Yufei Lin
Problem Set 1
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Problem Set 1

I. Propositions

Basic Properties of Equivalent:

(E0) If $a = b$, b can substitute a in any real formula

(E1) $\forall a$, $a = a$

(E2) $\forall a, b$, if $a = b$, then $b = a$

(E3) $\forall a, b, c$, if $a = b$, $b = c$, then $c = a$

Basic Properties of Numbers

(P1) $a + (b + c) = (a + b) + c$

(P2) $a + 0 = 0 + a = a$

(P3) $a + (-a) = (-a) + a = 0$

(P4) $a + b = b + a$

(P5) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

(P6) $a \cdot 1 = 1 \cdot a = a, 1 \neq 0$

(P7) $a \cdot a^{-1} = a^{-1} \cdot a = 1$, for $a \neq 0$

(P8) $a \cdot b = b \cdot a$

(P9) $a \cdot (b + c) = a \cdot b + a \cdot c$

(P10)For every number a, one and only one of the following holds:

$(i) a = 0$

$(ii) a \in P$

$(ii) (-a) \in P$

(P11) If a and b are in P, then $a + b$ is in P

(P12) If a and b are in P, then $a \cdot b$ is in P

II. Solutions

Questions 10, 13, 14, 16, 22, 24, 26

Question 6 $\forall a, b$: if $a \cdot b = 0$, then either $a = 0$ or $b = 0$

Proof:

First, suppose $a \cdot b = 0$. Then, let's assume $a = 0$, from Question 5, we have proved that 0 multiplies any number is 0, then $a \cdot b = 0$.

Then, let's assume $a \neq 0$. From Question 2 we have proved that if we multiply the same thing on both side of a equation, then the equation is still valid. Therefore, we could obtain, by multiplying a^{-1} on both sides of the equation:

$$a^{-1} \cdot (a \cdot b) = a^{-1} \cdot 0 = 0$$

By (P5), we can reformat the equation:

$$(a^{-1} \cdot a) \cdot b = 0$$

By (P7), we have $a^{-1} \cdot a = 1$

$$\therefore \ (a^{-1} \cdot a) \cdot b = 1 \cdot b = 0$$

By (P6) we have $1 \cdot b = b$,

$$\therefore \ b = 0$$

Question 7 $\forall a, b$: $(a+b)^2 = a^2 + 2ab + b^2$
Proof:
By definition of the exponentials we have $\forall a, a^2 = a \cdot a$
$\therefore$ we have $(a+b)^2 = (a+b) \cdot (a+b)$

$$\begin{aligned}
(a+b) \cdot (a+b) &= (a+b) \cdot a + (a+b) \cdot b \qquad \text{By(P9)} \\
&= a \cdot (a+b) + b \cdot (a+b) \qquad \text{By(P8)} \\
&= a \cdot a + a \cdot b + b \cdot a + b \cdot b \qquad \text{By(P9)} \\
&= a^2 + a \cdot b + a \cdot b + b^2 \qquad \text{By(P8)} \\
&= a^2 + (a \cdot b) \cdot 1 + (a \cdot b) \cdot 1 + b^2 \qquad \text{By(P6)} \\
&= a^2 + (a \cdot b) \cdot (1+1) + b^2 \qquad \text{By(P9)} \\
&= a^2 + (a \cdot b) \cdot 2 + b^2 \qquad \text{By(P9)} \\
&= a^2 + 2 \cdot (a \cdot b) + b^2 \qquad \text{By(P8)}
\end{aligned}$$

Question 9 $\forall a$: $(-1) \cdot a = -a$
Proof:

If we do the following:

$$\begin{aligned} a + (-1)\cdot a &= a\cdot 1 + (-1)\cdot a \qquad \text{By(P6)} \\ &= a\cdot 1 + (-1) \qquad \text{By(P9)} \\ &= a\cdot 0 \qquad \text{By(P3)} \\ &= 0 \qquad \text{By(Question 5)} \end{aligned}$$

$$\therefore\ a + (-1)\cdot a = 0$$

For $-a$, we have:

$$-a + a = 0 \qquad \text{By(P3)}$$

$$\therefore\ a + (-1)\cdot a = -a + a$$

Then we add $(-a)$ on both sides, by Question 1, the equation should still be valid

$$a + (-a) + (-1)\cdot a = -a + a + (-a)$$

$$(-1)\cdot a = -a \qquad \text{By(P3)}$$

Question 10 $\forall a, b$: $(-a)\cdot(-b) = a\cdot b$
Proof:

$$\begin{aligned} (-a)\cdot(-b) &= ((-1)\cdot a)\cdot((-1)\cdot b) \qquad \text{By(Question 9)} \\ &= (-1)\cdot a\cdot(-1)\cdot b \qquad \text{By(P5)} \\ &= (-1)\cdot(-1)\cdot a\cdot b \qquad \text{By(P8)} \\ &= a\cdot b \end{aligned}$$

$$\therefore (-a)\cdot(-b) = a\cdot b = a\cdot b \qquad \text{By(E1)}$$

Question 13 $\forall a, b, c$: if $a + c < b + c$, then $a < b$
Proof:

It's because $a+c<b+c$, therefore, $b+c-(a+c)\in P$

$$\begin{aligned}
b+c-(a+c) &= b+c+(-1)\cdot(a+c) \qquad \text{By(Question 9)}\\
&= b+c+(-1)\cdot a+(-1)\cdot c \qquad \text{By(P9)}\\
&= b+c+(-a)+(-c) \qquad \text{By(Question 9)}\\
&= b+(-a)+c+(-c) \qquad \text{By(P1)}\\
&= b+(-a)+(c+(-c)) \qquad \text{By(P1)}\\
&= b+(-a) \qquad \text{By(P3)}\\
&= b-a
\end{aligned}$$

