# Vv186 Honors Mathematics II

# Functions of a Single Variable

# Assignment 6

**Date Due: 8:00 AM, Thursday, the 5th of November 2020**

*There are no such things as applied sciences, only applications of science.*

Louis Pasteur, *Address*, September 11th, 1872

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This assignment has a total of **(29 Marks)**.

**Exercise 6.1**

Find the local and global maximum and minimum points[1] for the following functions defined on the given domain $I \subset \mathbb{R}$:

$$\begin{aligned} f(x) &= x^5 + x + 1 & I &= [-1, 1], \\ f(x) &= \frac{1}{x^5 + x + 1} & I &= [-1/2, 1], \\ f(x) &= \frac{x}{x^2 - 1} & I &= [0, 1) \cup (1, 5]. \end{aligned}$$

**(6 Marks)**

**Exercise 6.2**

Prove the following statement:[2] Let $I = [x_0, x_1] \subset \mathbb{R}$ or $I = [x_0, \infty)$ and $f, g \colon I \to \mathbb{R}$ be differentiable on the interior of $I$ and continuous on $I$. Assume that $f(x_0) \le g(x_0)$ and $f' \le g'$ on $\operatorname{int} I$. Then $f \le g$ on $I$.
**(2 Marks)**

**Exercise 6.3**

What is wrong[3] with the following use of l'Hopital's rule,

$$\lim_{x \to 1} \frac{x^3 + x - 2}{x^2 - 3x + 2} = \lim_{x \to 1} \frac{3x^2 + 1}{2x - 3} = \lim_{x \to 1} \frac{6x}{2} = 3?$$

**(2 Marks)**

**Exercise 6.4**

i) Show[4] that $f \colon \Omega \to \mathbb{R}$ is strictly convex on $I \subset \Omega$ if and only if for all $x, y \in I$ with $x < y$ and all $t \in (0, 1)$

$$f(tx + (1-t)y) < tf(x) + (1-t)f(y).$$

**(1 Mark)**

ii) Prove[5] that if $I = (a, b) \subset \Omega$ is an open interval and $f \colon \Omega \to \mathbb{R}$ is strictly convex on $I$, then $f$ is continuous on $I$.
**(2 Marks)**

iii) Give an example of a closed interval $[a, b]$ and a strictly convex function $f$ that is not continuous on $[a, b]$.
**(1 Mark)**

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[1] See *Spivak*, Ch. 11, Ex. 1
[2] See *Spivak*, Ch. 11, Ex. 27
[3] See *Spivak*, Ch. 11, Ex. 48
[4] See *Spivak*, App. to Ch. 11, Ex. 4
[5] See *Spivak*, App. to Ch. 11, Ex. 11

**Exercise 6.5**

Sketch the following functions on their maximal domains:

$$f(x) = \frac{x^2 + 2 + x}{1 - x}, \qquad g(x) = \frac{x + x^2}{1 + x^2}.$$

You will have to calculate their domain, range, local and global extrema, asymptotes (including slant asymptotes), domains of convexity and concavity and any other information necessary for deducing the shape of their graph.

**(6 Marks)**

**Exercise 6.6**

Let $[a, b] \subset \mathbb{R}$ be an closed interval and $f : [a, b] \to \mathbb{R}$ continuous and $n + 1$ times continuously differentiable on $(a, b)$. Let $x_0 \in (a, b)$ be a fixed point. The polynomial

$$T_n(x; x_0) := f(x_0) + \frac{1}{1!} f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2 + \cdots + \frac{1}{n!} f^{(n)}(x_0)(x - x_0)^n$$

is called the *Taylor polynomial of degree $n$ at $x_0$* for the function $f$.

i) Verify that the first $n$ derivatives of $f$ and $T_n$ coincide at $x_0$, i.e., $f^{(k)}(x_0) = T_n^{(k)}(x_0; x_0)$ for all $0 \le k \le n$.

ii) The Taylor polynomial can be used to approximate $f$ near $x_0$. Calculate $T_n(x; 0)$ for the function $f : [-1, 1] \to \mathbb{R}$, $f(x) = \sqrt{1 + x}$. Use $T_3$ to find an approximate value for $\sqrt{10} = 3\sqrt{1 + 1/9}$.
**(1 Mark)**

iii) Since the Taylor polynomial is only an approximation to $f$ on $I$, we write

$$f(x) = T_n(x; x_0) + r_n(x; x_0).$$

where $r_n$ is called the *remainder term*. Suppose that $\varphi : [a, b] \to \mathbb{R}$ is continuous and differentiable on $(a, b)$ with non-vanishing derivative. Show that for every $x \in [a, b]$ there exists some $\xi \in [x_0, x]$ (or $[x, x_0]$ if $x < x_0$) such that

$$r_n(x; x_0) = \frac{\varphi(x) - \varphi(x_0)}{\varphi'(\xi) n!} f^{(n+1)}(\xi)(x - \xi)^n$$

*Hint:* Apply the Cauchy mean value theorem to $f - T_n$ and $\varphi$.
**(2 Marks)**

iv) Using $\varphi(t) = x - t$, show that

$$r_n(x; x_0) = \frac{1}{n!} f^{(n+1)}(\xi)(x - \xi)^n (x - x_0)$$

(remainder term according to Cauchy).
**(2 Marks)**

v) Using $\varphi(t) = (x - t)^{n+1}$, show that

$$r_n(x; x_0) = \frac{1}{(n + 1)!} f^{(n+1)}(\xi)(x - x_0)^{n+1}$$

(remainder term according to Lagrange).
**(2 Marks)**

**Exercise 6.7**

A special case of *Jensen's inequality* is

$$\prod_{i=1}^{n} a_i^{\lambda_i} \le \sum_{i=1}^{n} \lambda_i a_i, \qquad \text{for } \sum_{i=1}^{n} \lambda_i = 1,\ a_i \ge 0,\ \lambda_i > 0,\ 1 \le i \le n. \tag{1}$$

We will prove this inequality later. For now, assume that (1) holds and deduce that

$$\sqrt[n]{a_1 \cdot a_2 \cdots a_n} \le \frac{a_1 + \cdots + a_n}{n}$$

for $a_1, \ldots, a_n \ge 0$. Apply this result to $\sqrt[n]{n} = \sqrt[n]{\sqrt{n} \cdot \sqrt{n} \cdot 1 \cdot 1 \cdots 1}$ to establish that $\lim_{n \to \infty} \sqrt[n]{n} = 1$. Deduce further that for any fixed number $x > 0$ the sequence $(a_n)$, $a_n = \sqrt[n]{x}$, converges to 1.
**(2 Marks)**