THE UNIVERSITY OF WARWICK

Department: Computer Science

Module Code: CS939

Module Title: Quantum Computing

Exam Paper Code: CS9390_A

Duration: 2 hours

Exam Paper Type: 24-hour window

STUDENT INSTRUCTIONS

1. Read all instructions carefully. We recommend you read through the entire paper at least once before writing.
2. There are **FOUR** questions. You must answer Question 1, as well as two out of Questions 2,3, and 4.
3. You should not submit answers to more than the required number of questions.
4. You should handwrite your answers either with paper and pen or using an electronic device with a stylus (unless you have special arrangements for exams which allow the use of a computer). Start each question on a new page and clearly mark each page with the page number, your student id and the question number. Handwritten notes must be scanned or photographed and all individual solutions collated into a single PDF with pages in the correct order.
5. Please ensure that all your handwritten answers are written legibly, preferably in dark blue or black ink. If you use a pencil ensure that it is not too faint to be captured by a scan or photograph.
6. Please check for legibility before uploading. It is your responsibility to ensure your work can be read.
7. Add your student number to all uploaded files.
8. You are permitted to access module materials, notes, resources, references and the Internet during the online assessment.
9. You must not communicate with any other candidate during the assessment period or seek assistance from anyone else in completing your answers. The Computer Science Department expects the conduct of all students taking this assessment to conform to the stated requirements. Measures will be in operation to check for possible misconduct. These will include the use of similarity detection tools and the right to require live interviews with selected students following the assessment.
10. By starting this assessment, you are declaring yourself fit to undertake it. You are expected to make a reasonable attempt at the assessment by answering the questions in the paper.

IMPORTANT INFORMATION

- We strongly recommend you use Google Chrome or Mozilla Firefox to access the Alternative Exams Portal.
- You are granted an additional 45 minutes beyond the stated duration of this assessment to allow for downloading/uploading your assessment, your files and any technical delays.
- Students with approved Alternative Exam Arrangements (Reasonable Adjustments) that permit extra time and/or rest breaks will have this time added on to the stated duration.
- You must have completed and uploaded the assessment before the 24-hour assessment window closes.
- Late submissions are not accepted.
- If you are unable to submit your assessment, you must submit Mitigating Circumstances immediately, attaching supporting evidence and your assessment. The Mitigating Circumstances Panel will consider the case and make a recommendation based on the evidence to the Board of Examiners.

SUPPORT DURING THE ASSESSMENT

Operational Support:

- Use the Alternative Exams Portal to seek advice immediately if during the assessment period:
  - you cannot access the online assessment
  - you believe you have been given access to the wrong online assessment

Technical Support:

- If you experience any technical difficulties with the Alternative Exam Portal please contact *helpdesk@warwick.ac.uk*.
- Technical support will be available between 09:00 and 17:00 BST for each examination (excluding Sunday).

Academic Support:

- If you have an academic query, contact the invigilator (using the 'Contact an Invigilator' tool in AEP) to raise your issue. Please be aware that two-way communication in AEP is not currently possible.
- Academic support will normally be provided for the duration of the examination (i.e. for a 2 hour exam starting at 09:00 BST, academic support will normally be provided between 09:00 and 11:45 BST). Academic support beyond this time is at the discretion of the department.

Other Support:

- If you can not complete your assessment for the following reasons submit Mitigating Circumstances immediately:
  - you lose your internet connection
  - your device fails
  - you become unwell and are unable to continue
  - you are affected by circumstances beyond your control

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Mandatory question (30 points):

1. (a) **(6 points)** Define the notion of *superposition* formally and explain the phenomenon that it captures.

   (b) **(4 points)** Provide a formal and complete definition of a mixed state and its density matrix.

   (c) **(2 points)** Explain what is the maximally mixed state.

   (d) **(18 points)** For each of the following statements, indicate whether it is **true** or **false**. You will be awarded 3 points for each correct answer and -3 points for each incorrect answer. (Should the sum of awarded points be negative, you will receive 0 points for this part of the question.)

      i. The Boolean Fourier expansion of a function $f : \{0,1\}^n \to \{0,1\}$ is a decomposition into the base of XOR characters $\{\chi_s\}_{s\in\{0,1\}^n}$ such that $\chi_s(x) = (-1)^{s\cdot x}$.

      ii. The state $|+\rangle$ is an entangled state.

