CSCB63 Tutorial, Week 2

Yufei Cui

January 14, 2019

1 Complexity

1.1 Big Notations

We define a series of notations to help us understand and describe the asymptotic relations between functions. We will use the conventional O, Ω, and Θ notations to represent a function's asymptotic upper, lower, and tight bound, respectively.

Definition. Let $g : \mathbb{N} \to \mathbb{R}_{\geq 0}$

$$O(g) = \{f \mid \exists\, c \in \mathbb{R}_{>0} \text{ and } n_0 \in \mathbb{N} \text{ such that, if } n \geq n_0 \text{ then } f(n) \leq c \cdot g(n)\} \tag{1}$$

$$\Omega(g) = \{f \mid \exists\, c \in \mathbb{R}_{>0} \text{ and } n_0 \in \mathbb{N} \text{ such that, if } n \geq n_0 \text{ then } c \cdot g(n) \leq f(n)\} \tag{2}$$

$$\Theta(g) = \{f \mid \exists\, c_1,\, c_2 \in \mathbb{R}_{>0},\, n_0 \in \mathbb{N} \text{ such that, if } n \geq n_0 \text{ then } c_1 \cdot g(n) \leq f(n) \leq c_2 \cdot g(n)\}^1 \tag{3}$$

Note. Notice how $O(g)$, $\Omega(g)$, and $\Theta(g)$ are all defined as sets, you can think of them as being family of functions. When we say "f is big-O of g" we really mean to say f belongs in the set $O(g)$, that is, $f \in O(g)$. Though it's often the case that textbooks use the notation $f = O(g)$ to mean the same thing. Refer to chapter 3.1 in the C.L.R.S for more detail.

Exercise 1.1. Using definitions 1, 2, and 3, confirm the following:

(a) $n \in O(n/2 - 1)$

(b) $n^2 - n + \sqrt{n}/3 \in \Omega(n^2)$

(c) $\log_2 n! \in \Theta(n \log_2 n)$

(d) $n \notin O(n^{1+\sin(n)})$

(e) $n \notin \Omega(n^{1-\cos(n)})$

[1] $\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$ is an equivalent definition.

Example 1.1. Below is a table with commonly seen functions distributed across the top row and left most column. A check mark is placed in the cell if and only if the column function is big-O of the row function.[2] [3]

(col) $\in O$ **(row)**	$\ln(n)$	$\lg(n)$	$\lg(n^2)$	$(\lg(n))^2$	n	$n\lg(n)$	2^n	2^{3n}
$\ln(n)$	✓	✓	✓	✓	✓	✓	✓	✓
$\lg(n)$	✓	✓	✓	✓	✓	✓	✓	✓
$\lg(n^2)$	✓	✓	✓	✓	✓	✓	✓	✓
$(\lg(n))^2$				✓	✓	✓	✓	✓
n					✓	✓	✓	✓
$n\lg(n)$						✓	✓	✓
2^n							✓	✓
2^{3n}								✓

Table 1: Big-O Table

Example 1.2. We will prove a selected few examples from Table 1

(a) $n \in O(n\lg(n))$

Proof. we want to find a constant c and a sufficiently large value n_0 so that we can bound n by $cn\lg(n)$

Choose $c = 1$, $n_0 = 2$
Suppose $n \geq n_0$

$$\begin{aligned} n &= 1 \cdot n \cdot 1 \\ &\leq 1 \cdot n \cdot \lg(n) && (n \geq n_0 = 2 \implies 1 \leq \lg(n)) \\ &= cn\lg(n) \end{aligned}$$

□

(b) $(\lg(n))^2 \in O(n)$

Proof.

[2] $\ln(n)$ is the natural log and $\lg(n)$ is $\log_2(n)$

[3] Your TA might have drawn a similar table where the check mark is placed if and only if the row function is big-O of the column function, they are equivalent.

