CS419/939

THE UNIVERSITY OF WARWICK

Summer 2022

Quantum Computing MOCK

Time allowed: 3 hours.

Exam type: Standard Examination.

Answer **BOTH** questions from Section A and **ONE** question from Section B.

Read carefully the instructions on the answer book.

Calculators are not allowed.

Section A Answer **BOTH** questions.

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1. (a) **(4 points)** Define a *Hermitian projector*.

   (b) **(12 points)** Give a full description of *projective measurement*.

   (c) **(14 points)** Suppose we apply the projective measurement $(|+\rangle\langle+| \otimes I, |-\rangle\langle-| \otimes I)$ to the quantum state $|\psi\rangle = \alpha_1 |00\rangle + \alpha_2 |01\rangle + \alpha_3 |10\rangle + \alpha_4 |11\rangle$. Compute the probability of each outcome and the post-measurement state in each case.

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2. (a) **(10 points)** Prove the no-cloning theorem for qubit states: there exists no 2-qubit unitary that maps $|\psi\rangle |0\rangle \mapsto |\psi\rangle |\psi\rangle$ for all quantum states $|\psi\rangle$.

   (b) **(5 points)** Show that it is possible to "clone" qubits in the computational basis: there exists a 2-qubit unitary that maps $|0\rangle |0\rangle \mapsto |0\rangle |0\rangle$ and $|1\rangle |0\rangle \mapsto |1\rangle |1\rangle$ (you may either draw a circuit or exhibit the 4×4 unitary matrix).

   (c) **(15 points)** Let $U_f^{\pm}$ be a *controlled* unitary transformation that "sign-implements" a function $f : \{0,1\}^n \to \{0,1\}$, i.e.:

   $$U_f^{\pm} |0\rangle |x\rangle = |0\rangle |x\rangle \text{ and } U_f^{\pm} |1\rangle |x\rangle = (-1)^{f(x)} |1\rangle |x\rangle .$$

   Show how to use $U_f^{\pm}$ and standard gates to obtain a *standard* implementation of f; that is, a unitary U_f such that

   $$U_f |b\rangle |x\rangle = |b \oplus f(x)\rangle |x\rangle \text{ for each } b \in \{0,1\}.$$

Section B Answer **ONE** question.

1. Denote by $|\theta\rangle$ the state $\cos(\theta)|0\rangle + \sin(\theta)|1\rangle$. In this question, you may use standard trigonometric identities without proof; e.g. the double-angle identities $\sin 2\mu = 2\sin\mu\cos\mu$ and $\cos 2\mu = 1 - 2\sin^2\mu$.)

   (a) **(2 points.)** Show that $\||\theta\rangle\| = 1$.

   (b) **(3 points.)** Show that the state $|\theta^\perp\rangle = \sin(\theta)|0\rangle - \cos(\theta)|1\rangle$ is orthogonal to $|\theta\rangle$.

   (c) **(5 points.)** Give a simplified expression for $|\langle +|\theta\rangle|^2$.

   (d) **(15 points.)** Show that $\max_{\theta\in[0,2\pi]}(|\langle +|\theta\rangle|^2 - |\langle 0|\theta\rangle|^2) = \frac{1}{\sqrt{2}}$. What value of θ achieves the maximum? (You may use without proof the fact that $\max_{\mu\in[0,2\pi]}(\sin\mu - \cos\mu) = \sqrt{2}$, with the maximum achieved at $\mu = 3\pi/4$.)

   (e) **(15 points.)** Suppose you are given the state $|0\rangle$ with probability $1/2$ and the state $|+\rangle$ with probability $1/2$. Using your answer to part (d), describe how to optimally distinguish the two states by measuring in the $(|\theta\rangle, |\theta^\perp\rangle)$ basis for some $\theta \in [0, 2\pi]$. What is the probability that you correctly detect which state you received?

2. Consider the following generalisation of Simon's problem: the input is $F : \{0,1\}^n \to \{0,1\}^n$, with the property that there is some unknown *subspace* $V \subseteq \{0,1\}^n$ such that $F(x) = F(y)$ if and only if there exists an element $v \in V$ such that $x = v \oplus y$. (In this case, a subspace V is simply a set such that $u \oplus v \in V$ whenever $u, v \in V$.)

   The goal is now to determine V. The usual definition of Simon's problem corresponds to the case where $V = \{0, s\}$ for unknown $s \in \{0,1\}^n$ (which is equivalent to finding s).

   (a) **(5 points)** Show that $\{0, s\}$ is a subspace of $\{0,1\}^n$ for any fixed s.

   (b) **(25 points)** Show that one run of Simon's algorithm produces a $z \in \{0,1\}^n$ that is orthogonal to V (i.e., $\sum_{i=0}^n z_i \cdot v_i = 0 \mod 2$ for every $v \in V$).

   (c) **(10 points)** Describe how to extend Simon's algorithm to solve this generalised version of the problem. Justify informally why your algorithm is correct.