Vv186 Honors Mathematics II

Sample Exercises for the First Midterm Exam

Suggested Solutions

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Exercise 1.
In the following exercises, mark the boxes corresponding to true statements with a cross (☒). In each case, it is possible that none of the statements are true or that more than one statement is true.

i) Let $A \subset \mathbb{R}$ be a non-empty set.

- ☐ If $\inf A$ exists, then $\underline{\lim}\, A$ exists.
- ☒ If $\underline{\lim}\, A$ exists, then $\inf A$ exists.
- ☐ $\underline{\lim}\, A$ exists if and only if A is bounded below.
- ☒ $\inf A$ exists if and only if A is bounded below.

ii) Let (a_n) be a sequence of real numbers.

- ☒ If (a_n) is convergent, then (a_n) is bounded.
- ☐ If (a_n) is bounded, then (a_n) is convergent.
- ☐ If $a_n \le a$ for some $a \in \mathbb{R}$, then $a_n \to a$.
- ☐ If $a_n \le a$ for some $a \in \mathbb{R}$ and (a_n) is strictly increasing, then $a_n \to a$.

(1/4 Mark) for each correctly checked or unchecked box.

Exercise 2.
Describe in words the difference between **an accumulation point of a sequence** and **a limit of a sequence**. How are these two concepts different? If you can, state some properties that they have in common or that serve to differentiate them from each other. Is one of them also always an example of the other? Give examples.
(5 Marks)

Solution. A limit of a sequence of numbers is a number such that for any given distance, all terms of the sequence eventually are within that distance of the limit. **(1 Mark)** An accumulation point is a number such that for any given distance and any given sequence term, there will always be some other succeeding sequence term within that distance of the accumulation point. **(1 Mark)**

After these two marks, a maximum of **(3 Marks)** can be obtained as follows:

Give **(1 Mark)**for any of the following statements:

- Any limit is also an accumulation point.
- A sequence can have at most a single limit, but may have zero or more accumulation points.
- An accumulation point is always a limit of a subsequence and vice-versa.
- If a sequence has a limit, it must be bounded, but the same statement is not if the wrd “limit” is replaced with “accumulation point”.
- If a Cauchy sequence has an accumulation point, that point is also the (unique) limit of the Cauchy sequence.

Give **(1/2 Mark)** for any coherent example to illustrate any of the above statements or the definitions above.

Exercise 3.

i) Consider the set $U \subset \mathbb{R}$, where $U = A \cup B \cup C$ with

$$A = \{x \in \mathbb{R} \colon 0 < x \leq 1\},$$
$$B = \{x \in \mathbb{R} \colon x = 2 - 1/n,\ n \in \mathbb{N} \setminus \{0\}\},$$
$$C = \{x \in \mathbb{R} \colon x = -1/n,\ n \in \mathbb{N} \setminus \{0\}\}.$$

State (without proof) $\min U$, $\max U$, $\inf U$, $\sup U$, $\underline{\lim}\, U$ and $\overline{\lim}\, U$ (if one or more of these do not exist, simply state this).

ii) Consider the sequence (a_n) given by

$$a_n = \frac{1}{2} + (-1)^n \frac{2+n}{2n}.$$

Calculate $\overline{\lim}\, a_n$ and $\underline{\lim}\, a_n$.

$(6 \times \frac{1}{2} + 2$ Marks)

Solution.

i)

$$\min U = -1, \qquad \max U \text{ does not exist},$$
$$\inf U = -1, \qquad \sup U = 2,$$
$$\underline{\lim}\, U = 0, \qquad \overline{\lim}\, U = 2.$$

$(6 \times \frac{1}{2}$ Mark)

ii) The subsequence (a_{2n}) converges to 1:

$$a_{2n} = \frac{1}{2} + (-1)^{2n} \frac{2+2n}{4n} = \frac{1}{2} + \frac{1}{2n} + \frac{1}{2} \xrightarrow{\to\infty} 1,$$

$(\frac{1}{2}$ Mark) while the subsequence (a_{2n+1}) converges to 0:

$$a_{2n+1} = \frac{1}{2} + (-1)^{2n+1} \frac{2+(2n+1)}{4n+2} = \frac{1}{2} - \frac{1}{2n+1} - \frac{1}{2} \xrightarrow{\to\infty} 0.$$

$(\frac{1}{2}$ Mark) These are the only two accumulation points. **$(\frac{1}{2}$ Mark)** Since $\overline{\lim}\, a_n$ is the largest and $\underline{\lim}\, a_n$ the smallest accumulation point, we have

$$\underline{\lim}\, a_n = 0, \qquad \overline{\lim}\, a_n = 1,$$

$(\frac{1}{2}$ Mark)

Exercise 4.

