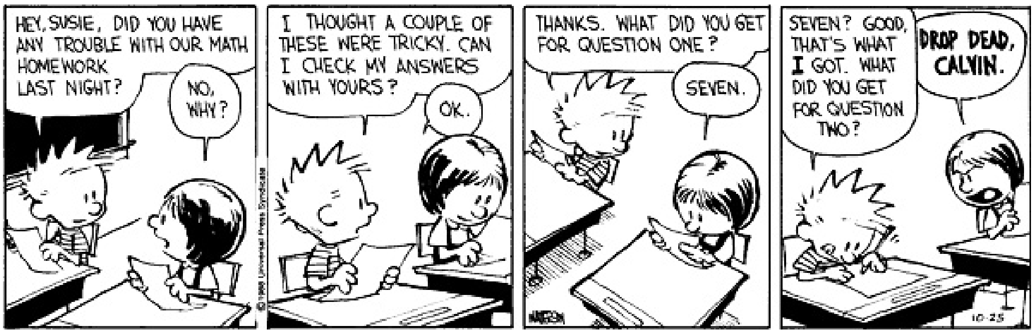
Vv186 Honors Mathematics II

Functions of a Single Variable

Assignment 8

Date Due: 8:00 AM, Thursday, the 19th of November 2020



Calvin & Hobbes by Bill Watterson

This assignment has a total of **(34 Marks)**.

Exercise 8.1
Consider the series

$$\sum_{n=1}^{\infty} \frac{2+(-1)^n}{2^{n-1}}.$$

Show that the ratio test is inconclusive and does not establish convergence or divergence. Then apply the root test to prove the convergence of the series. (Together with Exercise 8.4, this shows that the root test is "more powerful" than the ratio test.)
(3 Marks)

Exercise 8.2
Verify that the series

$$A := \sum_{k=0}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$$

is conditionally convergent. Then demonstrate that the Cauchy product of A with itself does not converge.
(2 Marks)

Exercise 8.3
Consider the series

$$\sum_{n=2}^{\infty} (-1)^n \left(\frac{1}{\sqrt{n}} + \frac{(-1)^n}{n} \right).$$

Show that $\frac{1}{\sqrt{n}} + \frac{(-1)^n}{n} > 0$ for all $n \geq 2$ so that the series is is alternating (i.e., if the nth summand is positive, the $(n+1)$st summand is negative). Show further that the series is not convergent. Does this contradict the Leibnitz criterion?
(2 Marks)

Exercise 8.4
The goal of this exercise is to prove that if

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = p, \qquad p \in [0,\infty], \tag{1}$$

for a sequence (a_n) of positive numbers, then

$$\lim_{n\to\infty} \sqrt[n]{a_n} = p.$$

(This shows that we can determine the radius of convergence for a power series by finding the limit $\lim_{n\to\infty} \frac{a_{n+1}}{a_n}$, if it exists.)

i) Let (1) hold with $p \in \mathbb{R}_+$. Let $x > p$. Show that
$$\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} < x.$$
Deduce that $\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} \le p$.
(3 Marks)

ii) Apply the same reasoning as in i) to show that $\underline{\lim_{n\to\infty}} \sqrt[n]{a_n} \ge p$. Deduce that $\lim_{n\to\infty} \sqrt[n]{a_n}$ exists and equals p.
(2 Marks)

iii) Show the statement for $p = 0$ and $p = \infty$.
(2 Marks)

Exercise 8.5
Find the radius of convergence ϱ of the following power series and analyze their convergence at $\pm\varrho$:
$$\sum_{n=3}^{\infty} \frac{n^3}{n^2-5} z^n, \qquad \sum_{n=1}^{\infty} (-1)^n \frac{n^2}{4^n + 3n} z^n, \qquad \sum_{n=1}^{\infty} \frac{1}{n!} z^n, \qquad \sum_{n=1}^{\infty} 2^{n^2+1} z^n, \qquad \sum_{n=1}^{\infty} (-1)^n \frac{n}{2^{n+1}} z^{2n},$$
Pay attention to the exponent in the last series!
(10 Marks)

Exercise 8.6
The *binomial series* is given by
$$B_\alpha(x) := \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n \qquad \binom{\alpha}{0} := 1, \qquad \binom{\alpha}{j} := \frac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!}, \qquad j \in \mathbb{N},\ \alpha \in \mathbb{R}.$$

i) Show that the binomial series has radius of convergence $\varrho = 1$ and that
$$(1+x)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} x^n, \qquad \text{for } |x| < 1.$$
Hint: Consider the derivative $\frac{d}{dx}[(1+x)^{-\alpha} B_\alpha(x)]$.
(3 Marks)

ii) Verify explicitly that for $\alpha \in \mathbb{N}$ the series is a finite sum and reduces to the binomial formula.
(2 Marks)

iii) Show that the binomial series gives
$$\sqrt{1+x} = 1 - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n} \binom{2n-2}{n-1} \frac{x^n}{n}, \qquad \text{for } |x| < 1.$$
(1 Mark)

iv) Show that at $x = -1$ the series diverges when $\alpha < 0$ and converges when $\alpha \ge 0$.
(1 Mark)

v) Show that at $x = 1$ the series diverges for $\alpha \le -1$.
(1 Mark)

vi) Show that at $x = 1$ the series with $\alpha > -1$ is alternating for sufficiently large n and converges.
Instructions:

- Show that $b_{n+1}/b_n < 0$ for $n > N_0$ for some $N_0 > 0$, so the series is alternating for $n > N_0$.
- Deduce that the sequence $|b_n|, |b_{n+1}|, \dots$ is decreasing.
- Show that $(|b_n|)_{n \ge N_0}$ converges to zero: Show that $|b_{n+1}/b_n| = 1 - (\alpha+1)/(n+1)$ and consider the quotient $|b_n/b_{N_0}| = |b_n/b_{n-1}| \cdot |b_{n-1}/b_{n-2}| \cdots |b_{N_0+1}/b_{N_0}|$. Prove that $1 - x \le \exp(-x)$ and use this estimate to estimate the product. Recall also that the harmonic series diverges.

(2 Marks)