THE UNIVERSITY OF WARWICK

MSc Examinations: Summer 2017

CS910: Foundations of Data Analytics

Time allowed: 2 hours.

Answer **SIX** questions only: **ALL THREE** from Section A and **THREE** from Section B.

Read carefully the instructions on the answer book and make sure that the particulars required are entered on **each** answer book.

Only calculators that are approved by the Department of Computer Science are allowed to be used during the examination.

Section A Answer **ALL** questions

1. The following questions relate to the log-log plot. [10]

   (a) Give a function $f(x)$ which does not appear as linear on a log-log plot. [3]

   (b) Give a function $f(x)$ which appears as linear on a log-linear plot (the x-axis uses a logarithmic scale and the y-axis uses a linear scale). [3]

   (c) Consider the function $f(x)$ shown on the log-log plot from Figure 1. Give the expression of $f(x)$. [4]

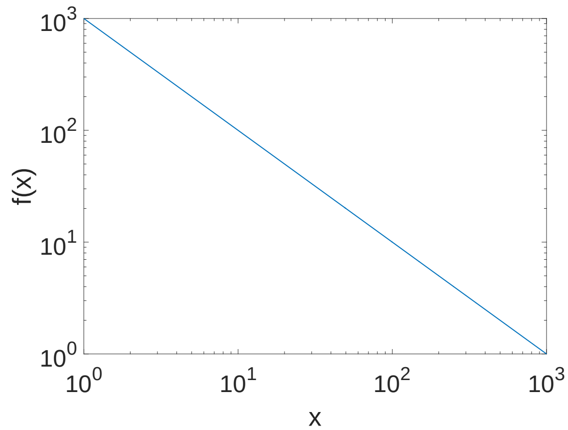

Figure 1: The ~~log-log~~ plot of $f(x)$

Solution: *Comprehension – requires student to show understanding of concepts*

(a)

$$f(x) = e^{-x}$$

(b)

$$f(x) = -\log x + 3$$

(c) The function $f(x)$ must satisfy

$$\log f(x) = -\log x + 3\,,$$

which yields

$$f(x) = 10^3 x^{-1}.$$

2. Consider the strings s_1="saturday" and s_2="sunday". [10]

   (a) Compute the Hamming distance between s_1 and s_2. [2]

(b) Compute the (text) edit distance between s_1 and s_2. [2]

(c) Could the cosine similarity distance between s_1 and s_2 be 0? Justify! [3]

(d) Could the cosine similarity distance between s_1 and s_2 be different than 0? Justify! [3]

Solution: *Application – student needs to apply techniques they have learned*

(a) Undefined because s_1 and s_2 have different lengths.

(b) 3

(c) Yes. We could consider two dimensions corresponding to the two words. The encodings will then be $[1\ 0]$ and $[0\ 1]$ and hence the cosine similarity would be 0.

(d) Yes. We could consider multiple dimensions, each corresponding to an alphabetical letter. Given the overlap between s_1 and s_2 in terms of letters, the cosine similarity would be non zero.

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3. Consider n paired observations (x_i, y_i) of some random variables X and Y. [20]

(a) Provide a full derivation of a linear regression model

$$y = ax$$

using the principle of least squares. The answer should include the expression of the parameter a in terms of X and Y. [10]

(b) Fully simplify the sum of squares of the residuals [3]

$$\sum_{i=1}^{n}(y_i - ax_i)^2$$

for the value of a obtained in (a).

(c) Assume that your data satisfies $y_i = x_i^2\ \forall i = 1 \ldots n$. Which of the following two regression models would best fit the data?

$$\begin{aligned} &\text{Model 1:} \quad y = ax + b^2 x \\ &\text{Model 2:} \quad y = ax \end{aligned}$$

Justify! (Note that a and b are regression parameters.) [7]

Solution: *Bookwork – primarily requires recollection of taught concepts*

(a)

$$a = \frac{E[XY]}{E[X^2]} .$$

The derivations were shown in the slides.

(b)

$$n\left(E[Y^2] - \frac{E[XY]^2}{E[X^2]}\right) .$$

The derivations were shown in the slides.

