THE UNIVERSITY OF WARWICK

LEVEL 7 Open Book Assessment [3 hours]

Department of Computer Science

CS4092 ALGORITHMIC GAME THEORY

Instructions
1. Read all instructions carefully and read through the entire paper at least once before you start writing.
2. There are **SIX** questions. You should **attempt FOUR questions**. **Indicate clearly which four** questions you have attempted. You should not submit answers to more than the required number of questions.
3. All questions will carry the same number of marks unless otherwise stated.
4. You should handwrite your answers either with paper and pen or using an electronic device with a stylus (unless you have special arrangements for exams which allow the use of a computer). Start each question on a new page and clearly mark each page with the page number, your student id and the question number. Handwritten notes must be scanned or photographed and all individual solutions should (if you possibly can) be collated into a single PDF with pages in the correct order. You must upload two files to the AEP: your PDF of solutions and a completed cover sheet. You must click **FINISH ASSESSMENT** to complete the submission process. After you have done so you will not be able to upload anything further.
5. Please ensure that all your handwritten answers are written legibly, preferably in dark blue or black ink. If you use a pencil ensure that it is not too faint to be captured by a scan or photograph.
6. Please check the legibility of your final submission before uploading. It is your responsibility to ensure that your work can be read.
7. You are allowed to access module materials, notes, resources, references and the internet during the assessment.

8. You should not try to communicate with any other candidate during the assessment period or seek assistance from anyone else in completing your answers. The Computer Science Department expects the conduct of all students taking this assessment to conform to the stated requirements. Measures will be in operation to check for possible misconduct. These will include the use of similarity detection tools and the right to require live interviews with selected students following the assessment.

9. By starting this assessment you are declaring yourself fit to undertake it. You are expected to make a reasonable attempt at the assessment by answering the questions in the paper.

Please note that:

– You must have completed and uploaded your assessment before the 24 hour assessment window closes.

– You have an additional 45 minutes beyond the stated duration of this assessment to allow for downloading and uploading the assessment, your files and technical delays.

– For further details you should refer to the AEP documentation.

Use the AEP to seek advice immediately if during the assessment period:

- you cannot access the online assessment;
- you believe you have been given access to the wrong online assessment;

Please note that technical support is only available between 9AM and 5PM (BST).

Invigilator support will be also be available (via the AEP) between 9AM and 5PM (BST).

Notify Dcs.exams@warwick.ac.uk as soon as possible if you cannot complete your assessment because:

- you lose your internet connection;
- your device fails;
- you become unwell and are unable to continue;
- you are affected by circumstances beyond your control (e.g. fire alarm).

Please note that this is for notification purposes, it is not a help line.

Your assessment starts below.

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1. (a) Consider a combinatorial auction over items $\{ A, B, C, D \}$ and with five single-minded bidders. The auction uses the VCG mechanism and the following bids are being submitted:

- bidder 1 submits bid $(\{ A, B \}, 2)$;
- bidder 2 submits bid $(\{ A, C \}, 3)$;
- bidder 3 submits bid $(\{ B \}, 2)$;
- bidder 4 submits bid $(\{ C \}, 2)$;
- bidder 5 submits bid $(\{ C, D \}, 2)$.

Perform the following two tasks.

i. Determine the winners of the auction above and justify your answer. [5]

ii. Determine all the payments and give the amount that the auctioneer obtains. [5]

(b) In a combinatorial auction over items $\{ A, B, C, D \}$ that is using the VCG mechanism, the following bids are being submitted:

- bidder 1 submits bid $(\{ A, B \}, 1)$
- bidder 2 submits bid $(\{ A, C \}, 1)$
- bidder 3 submits bid $(\{ C, D \}, 1)$

A fourth bidder wants to manipulate this auction by entering it under two false names ("bidder" 4 and "bidder" 5). Help them by giving two extra bids that will win them all four items without having to pay anything to the auctioneer. [5]

(c) Consider a single-item auction with at least three bidders. Give an example showing that awarding the item to the highest bidder, at the price equal to the third-highest bid, yields an auction that is not incentive compatible. [5]

(d) Suppose there are 7 identical copies of an item and $n > 7$ bidders, and assume that each bidder can receive at most one copy. Give an auction mechanism for this scenario that is inspired by, and generalizes, the second-price auction. Prove that your auction mechanism is incentive compatible. [5]

2. Consider the following two-player game. The utilities of the two players in each of the twelve pure strategy profiles are given in the corresponding cell of the following table as pairs of numbers: the first number is the utility of the row player and the second number is the utility of the column player.

