VV186: Honors Mathematics

Math Foundations

Xingjian Zhang

Univerity of Michigan-Shanghai Jiao Tong University Joint Institute

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About Exam

- 30 pts, possibly contain
  1. Multiple choice questions (The choices can be all wrong.)
  2. Calculation question
  3. Proof question
  4. Explanation question
  5. ...
- 100 mins
- $100/30 = 3.3$ mins/pt. Therefore, you probably don't want to spend more than 5 mins on 1 pt.
- Don't panic if you cannot figure out all some specific question. Just skip it and do it later.
- The questions may not be arranged in the order of difficulty.

Checking List

- Truth Table
- Prove by Contraposition
- Logical/Sets Operation
- Properties of Natural/Rational/Complex Numbers
- **Mathematical Induction**

Part 0 is **not** the key point of the exam. However, you should still go through these concepts. Though they might not be directly tested, they could occur somewhere in your exam paper.

Exercise 1.2

i) Let a, b be statements. Write out the truth tables to prove *de Morgan's rules*:

$$\neg(a \wedge b) \equiv \neg a \vee \neg b, \qquad \neg(a \vee b) \equiv \neg a \wedge \neg b.$$

(2 Marks)

Notice that the last column of the truth table should be an evaluation on equivalence. For example, for the first question, you should evaluate $\neg(a \wedge b) \Leftrightarrow \neg a \vee \neg b$. It is supposed to be true for the whole column. Only through this fact can you conclude that $\neg(a \wedge b) \equiv \neg a \vee \neg b$.

Please distinguish equivalence ($\Leftrightarrow$) and logical equivalence ($\equiv$). The former one is a binary operation, while the later one indicates a relationship between two compound statements. To be more precise, logical equivalence indicates that the equivalence between two compound statements is a tautology.

Contraposition

The famous tautology

$$(A \Rightarrow B) \equiv (\neg B \Rightarrow \neg A)$$

is useful in some situation. If you are asked to prove some statement $A \Rightarrow B$ but you cannot find a simple way, you can try to prove by contraposition.

Exercise

Let $x \in \mathbb{Z}$. If $x^2 - 6x + 5$ is even, then x is odd.

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Let $x \in \mathbb{Z}$. If $x^2 - 6x + 5$ is even, then x is odd.

Proof: Suppose that x is even. Then we want to show that $x^2 - 6x + 5$ is odd. Write $x = 2a$ for some $a \in \mathbb{Z}$, and plug in:

$$\begin{aligned} x^2 - 6x + 5 &= (2a)^2 - 6(2a) + 5 \\ &= 4a^2 - 12a + 5 \\ &= 2\left(2a^2 - 6a + 2\right) + 1 \end{aligned}$$

Thus $x^2 - 6x + 5$ is odd.

Mathematical Induction

Often one wants to show that some statement frame $A(n)$ is true for all $n \in \mathbb{N}$ with $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Mathematical induction works by establishing two statements:

(I) $A(n_0)$ is true.

(II) $A(n+1)$ is true whenever $A(n)$ is true for $n \geq n_0$, i.e.,

$$\forall_{\substack{n \in \mathbb{N} \\ n \geq n_0}} \big(A(n) \Rightarrow A(n+1)\big)$$

Remark: Mathematical Induction is basically the only relatively important concept in Part 0 of VV186. Please see Ex5, Ex6 and Ex7 in sample exam and see the related rubric.

Let a_n be the following expression with n nested radicals:

$$a_n = \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2 + \sqrt{2}}}}$$

Prove that $a_n = 2\cos\frac{\pi}{2^{n+1}}$

Exercise

Proof: Note that a_n can be defined recursively like this: $a_1 = \sqrt{2}$, and $a_{n+1} = \sqrt{2 + a_n}$ for $n \geq 1$. We proceed by induction. For $n = 1$ we have in fact $a_1 = \sqrt{2}$, and $2 \cos \frac{\pi}{4} = 2 \cdot \frac{1}{\sqrt{2}} = \sqrt{2}$.

Next, assuming the result is true for some $n \geq 1$, we have

$$\begin{aligned}
a_{n+1} = \sqrt{2 + a_n} &= \sqrt{2 + 2\cos\frac{\pi}{2^{n+1}}} \\
&= \sqrt{2 + 2\cos 2\frac{\pi}{2^{n+2}}} \\
&= \sqrt{2 + 2\left(2\cos^2\frac{\pi}{2^{n+2}} - 1\right)} \\
&= \sqrt{4\cos^2\frac{\pi}{2^{n+2}}} \\
&= 2\cos\frac{\pi}{2^{n+2}}
\end{aligned}$$

By induction, we conclude that $a_n = 2\cos\frac{\pi}{2^{n+1}}$.

Exercise

We define recursively the *Ulam* numbers by setting $u_1 = 1, u_2 = 2$, and for each subsequent integer n, we set n equal to the next *Ulam* number if it can be written uniquely as the sum of two different *Ulam* numbers; e.g.: $u_3 = 3, u_4 = 4, u_5 = 6$, etc. Prove that there are infinitely many *Ulam* numbers.

Proof:

We prove there are infinitely many *Ulam* numbers by induction.

For $n = 1, 2,\ u_1 = 1,\ u_2 = 2$ are defined.

Next, we assume if we have found the first $m \geq 2$ *Ulam* numbers, we can always find u_{m+1} that satisfy the recursive definition. Let $U_m = \{u_1, u_2, \ldots, u_m\}\,(m \geq 2)$ be the first m *Ulam* numbers (written in increasing order). Let S_m be the set of integers greater than u_m that can be written uniquely as the sum of two different Ulam numbers from U_m. The next Ulam number u_{m+1} is precisely the minimum element of S_m, unless S_m is empty, but it is not because $u_{m-1} + u_m \in S_m$.

Good Luck!