Yufei Lin
Problem Set 7
Nov 14^{th} 2019

Problem Set 7

Question 1
Suppose that $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, and $F : \mathbb{R} \to \mathbb{R}$, suppose that f and g are differentiable, and suppose that

$$f' = F$$

and

$$g' = F.$$

Then there exists a $c \in \mathbb{R}$ such that

$$\forall x \in \mathbb{R} : \quad g(x) = f(x) + c.$$

Proof:
Suppose $f' = F$ and $g' = F$. Thus, we have $f' - g' = (f-g)' = F - F = 0$. Therefore, $f - g = c$. Then, we know $f(x) = g(x) + c$.

Question 2
Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable, and suppose that

$$\forall x \in \mathbb{R} : f'(x) = 0.$$

Then prove that there exists a $c \in \mathbb{R}$ such that

$$\forall x \in \mathbb{R} : f(x) = c.$$

Proof:
Suppose $f'(x) = 0$. Then by mean value theorem, given $a, b \in \mathbb{R}, a < b$ such that there exists $c, f'(c) = \frac{f(b)-f(a)}{b-a}$. Therefore, if $f'(x) = 0$, then $f(b) - f(a) = 0$. Thus, $f(b) = f(a)$. This means $f(x)$ is constant. $f(x) = c$.

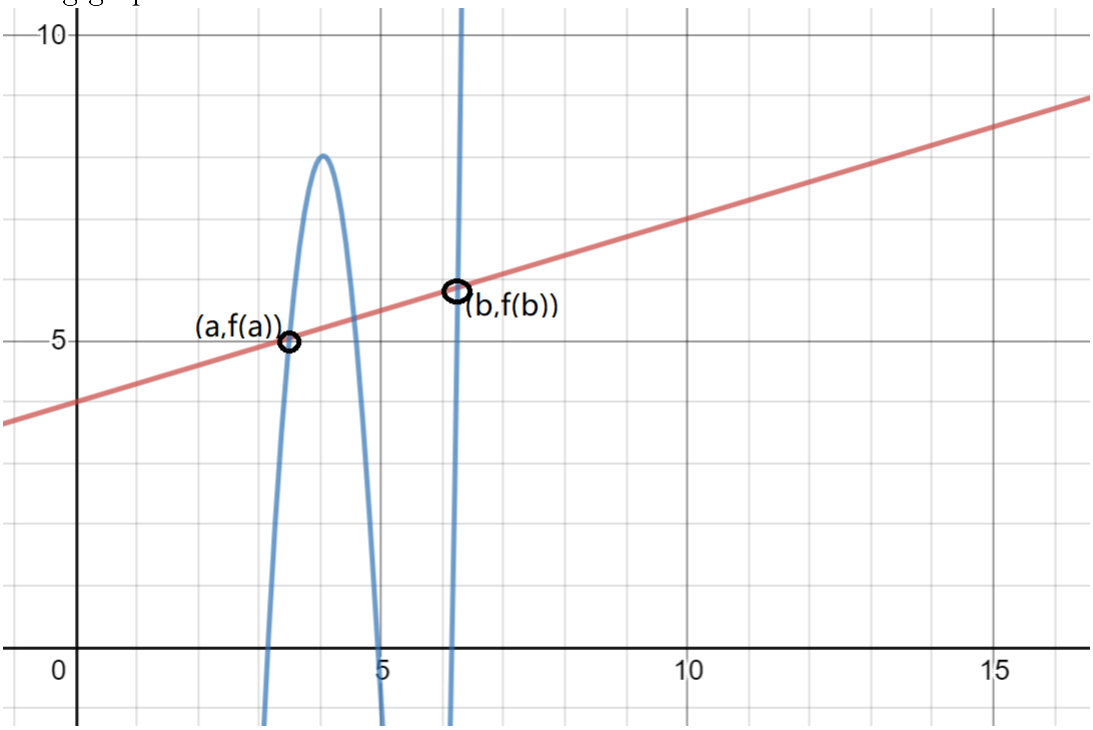
Question 3

Suppose that $f:\mathbb{R}\to\mathbb{R}$ is differentiable, and suppose that $a,b\in\mathbb{R}$, with $a<b$. Then there exists a $c\in\mathbb{R}$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

Proof:

Suppose $a,b\in\mathbb{R}$, and $f(x)$ is differentiable $\forall x, x\in\mathbb{R}$. Therefore, we could have the following graph:



Thus, we can show that if we can find a straight line function such that $(a,f(a))$ and $(b,f(b))$ is on the graph we can thus proof if the slope exist then, there must be a $f'(c)$ exist such that $f'(c)=\frac{f(b)-f(a)}{b-a}$.

