CS9290_B

THE UNIVERSITY OF WARWICK

Algorithmic Game Theory

Standard Examination: Summer 2022

Time allowed: 3 hours.

Answer exactly **FOUR** questions.

Read carefully the instructions on the answerbook. Clearly mark any rough work.

Calculators are not allowed.

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1. (a) Solve the following game using the iterated elimination of dominated strategies. Explain your answer step by step. The numbers in the lower left corners denote the payoff for player I and the numbers in the upper right corners denote the payoff for player II.

I \ II	X	Y	Z
A	~~1~~ ~~2~~	4 5	~~2~~ ~~3~~
B	~~2~~ ~~-1~~	~~1~~ ~~3~~	~~3~~ ~~2~~
C	~~4~~ ~~1~~	~~1~~ ~~2~~	~~0~~ ~~4~~

S={A,Y}

[7]

(b) Give a Nash equilibrium of the following two-player zero-sum game. Explain your answer. The values given, are the payoffs for player I. Player I wants to maximize her payoff and player II wants to minimize player I's payoff.

I \ II	X	Y
A	3	5
B	4	2
C	5	1

Y=-2X+5
Y=2X+2
X=3/4, Y=3.5
(3/4, 3.5)

XAY=3.5
C=0
A+B=1
S=(1/2,1/2,0)

[9]

(c) Either prove or disprove the following claim: If a two-player zero-sum game has a Nash equilibrium in which the first player receives a payoff of 8, then every mixed strategy profile in which the first player receives a payoff of 8 is a Nash equilibrium. [9]

8,8
8,8

2. (a) We want to use methods from the Lemke-Howson algorithm to find Nash equilibria in the following 2×3 two-player game. The numbers in the lower left corners denote the payoff for player I and the numbers in the upper right corners denote the payoff for player II.

	I \ II	3	4	5
x	1	$\underline{5}$ $\underline{1}$	2 $\underline{2}$	4 2
1-x	2	0 0	$\underline{2}$ 0	1 $\underline{4}$

Draw player I's mixed strategy simplex and subdivide it into best response regions of player II. [9]

(b) For the game in 2a, draw player II's mixed strategy simplex and subdivide it into best response regions of player I. [9]

(c) Label the facets of the mixed strategy simplices in 2a and 2b and use the resulting diagrams to identify *all* Nash equilibria of the game in 2a. Explain your answer. [7]

3.MSNE: a probability distribution over Si; PSNE: u>=u;

3. (a) Define the concept of a mixed Nash equilibrium and a pure Nash equilibrium. What are the differences? Your answer should include a statement of the theorem of Nash. [7]

   (b) Give a two-player zero-sum game in which each player has exactly two strategies and for which the only Nash equilibrium is the mixed strategy profile $((2/5, 3/5), (2/3, 1/3))$. Justify your answer. [6]

   (c) Consider an n-player game given by n payoff functions, where each player has m strategies and each of the n payoff functions is given as a table of size m^n numbers. Design an algorithm that finds a pure Nash equilibrium in such a game in time $\mathcal{O}(n^2 m^n)$. [5]

   (d) Consider an n-player game given by n payoff functions, where each player has 2 strategies and each of the n payoff functions is given as a table of size 2^n numbers. Show that without any further assumption about the game, no algorithm can decide if the game has a pure Nash equilibrium without inspecting at least 2^n of the numbers provided in the input. [7]

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4. (a) Consider a three player game in which the finite sets of strategies of the players are S_1, S_2, and S_3, respectively. Define the concept of a correlated equilibrium in such a game. [5]

(b) Consider the three player game in which player I has two strategies 1 and 2, player II has two strategies A and B, player III has three strategies L, M, and R, and the following three tables (read from left to right) describe the payoffs of the three players when player III picks strategy L, M, and R, respectively.

