Vv186 Honors Mathematics II

Mid 1 Review: Limits and Continuity

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1. Limit of Function
2. Landau Notation
3. Continuity
4. Continuous Functions
5. Inverse Function
6. Uniform Continuity

Definitions

2.4.1. Definition. Let f be a real- or complex-valued function defined on a subset of $\mathbb{R}$ that includes some interval (a,∞), $a \in \mathbb{R}$. Then f converges to $L \in \mathbb{C}$ as $x \to \infty$, written

$$\lim_{x\to\infty} f(x) = L \qquad :\Leftrightarrow \qquad \underset{\varepsilon>0}{\forall}\ \underset{C>0}{\exists}\ \underset{x>C}{\forall}\ |f(x)-L| < \varepsilon. \tag{2.4.1}$$

2.4.3. Definition. Let f be a real- or complex-valued function defined on a subset $\Omega \subset \mathbb{R}$ and let x_0 be an accumulation point of Ω. Then the limit of f as $x \to x_0$ is equal to $L \in \mathbb{C}$, written

$$\lim_{x\to x_0} f(x) = L \quad :\Leftrightarrow \quad \underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x\in\Omega\setminus\{x_0\}}{\forall}\ |x-x_0| < \delta \Rightarrow |f(x)-L| < \varepsilon.$$

A Useful Theorem

2.4.9. Theorem. Let f be a real- or complex-valued function defined on a subset $\Omega \subset \mathbb{R}$ and let x_0 be an accumulation point of Ω. Then

$$\lim_{x \to x_0} f(x) = L \qquad \Leftrightarrow \qquad \underset{\substack{(a_n) \\ a_n \in \Omega \setminus \{x_0\}}}{\forall} \left(a_n \xrightarrow{n\to\infty} x_0 \Rightarrow f(a_n) \xrightarrow{n\to\infty} L \right)$$

A similar result holds for $x_0 = \pm\infty$.

While the definition is mostly used to prove that a limit **exists**, this theorem is generally used to prove that a limit **doesn't exist** (by constructing two sequences that converge to different limits).

Limit Operations and Common Limits

2.4.5. Theorem. Let f and g be real- or complex-valued functions and x_0 an accumulation point of $\operatorname{dom} f \cap \operatorname{dom} g$ such that $\lim\limits_{x\to x_0} f(x)$ and $\lim\limits_{x\to x_0} g(x)$ exist. Then

1. $\lim\limits_{x\to x_0}(f(x)+g(x)) = \lim\limits_{x\to x_0} f(x) + \lim\limits_{x\to x_0} g(x)$,
2. $\lim\limits_{x\to x_0}(f(x)\cdot g(x)) = \big(\lim\limits_{x\to x_0} f(x)\big)\big(\lim\limits_{x\to x_0} g(x)\big)$,
3. $\lim\limits_{x\to x_0} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to x_0} f(x)}{\lim\limits_{x\to x_0} g(x)}$ if $\lim_{x\to x_0} g(x) \neq 0$.

These statements remain true if $x_0 = \pm\infty$.

The starting point to use these operations are some common limits:

1. $\lim_{x\to x_0} f(x) = f(x_0)$ (if f is defined at x_0)
2. $\lim_{x\to\infty} x^{-p} = 0$, $p > 0$
3. $\lim_{x\to\infty} c = c$
4. $\lim_{x\to x_0/\infty} f(x) = \infty \Leftrightarrow \lim_{x\to x_0/\infty} 1/f(x) = 0$
5. $\ldots$

Limit Substitution Rule

2.5.10. Theorem. Let f, g be real functions such that $\lim_{x \to x_0} g(x) = L$ exists and f is continuous at $L \in \operatorname{dom} f$. Then

$$\lim_{x \to x_0} f(g(x)) = f(L).$$

Example:

If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $\lim_{x \to 0} \frac{f(x)}{x} = \alpha$ for some $\alpha \in \mathbb{R}$, calculate

i) $\lim_{x \to 0} \frac{f(2x)}{x}$,

ii) $\lim_{x \to 0} \frac{[f(2x)]^2}{x^2}$,

iii) $\lim_{x \to 0} \frac{[f(2x)]^2}{x}$.

Use common sense when calculating limits!

One Sided Limits

2.4.6. Definition. Let f be a real- or complex-valued function defined on a subset $\Omega \subset \mathbb{R}$ and let x_0 be an accumulation point of Ω.

Then the limit of f as x converges to x_0 from above is equal to $L \in \mathbb{C}$,

$$\lim_{x \searrow x_0} f(x) = L \quad :\Leftrightarrow \quad \underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x\in\Omega\setminus\{x_0\}}{\forall}\ 0 < x - x_0 < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

Analogously, the limit of f as x converges to x_0 from below is equal to $L \in \mathbb{C}$,

$$\lim_{x \nearrow x_0} f(x) = L \quad :\Leftrightarrow \quad \underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x\in\Omega\setminus\{x_0\}}{\forall}\ 0 < x_0 - x < \delta \Rightarrow |f(x) - L| < \varepsilon.$$

2.4.7. Remark. Clearly, $f(x) \to L$ as $x \to x_0$ if and only if $f(x) \to L$ as $x \searrow x_0$ and $f(x) \to L$ as $x \nearrow x_0$.