(c) Both models fit the data equally well. The reason is that the solutions for the linear regressions would be identical. Concretely, Model 2 yields some value for a, which could be further expressed in terms of $c + d^2$.

Section B Choose **THREE** questions.

4. The following questions relate to random variables. [20]

   (a) Give a random variable whose median is strictly smaller than its mean. Justify! [2]

   (b) Give a random variable whose median is strictly smaller than its mode. Justify! [3]

   (c) Prove that for any random variable X the following holds [3]

$$Var[X] = E[X^2] - (E[X])^2 \ .$$

   (d) Give a random variable X with at least two values, each having positive probability, such that [5]

$$E\left[\frac{1}{X}\right] = \frac{1}{E[X]} \ .$$

   (e) Prove that for any random variables X and Y such that $E[X] = E[Y] = 0$ the following holds [7]

$$(E[XY])^2 \leq E[X^2]E[Y^2] \ .$$

Solution: *Comprehension – requires student to show understanding of concepts*

(a) The exponential distribution with density $f(x) = \lambda e^{-\lambda x}$ for $x \geq 0$ and $\lambda > 0$. The median is $\frac{\ln 2}{\lambda}$ whereas the mean is $\frac{1}{\lambda}$.

(b) Take the random variable

$$X = \begin{pmatrix} 1 & 2 & 3 \\ 0.25 & 0.25 & 0.5 \end{pmatrix} \ .$$

The median is 2 whereas the mode is 3.

(c) One can write

$$\begin{aligned} Var[X] &= E[(X - E[X])^2] \\ &= E[X^2 - 2XE[X] + (E[X])^2] \\ &= E[X^2] - 2E[X]E[X] + (E[X])^2 \\ &= E[X^2] - (E[X])^2 \ . \end{aligned}$$

(d) Take the random variable

$$X = \begin{pmatrix} -1 & \frac{1}{2} & 2 \\ \frac{1}{9} & \frac{4}{9} & \frac{4}{9} \end{pmatrix} \ ,$$

for which

$$E[X] = -\frac{1}{9} + \frac{2}{9} + \frac{8}{9} = 1 = -\frac{1}{9} + \frac{8}{9} + \frac{2}{9} = E\left[\frac{1}{X}\right] \ .$$

(e) For some real number a one has

$$0 \leq E[(X - aY)^2] = E[X^2] - 2aE[XY] + a^2E[Y^2] \, .$$

Choose $a = \frac{E[XY]}{E[Y^2]}$ and the claim follows.

5. Consider running a k-NN classifier using Euclidean distance on the data set from Figure 2, whereby each points belongs to one of two classes: $+$ and $\circ$. [20]

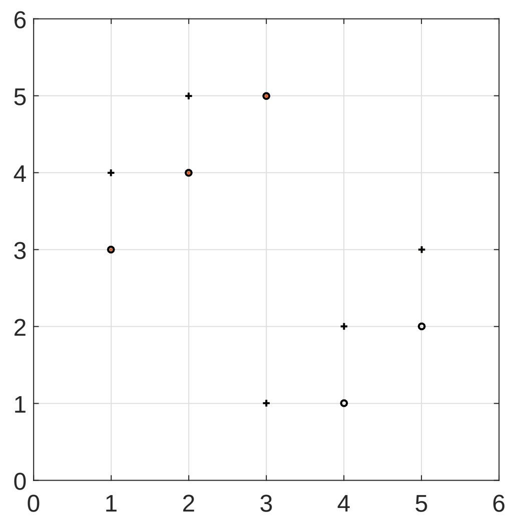

Figure 2: Points belonging to two classes

(a) What is the 10-fold cross validation error when $k = 1$? [5]

(b) Which of the values $k \in \{3, 4, 5, 9\}$ yields the minimum number of 10-fold cross validation errors? [7]

(c) Give a distance metric, instead of the Euclidean distance, such that the 10-fold cross validation error of 1-NN is $\frac{4}{10}$. [8]

Solution: *Application – student needs to apply techniques they have learned*

(a) Each point is misclassified and hence the error is 1.