1/3	0
0	2/3

I \ II	A	B
1	0 8 0	0 0 0
2	0 0 1	0 0 0

I \ II	A	B
1	5 6 4	1 0 5
2	2 0 4	5 4 6

I \ II	A	B
1	0 0 0	0 0 0
2	1 0 0	0 7 0

The three numbers in each cell of the table indicate the respective payoffs of the three players. The number in the bottom left is the payoff of player I, the number in the top right is the payoff of player II, and the number in the center is the payoff of player III.

For example, if player I picks strategy 2, player II picks strategy B, and player III picks strategy M (that is, the middle table), then the payoff of player I is 6, the payoff of player II is 5, and the payoff of player III is 4.

Argue that the probability distribution σ such that $\sigma(1,\mathrm{A},\mathrm{M}) = 1/3$ and $\sigma(2,\mathrm{B},\mathrm{M}) = 2/3$ is a correlated equilibrium. [7]

1/3(4+6+5)+
2/3(6+4+5)

(c) Calculate the social payoff of the correlated equilibrium σ from part (b). Give another correlated equilibrium in the three player game from part (b) whose social payoff is strictly smaller than that of σ. [6]

(d) Prove that if a correlated equilibrium in a two player game is a product distribution determined by two mixed strategies then the two mixed strategies are a Nash equilibrium. [7]

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5. (a) Consider the combinatorial auction with four items A, B, C, and D, and five single-minded bidders who submit the following bids: $(\{A,B\},2)$, $(\{A,C\},3)$, $(\{B\},2)$, $(\{C\},2)$, and $(\{C,D\},2)$, respectively. Calculate the winners in this auction if the VCG mechanism is used. [6]

   (b) In the auction from part (a), calculate the payments of the five bidders and the auctioneer's revenue. [7]

   (c) The University is planning to construct a new building for the Sciences departments. There are several builders who can carry out this project and each builder i has a cost $c_i > 0$ of doing so. The costs may differ between builders depending on their efficiency and specialization, and only the builders themselves know their own costs.

   You may assume that builders aim to maximize their profit, which is the difference between what they are paid by the University and their cost if they are awarded the contract, and which is 0 if they are not awarded the contract.

   The University wants to award the contract to the builder whose cost is the smallest. Design an incentive compatible auction mechanism to achieve this goal, in which the University is the auctioneer and the builders are the bidders. [6]

   (d) Consider combinatorial auctions with m items and n bidders, in which every bidder submits a list of bids of size $2^m - 1$, one bid for each of the $2^m - 1$ non-empty bundles of items. Design an algorithm which computes the largest social welfare of an allocation of bundles to bidders, and whose running time is polynomial in $n \cdot 2^m$. [6]

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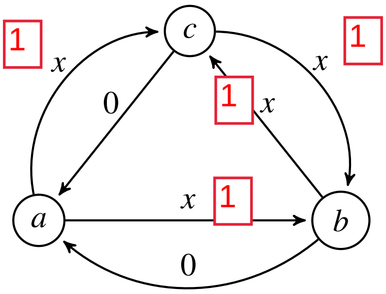
6. (a) Consider a non-atomic congestion game in which there are three parallel edges from the source vertex to the sink vertex, whose latency functions are $3x^2$, 3, and 2, respectively, and the demand for traffic from the source to the sink is 1.

   Calculate a flow that minimizes the social cost. [7]

   (b) Calculate a Wardrop equilibrium in the congestion game from part (a) and the price of anarchy. [6]

   (c) Consider the following atomic congestion game with four players: player 1 wants to send one unit of flow from vertex a to vertex b, player 2 wants to send one unit of flow from a to c, player 3 wants to send one unit of flow from b to c, and player 4 wants to send one unit of flow from c to b.



   The flows of each of the players are unsplittable and the latency functions of the six edges are as indicated in the figure. Observe that each of the four players can pick one of two paths: a one-hop one or a two-hop one; for example, player 3 can pick the one-hop path (b,c) or the two-hop path (b,a,c).

   Prove that the flow in which each of the four players picks a two-hop path is a Wardrop equilibrium. [6]

   (d) Calculate the social cost of the flow in which each of the four players picks a two-hop path.

   Prove that the price of anarchy of the atomic congestion game from part (c) is at least $5/2$. [6]