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Problem Set 2
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Problem Set 2

Chapter 2
1.(i) $1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.
Proof:
Let $n = 1$, then we have on the left hand side:

$$1^2 = 1$$

Then, on the right hand side:

$$\begin{aligned}\frac{1 \times (1+1) \times (2 \cdot 1 + 1)}{6} &= \frac{1 \times 2 \times 3}{6} \\ &= \frac{6}{6} \\ &= 1\end{aligned}$$

Therefore, left hand side equals to right hand side. This claim holds for 1.
Then, assume if $n = k$, and $1^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$.
Let $n = k + 1$, on the left hand side, we would have:

$$\begin{aligned}1^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\ &= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1) \cdot (k+1)}{6} \\ &= \frac{((k+1)(2k^2+k) + (6k+6)(k+1))}{6} \\ &= \frac{(k+1)(2k^2+k+6k+6)}{6} \\ &= \frac{(k+1)(2k^2+7k+6)}{6} \\ &= \frac{(k+1)(2k+3)(k+2)}{6}\end{aligned}$$

$$
\begin{aligned}
&= \frac{(k+1)(2(k+1)+1)((k+1)+1)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}
\end{aligned}
$$

And on the right hand side, we would have:

$$\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$\therefore \text{Left hand side equals to right hand side}$$

The claim holds.

1.(ii) $1^3 + \cdots + n^3 = (1 + \cdots + n)^2$.

Proof:

Let $n = 1$, then we would have on the left hand side:

$$1^3 = 1$$

On the right hand side:

$$1^2 = 1$$

Therefore, left hand side equals to right hand side this claim holds for 1.

Then, assume if $n = k$, and $1^3 + \cdots + k^3 = (1 + \cdots + k)^2$.

Let $n = k + 1$, on the left hand side, we would have:

$$
\begin{aligned}
1^3 + \cdots + k^3 + (k+1)^3 &= (1 + \cdots + k)^2 + (k+1)^3 \\
&= (\frac{k(k+1)}{2})^2 + (k+1)^3 \\
&= (\frac{k^2(k+1)^2}{4}) + (k+1) \cdot (k+1)^2 \\
&= \frac{k^2}{4} \cdot (k+1)^2 + (k+1) \cdot (k+1)^2 \\
&= (k+1)^2 \cdot (\frac{k^2}{4} + (k+1)) \\
&= (k+1)^2 \cdot (\frac{k^2}{4} + \frac{4(k+1)}{4}) \\
&= (k+1)^2 \cdot (\frac{k^2}{4} + \frac{4k+4}{4})
\end{aligned}
$$

$$
\begin{aligned}
&= (k+1)^2 \cdot (\frac{k^2+4k+4}{4} \\
&= (k+1)^2 \cdot (\frac{(k+2)^2}{4} \\
&= (k+1)^2 \cdot (\frac{(k+2)}{2})^2 \\
&= (k+1)^2 \cdot (\frac{((k+1)+1)}{2})^2 \\
&= (\frac{(k+1)((k+1)+1)}{2})^2 \\
&= (1+\cdots+(k+1))^2
\end{aligned}
$$

And on the right hand side, we would have:

$$(1+\cdots+(k+1))^2$$

$$\therefore \text{Left hand side equals to right hand side}$$

The claim holds.

2.(i)

$$\sum_{i=1}^{n}(2i-1) = n^2$$

2.(ii)

$$
\begin{aligned}
\sum_{i=1}^{n}(2i-1)^2 &= \sum_{i=1}^{2n}(i)^2 - \sum_{i=1}^{n}(2i)^2 \\
&= \sum_{i=1}^{2n}(i)^2 - 4\sum_{i=1}^{n}(i)^2 \\
&= \frac{(2n)((2n)+1)(2(2n)+1)}{6} \\
&= \frac{(2n)(2n+1)(4n+1) - 4n(n+1)(2n+1)}{6} \\
&= \frac{(2n)(2n+1)(4n+1-2n-2)}{6} \\
&= \frac{(2n)(2n+1)(2n-1)}{6}
\end{aligned}
$$

3.(a)

On the right hand side we have:

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-(k-1))!} + \frac{n!}{k!(n-k)!} \\
&= \frac{k \cdot n! + (n-k+1) \cdot n!}{k!(n-k+1))!} \\
&= \frac{(k+n-k+1)n!}{k!((n+1)-k))!} \\
&= \frac{(n+1)!}{k!((n+1)-k))!} \\
&= \binom{n+1}{k}
\end{aligned}$$

$\therefore$ Left hand side is the same as the right hand side.

