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Problem Set 6
Nov $4^{th}$ 2019

# Problem Set 6

**Question 1**
Prove that if $f(x)$ is differentiable at $a$, then it is continuous at $a$.
**Proof:**
Suppose $f(x)$ is differentiable at $a$. Then, $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = L$ and $f(a)$ exist. Since $h$ is constant $\lim_{h\to 0} h$ exist. Then,

$$\begin{aligned}\lim_{h\to 0} h \cdot \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0} h \cdot L \\ &= 0 \cdot L \\ &= 0 \\ &= \lim_{h\to 0}(f(a+h)-f(a))\end{aligned}$$

Then we know $\lim_{h\to 0} f(a+h) = \lim_{h\to 0} f(a)$. Thus, $\lim_{x\to a} f(x) = f(a))$.

**Question 2**
Prove if $\forall x, f(x) = c, c \in \mathbb{R}$, then $\forall a : f'(a) = 0$.
**Proof:**
Suppose $a$ is any number, then $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$.

$$\begin{aligned}\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0}\frac{c-c}{h} \\ &= \lim_{h\to 0} 0 \\ &= 0\end{aligned}$$

Therefore, $f'(a) = 0$.

**Question 3**
Prove if $f(x) = x$, then $\forall a : f'(a) = 1$.
**Proof:**

Suppose $a$ is any number, then $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

$$\begin{aligned}
\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \to 0} \frac{a+h-a}{h} \\
&= \lim_{h \to 0} \frac{h}{h} \\
&= \lim_{h \to 0} 1 \\
&= 1
\end{aligned}$$

Therefore, $f'(a) = 0$.

**Question 4**

Prove that if $f(x)$ and $g(x)$ are differentiable at $a$, then $f(x) + g(x)$ is differentiable at $a$ and $(f+g)'(a) = f'(a) + g'(a)$.

**Proof:**

Suppose $f(x)$ and $g(x)$ are differentiable at $a$, then $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ and $g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$. Thus,

$$\begin{aligned}
f'(a) + g'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} + \lim_{h \to 0} \frac{g(a+h) - g(a)}{h} \\
&= \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} + \frac{g(a+h) - g(a)}{h}\right) \\
&= \lim_{h \to 0} \frac{f(a+h) - f(a) + g(a+h) - g(a)}{h} \\
&= \lim_{h \to 0} \frac{(f(a+h) + g(a+h)) - (f(a) + g(a))}{h} \\
&= \lim_{h \to 0} \frac{(f+g)(a+h) - (f+g)(a)}{h} \\
&= (f+g)'(a)
\end{aligned}$$

Thus, $(f+g)'(a) = f'(a) + g'(a)$.

**Question 5**

Prove that if $f(x)$ and $g(x)$ are differentiable at $a$, then $f(x) \cdot g(x)$ is differentiable at $a$ and $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$.

**Proof:**

Suppose $f(x)$ and $g(x)$ are differentiable at $a$, then $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ and $g'(a) =$

$\lim_{h\to 0}\frac{g(a+h)-g(a)}{h}$. Also, we know both $f(a)$ and $g(a)$ exist. Thus,

$$\begin{aligned}
(f\cdot g)'(a) &= \lim_{h\to 0}\frac{(f\cdot g)(a+h)-(f\cdot g)(a)}{h}\\
&= \lim_{h\to 0}\frac{(f(a+h)g(a+h)-f(a)g(a)}{h}\\
&= \lim_{h\to 0}\frac{(f(a+h)g(a+h)-f(a)g(a+h)+f(a)g(a+h)-f(a)g(a)}{h}\\
&= \lim_{h\to 0}\frac{((f(a+h)-f(a))g(a+h)+f(a)(g(a+h)-g(a))}{h}\\
&= \lim_{h\to 0}\frac{((f(a+h)-f(a))g(a+h)}{h}+\lim_{h\to 0}\frac{f(a)(g(a+h)-g(a))}{h}\\
&= \lim_{h\to 0}\frac{((f(a+h)-f(a))}{h}\cdot\lim_{h\to 0}g(a+h)+\lim_{h\to 0}\frac{(g(a+h)-g(a))}{h}\cdot\lim_{h\to 0}f(a)\\
&= f'(a)\cdot\lim_{h\to 0}g(a+h)+g'(a)\cdot\lim_{h\to 0}f(a)\\
&= f'(a)\cdot g(a)+g'(a)\cdot f(a)
\end{aligned}$$

Thus, $(f\cdot g)'(a)=f'(a)\cdot g(a)+g'(a)\cdot f(a)$.