Suppose that f is a function such that $f \circ g = g \circ f$ for *all* functions g. Prove that $f(x) = x$, i.e., f is the identity function.

(2 Marks)

Solution. Let g be a constant function, $g(x) = c$ for all x and some c. Then

$$(f \circ g)(x) = (g \circ f)(x) = c$$

for all x. It follows that f is either constant, $f(x) = c$, or the identity, $f(x) = x$ for all x. **(1 Mark)** Now choose another function $\tilde{g}$ such that $\tilde{g}(x) = \tilde{c}$ for some $\tilde{c} \neq c$. Then

$$(f \circ \tilde{g})(x) = (\tilde{g} \circ f)(x) = \tilde{c} \neq c,$$

which contradicts $f(x) = c$ and therefore $f(x) = x$ for all x. **(1 Mark)**

Exercise 5.
Prove the following statement using induction in n:

$$\sum_{j=1}^{n} x^{n-j}y^{j-1} = \frac{x^n - y^n}{x-y}, \qquad x, y \in \mathbb{R},\, x \neq y,\, n \geq 1.$$

(3 Marks)

Solution. Award **1/2 Mark** for checking that the statement is true for $n = 1$:
$A(n = 1)$: $\quad \sum_{j=1}^{1} x^{1-j}y^{j-1} = x^0y^0 = 1 = \frac{x^1-y^1}{x-y}$
Award **1/2 Mark** for saying that "Assuming the statement is true for n, we now show that it is true for $n+1$" or some equivalent remark. Award **2 Marks** for then succesfully proving this as follows:
$A(n) \Rightarrow A(n+1)$:

$$\begin{aligned}\sum_{j=1}^{n+1} x^{n+1-j}y^{j-1} &= x\sum_{j=1}^{n+1} x^{n-j}y^{j-1} = x\Big(x^{-1}y^n + \sum_{j=1}^{n} x^{n-j}y^{j-1}\Big) \\ &= y^n + x\frac{x^n - y^n}{x-y} = \frac{y^n(x-y) + x^{n+1} - xy^n}{x-y} = \frac{x^{n+1} - y^{n+1}}{x-y}\end{aligned}$$

Exercise 6.
Prove (e.g., with mathematical induction), that the sequence

$$\sqrt{2},\ \sqrt{2+\sqrt{2}},\ \sqrt{2+\sqrt{2+\sqrt{2}}},\ \ldots$$

is increasing and bounded. (Be sure to argue carefully and precisely.) Then calculate the limit of the sequence.
(6 Marks)

Solution. The sequence can be defined recursively:

$$a_0 = 0, \qquad a_{n+1} = f(a_n) = \sqrt{2 + a_n} \quad \text{for } n \in \mathbb{N}.$$

($\frac{1}{2}$ Mark) We surmise that $a_n < 2$. We prove this by induction: clearly, $a_0 < 2$. Furthermore, if $a_n < 2$, then

$$a_{n+1} = \sqrt{2 + a_n} < \sqrt{2 + 2} = \sqrt{4} = 2.$$

Hence we have shown that $a_n < 2$ for all $n \in \mathbb{N}$. **(1 Mark)** Furthermore, $a_n > 0$ for all n, so (a_n) is bounded. **($\frac{1}{2}$ Mark)**
Next, we show that (a_n) is increasing:

$$a_{n+1} = \sqrt{2 + a_n} > \sqrt{a_n + a_n} = \sqrt{2a_n} > \sqrt{a_n \cdot a_n} = a_n.$$