(b) For each $k \in \{3, 5, 9\}$, each point is misclassified. When $k = 4$ some points may be correctly classified, depending on how 4-NN handles ties.

(c) The inverse of the Euclidean distance.

6. Consider the data from the table below in which the attribute A is binary, whereas the values $a_i \in \{Y, N\}$ are unknown. [20]

Name	Sex	A
Alex	M	a_1
Mary	F	a_2
Alex	F	a_3
Alex	F	a_4
John	M	a_5
Zoe	F	a_6
Nina	F	a_7
Dan	M	a_8

(a) Ignoring the attribute A, what would a Naïve Bayes Classifier predict on the input Alex, i.e., M or F? Justify! [4]

(b) Again, ignoring the attribute A, build a decision tree classifier which would predict M on the input Alex. [5]

(c) Determine some values a_i such that a Naïve Bayes Classifier would predict M on the input (Alex,Y) . [6]

(d) What is the key advantage of the Naïve Bayes Classifier over decision tree classifiers? What is its weakness? [5]

Solution: *Application – student needs to apply techniques they have learned*

(a) We have

$$P(Alex|M)P(M) = \frac{1}{3}\frac{3}{8} = \frac{1}{8}$$

and

$$P(Alex|F)P(F) = \frac{2}{5}\frac{5}{8} = \frac{2}{8}$$

and hence the prediction is F.

(b) Choose the attribute Name at the root and the binary split: Alex and the rest of names. Label the leaf with the minority class. Note that the leaf Alex will be labelled with M.

(c) Consider

Name	Sex	A
Alex	M	Y
Mary	F	N
Alex	F	Y
Alex	F	Y
John	M	Y
Zoe	F	N
Nina	F	N
Dan	M	Y

Continued

in which case we would have

$$P(Alex|M)P(Y|M)P(M) = \frac{1}{3}1\frac{3}{8} = \frac{1}{8}$$

and

$$P(Alex|F)P(Y|F)P(F) = \frac{2}{5}\cancel{0}\frac{5}{8} = \cancel{0}$$

and hence the prediction is M.

(d) The ability to account for correlation amongst attributes.

The assumption that each attribute is conditionally independent of others.

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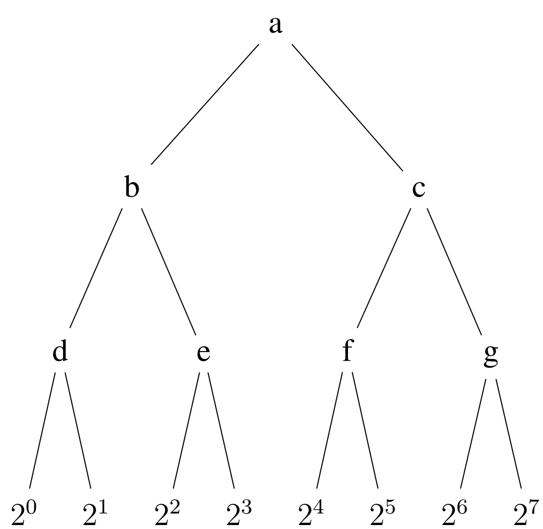
7. Consider the points $2^0, 2^1, 2^2, \ldots, 2^{2^n-1}$ for some $n \geq 1$. [20]

(a) Sketch the clustering trees produced by hierarchical clustering with Euclidean distance and the following inter-cluster distances: single-link, complete-link, and average-link. (Recall that for single-link $d(X, Y) := \min d(x \in X, y \in Y)$, whereas for complete and average-link the '$\min$' is replaced by '$\max$' and 'avg', respectively.) [12]

(b) Replace the Euclidean distance metric from (a) by another distance *function* (which does not necessarily have to obey the *metric* rules) such that hierarchical clustering with single-link would produce a full binary tree (each node, except for the leaves, has exactly two children). For instance, if $n = 3$, the tree below would be produced. (Note: for two points x and y you need to construct a function $d(x, y)$ obeying the requirements).

[8]



Solution: *Application – student needs to apply techniques they have learned*

(a) All three inter-cluster distances yield the figure below.

(b) $d(a,b) = \frac{\max(a,b)}{\min(a,b)}$