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Problem Set 3
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Problem Set 3

Question

Prove that if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a} f(x)\cdot g(x) = L\cdot M$.

Proof:

We know $\forall x$ such that $0 < |x-a| < \delta_1, |f(x)-L| < \epsilon_1$, and such that $0 < |x-a| < \delta_2, |g(x)-M| < \epsilon_2$. In order for these two ineqaulities to hold at the same time, we need to have $0 < |x-a| < \delta = \text{Min}(\delta_1,\delta_2)$. Then, we would have $|g(x)-M|\cdot|f(x)-L| < \epsilon_1\cdot\epsilon_2$, based on the theorem that if $0 < a, b$ and $a < c, b < d$ then $ab < cd$. Also, from the theorem that $|a|\cdot|b| = |ab|$. Then we would have the following:

$$|g(x)-M|\cdot|f(x)-L| < \epsilon_1\epsilon_2$$

$$|f(x)\cdot g(x) - M\cdot f(x) - L\cdot g(x) + LM| < \epsilon_1\epsilon_2$$

By another theorem that $|a|+|b| < |a+b|$, we could on both sides of the inequality add $|M\cdot f(x) + L\cdot g(x) - 2LM|$. Also, we know that $0 < \epsilon_1, \epsilon_2$. Thus, $\epsilon_1\epsilon_2 = |\epsilon_1\epsilon_2|$. Then, we would have for the inequality:

$$|f(x)\cdot g(x) - M\cdot f(x) - L\cdot g(x) + LM| + |M\cdot f(x) + L\cdot g(x) - 2LM| \tag{1}$$

$$|f(x)\cdot g(x) - M\cdot f(x) - L\cdot g(x) + LM + M\cdot f(x) + L\cdot g(x) - 2LM| \tag{2}$$

$$|\epsilon_1\epsilon_2| + |M\cdot f(x) + L\cdot g(x) - 2LM| \tag{3}$$

Where (2) < (1) and (1) < (3). Thus, (2) < (3).

Then we have:

$$(2) < |\epsilon_1\epsilon_2| + |M\cdot f(x) + L\cdot g(x) - LM - LM|$$

$$(2) < |\epsilon_1\epsilon_2| + |M\cdot f(x) - LM + L\cdot g(x) - LM|$$

$$(2) < |\epsilon_1\epsilon_2| + |M\cdot(f(x)-L) + L\cdot(g(x)-M)|$$

$$(2) < |\epsilon_1\epsilon_2| + |M(f(x)-L) + L(g(x)-M)|$$

Also, we know from the definition of a limit that $|f(x)-L| < \epsilon_1$, $|g(x)-M| < \epsilon_2$. We would therefore have:

$$|M| \cdot |f(x) - L| < |M| \cdot \epsilon_1$$
$$|L| \cdot |g(x) - M| < |L| \cdot \epsilon_2$$
$$\therefore |M \cdot (f(x) - L)| < |M \cdot \epsilon_1|, |L \cdot (g(x) - M)| < |L \cdot \epsilon_2|$$

Thus, we would have a new inequality:

$$|\epsilon_1\epsilon_2| + |M(f(x) - L)| + |L(g(x) - M)| < |\epsilon_1\epsilon_2| + |M \cdot \epsilon_1 + L \cdot \epsilon_2|$$

From there, we could say that $|f(x) \cdot g(x) - LM| < |\epsilon_1\epsilon_2| + |M \cdot \epsilon_1 + L \cdot \epsilon_2| \leq \epsilon$
Assume $\epsilon_1\epsilon_2 \leq \frac{\epsilon}{2}$. From there we know, $\epsilon_1, \epsilon_2 \leq \sqrt{\frac{\epsilon}{2}}$. On the other hand, from $|M \cdot \epsilon_1 + L \cdot \epsilon_2| \leq \frac{\epsilon}{2}$. We know that $|M \cdot \epsilon_1 + L \cdot \epsilon_2| \leq |M \cdot \epsilon_1| + |L \cdot \epsilon_2|$. In order for the previous inequality to hold, we assign $|M \cdot \epsilon_1| < \epsilon/4$ and $|L \cdot \epsilon_2| < \epsilon/4$. Therefore, $\epsilon_2 < \frac{\epsilon}{4 \cdot |M|}$ and $\epsilon_1 < \frac{\epsilon}{4 \cdot |L|}$, where $L, M \neq 0$. If we have $\epsilon_1 < \text{Min}(\frac{\epsilon}{4 \cdot |L|}, \sqrt{\frac{\epsilon}{2}})$ and $\epsilon_1 < \text{Min}(\frac{\epsilon}{4 \cdot |M|}, \sqrt{\frac{\epsilon}{2}})$. Also, if we have L or M equal to 0. We can just say $\epsilon_1 < \sqrt{\frac{\epsilon}{2}}$ and $\epsilon_2 < \sqrt{\frac{\epsilon}{2}}$. Then, we have ϵ to be a very small number. We then have $\lim_{x \to a} f(x) \cdot g(x) = L \cdot M$.