Therefore, $b+c-(a+c)=b-a\in P$. Thus, $a<b$.
Question 14 $\forall a,b$: if $a<0$, $b<0$,then $a\cdot b>0$
Proof:
Suppose $a<0$,$b<0$, then $0-a, 0-b\in P$ From (P2) we have

$$\begin{aligned}
0-a &= 0+(-a)\\
&= -a
\end{aligned}$$

And similar for b, $0-b=-b$

$$\therefore -a,-b\in P$$

From (P12), we can have because $-a,-b\in P$, then $-a\cdot(-b)\in P$ From Question 10 we know, $\forall a,b$: $(-a)\cdot(-b)=a\cdot b$

$$\therefore a\cdot b\in P$$

From (P2), we have $a\cdot b+(-0)=a\cdot b-0\in P$

$$\therefore a\cdot b>0$$

Question 16 $\forall a,b$: $a\cdot b>0$, then either $a>0$ and $b>0$ or $a<0$ and $b<0$
Proof:
Suppose $a=0$, from Question 5, we have $\forall a$,$a\cdot 0=0$. And 0 cannot be greater than 0. Therefore, $a\neq 0$.
Assume $a<0$.
Suppose $b>0$. From Question 15, we have $\forall a,b$ if $a<0,b>0$, then $a\cdot b<0$. Thus, $a<0$ and $b<0$.
Assume $a>0$.
Suppose$b<0$. Because we can use our symbols interchangeably, from Question 15, we can

have $\forall a, b$ if $b < 0, a > 0$, then $a \cdot b < 0$. Thus, $a > 0$ and $b > 0$.

Question 22 $\forall a, b, c$: if $a < b$ and $c > 0$, then $a \cdot c < b \cdot c$

Proof:

It's because $a < b$, therefore, $b - a \in P$. Also, because $c > 0$, meaning $c - 0 = c \in P$. Therefore, $c, (b-a) \in P$. From (P12), if both c and $(b-a)$ belong to P, then $c \cdot (b-a) \in P$. Then, we have

$$\begin{aligned} c \cdot (b-a) &= c \cdot (b + (-a)) \\ &= c \cdot b + c \cdot (-a)) \qquad \text{By(P9)} \\ &= c \cdot b - c \cdot a \end{aligned}$$

Therefore, $c \cdot b - c \cdot a \in P$. So, $c \cdot a < c \cdot b$

Question 24 $\forall a, b, c$: if $a < b$ and $c < 0$, then $b \cdot c < b \cdot a$.

Proof:

It's because $a < b$, therefore, $b - a \in P$. Also, because $c < 0$, meaning $0 - c = -c \in P$. Therefore, $-c, (b-a) \in P$. From (P12), if both $-c$ and $(b-a)$ belong to P, then $-c \cdot (b-a) \in P$.

Then, we have

$$\begin{aligned} -c \cdot (b-a) &= -c \cdot (b + (-a)) \\ &= -c \cdot b + (-c) \cdot (-a)) \qquad \text{By(P9)} \\ &= -c \cdot b + c \cdot a \end{aligned}$$

Therefore, $c \cdot a - c \cdot b \in P$. So, $c \cdot b < c \cdot a$

Question 26 $\forall a, b$: $|a+b| \leq |a| + |b|$

Proof:

First, assume $a, b \geq 0$

$$\therefore |a+b| = a+b, |a| + |b| = a+b$$

$a + b = a + b$, meaning $|a+b| = |a| + |b|$ The assumption holds for $a, b \geq 0$

Then, assume $a, b < 0$

$|a+b| = -(a+b) = -a - b$ (By(P9)), $|a| + |b| = -a - b$ Therefore, we have $|a+b| = -a - b = |a| + |b|$. And this assumption holds when $a, b < 0$.

Assume, $a \geq 0, b \leq 0$, and $|a| \geq |b|$, it would be the same situation, when $b \geq 0, a \leq 0$, and $|b| \geq |a|$.

It is because $|a| \geq |b|$, then $a + b \geq 0$. It means $|a+b| = a+b$, and $|a| + |b| = a - b$. Both calculations are absolute values, meaning both of them are greater than 0.

Then, we have

$$\begin{aligned}
a-b-(a+b) &= a-b-a-b \qquad \text{By(P9)} \\
&= a-a-b-b \qquad \text{By(P4)} \\
&= -b-b \qquad \text{By(P3)} \\
&= -b+(-b) \\
&= -1\cdot b+(-1)\cdot b \\
&= (-1+(-1))\cdot b \qquad \text{By(P9)} \\
&= -2\cdot b \qquad \text{By(P9)}
\end{aligned}$$

$$\therefore a-b-(a+b) = -2b \geq 0$$

$$\therefore |a|+|b| = a-b \geq |a+b| = a+b$$

Assume, $a \geq 0, b \leq 0$, and $|a| \leq |b|$, it would be the same situation, when $b \geq 0, a \leq 0$, and $|b| \leq |a|$.
It is because $|a| \leq |b|$, then $a+b \leq 0$. It means $|a+b| = -(a+b)$, and $|a|+|b| = a-b$. Both calculations are absolute values, meaning both of them are greater than 0.

Then, we have

$$\begin{aligned}
a-b-(-(a+b)) &= a-b+(a+b) \\
&= a+a+b-b \qquad \text{By(P4)} \\
&= a+a \qquad \text{By(P3)} \\
&= 1\cdot a+1\cdot a \\
&= (1+1)\cdot a \qquad \text{By(P9)} \\
&= 2\cdot a
\end{aligned}$$

$$\therefore a-b-(-(a+b)) = 2a \geq 0$$

$$\therefore |a|+|b| = a-b \geq |a+b| = a+b$$