Big-O Symbol

2.4.12. Definition. Let f, ϕ be real- or complex-valued functions defined on a subset $\Omega \subset \mathbb{R}$ and let x_0 be an accumulation point of Ω. We say that

$$f(x) = O(\phi(x)) \qquad \text{as } x \to x_0$$

if and only if

$$\underset{C>0}{\exists}\ \underset{\varepsilon>0}{\exists}\ \underset{x\in\Omega}{\forall} \quad |x - x_0| < \varepsilon \quad \Rightarrow \quad |f(x)| \le C|\phi(x)| \tag{2.4.2}$$

2.4.23. Theorem. Let f, ϕ be a real- or complex-valued functions defined on a subset $\Omega \subset \mathbb{R}$ and let x_0 be an accumulation point of Ω. If $x_0 \in \Omega$, we require $\phi(x_0) > 0$. Suppose that exists some $C \ge 0$ such that

$$\lim_{x\to x_0} \frac{|f(x)|}{|\phi(x)|} = C. \tag{2.4.4}$$

Then $f(x) = O(\phi(x))$ as $x \to x_0$.

Example:

$$x^3 + 2x^2 - x + 1 = O(x^3) = O(x^4), \qquad x \to \infty$$

Little-o Symbol

2.4.17. Definition. Let f, ϕ be real- or complex-valued functions defined on a subset $\Omega \subset \mathbb{R}$ and let x_0 be an accumulation point of Ω. We say that

$$f(x) = o(\phi(x)) \qquad \text{as } x \to x_0$$

if and only if

$$\underset{C>0}{\forall}\ \underset{\varepsilon>0}{\exists}\ \underset{x\in\Omega\setminus\{x_0\}}{\forall} \quad |x - x_0| < \varepsilon \quad \Rightarrow \quad |f(x)| < C|\phi(x)| \tag{2.4.3}$$

2.4.24. Theorem. Let f, ϕ be a real- or complex-valued functions defined on an interval $I \subset \mathbb{R}$ and let $x_0 \in \overline{I}$. Then

$$\lim_{x\to x_0} \frac{|f(x)|}{|\phi(x)|} = 0 \qquad \Leftrightarrow \qquad f(x) = o(\phi(x)) \text{ as } x \to x_0. \tag{2.4.5}$$

Example:

$$x^3 + 2x^2 - x + 1 = o(x^4), \qquad x \to \infty$$

For little-o (and sometimes big-O), it's more easy to use the definition by limits then the definition by inequalities.

Common Properties

- $o(f(x)) = O(f(x))$
- $O(f(x)) + O(g(x)) = O(|f(x)| + |g(x)|)$
- $O(f(x))O(g(x)) = O(f(x)g(x))$
- $O(O(f(x))) = O(f(x))$
- $o(O(f(x))) = o(f(x))$

Definition

2.5.1. Definition. Let $\Omega \subset \mathbb{R}$ be any set and $f\colon \Omega \to \mathbb{R}$ be a function defined on Ω. Let $x_0 \in \Omega$. We say that f is ***continuous at x_0*** if

$$\lim_{x \to x_0} f(x) = f(x_0).$$

If $U \subset \Omega$, we say that f is ***continuous on U*** if f is continuous at every $x_0 \in U$.

We say that f is ***continuous on its domain***, or simply ***continuous***, if f is continuous at every $x_0 \in \Omega$.

Three points:

1. $\lim_{x \to x_0} f(x)$ exists
2. $f(x_0)$ exists
3. $\lim_{x \to x_0} f(x) = f(x_0)$

Manipulating Continuous Functions

If f and g are continuous, then on points when taking function makes sense:

- $f + g$ is continuous
- $f \cdot g$ is continuous
- f/g is continuous $(g \neq 0)$
- $f \circ g$ is continuous

Most elementary functions are continuous except on some certain points.

Theorems for Continuous Functions

The two most important theorems for continuous functions are as follows:

2.5.13. Bolzano Intermediate Value Theorem. Let $a < b$ and $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Then for $y \in [\min\{f(a), f(b)\}, \max\{f(a), f(b)\}]$ there exists some $x \in [a,b]$ such that $y = f(x)$.

2.5.17. Theorem. Let $a < b$ and $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Then there exists a $y \in [a,b]$ such that $f(x) \le f(y)$ for all $x \in [a,b]$.

Hence $\max\{f(x) : x \in [a,b]\}$ exists. Colloquially, we say that "a continuous function attains its maximum".

Intermediate Value Theorem

The **Intermediate Value Theorem** states that every value between the end points can be attained for a continuous function.

