\forall} \quad \underset{N\in\mathbb{N}}{\exists} \quad \underset{n>N}{\forall} \quad a_n > C \Leftrightarrow \lim_{n\to\infty} a_n = \infty$$

- Divergent to minus infinity:

$$\underset{C<0}{\forall} \quad \underset{N\in\mathbb{N}}{\exists} \quad \underset{n>N}{\forall} \quad a_n < C \Leftrightarrow \lim_{n\to\infty} a_n = -\infty$$

Properties of Convergent Sequences

- They can only have one limit.
- They are bounded.
- Every subsequence converge to the same value as the sequence
  1. If there is a subsequence which is not convergent then the sequence is not convergent
  2. If two subsequences have different convergent values, then the sequence is not convergent.

Properties of Convergent Sequences

If $\{a_n\}$ and $\{b_n\}$ are convergent,

$$\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$$

$$\lim_{n\to\infty} a_n b_n = \lim_{n\to\infty} a_n \lim_{n\to\infty} b_n,$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim\limits_{n\to\infty} a_n}{\lim\limits_{n\to\infty} b_n}.$$

Properties of Convergent Sequences

- $a_n \to a \quad \Leftrightarrow \quad a_n - a \to 0$
- $a_n \to 0 \quad \Leftrightarrow \quad |a_n| \to 0$
- $\lim_{n\to\infty} a_n = a, \alpha < a < \beta \Rightarrow \alpha < a_n < \beta$ for sufficient large n.
- $\lim_{n\to\infty} a_n = a, \lim_{n\to\infty} b_n = b, a < b \Rightarrow a_n < b_n$ for sufficient large n.
- $\lim_{n\to\infty} a_n = a, \lim_{n\to\infty} b_n = b, a_n \leq b_n$ for sufficient large n$\Rightarrow a \leq b$

Squeeze theorem

$$a_n < b_n < c_n, n \in N^*, \lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = a \Rightarrow \lim_{n\to\infty} b_n = a$$

Complex Sequence

r_n: real sequence with $\lim_{n\to\infty} r_n = 0$

z_n: complex sequence with $|z_n| < r_n$ for sufficiently large n,
we have $z_n \to 0$.

Complex Sequence

z_n: complex sequence with $z_n = x_n + iy_n$

$z = x + iy \in \mathbb{C}$

we have $z_n \to z \quad \Longleftrightarrow \quad (x_n \to x \quad \text{and} \quad y_n \to y)$

Definition

A real sequence is

- increasing: $\underset{n\in\mathbb{N}}{\forall} a_n \leq a_{n+1}$
- decreasing: $\underset{n\in\mathbb{N}}{\forall} a_n \geq a_{n+1}$
- strictly increasing: $\underset{n\in\mathbb{N}}{\forall} a_n > a_{n+1}$
- strictly decreasing: $\underset{n\in\mathbb{N}}{\forall} a_n < a_{n+1}$
- monotonic if either increasing or decreasing

Properties

- Monotonous Bounded sequences all have limits.
- Every real sequence has a monotonous subsequence.

Exercise

$a_1 = 1$, $a_{n+1} = 3 - \frac{1}{a_n}$

1. discuss its monotonicity
2. prove it converges
3. find its limit

Solution

(1) $a_{n+1} - a_n = 3 - \frac{1}{a_n} - a_n = \frac{1}{a_n}(-a_n^2 + 3a_n - 1)$

$-a_n^2 + 3a_n - 1 \geq 0 \Rightarrow \frac{3-\sqrt{5}}{2} \leq a_n \leq \frac{3+\sqrt{5}}{2}$. (1')

In order to let $a_{n+1} > a_n$, we must let $a_n \in [\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}]$.

We use induction to prove this for all $n \in \mathbb{N}^+$.

$n=1$: $a_1 = 1 > \frac{3-\sqrt{5}}{2}$

Assume $\frac{3-\sqrt{5}}{2} \leq a_n \leq \frac{3+\sqrt{5}}{2}$, $a_{n+1} \geq 3 - \frac{2}{3-\sqrt{5}} = \frac{3-\sqrt{5}}{2}$

$a_{n+1} \leq 3 - \frac{2}{3+\sqrt{5}} = \frac{3+\sqrt{5}}{2}$. (2')

So the induction has completed. So $(a_n) \nearrow$.