Choose $c = 1$, $n_0 = 3$
Suppose $n \geq n_0$

$$\begin{aligned}
(\lg(n))^2 &= \lg(n) \cdot \lg(n) \\
&\leq \sqrt{n} \cdot \sqrt{n} && (n \geq n_0 = 3 \implies \lg(n) \leq \sqrt{n}) \\
&= n \\
&= 1 \cdot n \\
&= cn
\end{aligned}$$

$\square$

(c) $n \lg(n) \notin O(n)$

Proof. Suppose on the contrary that the statement is true. That is, $n \lg(n) \in O(n)$, then there exist some c and some n_0 such that if $n \geq n_0$ then $n \lg(n) \leq cn$

Let $n \geq \max\{n_0,\, 1,\, 2^c + 1\}$

$$\begin{aligned}
& n \lg(n) \leq cn && \text{(by definition of big-O)} \\
\implies & \lg(n) \leq c && (n \geq 1 \implies n \neq 0) \\
\implies & 2^{\lg(n)} \leq 2^c && \text{(raise both sides to the power of 2)} \\
\implies & n \leq 2^c && \text{(log property)}
\end{aligned}$$

We let $n \geq \max\{n_0,\, 1,\, 2^c + 1\} \implies n \geq 2^c + 1$ but $n \leq 2^c$. That is a contradiction as there's no such $c \in \mathbb{R}_{\geq 0}$ such that $2^c + 1 \leq n \leq 2^c$. Therefore, our original assumption must have been false, so $n \lg(n) \notin O(n)$. $\square$

(d) Using the limit theorems, prove $2^{3n} \notin O(2^n)$

Proof. One of the limit theorem states

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \infty \implies f(n) \notin O(g(n))$$

Applying the above theorem

$$\begin{aligned}
& \lim_{n\to\infty} \frac{2^{3n}}{2^n} \\
&= \lim_{n\to\infty} \frac{(2^n)^3}{2^n} && \text{(by exponent law)} \\
&= \lim_{n\to\infty} \frac{(2^n)(2^n)\cancel{(2^n)}}{\cancel{2^n}} \\
&= \lim_{n\to\infty} (2^n)^2 \\
&= \infty
\end{aligned}$$

Therefore, $2^{3n} \notin O(2^n)$ by the limit theorem. $\square$

Example 1.3. We'll take our understanding of the definitions one step further and create a Big-Theta table similar to Table 1

(col) $\in \Theta$ **(row)**	$\ln(n)$	$\lg(n)$	$\lg(n^2)$	$(\lg(n))^2$	n	$n\lg(n)$	2^n	2^{3n}
$\ln(n)$	✓	✓	✓					
$\lg(n)$	✓	✓	✓					
$\lg(n^2)$	✓	✓	✓					
$(\lg(n))^2$				✓				
n					✓			
$n\lg(n)$						✓		
2^n							✓	
2^{3n}								✓

Table 2: Big-Theta Table

Example 1.4. Big Omega/Theta proofs

(a) $3n^2 - 4n \in \Omega(n^2)$

Proof.
Choose $c = 1$, $n_0 = 2$
Suppose $n \geq n_0$

$$\begin{aligned} 3n^2 - 4n &\geq 3n^2 - 2n^2 && \text{(see note below)} \\ &= n^2 \\ &= cn^2 \end{aligned}$$

Note. We want an n_0 such that $-4n \geq -2n^2$, we will try to find the range of n_0 by simplifying the inequality

$$\begin{aligned} &-4n \geq -2n^2 \\ \implies &\ 4n \leq 2n^2 \\ \implies &\ 2n \leq n^2 \\ \implies &\ 2 \leq n && \text{(we've found our } n_0\text{)} \end{aligned}$$