Two theorems stem from the intermediate value theorem:

2.5.12. Theorem. Let $a < b$ and $f : [a, b] \to \mathbb{R}$ be a continuous function with $f(a) < 0 < f(b)$. Then there exists some $x \in [a, b]$ such that $f(x) = 0$.

2.5.14. Theorem. Let $f : [a, b] \to \mathbb{R}$ be a continuous function with $\operatorname{ran} f \subset [a, b]$. Then f has a fixed point, i.e., there exists some $x \in [a, b]$ such that $f(x) = x$.

Continuous Function Attains its Extremum

Theorem **2.5.17** states that every continuous function on a **closed interval** attains its extremum (maximum/minimum).

We can furthermore state that the image of a function on an closed interval is also an closed interval:

2.5.25. Corollary. Let $\Omega \subset \mathbb{R}$ and $f\colon \Omega \to \mathbb{R}$ continuous. Suppose that $I \subset \Omega$ is a closed interval. Then the image

$$f(I) = \Big\{y \in \mathbb{R}\colon \underset{x \in I}{\exists} f(x) = y\Big\}$$

is also a closed interval.

Exercise

Prove that the equation

$$x^2 + |x|^{5/2} - 4x + 1 = 0, \qquad x \in \mathbb{R},$$

has a solution in the interval $(1, 2)$.

Exercise

Suppose that $f \colon \mathbb{R} \to (0, \infty)$ is continuous and satisfies $\lim_{x \to -\infty} f(x) = \lim_{x \to \infty} f(x) = 0$.

i) Show that f attains its maximum, i.e., there exists some $x_0 \in \mathbb{R}$ such that $f(x_0) \geq f(x)$ for all $x \in \mathbb{R}$.
(2 Marks)

ii) Let x_0 be given as in i) above and let $y_0 := f(x_0)$. Show that $\operatorname{ran} f = (0, y_0]$, i.e., for every $\eta \in (0, y_0]$ there exists some $\xi \in \mathbb{R}$ such that $f(\xi) = \eta$.
(2 Marks)

-jectives

2.5.18. Definition. Let $\Omega, \widetilde{\Omega} \subset \mathbb{R}$ and $f \colon \Omega \to \widetilde{\Omega}$ a function. We say that f is

- ***injective*** if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ for all $x_1, x_2 \in \Omega$;
- ***surjective*** if for every $y \in \widetilde{\Omega}$ there exists an $x \in \Omega$ such that $f(x) = y$ (i.e., if $\operatorname{ran} f = \widetilde{\Omega}$);
- ***bijective*** if for every $y \in \widetilde{\Omega}$ there exists a ***unique*** $x \in \Omega$ such that $f(x) = y$ (i.e., f is injective and surjective).

Injective can be seen as "one to one"
Surjective can be seen as "onto"

Bijectivity and Monotonicity

2.5.19. Theorem. Let $a, b \in \mathbb{R} \cup \{-\infty\} \cup \{\infty\}$ such that $a < b$. Let $f : (a, b) \to \mathbb{R}$ be strictly increasing and continuous. Then

$$\alpha := \lim_{x \searrow a} f(x) \geq -\infty, \qquad \beta := \lim_{x \nearrow b} f(x) \leq \infty,$$

exist and $f : (a, b) \to (\alpha, \beta)$ is bijective. Furthermore, the inverse function f^{-1} is also continuous and strictly increasing and, furthermore,

$$\lim_{y \searrow \alpha} f^{-1}(y) = a, \qquad \lim_{y \nearrow \beta} f^{-1}(y) = b. \tag{2.5.1}$$

2.5.20. Theorem. Let $I \subset \mathbb{R}$ be an interval and $\widetilde{\Omega} \subset \mathbb{R}$ a set. If $f : I \to \widetilde{\Omega}$ is continuous and bijective, then f is strictly monotonic on I.

Basically speaking, for continuous functions:

$$\text{strictly monotonic} \;\Leftrightarrow\; \text{bijective} \;\Leftrightarrow\; \text{invertible}$$

Definition and Theorems

2.5.23. Definition. Let $I \subset \mathbb{R}$ be an interval and $f : \Omega \to \mathbb{R}$ a function with $I \subset \Omega$. Then f is called ***uniformly continuous on I*** if and only if

$$\underset{\varepsilon>0}{\forall}\ \underset{\delta>0}{\exists}\ \underset{x,y\in I}{\forall} |x-y| < \delta \Rightarrow |f(x) - f(y)| < \varepsilon.$$

A uniformly continuous function loose speaking is a continuous function that does not increase or decrease "too rapidly".

The one important theorem on uniform continuity is as follows:

2.5.24. Theorem. Let $f : \Omega \to \mathbb{R}$ a function with $I = [a, b] \subset \Omega$. If f is continuous on $[a, b]$ then f is also uniformly continuous on $[a, b]$.

Good luck!