Solution

(2). $1 \le a_n < \frac{3+\sqrt{5}}{2} \Rightarrow (a_n)$ bounded; $(a_n)\nearrow \Rightarrow (a_n)$ monotonic

So it converges. (1')

(3) As the limit exists, for sufficiently large n,

we have $a = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1} = 3 - \lim_{n\to\infty} \frac{1}{a_n}$ (1')

$\Rightarrow a = \frac{3-\sqrt{5}}{2}$ or $a = \frac{3+\sqrt{5}}{2}$. As $(a_n)\nearrow$ & $a_1 = 1$,

we have $a = \frac{3+\sqrt{5}}{2}$. (1')

Accumulation Points

Abstract definition: $\underset{\varepsilon>0}{\forall}\underset{N\in\mathbb{N}}{\forall}\underset{n>N}{\exists}\,|a_n - a| < \varepsilon$

Accumulation Points

Concrete definition: All of the limits of the subsequences.

Note:

1. The number of subsequences can be infinite. I made a mistake during my RC3
2. If a sequence converges, then the limit is the only accumulation point.
3. Every bounded real sequence has an accumulation point.

Exercise

Let (a_n) be a sequence of real numbers.

☐ If (a_n) is convergent, then (a_n) is bounded.

☐ If (a_n) is bounded, then (a_n) is convergent.

☐ If $a_n \leq a$ for some $a \in \mathbb{R}$, then $a_n \to a$

☐ If $a_n \leq a$ for some $a \in \mathbb{R}$ and (a_n) is strictly increasing, then $a_n \to a$.

Solution

1. (a_n) convergent $\Rightarrow$ (a_n) bounded.

2. (a_n) bounded $\Rightarrow$ (a_n) convergent subsequence $\not\Leftrightarrow$ (a_n) convergent

   (may have more than 1 accumulation points).

3. $a_n \le a \Rightarrow a_n$ bounded above $\overset{?}{\not\Rightarrow}$ $a_n \to a$.

   i.e. $a_n = 1 - \frac{1}{n} \to 1$, $a_n \le 2$, $a \to 2$?

4. $a_n \le a$, $a_n \nearrow$ Same as 3.

limsup and *liminf* for Sets

- limsup: infimum of almost upper bounds
- liminf: supreme of almost lower bounds

Exercise

Let A be an infinite bounded set. Prove that

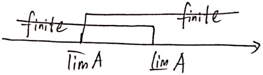
(a) $\underline{\lim} A \leq \overline{\lim} A$

(b) $\overline{\lim} A \leq \sup A, \underline{\lim} A \geq \inf A$

(c) If $\overline{\lim} A < \sup A$, then $\max A$ exists. If $\underline{\lim} A > \inf A$, then $\min A$ exists.

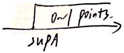
Solution

(a). If $\underline{\lim}\, A > \overline{\lim}\, A$



$\Rightarrow$ A: finite set.

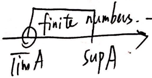
Contradictory.

(b). $\sup A$: 

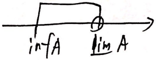
· B. the set of almost upper bounds.

$\sup A \in B$, $\overline{\lim}\, A = \inf B < \sup A$.

$\inf A$. is similar.

(c). 

$\longrightarrow$ must find a maximum value $\Rightarrow$ $\max A$ exists.



similar.

limsup and *liminf* for Sequences

Definition provided by hss:

- limes superior: $\overline{\lim_{n\to\infty}}\, x_n = \lim_{n\to\infty} \sup\{x_m : m \geq n\} = \lim_{n\to\infty} \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\}$
- limes inferior: $\underline{\lim_{n\to\infty}}\, x_n = \lim_{n\to\infty} \inf\{x_m : m \geq n\} = \lim_{n\to\infty} \inf\{x_n, x_{n+1}, x_{n+2}, \ldots\}$

limsup and *liminf* for Sequences

Another equivalent definition:

- A: the set of all accumulation points
- $\varliminf_{n\to\infty} x_n = \min A$
- $\varlimsup_{n\to\infty} x_n = \max A$
- Why are they equivalent?

