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Problem Set 3
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Question
Prove that if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a} f(x) \cdot g(x) = L \cdot M$.
Proof:
We know $\forall x$, such that $0 < |x-a| < \delta_1, |f(x)-L| < \epsilon_1$, and $0 < |x-a| < \delta_2, |g(x)-M| < \epsilon_2$. Then, we would have $|g(x) - M| \cdot |f(x) - L| < \epsilon_1 \cdot \epsilon_2$, based on the theorem that if $0 < a, b$ and $a < c, b < d$ then $ab < cd$. Also, from the theorem that $|a| \cdot |b| = |ab|$. Then we would have the following:

$$|g(x) - M| \cdot |f(x) - L| < \epsilon_1\epsilon_2$$

$$|f(x) \cdot g(x) - M \cdot f(x) - L \cdot g(x) + LM| < \epsilon_1\epsilon_2$$

By another theorem that $|a| + |b| = |a + b|$, we could on both sides of the inequality add $|M \cdot f(x) + L \cdot g(x) - 2LM|$. Also, we know that $0 \le \epsilon_1, \epsilon_2$. Thus, $\epsilon_1\epsilon_2 = |\epsilon_1\epsilon_2|$. Then, we would have for the inequality:

$$|f(x)\cdot g(x)-M\cdot f(x)-L\cdot g(x)+LM|+|M\cdot f(x)+L\cdot g(x)-2LM| < |\epsilon_1\epsilon_2|+|M\cdot f(x)+L\cdot g(x)-2LM|$$

$$|f(x)\cdot g(x)-M\cdot f(x)-L\cdot g(x)+LM+M\cdot f(x)+L\cdot g(x)-2LM| < |\epsilon_1\epsilon_2+M\cdot f(x)+L\cdot g(x)-2LM|$$

$$|f(x) \cdot g(x) - LM| < |\epsilon_1\epsilon_2 + M \cdot f(x) + L \cdot g(x) - LM - LM|$$

$$|f(x) \cdot g(x) - LM| < |\epsilon_1\epsilon_2 + M \cdot f(x) - LM + L \cdot g(x) - LM|$$

$$|f(x) \cdot g(x) - LM| < |\epsilon_1\epsilon_2 + M \cdot (f(x) - L) + L \cdot (g(x) - M)|$$

$$|f(x) \cdot g(x) - LM| < |\epsilon_1\epsilon_2| + |M(f(x) - L)| + |L(g(x) - M)|$$

Also, we know from the definition of a limit that $|f(x) - L| < \epsilon_1$, $|g(x) - M| < \epsilon_2$. We would therefore have:

$$|M| \cdot |f(x) - L| < |M| \cdot \epsilon_1$$
$$|L| \cdot |g(x) - M| < |L| \cdot \epsilon_2$$
$$\therefore |M \cdot (f(x) - L)| < |M \cdot \epsilon_1|, |L \cdot (g(x) - M)| < |L \cdot \epsilon_2|$$

Thus, we would have a new inequality:

$$|\epsilon_1\epsilon_2| + |M(f(x) - L)| + |L(g(x) - M)| < |\epsilon_1\epsilon_2| + |M \cdot \epsilon_1| + |L \cdot \epsilon_2|$$

From there, we could say that $|f(x) \cdot g(x) - LM| < |\epsilon_1\epsilon_2| + |M \cdot \epsilon_1| + |L \cdot \epsilon_2| = \epsilon$
We then have $\lim_{x\to a} f(x) \cdot g(x) = L \cdot M$.