Let the straight line's function be $h(x)$, then $h(x)=\frac{f(b)-f(a)}{b-a}\cdot x+t, t\in\mathbb{R}$. Since $h(a)=f(a), h(b)=f(b)$ then, $h(x)=\frac{f(b)-f(a)}{b-a}\cdot x+\frac{b(f(a))-a(f(b))}{b-a}$. Then we have $g(x)=f(x)-h(x)$. $g(a)=g(b)=0$. Thus, by Rolle's theorem, if $g(a)=g(b)=0$ then, $\exists c$, such that $g'(c)=0$. Then, we have $g'(c)=f'(c)-h'(c)=f'(c)-\frac{f(b)-f(a)}{b-a}=0$. Therefore, $f'(c)=\frac{f(b)-f(a)}{b-a}$.

Question 4

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable, and suppose that $a, b \in \mathbb{R}$, with $a < b$. Suppose also that $f(a) = f(b)$. Then there exists a $c \in \mathbb{R}$ such that

$$f'(c) = 0.$$

Proof:

(I) Maximum inside the range

Suppose $a < b$, and $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then there must be a $c \in [a, b]$ such that $f(c) \geq f(x), \forall x, x \in [a, b]$. Thus, we have a local maximum in range $[a, b]$. Based on Question 5, we know that if we have a local maximum at c, then $f'(c) = 0$.

(II) Maximum at the end of the range

Suppose $a < b$, and $f : \mathbb{R} \to \mathbb{R}$ is differentiable, and we have $f(a) = f(b)$ and both of them are the maximum within the range. Since a, b are the maximum point of the function in that range, then there must exist a c such that $f(c)$ is the smallest within the range. Also, we know that f is differentiable everywhere. Then we have $\lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$ exist and $\lim_{h\to 0+} \frac{f(c+h) - f(c)}{h} = \lim_{h\to 0-} \frac{f(c+h) - f(c)}{h}$. On the left hand side of the above equality, because we know $f(c + h) > f(c)$, we have a positive numerator and a positive h. Then, $\lim_{h\to 0+} \frac{f(c+h) - f(c)}{h} \geq 0$. For the right hand side, we know $f(c - h) > f(c)$, then $\lim_{h\to 0-} \frac{f(c+h) - f(c)}{h} \leq 0$. Since left hand side is equal to the right hand side, we have $f'(c) = 0$.

Question 5

Suppose that $f : \mathbb{R} \to \mathbb{R}$ has a local maximum at $c \in \mathbb{R}$, and suppose that f is differentiable at c. Then we have

$$f'(c) = 0.$$

Proof:

Suppose f is differentiable at c and $f(c)$ is a local maximum. Then, we know $\lim_{h\to 0} \frac{f(c+h) - f(c)}{h}$ exist and $\lim_{h\to 0+} \frac{f(c+h) - f(c)}{h} = \lim_{h\to 0-} \frac{f(c+h) - f(c)}{h}$. Therefore, we should be looking at the two sides of this local maximum. Such that we will have $\lim_{h\to 0+} \frac{f(c+h) - f(c)}{h}$. The numerator is negative because $f(c)$ is local maximum, and h is positive. Then, we know $\lim_{h\to 0+} \frac{f(c+h) - f(c)}{h} \leq 0$. On the other hand, $\lim_{h\to 0-} \frac{f(c+h) - f(c)}{h}$. The denominator is negative because $f(c)$ is local maximum, and h is negative. Then, we have

$\lim_{h \to 0-} \frac{f(c+h) - f(c)}{h} \geq 0$. Thus, $\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = 0$.

Question 6

Suppose that $f : \mathbb{R} \to \mathbb{R}$, and $a \in \mathbb{R}$. Then the limit of f at a exists and equals L if, and only if, both the right- and left-handed limits of f at a exist and they both equal L.

Proof:

Suppose $\lim_{x \to a} f(x) = L$. Then, $\forall \epsilon > 0, \exists \delta > 0$, such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

(I) Look at the right-handed limit first:

If $a < x < a + \delta$, then we have $0 < |x - a| < \delta$. Therefore, $|f(x) - L| < \epsilon$.

(II) Look at the left-handed limit:

If $a > x > a - \delta$, then we have $0 < |x - a| < \delta$. Therefore, $|f(x) - L| < \epsilon$.

Therefore, if $\lim_{x \to a} f(x) = L$, then both the right- and left-handed limits of f at a exist and they both equal L.

Conversely, suppose $\lim_{x \to a+} f(x) = \lim_{x \to a-} f(x) = L$. Then we know that $\forall \epsilon > 0, \exists \delta_1 > 0$, such that if $a < x < a + \delta_1$, then $|f(x) - L < \epsilon|$ and $\forall \epsilon > 0, \exists \delta_2 > 0$, such that if $a - \delta_2 < x < a$, then $|f(x) - L < \epsilon|$. Let $\delta = min(\delta_1, \delta_2)$. Then we can have $\forall \epsilon$, if $a < x < a + \delta$, then $|f(x) - L < \epsilon|$ and if $a - \delta < x < a$, then $|f(x) - L < \epsilon|$. Therefore, if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Thus, $\lim_{x \to a} f(x) = L$ if and only if both right-handed and left-handed limit has the same value.