(1 Mark) Since (a_n) is bounded above and increasing, it follows that a limit a of (a_n) exists. **($\frac{1}{2}$ Mark)** Since the function $f(x) = \sqrt{2+x}$ is continuous on the metric space $(\mathbb{R}_+, \varrho)$, where $\mathbb{R}_+ = \{x \in \mathbb{R} \colon x > 0\}$ and $\varrho(x, y) = |x - y|$, it follows that $a = f(a)$, **($\frac{1}{2}$ Mark)**i.e.,

$$a = \sqrt{2 + a} \quad \Leftrightarrow \quad a^2 = 2 + a \quad \Leftrightarrow \quad a^2 - a - 2 = 0.$$

This equation is solved by $a = 2$ and and $a = -1$. **($\frac{1}{2}$ Mark)** Since $a_n > 0$ for all n, it follows that $a_n \to 2$ as $n \to \infty$.**($\frac{1}{2}$ Mark)**

Exercise 7.
Show that the sequence defined by

$$a_1 = 2, \qquad a_{n+1} = \frac{1}{3 - a_n}, \qquad (n \geq 1)$$

satisfies $0 < a_n \leq 2$ and is decreasing. Deduce that the sequence is convergent and find its limit.
(4 Marks)

Solution. We first show that $0 < a_n \leq 2$ for all $n \geq 1$ by using induction. The statement is clearly true for $n = 1$. Now assume that it is true for n. Then $1 < 3 - a_n < 3$ and

$$a_{n+1} = \frac{1}{3 - a_n} \leq \frac{1}{3 - 2} = 1 < 2$$

and

$$a_{n+1} = \frac{1}{3 - a_n} \geq \frac{1}{3 - 0} = \frac{1}{3} > 0.$$

Hence the statement is then true for $n + 1$. This completes the proof by induction and shows that $0 < a_n \leq 2$ for all $n \geq 1$. **(1 Mark)**
We next show that $a_{n+1} \leq a_n$. In particular,

$$a_{n+1} \leq a_n \quad \Leftrightarrow \quad \frac{1}{3 - a_n} \leq a_n \quad \Leftrightarrow \quad 0 \leq -a_n^2 + 3a_n - 1 \tag{1}$$

(1 Mark) The zeroes of the polynomial $x^2 - 3x + 1$ are $3/2 \pm \sqrt{5/4}$ so we need to show that in fact $a_n > (3 - \sqrt{5})/2$. Again by induction, assume that $a_n > (3 - \sqrt{5})/2$. Then

$$a_{n+1} = \frac{1}{3 - a_n} \geq \frac{1}{3 - (3 - \sqrt{5})/2} = \frac{2}{6 - (3 - \sqrt{5})} = \frac{2}{3 + \sqrt{5}} = \frac{2(3 - \sqrt{5})}{3^2 - 5} = (3 - \sqrt{5})/2,$$

so we have shown that $(3 - \sqrt{5})/2 \leq a_n \leq 2 < (3 + \sqrt{5})/2$ for all $n \geq 1$. But then by (1), $a_{n+1} \leq a_n$, and (a_n) is decreasing. Since (a_n) is also bounded, a limit a exists. **(1 Mark)**
The limit a is a fixed point of the recursion equation, so by our above calculations

$$a = \frac{1}{3 - a} \quad \Leftrightarrow \quad 0 = -a^2 + 3a - 1 \quad \Leftrightarrow \quad a = (3 - \sqrt{5})/2$$

where the other root of the quadratic equation has been discarded because $a_n \leq 2 < (3 + \sqrt{5})/2$ for all n. **(1 Mark)**

Exercise 8.
Using the ε-δ definition of continuity, show that the function f given by $f(x) = 1/\sqrt{x}$ is continuous at $x = 1$. **(3 Marks)**

Solution. The function f is defined at $x = 1$ and $f(1) = 1$. **(1/2 Mark)** We hence need to show that

$$\lim_{x \to 1} f(x) = 1.$$

(1/2 Mark) By the definition of the limit, we need to show that

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x\in\mathbb{R}\setminus\{1\}}{\forall} : |x - 1| < \delta \Rightarrow \left| \frac{1}{\sqrt{x}} - 1 \right| < \varepsilon.$$

