CS419/939: Quantum Computing – Assignment 2

LaTeX submissions are preferred; handwritten submissions are acceptable, so long as your handwriting is clear. You may work in groups but you must write up your solutions individually. Questions marked with (*) are optional (and more challenging): you will get feedback but they do not count towards the final mark. You do not need to show your work unless the question asks you to, but you may receive part marks for correct working even if you get the wrong answer.

Problem 1 (10 marks)

Express the state of the following composite systems in the computational basis.

1. $|+\rangle \otimes |-\rangle$.
2. $|0\rangle \otimes |-\rangle \otimes |+\rangle$.
3. $\mathsf{CNOT}(|+\rangle \otimes |-\rangle)$ (where the first qubit is the control qubit).

In each of the systems above, compute the probability of obtaining the all-zero outcome (i.e. 00 or 000) when measuring in the computational basis. Show your work.

Problem 2 (15 marks)

Recall that in the quantum teleportation protocol, Alice measures her qubits in the *Bell basis*:

$$\left|\Phi^+\right\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \qquad \left|\Phi^-\right\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}} \qquad \left|\Psi^+\right\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}} \qquad \left|\Psi^-\right\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}}$$

Show how to implement a measurement in the Bell basis using only CNOT and H gates and measurements in the computational basis. You must show that your circuit has the correct distribution of outcomes and yields the correct post-measurement state.

Problem 3 (15 marks)

Compute the density matrices of the following mixed states:

1. $\begin{cases} |0\rangle & \text{with probability } 2/3 \\ |-\rangle & \text{with probability } 1/3 \end{cases}$

2. $\begin{cases} |00\rangle & \text{with probability } 1/2 \\ |10\rangle & \text{with probability } 1/4 \\ |11\rangle & \text{with probability } 1/4 \end{cases}$

3. $\begin{cases} i\,|0\rangle & \text{with probability } 1 \end{cases}$

4. $\begin{cases} |00\rangle & \text{with probability } 1/4 \\ |01\rangle & \text{with probability } 1/4 \\ \frac{1}{\sqrt{2}}(|00\rangle + i\,|11\rangle) & \text{with probability } 1/2 \end{cases}$

In each case, if the density matrix you compute is not diagonal, diagonalise it (i.e., write it as $\sum_i \lambda_i |\phi_i\rangle\langle\phi_i|$ for orthogonal unit vectors $|\phi_i\rangle$). You may use a computer to find the eigenvectors and eigenvalues, but you should write your answers in an exact form. (WolframAlpha can do this for you, for example.)

Problem 4 (30 marks)

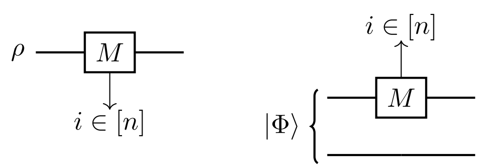
In this problem, we will explore a useful tool in quantum information called *purification.* We start with a density matrix ρ on system A; recall that ρ is Hermitian, has positive eigenvalues, and $\mathrm{Tr}(\rho) = 1$.

1. Let $|\phi_1\rangle, \ldots, |\phi_n\rangle$ be orthonormal eigenvectors of ρ. For each i, let λ_i be the eigenvalue corresponding to $|\phi_i\rangle$; i.e. $\rho|\phi_i\rangle = \lambda_i |\phi_i\rangle$. Show that $|\Phi\rangle_{AB} = \sum_{i=1}^n \sqrt{\lambda_i}\, |\phi_i\rangle_A \otimes |\phi_i\rangle_B$ is a (pure) quantum state on the system AB; i.e., $\||\Phi\rangle_{AB}\| = 1$. We call $|\Phi\rangle$ a *purification* of ρ. ((*) Is $|\Phi\rangle_{AB}$ an entangled state?)

   (The subscripts A, B, AB are there just to help keep track of which state is on system A, on the ancilla system B, and on both systems, respectively.)

2. Let $\rho = I_2/2$, where I_2 is the 2×2 identity matrix; i.e., the maximally mixed state on one qubit. Give a purification $|\Phi\rangle$ of ρ. ($|\Phi\rangle$ should be a two-qubit state.) Is $|\Phi\rangle$ separable? Justify your answer.

3. Let $M = (\Pi_i)_{i=1}^n$ be any projective measurement on system A. Show that for any state ρ and a purification $|\Phi\rangle$ of ρ, the distribution of measurement outcomes is the same in each of the two circuits below.



4. Show that for any state $|\phi\rangle$ on one qubit, there is a projective measurement that distinguishes $\rho = I_2/2$ from $|\phi\rangle$. This shows that adding system B is necessary to achieve the result of part 3.

Problem 5 (30 marks)

Follow the link on the course webpage to the Problem 4 circuit. The circuit opens in IBM Quantum Composer, a tool for simulating quantum circuits. The circuit has two `input` wires, one ancilla wire `anc[0]` and one output wire `out[0]`.

1. What function of the inputs does the circuit compute on the wire `anc[0]`? What function f of the inputs does the circuit compute on the wire `out[0]`? Is this function constant or balanced?

2. Test the circuit on the superposition of all inputs by doing the following. First, add Hadamard (H) gates to both `input` wires on the left (*before* the rest of the circuit). Then, add a measurement (the symbol is a dial with a z) to `out[0]`. What do you see in the "Probabilities" view? Next, change the measurement so that it measures `anc[0]`. You can do this by clicking the measurement and pressing Ctrl-E (or Cmd-E), and changing "qbit" to `anc[0]`. What do you see?

3. Next we will run the Deutsch-Josza algorithm on f. To do this, add more Hadamard (H) gates to the `input` wires, this time on the right (after the rest of the circuit). Add a second measurement output by clicking the wire label "meas1" and then clicking "+". Then delete the measurement from the output wire and add a measurement to each of the input wires, on the right. Finally, add two gates to the output wire (on the left) to change its state from $|0\rangle$ to $|-\rangle$. What do you see?

4. Fix the circuit by adding one gate, so that the Deutsch-Josza algorithm gives the correct output. Take a screenshot of the final circuit and include it with your submission.