Vv186 Honors Mathematics II

Functions of a Single Variable

Assignment 2

Date Due: 2:00 PM, Wednesday, the 30th of September 2020

Investigation may be likened to the long months of pregnancy, and solving a problem to the day of birth. To investigate a problem is, indeed, to solve it.

Mao Zedong, *Oppose Book Worship*[1]

This assignment has a total of **(44 Marks)**.

Exercise 2.1

i) Prove[2] that if $m, n \in \mathbb{N}^*$ and $m^2/n^2 < 2$, then $(m+2n)^2/(m+n)^2 > 2$; show, moreover, that

$$\frac{(m+2n)^2}{(m+n)^2} - 2 < 2 - \frac{m^2}{n^2}.$$

(2 Marks)

ii) Prove the same results with all the inequality signs reversed.
(2 Marks)

iii) Prove that if $(m/n)^2 < 2$, then there is another rational number m'/n' with $(m/n)^2 < (m'/n')^2 < 2$. (This means that $\max U_1$,

$$U_1 = \{a \in \mathbb{Q} \colon a^2 < 2\},$$

does not exist in $\mathbb{Q}$.)
(2 Marks)

iv) Show that $\min U_2$, where $U_2 = \{a \in \mathbb{Q} \colon a > 0 \wedge a^2 > 2\}$, does not exist in $\mathbb{Q}$.
(2 Marks)

v) Show directly that $\inf U_2$ and $\sup U_1$ do not exist in $\mathbb{Q}$: for any given lower bound of U_2, there always exists a slightly larger lower bound and a similar situation holds for the upper bounds of U_1.
(2 Marks)

Exercise 2.2

Give (without proof) the maximum, minimum, supremum and infimum (if they exist) of the following sets.

$$i) \quad \{1 + 2^{-n} \colon n \in \mathbb{N} \setminus \{0\}\}, \qquad\qquad ii) \quad \left\{(-1)^n + \frac{1}{n^2} \colon n \in \mathbb{N} \setminus \{0\}\right\}$$

(4 Marks)

Exercise 2.3

In the lecture, we proved the uniqueness and the first half of the existence in the theorem

Theorem: For any $x > 0$, the set $\mathbb{R}$ contains exactly one positive solution y to the equation $y^2 = x$.

Complete the proof, i.e., for $y = \inf\{t \in \mathbb{R} \colon t > 0 \wedge t^2 > x\}$ show that the assumption $y^2 < x$ leads to a contradiction. *Hint:* use a different strategy than the one used to eliminate the possibility that $y^2 > x$. If $y^2 < x$, show that $(y + \varepsilon)^2 < x$ for sufficiently small $\varepsilon > 0$. How small does ε have to be? Why does this lead to a contradiction?
(3 Marks)

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[1] 调查就像“十月怀胎”，解决问题就像“一朝分娩”。调查就是解决问题。 See 反对本本主义. Translation taken from the Selected Works of Mao Zedong at http://www.marxists.org

[2] See *Spivak*, Ch. 2, Ex. 16.

Exercise 2.4
A number x is called an *almost upper bound* for a set $A \subset \mathbb{R}$ if there are only finitely many numbers $y \in A$ with $y \geq x$.[3] An *almost lower bound* is defined similarly.

i) State (without proof) all amost upper and almost lower bounds for the sets

a) $\{1 + 2^{-n} : n \in \mathbb{N}^*\}$, b) $\left\{(-1)^n + \dfrac{1}{n^2} : n \in \mathbb{N}^*\right\}$

c) $\left\{\dfrac{1}{n} : n \in \mathbb{Z} \setminus \{0\}\right\}$, d) $\left\{x \in \mathbb{Q} : 0 \leq x \leq \sqrt{2}\right\}$

(4 Marks)

ii) Suppose that X is a bounded infinite set. Prove that the set Y of all almost upper bounds of X is nonempty, and bounded below.
(2 Marks)

iii) By (P13), the infimum $\inf Y$ exists; this number is called the *limit superior* of X and denoted by $\limsup X$ or $\overline{\lim} X$. Find the limit superior for the sets given in i).
(2 Marks)

iv) Formulate a definition for the *limit inferior* $\underline{\lim} X$ and find the limit inferior for the sets given in i).
(2 Marks)

v) Let A be an infinite bounded set.[4] Prove that

(a) $\underline{\lim} A \leq \overline{\lim} A$,
(2 Marks)

(b) $\overline{\lim} A \leq \sup A$, $\underline{\lim} A \geq \inf A$.
(2 Marks)

(c) If $\overline{\lim} A < \sup A$, then $\max A$ exists. If $\underline{\lim} A > \inf A$, then $\min A$ exists.
(2 Marks)

Exercise 2.5

i) For complex numbers $z_1, z_2 \in \mathbb{C}$, prove that

$$|z_1 + z_2| \leq |z_1| + |z_2| \qquad \text{and} \qquad |z_1 z_2| = |z_1||z_2|.$$

(4 Marks)

ii) Which complex numbers $z \in \mathbb{C}$ satisfy the inequality $|z + 2| \leq |z - 1|$?
(2 Marks)

iii) Prove the *parallelogram law* $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$, $z_1, z_2 \in \mathbb{C}$.
(2 Marks)

iv) Show that in $\mathbb{C}$ it is impossible to define a set P satisfying properties analogous to (P10)-(P12) for real numbers.
(3 Marks)

[3] See *Spivak*, Ch. 8, Ex. 18. "Finitely many" means zero or more, but not an infinite number.
[4] See *Spivak*, Ch. 8, Ex. 19. An infinite set is a set with an infinite number of elements.