(1/2 Mark) Fix $\varepsilon > 0$. We will now find some δ such that $|x - 1| < \delta \Rightarrow \left| \frac{1}{\sqrt{x}} - 1 \right| < \varepsilon$ for all $x \in \mathbb{R} \setminus \{1\}$. Assume that $\delta < 1/2$. Then $|x - 1| < \delta$ implies that $1/2 < x < 3/2$. Therefore,

$$\left| \frac{1}{\sqrt{x}} - 1 \right| = \frac{|1 - \sqrt{x}|}{\sqrt{x}} = \frac{|1 - x|}{\sqrt{x}(1 + \sqrt{x})} < \frac{1}{\sqrt{2}} |1 - x| < |1 - x|.$$

Hence if we choose $\delta = \min(1/2, \varepsilon)$, we have that $|x - 1| < \delta$ implies

$$\left| \frac{1}{\sqrt{x}} - 1 \right| < |1 - x| < \varepsilon.$$

This proves $\lim_{x \to 1} f(x) = 1$ and hence f is continuous at $x = 1$. **(3/2 Marks)**

Exercise 9.

If $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\lim_{x\to 0} \frac{f(x)}{x} = \alpha$ for some $\alpha \in \mathbb{R}$, calculate

i) $\lim_{x\to 0} \frac{f(2x)}{x}$,

ii) $\lim_{x\to 0} \frac{[f(2x)]^2}{x^2}$,

iii) $\lim_{x\to 0} \frac{[f(2x)]^2}{x}$.

(3 × 1 Mark)

Solution.

i) $\lim_{x\to 0} \frac{f(2x)}{x} = 2 \lim_{x\to 0} \frac{f(2x)}{2x} = 2\alpha.$

ii) $\lim_{x\to 0} \frac{[f(2x)]^2}{x^2} = 4 \lim_{x\to 0} \left(\frac{f(2x)}{2x} \frac{f(2x)}{2x} \right) = 4 \left(\lim_{x\to 0} \frac{f(2x)}{2x} \right)^2 = 4\alpha^2.$

iii) $\lim_{x\to 0} \frac{[f(2x)]^2}{x} = \lim_{x\to 0} \left(x \frac{[f(2x)]^2}{x^2} \right) = \left(\lim_{x\to 0} x \right) \left(\lim_{x\to 0} \frac{[f(2x)]^2}{x^2} \right)^2 = 0 \cdot 4\alpha^2 = 0.$

(3 × 1 Mark)

Exercise 10.
Prove that the equation

$$x^2 + |x|^{5/2} - 4x + 1 = 0, \qquad x \in \mathbb{R},$$

has a solution in the interval $(1, 2)$.
(2 Marks)

Solution. Note that the left-hand side of the equation is a continuous function of x on $[1, 2]$. At $x = 1$ the left-hand side is

$$x^2 + |x|^{5/2} - 4x + 1\big|_{x=1} = 1 + 1 - 4 + 1 = -1 < 0$$

while at $x = 2$ the left-hand side is

$$x^2 + |x|^{5/2} - 4x + 1\big|_{x=2} = 4 + 4\sqrt{2} - 8 + 1 > 0$$

so by the intermediate value theorem there exists an $x \in (1, 2)$ such that the left-hand side vanishes.

Exercise 11.
Let $a > 0$ and $f\colon [0, 2a] \to \mathbb{R}$ be a continuous function with $f(0) = f(2a)$. Prove that there exists some $c \in [0, a]$ such that $f(c) = f(c + a)$.
(4 Marks)

Solution. Instead of f we consider the function $g\colon [0, 2a] \to \mathbb{R}$, $g(x) = f(x) - f(0)$. Any c such that $g(c) = g(c+a)$ will also give $f(c) - f(0) = f(c + a) - f(0)$ and thence $f(c) = f(c + a)$. We have $g(0) = g(2a) = 0$. **(1 Mark)**
We consider the function $h\colon [0, a] \to \mathbb{R}$ defined by $h(x) = g(x) - g(x + a)$. Then $h(0) = g(0) - g(a) = -g(a)$ and $h(a) = g(a) + g(2a) = g(a) + g(0) = g(a)$. **(1 Mark)**
Now $h(x) = 0 \Leftrightarrow g(x) = g(x + a) \Leftrightarrow f(x) = f(x + a)$, so we are looking for a zero of h.
If $g(a) = 0$, then $h(a) = 0$ and we have found that $g(a) = g(2a)$ and $f(a) = f(a + a)$, i.e., $c = a$. **(1 Mark)**
Assume that $f(a) > 0$. Then $h(0) < 0$ and $h(a) > 0$. Thus for some $c \in [0, a]$ we have $h(c) = 0$ and therefore $f(c) = f(c + a)$. The same argument holds when $f(a) < 0$. **(1 Mark)**