Proof

They are equivalent: $\longrightarrow$ any accumulation point. in A.

I. prove $\underline{\lim}_{n\to\infty} X_n \le a \le \overline{\lim}_{n\to\infty} A$

$y_k = \inf_{n\ge k} X_n$, $z_k = \sup_{n\ge k} X_n$,

$a \in A \ni . \exists \lim_{k\to\infty} X_{n_k} = a$. (squeeze)

Since $y_k \le X_{n_k} \le z_k$ for all $k \in N$

we have $\lim_{k\to+\infty} y_k \le \lim_{k\to\infty} X_{n_k} \le \lim_{k\to\infty} z_k$.

So $\underline{\lim}_{n\to\infty} X_n \le a \le \overline{\lim}_{n\to\infty} X_n$

Proof

I. prove $\underline{\lim}_{n\to\infty} X_n$ & $\overline{\lim}_{n\to\infty} X_n$ are also accumulation points.

(X_n). has $\inf_{n\geq k} X_n = y_k$. We first find a $n_{k-1} \in \mathbb{N}$,

then find a $m_k > n_{k-1}$, then find a $n_k \in \mathbb{N}$, $n_k \geq m_k$.

which makes $\underset{=\inf_{n\geq m_k}(X_n)}{y_{m_k}} \leq X_{n_k} \leq y_{m_k} + \frac{1}{k}$ $\quad X_{n_k} \in \{X_{m_k}, X_{m_k+1}, \cdots\}$.

(Recall the definition of infimum: ① $X \geq \alpha$ ② $\forall \varepsilon > 0$. $\exists X_\varepsilon$, $X_\varepsilon < \alpha + \varepsilon$)

So $\lim_{k\to\infty} X_{n_k} = \lim_{k\to\infty} y_{m_k}$ (As y_n converge, y_{m_k} as a subsequence, converge to the same point)

$\lim_{k\to\infty} X_{n_k}$ = one accumulation point

$\lim_{k\to\infty} y_{m_k} = \underline{\lim}_{n\to\infty} X_n$.

$\overline{\lim}_{n\to\infty} X_n$ is similar.

Some Properties

These three sentences are equivalent for a bounded real sequence x_n:

1. $\varliminf\limits_{n\to\infty} x_n = \varlimsup\limits_{n\to\infty} x_n = a$
2. $\lim\limits_{n\to\infty} x_n = a$
3. x_n only has one accumulation point a

Proof

② ⟶ ③: obvious. (If a sequence converges, then the limit is the only accumulation point)

③ ⟶ ①: obvious. (If a is the only accumulation point, then $\max A = \min A = a$,)

① ⟶ ②: $y_n = \inf_{k \geq n} \{x_k\}$, $z_n = \sup_{k \geq n} \{x_k\}$.

$$y_n \leq x_n \leq z_n.$$

Since $\lim_{n\to\infty} y_n = \lim_{n\to\infty} z_n = a$. (squeeze)

$$\lim_{n\to\infty} x_n = a$$

Some Properties

If we treat the all the elements of sequence as a set, then the definition for sets and for sequences are actually equivalent

- $\overline{\lim_{n\to\infty}} x_n = \overline{\lim} \operatorname{ran}(x_n)$
- $\underline{\lim_{n\to\infty}} x_n = \underline{\lim} \operatorname{ran}(x_n)$

Actually these three definitions all describes the minimum point to let its right hand side only has finite number of values and the maximum point to let its left hand side only has finite number of values

Exercise

Let $A \subset \mathbb{R}$ be a non-empty set.

☐ If $\inf A$ exists, then $\underline{\lim} A$ exists.

☐ If $\underline{\lim} A$ exists, then $\inf A$ exists.

☐ $\underline{\lim} A$ exists if and only if A is bounded below.

☐ $\inf A$ exists if and only if A is bounded below.

Proof

Set A as a finite set, then we can find both the wrong answers.

Thank You For Your Listening
Good Luck!