Vv186 Honors Mathematics II

Functions of a Single Variable

Assignment 5

Date Due: 8:00 AM, Thursday, the 29th of October 2020

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Mathematical proofs, like diamonds, are hard as well as clear, and will be touched with nothing but strict reasoning.

John Locke in *Second Reply to the Bishop of Worcester*

This assignment has a total of **(20 Marks)**.

Exercise 5.1
Let $f\colon \mathbb{R} \to \mathbb{R}$ be a real function.[1]

i) Prove that if f is even, then f' is odd.

ii) Prove that if f is odd, then f' is even.

(2 Marks)

Exercise 5.2
Let $f\colon \mathbb{R} \to \mathbb{R}$ be a real function.[2]

i) Let $\alpha > 1$. If f is continuous at $x = 0$ and $f(x) = O(x^\alpha)$ as $x \to 0$, prove that f is differentiable at $x = 0$.

ii) Let $\beta \leq 1$. If $f(x) \geq |x|^\beta$ and $f(0) = 0$, prove that f is not differentiable at $x = 0$.

(4 Marks)

Exercise 5.3
Differentiate the following functions $f\colon \Omega \to \mathbb{R}$, where Ω is their maximal domain:

$$f(x) = \sqrt{x}(1-x), \qquad f(x) = \frac{x^2 - 2\sqrt{x}}{x}, \qquad f(x) = \sqrt{3}x - \sqrt{2x}, \qquad f(x) = \sqrt[3]{x^2} - \sqrt[2]{x^3}$$

$$f(x) = (x^2+1)\sqrt[3]{x^2+2}, \qquad f(x) = \sqrt{x + \sqrt{x + \sqrt{x}}}, \qquad f(x) = \frac{(x-1)^4}{(x^2+2x)^5}, \qquad f(x) = \frac{1}{\sqrt{1+x^2}}$$

(8 Marks)

Exercise 5.4
Use mathematical induction to show the *Leibniz rule*[3] for the nth derivative of the product of two functions f, g that are n times differentiable at $x \in \mathbb{R}$:

$$(f \cdot g)^{(n)}(x) = \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(x) g^{(n-k)}(x).$$

(2 Marks)

Exercise 5.5

Calculate[4] the 100th derivative of the real functions $(x^2+3x+2)^{-1}$ and $\dfrac{x^2+1}{x^3-x}$.

(4 Marks)

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[1] See *Spivak*, Ch. 9, Ex. 23, 24

[2] See *Spivak*, Ch. 9, Ex. 16, 17

[3] See *Spivak*, Ch. 11, Ex. 18

[4] Taken from V.I. Arnol'd *A Mathematical Trivium*, Russian Math Surveys, 46:1 (1991). Note the informal approach to writing a function (which is typical for the Russian school of mathematics) and compare this with Spivak's comments on denoting a function.