	X	Y	Z
A	$-2, 4$	$0, 1$	$-2, 0$
B	$0, 0$	$1, 0$	$-1, 1$
C	$-1, 7$	$5, 10$	$-3, 12$
D	$1, -1$	$0, -6$	$-4, -4$

(a) Apply iterated elimination of strictly dominated strategies to reduce the game to a 2×2 two-player game. For every elimination step, explain which strategy strictly dominates which other strategy. [6]

(b) Consider the 2×2 game obtained in part (a). Find all Nash equilibria in this game. Explain why those are Nash equilibria. [6]

(c) Consider the 2×2 game obtained in part (a). Write down the constraints that define the set of correlated equilibria in this game. [6]

(d) Consider the 2×2 game obtained in part (a). Write down the social utility function and then derive a socially optimal correlated equilibrium and its social utility value. [7]

3. (a) Either give a valid coloring of a triangulation of a triangle in which exactly two triangle cells have all three colors, or argue why this is not possible. [6]

(b) We say that a coloring of a triangulation of a triangle is almost valid if a valid coloring can be obtained from it by changing the color of just one vertex. Either give an almost valid triangulation of a triangle such that exactly two triangle cells have all three colors, or argue why this is not possible. [6]

(c) Give a continuous function from a convex and bounded set into itself that does not have a fixed point. Explain why it does not contradict Brouwer's fixed point theorem. [6]

(d) We say that two subsets C and D of $\mathbb{R}^n$ that are homeomorphic if there exists a bijection $h : C \to D$ such that both h and its inverse h^{-1} are continuous.

Prove that if every continuous function from C to itself has a fixed point, then every continuous function from D to itself has a fixed point. [7]

4. (a) Consider the following 3-player game. Player I has strategies T and B, player II has strategies ℓ and r, and player III has strategies $\mathcal{L}$ and $\mathcal{R}$. The utilities of the three players in each of the eight pure strategy profiles are given in the corresponding cell of the following two tables as triples of numbers: the first number is the utility of player I, the second one is the utility of player II, and the last one is the utility of player III.

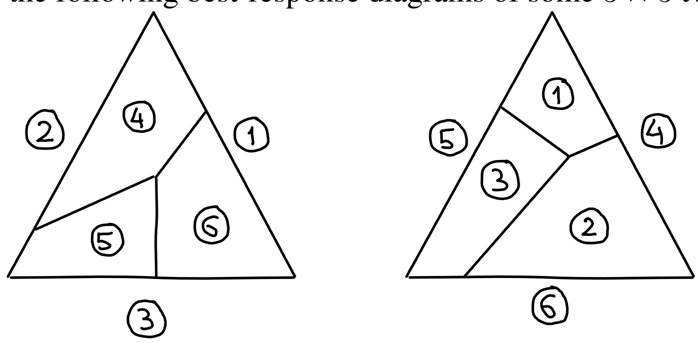
$\mathcal{L}$	ℓ	r
T	$3,4,4$	$1,3,3$
B	$8,1,4$	$2,0,6$

$\mathcal{R}$	ℓ	r
T	$4,0,5$	$0,1,6$
B	$5,1,3$	$1,2,5$

i. Give all pairs of pure strategies where one strategy strictly, or weakly, dominates the other. [4]

ii. Apply iterated elimination of strictly dominated strategies to this game. What are the Nash equilibria of the game? [4]

(b) Consider the following best-response diagrams of some 3×3 two-player game.

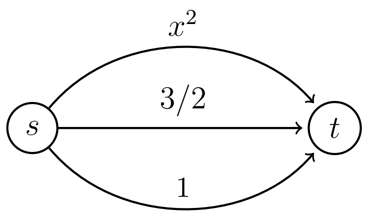


List all pairs of vertices in the two diagrams that correspond to Nash equilibria and, for each of those equilibria, explain which strategies are played with non-zero probability in it. [6]

(c) Prove that a 2-player game with utility matrices A and B has the same set of equilibria as the game obtained from it by adding 2020 to each entry in column 42 of matrix A. [6]

(d) Prove that for every pure Nash equilibrium E in a two-player game, there is a label that can be dropped in the first step of the Lemke-Howson algorithm, such that the algorithm will find equilibrium E after just two steps. [5]

5. Consider the following congestion game in which the amount of traffic that needs to be sent from the source vertex s to the destination vertex t is 1.



(a) Calculate a Wardrop equilibrium flow and compute its social cost. [5]

(b) Calculate a flow that has the smallest social cost and give its cost. [5]

(c) Calculate the price of anarchy. [3]

(d) Give an example of a flow network in which the price of anarchy is at least $3/2$. [6]

(e) Prove that if all edge latency functions in a flow network are linear functions without offsets (that is, if they are of the form $\ell_e(x) = a_e \cdot x$ for some $a_e > 0$) then the price of anarchy is 1. [6]

6. (a) Give an example of a two-player zero-sum game in which each player has at least three pure strategies, and in which there is no pure Nash equilibrium. Explain why there is no pure Nash equilibrium in your game. [4]

(b) Find a Nash equilibrium in the following zero-sum game. The values given are utilities of player I.

	X	Y
A	3	5
B	4	3
C	5	2

[6]

(c) Give a 2×2 zero-sum game whose only Nash equilibrium is the mixed strategy profile $\left(\left(\frac{1}{4}, \frac{3}{4}\right), \left(\frac{1}{3}, \frac{2}{3}\right)\right)$. [7]

(d) In a two-player game with utility matrices (A, B), we say that:

- a mixed strategy of the row player is their maximin strategy if it is a maximin strategy in the zero-sum game A;
- a mixed strategy of the column player is their maximin strategy if it is a maximin strategy in the zero-sum game B^T.

Compute a maximin strategy for each of the two players in the following game:

	3	4
1	$3, 4$	$1, 6$
2	$2, 5$	$4, 1$

What are the corresponding maximin values for each of the two players? [8]