Question
Suppose that $\lim\limits_{x \to a} f(x)$ exists, and that $\lim\limits_{x \to a} f(x) = L$. Suppose M is any number. Then prove that $\lim\limits_{x \to a}(Mf(x))$ exists, and $\lim\limits_{x \to a}(Mf(x)) = M \lim\limits_{x \to a} f(x)$.
Proof:
Suppose $\epsilon_1 = \frac{\epsilon}{|M|}$ where $|M| \neq 0$ and $\forall x$ such that $0 < |x - a| < \delta, |f(x) - L| < \epsilon$. From the theorem that $|a| \cdot |b| = |ab|$. Thus, $|M| \cdot |f(x) - L| = |M(f(x) - L)|$. Then, we know that

$$M \cdot |f(x) - L| = |M(f(x) - L)| < |M| \cdot \epsilon$$

$$|M \cdot f(x) - LM| < |M \cdot \epsilon| = \epsilon$$

Then we could have $|M \cdot f(x) - LM|$ be a small number and therefore, $\lim_{x \to a}(M \cdot f(x)) = L \cdot M$. Also, because $\lim_{x \to a} f(x)$ is a number, then we know that $M \cdot \lim_{x \to a} f(x) = M \cdot L$. Therefore, $M \cdot \lim_{x \to a} f(x) = \lim_{x \to a}(M \cdot f(x))$.
If we have $M = 0$, then we know that on the left hand side, we are finding the limit of 0, which is 0. And on the right hand side, we have $0 \cdot L = 0$. Therefore, the theorem still holds.

Question
Show that a function cannot have two different limits at a. That is, if $\lim\limits_{x \to a} f(x)$ exists, and $\lim\limits_{x \to a} f(x) = L$, and $\lim\limits_{x \to a} f(x) = M$, then we must have $L = M$.

Proof:

Suppose we have $L < M$ and we assign $0 < \epsilon \leq \frac{M-L}{2}$ such that $2\epsilon \leq M-L$ and $L+\epsilon \leq M-\epsilon$ therefore, $(M-\epsilon, M+\epsilon) \cap (L-\epsilon, L+\epsilon) = \emptyset$. Also, $\forall x$ such that $0 < |x-a| < \delta, |f(x)-L| < \epsilon$ and $|f(x) - M| < \epsilon$. From that we have $L - \epsilon < f(x) < L + \epsilon$ and $M - \epsilon < f(x) < M + \epsilon$. Then, we know that we need to have two different $f(x)$ in order to have $f(x)$ to be in two different ranges. This means $f(x)$ is not a one-to-one relationship, thus not a function. Therefore, for $f(x)$ there should be only one possible function. Then, if $\lim_{x \to a} f(x)$ exists, and $\lim_{x \to a} f(x) = L$, and $\lim_{x \to a} f(x) = M$, then we must have $L = M$.

Chapter 5. #8

(i) Counter Example

For instance, if we let $f(x) = \frac{1}{x^2}$ and $g(x) = -\frac{1}{x^2}$, then we have $f(x)+g(x) = \frac{1}{x^2}+(-\frac{1}{x^2}) = 0$. And if $x \to 0$, then both $\lim_{x\to0} f(x)$ and $\lim_{x\to0} g(x) = L$ do not exist, but $\lim_{x\to0}(f(x) + g(x)) = \lim_{x\to0} 0 = 0$ which do exist. Therefore, a counter example.

If we were to generalize this situation, for any polynomial $f(x)$ that does not contain a constant, let $g(x) = \frac{1}{f(x)}$ and $h(x) = -\frac{1}{f(x)}$. As $x \to 0, \lim_{x\to0} f(x)$ and $\lim_{x\to0} g(x) = L$ do not exist, but $\lim_{x\to0}(f(x) + g(x)) = \lim_{x\to0} 0 = 0$ which do exist.

(ii) Proof

Assume, $\lim_{x\to0} g(x)$ does not exist. From the thorem I, we have proved that $\lim_{x\to0}(f(x)+g(x)) = \lim_{x\to0} f(x) + \lim_{x\to0} g(x)$. It is because $\lim_{x\to0}(f(x) + g(x))$ exist, and $\lim_{x\to0} f(x)$ exist. By definition of a real number such that the differece between any two real number is a real number. Thus, $\lim_{x\to0}(f(x)+g(x)) - \lim_{x\to0} f(x)$ exist, which contradicts with our assumption. Therefore, $\lim_{x\to0} g(x)$ does not exist.

(iv) Counter Example

Let $f(x) = x$ and $g(x) = \sqrt{x}$, such that for $x \to -1$, $f(x)$ has a limit and $g(x)$ does not exist. However, for $f(x) \cdot g(x) = x \cdot \sqrt{x} = 1$, $\lim_{x\to-1}(f(x) \cdot g(x)) = \lim_{x\to-1} 1 = 1$, which means the limit of $f(x)g(x)$ exist.