CS419/939: Quantum Computing – Assignment 1

LaTeX submissions are strongly preferred; handwritten submissions are acceptable. You may work in groups but you must write up your solutions individually. Questions marked with (*) are optional (and more challenging).

Problem 1 (10 marks)

Consider the following unitary matrices, known as the Pauli matrices.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

1. We can write X as a linear combination of outer products: $X = |1\rangle\langle 0| + |0\rangle\langle 1|$. Write Y and Z as a linear combination of outer products.
2. For each one of the Pauli matrices, use Dirac notation to find where each transformation maps the states $|0\rangle, |1\rangle, |-\rangle, |+\rangle$. (That is, compute $X|0\rangle$, $X|1\rangle$, $X|-\rangle$, $\ldots$, $Z|-\rangle$, $Z|+\rangle$.)

Problem 2 (15 marks)

Consider the state $|\psi\rangle = \sqrt{\frac{3}{5}}|0\rangle + \sqrt{\frac{2}{5}}|1\rangle$.

1. Compute the inner product $\langle\psi|\psi\rangle$.
2. Compute the **matrix** $|\psi\rangle\langle\psi|$.
3. What will be the outcome of measuring ψ in the computational ($\{|0\rangle, |1\rangle\}$) basis?
4. What will be the outcome of measuring ψ in the $\{|+\rangle, |-\rangle\}$ basis?

Problem 3 (15 marks)

Consider the the rotation gate

$$R_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

1. If we measure $R_\theta|0\rangle$ in the $\{|0\rangle, |1\rangle\}$-basis, what is the probability we will observe $|1\rangle$?
2. If we measure $R_\theta|1\rangle$ in the $\{|+\rangle, |-\rangle\}$-basis, what is the probability we will observe $|+\rangle$?

Problem 4 (40 marks)

In this problem you will simulate the Elitzur-Vaidman bomb testing strategy seen in class.

Recall that the bomb is represented by a computational basis measurement: if the outcome is $|0\rangle$ we obtain the (measured) qubit $|0\rangle$ on the other side, and if the outcome is $|1\rangle$ the bomb explodes.

In its simplest form, the experiment is as follows: starting from a qubit in the state $|0\rangle$,

1. apply a Hadamard gate H;
2. send the qubit through the black box;
3. apply H again;
4. perform a computational basis measurement.

Assuming no explosion occurs, we can guess the contents of the box: if there was nothing in it, two H gates transform $|0\rangle$ back to $|0\rangle$; whereas a bomb performs a measurement in between, so the last measurement outcome is $|1\rangle$ with probability $1/2$.

This experiment is implemented in Qiskit in the Jupyter notebook `elitzur-vaidman-bomb.ipynb`. Note that this is not quite a satisfying solution: in the presence of a bomb, we are as likely to explode as not to, and even when we don't the presence of the bomb is only detected with 50% probability.

Your task is to make the experiment safer: given an error parameter ε (denoted `eps` in the notebook), you should implement a variation on the above experiment that detects a bomb *without exploding it* with probability $1-\varepsilon$. (Your experiment should also correctly detect the absence of a bomb with probability 1.) You are allowed to use rotation gates with any angle θ, and to use the black box as many times as you want.

Your submission must include the file `elitzur-vaidman-bomb.ipynb` **edited only in the highlighted sections within the `elitzur_vaidman` function**. (Apart from visualisations that you may choose to add in the last cell, which will not be marked.)

We recommend using IBM Quantum Lab, which requires creating an account. To work on this problem in IBM Quantum Lab, upload `elitzur-vaidman-bomb.ipynb` to your workspace. You can also choose to install qiskit locally (see this guide).

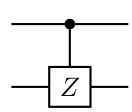
Problem 5 (20 marks)

This problem considers the task of designing quantum circuits that implement a given unitary using a restricted set of gates.

1. The controlled-Z gate is a 2-qubit gate with the following matrix representation:

$$\mathsf{cZ} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

   Show how to implement cZ using H and CNOT gates.

2. The controlled-Z gate is written as follows: 

   Show how to implement CNOT using H and controlled-Z gates.

3. A SWAP gate interchanges two qubits, mapping $|ab\rangle \mapsto |ba\rangle$ for all $a, b \in \{0, 1\}$. Show how to implement a SWAP gate using CNOT gates. (Hint: either qubit can be the control qubit!)

4. (*) Show that it is impossible to implement the CNOT gate from *any* combination of single-qubit unitaries.