Vv186 Honors Mathematics II

Functions of a Single Variable

Assignment 10

Date Due: 8:00 AM, Thursday, the 3rd of December 2020

"Mathematicians are like Frenchmen: whatever you say to them they translate into their own language and forthwith it is something entirely different." [1]

Johann Wolfgang von Goethe (1749-1832)

This assignment has a total of **(15 Marks)**.

Exercise 10.1
The goal of this exercise[2] is to calculate the regulated integral of $f\colon [a,b] \to \mathbb{R}$, $f(x) = x^p$ where $p > 0$ and $a > 0$.

i) We first find a partition $\mathcal{P}_n = (a_0, \dots, a_n)$, $n \in \mathbb{N} \setminus \{0\}$, of $[a,b]$ such that the *ratios* of the n partition points are equal, i.e., we require a_i/a_{i-1}, $i = 1, \dots, n$, to be the same for all i. (This is in contradistinction to the examples in class, where we set $a_i - a_{i-1} = 1/n$.) Show that for such a partition we have

$$a_i = a \cdot c^{i/n}, \qquad c = \frac{b}{a}, \qquad i = 0, \dots, n.$$

(2 Marks)

ii) Define step functions f_n, $n \in \mathbb{N} \setminus \{0\}$, on $[a,b]$ by setting $f_n(a) = a^p$ and

$$f_n(x) = a^p \cdot c^{pi/n}, \qquad x \in (a_{i-1}, a_i], \qquad i = 1, \dots, n.$$

Show that the sequence (f_n) converges uniformly to f.
(3 Marks)

iii) Calculate the integral $I(f_n)$ and show that

$$\lim_{n\to\infty} I(f_n) = \frac{b^{p+1} - a^{p+1}}{p+1}.$$

(3 Marks)

Exercise 10.2
Prove Theorem 4.1.17 of the lecture, i.e., that $\mathrm{PC}([a,b]) \subset \mathrm{Reg}([a,b])$.
(3 Marks)

Exercise 10.3
The goal of this exercise is to prove Theorem 4.1.20 of the lecture. Suppose that (f_n) is a sequence of regulated functions on $[a,b]$ that converges uniformly to a function f.

i) Prove that for any $\varepsilon > 0$ there exists a step function φ such that $\|f - \varphi\|_\infty < \varepsilon$ and conclude that f is regulated.
(2 Marks)

ii) Show that

$$\int_a^b f_n \xrightarrow{n\to\infty} \int_a^b f. \tag{1}$$

(2 Marks)

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[1] "Die Mathematiker sind eine Art Franzosen; redet man mit ihnen, so übersetzen sie es in ihre Sprache, und dann ist es alsobald ganz etwas anderes."

[2] compare with *Spivak*, Chapter 13, Exercise 4

Exercise 10.4

Use the Mean Value Theorem (of differential calculus) to prove the *Mean Value Theorem of integral calculus*: Let $[a, b] \subset \mathbb{R}$ be a closed interval and $f\colon [a, b] \to \mathbb{R}$ a continuous real function. Then there exists a $\xi \in (a, b)$ such that

$$\int_a^b f(x)\,dx = (b - a)f(\xi).$$

(2 Marks)