$\square$

(b) $f(n) = 6n^5 + n^2 - n^3 \in \Theta(n^5)$

Proof. There are many ways to prove this, here's one way

Choose $c_1 = 5$, $c_2 = 7$, $n_0 = 1$
Suppose $n \geq n_0$

$$\begin{aligned} 6n^5 + n^2 - n^3 &\leq 6n^5 + n^2 \\ &\leq 6n^5 + n^5 && (n \geq 1 \implies n^2 \leq n^5) \\ &= 7n^5 \\ &= c_1 n^5 \end{aligned}$$

$$\begin{aligned} 6n^5 + n^2 - n^3 &\geq 6n^5 - n^3 \\ &\geq 6n^5 - n^5 && (n \geq 1 \implies -n^3 \geq -n^5) \\ &= 5n^5 \\ &= c_2 n^5 \end{aligned}$$

Therefore, whenever $n \geq 1 \implies c_1 n^5 \leq f(n) \leq c_2 n^5$ $\square$

Exercise 1.2. Let functions $f, g, h : \mathbb{N} \to \mathbb{R}_{\geq 0}$. Prove the following properties:

(a) $f \in O(g) \wedge g \in O(h) \implies f \in O(h)$

(b) $f \in \Omega(g) \wedge g \in \Omega(h) \implies f \in \Omega(h)$

(c) $f \in \Theta(g) \wedge g \in \Theta(h) \implies f \in \Theta(h)$

(d) $f \in O(g) \iff g \in \Omega(f)$

(e) $f \in \Theta(g) \iff g \in \Theta(f)$

(f) $f + g \in O(\max\{f, g\})$

(g) $f \cdot g \in O(f \cdot g)$

(h) $\exists\, u, v : \mathbb{N} \to \mathbb{R}_{\geq 0}, u \notin O(v) \wedge v \notin O(u)$ [Hint: Exercise 1.1]

Example 1.5. Exercise 1.2 can be a little intimidating for some of you who haven't seen these or has forgotten them, so we will cover some of them.

(a) $f \in O(g) \land g \in O(h) \implies f \in O(h)$

This is called the *transitive* property. This intuitively makes sense if you consider the implication: f can be asymptotically bounded above by g and similarly g can be bounded by h then you would imagine that f can be also bounded above by h.

Proof. Suppose $f \in O(g) \land g \in O(h)$. That is, $\exists\, c_g, n_g$ such that $f(n) \leq c_g g(n)$ whenever $n \geq n_g$. Likewise, $\exists\, c_h, n_h$ such that $g(n) \leq c_h h(n)$ whenever $n \geq n_h$.

Choose $c = max\{c_g, c_h, c_g \cdot c_h\}$, $n_0 = max\{n_g, n_h\}$
Suppose $n \geq n_0$

$$\begin{aligned}
f(n) &\leq c_g g(n) && \text{(by definition of } f \in O(g)) \\
&\leq c_g \cdot (c_h h(n)) && \text{(by definition of } g \in O(h)) \\
&\leq (c_g \cdot c_h)\, h(n) \\
&\leq c h(n)
\end{aligned}$$

□

(f) $f + g \in O(\max\{f, g\})$

This is called the *sum* property.[4] This property is a more formal way of saying we are allowed to "ignore the lower order terms" of a function when we deal with Big-O. For example, $n^2 + n + \log n \in O(n^2)$.

Proof. Choose $c = 2$, $n_0 \geq 0$ and suppose $n \geq n_0$

$$\begin{aligned}
f(n) + g(n) &\leq max\{f(n), g(n)\} + max\{f(n), g(n)\} \\
&\leq 2 \cdot max\{f(n), g(n)\} \\
&\leq c \cdot max\{f(n), g(n)\}
\end{aligned}$$

□

[4] The more general version of the sum rule is: $f_1 \in O(g_1) \land f_2 \in O(g_2) \implies f_1 + f_2 \in O(\max\{g_1, g_2\})$. Similarly, $f_1 \in O(g_1) \land f_2 \in O(g_2) \implies f_1 \cdot f_2 \in O(g_1 \cdot g_2)$ is a general version of the product rule, which is question (g) in Exercise 1.2.