Exercise 12.
Let $f\colon (0,1) \to \mathbb{R}$ be uniformly continuous on $(0,1)$. Show that $\lim_{x \searrow 0} f(x)$ exists.
(3 Marks)

Solution. Since f is uniformly continuous on $(0,1)$ we know that

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x,y\in(0,1)}{\forall} |y-x| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

(1/2 Mark) Let (x_n) be an arbitary sequence in $(0,1)$ with $x_n \to 0$. We will show first that the sequence $(f(x_n))$ converges. Since $(f(x_n))$ is a sequence of real numbers, it is sufficient to show that $(f(x_n))$ is a Cauchy sequence. **(1/2 Mark)**

Fix $\varepsilon > 0$. We will show that there exists an $N \in \mathbb{N}$ such that $|f(x_n) - f(x_m)| < \varepsilon$ for all $n, m > N$. Since $x_n \to 0$ we know that (x_n) is Cauchy, i.e., for any $\delta > 0$ there exists an M such that $|x_n - x_m| < \delta$ for all $n, m > M$. Choose δ so that $|x_n - a_m| < \delta$ implies $|f(x_n) - f(x_m)| < \varepsilon$ for all $n, m \in \mathbb{N}$. (This is possible by the uniform continuity of f.). Then choose M large enough so that $|x_n - x_m| < \delta$ for all $n, m > M$. Then for this M, $|f(x_n) - f(x_m)| < \varepsilon$ for all $n, m > M$. This proves that $(f(x_n))$ is Cauchy and converges. **(1 Mark)**

Thus, for any sequence (x_n) with $x_n \to 0$ the sequence $f(x_n)$ converges. It remains to show that all sequences $f(x_n)$ converge to the same limit. Suppose two sequences (x_n) and (y_n) both converge to 0. Then the sequence (z_n) given by $z_{2n} = x_n$, $z_{2n+1} = y_n$ also converges to zero. We know that then $f(z_n)$ converges. But this is only possible if $f(x_n)$ and $f(y_n)$ both converge to the same limit. (Since $\lim_{n\to\infty} f(x_n)$ and $\lim_{n\to\infty} f(y_n)$ are two accumulation points of $f(z_n)$.) **(1/2 Mark)**

Hence, for any sequence (x_n) in $(0,1)$ with $x_n \to 0$ the limit $\lim_{n\to\infty} f(x_n)$ exists and is independent of (x_n). This implies that the limit $\lim_{x\searrow 0} f(x)$ exists. **(1/2 Mark)**

Exercise 13.
Let $f\colon \mathbb{R} \to \mathbb{R}$ be continuous. Suppose that $A \subset \operatorname{ran} f$ is an open set. Prove that then the *pre-image*

$$f^{-1}(A) := \{x \in \mathbb{R} \colon f(x) \in A\}$$

is also an open set.
(4 Marks)

Solution. Let $x_0 \in f^{-1}(A)$. We need to show that for some $\delta > 0$ the open ball $B_\delta(x_0) = \{x \in \mathbb{R} \colon |x - x_0| < \delta\}$ is a subset of $f^{-1}(A)$. We know that $f(x_0) \in A$ by definition. Since A is open, there exists an $\varepsilon > 0$ such that $B_\varepsilon(f(x_0)) \subset A$. The continuity of f implies that for this ε there exists some $\delta > 0$ (which we now fix) such that $|x - x_0| < \delta$ implies $|f(x) - f(x_0)| < \varepsilon$. In other words, for this delta we see that if $x \in B_\delta(x_0)$, then $f(x) \in B_\varepsilon(f(x_0)) \subset A$. But this means that $x \in B_\delta(x_0)$ implies $x \in f^{-1}(A)$ by definition, i.e., $B_\delta(x_0) \subset f^{-1}(A)$. Hence $f^{-1}(A)$ is open.