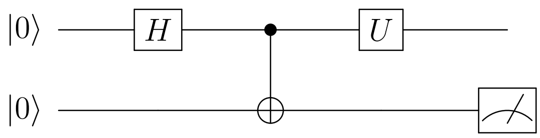
      iii. Grover's search algorithm admits an exponential speedup over classical algorithms.

      iv. The circuit of the Bernstein-Vazirani algorithm uses exponentially fewer gates than the best classical circuit for this problem.

      v. There exists a quantum algorithm that given bit string $x \in \{0,1\}^n$, clones and outputs two copies of x.

Choose TWO out of the following THREE questions (35 points per question):

2. Let U be an arbitrary unitary acting on 1 qubit (i.e., when applying U to a qubit state $|\psi\rangle$, we get $U|\psi\rangle$), and consider the following circuit.



In the following, provide explanations for your calculations.

(a) **(5 points)** What is the state of the qubits immediately before applying unitary U?

(b) **(5 points)** What is the state immediately after U is applied (but before the measurement)?

(c) **(10 points)** Suppose the measurement is made in the computational basis. What are the outcome probabilities and quantum states obtained after the measurement?

(d) **(10 points)** Now suppose that the measurement is made in the $\{|+\rangle, |-\rangle\}$ basis. What are the measurement probabilities and post-measurement quantum states in this case?

(e) **(5 points)** List the possible post-measurement quantum states if the measurement is made in any basis $\{|\psi_0\rangle, |\psi_1\rangle\}$ where $|\psi_0\rangle$ and $|\psi_1\rangle$ have *real* amplitudes.

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3. In this exercise, you will show how quantum states in a restricted subset may be teleported more economically (by communicating 1, rather than 2, classical bits). Let $\theta \in [0, 2\pi)$, define the unitary $U_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and set $|\psi\rangle = U_\theta |0\rangle$ as well as $|\varphi\rangle = U_\theta |1\rangle$.

   (a) **(5 points)** Show that $ZX |\varphi\rangle = |\psi\rangle$.

   (b) **(10 points)** Show that the Bell state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)$ can also be written as $\frac{1}{\sqrt{2}}(|\psi\rangle \otimes |\psi\rangle + |\varphi\rangle \otimes |\varphi\rangle)$.

   (c) **(15 points)** Suppose Alice and Bob each hold a qubit of an EPR pair. If Alice applies U_θ^{-1} to her qubit and then measures in the computational basis, what is the state of Bob's qubit if Alice's measurement outcome is 0, and what is his state if her outcome is 1?

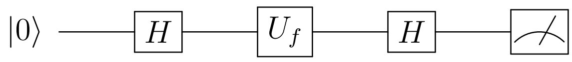
   (d) **(5 points)** Suppose Alice knows θ but Bob does not. Outline a protocol that uses one EPR pair and one classical bit of communication, at the end of which Bob holds a qubit in the state $|\psi\rangle$.

4. This exercise analyses the consequences of not restoring ancillas to their initial state. Let $f : \{0,1\} \to \{0,1\}$ be a (1-bit) function and U_f be the phase query that implements f, i.e.,

$$U_f \left|x\right\rangle = (-1)^{f(x)} \left|x\right\rangle .$$

   (a) **(5 points)** Show that f is either constant or balanced.

   (b) **(10 points)** Compute the outcome probabilities of the following circuit in the case where f is constant and in the case where f is balanced.



   (c) **(20 points)** Suppose that, instead of U_f, we have access to a query unitary that uses an ancilla qubit but does not return it to its original state. More precisely, let V_f be the two-qubit unitary defined below.

$$V_f \left|x\right\rangle \left|y\right\rangle = (-1)^{f(x)} \left|x\right\rangle \left|x \oplus y\right\rangle .$$

   Compute the outcome probabilities of the following circuit (which executes the algorithm of (b), ignoring the ancilla) in the case where f is constant and in the case where f is balanced.