Exercise 14.
Suppose the function $f\colon \mathbb{R} \to \mathbb{R}$ is such that $xf(x) + f(2-x) = x^2$.

i) Find an expression for $f(x)$.

ii) Is f continuous everywhere? Explain your answer.

(2 + 2 Marks)

Solution.

i) Setting $x = 2 - y$, $x, y \in \mathbb{R}$, we have

$$(2-y)f(2-y) + f(y) = (2-y)^2$$

Since $f(2-y) = y^2 - yf(y)$ this yields

$$(2-y)(y^2 - yf(y)) + f(y) = (2-y)^2$$

(1/2 Mark) or

$$(1 - 2y + y^2)f(y) = (1-y)^2 = (2-y)(2-y-y^2)$$

If $y \neq 1$ **(1/2 Mark)** we can divide by $y-1$ and obtain

$$f(y) = \frac{(2-y)(2+y)}{1-y} = \frac{4-y^2}{1-y}.$$

for $y \in \mathbb{R} \setminus \{1\}$. **(1/2 Mark)** We treat the case $y = 1$ separately: by the original expression, we have

$$1 \cdot f(1) + f(2-1) = 1,$$

so $f(1) = 1/2$. **(1/2 Mark)** Thus,

$$f(x) = \begin{cases} \frac{4-y^2}{1-y} & x \neq 1, \\ 1/2 & x = 1. \end{cases}$$

ii) Since $\lim_{x \to 1} f(x) = \lim_{x \to 1} (4-x^2)/(1-x)$ does not exist, f is not continuous at $x = 1$ and hence not continuous everywhere. **(2 Marks)**

Exercise 15.
Let $\Omega \subset \mathbb{R}$, $I \subset \Omega$ an interval and $f\colon \Omega \to \mathbb{R}$ a real function. Then f is called *Lipschitz-continuous on I* if there exists a constant $L > 0$ (called a *Lipschitz constant*) such that

$$|f(x) - f(y)| \leq L|x - y| \qquad \text{for all } x, y \in I.$$

i) Show that if f is Lipschitz-continuous on I, f is also uniformly continuous on I.

ii) Why is the function $f\colon [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$ uniformly continuous on $[0, 1]$?

iii) Show that $f\colon [0, \infty) \to \mathbb{R}$, $f(x) = \sqrt{x}$ is not Lipschitz-continuous on $[0, 1]$.

(3+1+2 Marks)

Solution.

i) In order to show that f is uniformly continuous on I, we need to show that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in I$

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

(1 Mark) Fix $\varepsilon > 0$. **(1/2 Mark)** Then let $\delta = \varepsilon/L$. **(1/2 Mark)** Since f is Lipschitz-continuous, it follows that

$$|f(x) - f(y)| \leq L|x - y| < L\delta = L\frac{\varepsilon}{L} = \varepsilon$$

whenever $|x - y| < \delta$. Since δ depends only on ε and not on $x, y \in I$, it follows that f is uniformly continuous on I. **(1 Mark)**

ii) Since $[0, 1]$ is a closed interval and f is continuous on $[0, 1]$, we know that f is uniformly continuous on this interval. **(1 Mark)**

iii) We show that f is not Lipschitz-continuous, i.e., that there exists no constant L such that

$$|\sqrt{x} - \sqrt{y}| \leq L|x - y| \qquad \text{for all } x, y \in [0, 1].$$

This is equivalent to showing that for all $L > 0$ there exist $x, y \in [0, 1]$ such that

$$|\sqrt{x} - \sqrt{y}| > L|x - y|.$$

(1 Mark) Fix $L > 0$ and choose $y = 0$. Then we want to show that there exists an $x \in [0, 1]$ such that

$$\sqrt{x} > Lx, \qquad \text{or, equivalently,} \qquad 1 > L\sqrt{x}.$$

To achieve this, we need simply choose some $x \in [0, 1]$ such that $0 < x < 1/L^2$. Since this is always possible, we have shown that f is not Lipschitz-continuous. **(1 Mark)**