3.(b)

$\forall n, n \in \mathbb{Z}_{\geq 0}$,we would have by definition $\binom{n}{n} = \binom{n}{0} = 1$, which is a natural number.
Let $n = 0$, we would have $\binom{0}{0} = 1$, and it is a natural number. The claim holds for 0.
Then, assume $n = m$ and $\forall k, k \leq m, \binom{m}{k}$ is natural number.
Let $n = m + 1$, we would have $\forall k, k < m + 1, \binom{m+1}{k} = \binom{m}{k-1} + \binom{m}{k}$ from the theorem we prove in (a). It is because from the assumption above, we have both $\binom{m}{k-1}, \binom{m}{k}$ are natural numbers. Therefore, $\binom{m+1}{k}$ is a natural number. And $\binom{m+1}{m+1} = m + 10 = 1$ by definition of a binomial coefficient. Therefore, the claim holds for all $n \in \mathbb{Z}_{\geq 0}$.

3.(c)

When choosing k things out of n things, we are looking at the arrangement of first k things in an ordered list of n things without considering their order. THerefore, first we are arranging all n things, then we have $n!$ ways to do the arrangment. Then, we choose the first k things and find their arrangements. Therefore, we are overcounting all such arrangements by $(n-k)!$times, since we do not care about the rest $n-k$ things' arrangements. Then we would have $\frac{n!}{(n-k)!}$ ways of arrangement left. However, we are still overcounting the arrangements $k!$ times because we do not need to know the order fo the arrangements of these k things. Therefore, we have $\frac{n!}{k! \cdot (n-k)!}$ in total.

3.(d)

Let $n = 1$, then we have on the left hand side: $(a + b)^1 = a + b$ and on the right hand side: $\binom{1}{0}a^1 + \binom{1}{1}b^1 = a + b$. We have left hand side equals to the right hand side. The claim holds for $n = 1$.
Let $n = m$, such that $(a + b)^m = a^m + \binom{m}{1}a^{m-1}b + \binom{m}{2}a^{m-2}b^2 + \cdots + \binom{m}{m-1}ab^{m-1} + b^m$

Let $n = m + 1$, on the left hand side we would have:

$$
\begin{aligned}
(a+b)^{m+1} &= (a+b)^m \cdot (a+b) \\
&= (a^m + \binom{m}{1}a^{m-1}b + \binom{m}{2}a^{m-2}b^2 + \cdots + \binom{m}{m-1}ab^{m-1} + b^m) \cdot (a+b) \\
&= a \cdot \left(\binom{m}{0}a^m + \binom{m}{1}a^{m-1}b + \binom{m}{2}a^{m-2}b^2 + \cdots + \binom{m}{m-1}ab^{m-1} + \binom{m}{m}b^m\right) \\
&\quad + b \cdot \left(\binom{m}{0}a^m + \binom{m}{1}a^{m-1}b + \binom{m}{2}a^{m-2}b^2 + \cdots + \binom{m}{m-1}ab^{m-1} + \binom{m}{m}b^m\right) \\
&= \binom{m}{0}a^{m+1} + \binom{m}{1}a^m b + \cdots + \binom{m}{m-1}a^2b^{m-1} + \binom{m}{m}ab^m \\
&\quad + \binom{m}{0}a^m b + \binom{m}{1}a^m b^2 + \cdots + \binom{m}{m-1}ab^m + \binom{m}{m}b^{m+1} \\
&= \binom{m}{0}a^{m+1} + \left(\binom{m}{0} + \binom{m}{1}\right)a^m b + \cdots + \left(\binom{m}{m-1} + \binom{m}{m}\right)ab^m + \binom{m}{m}b^{m+1}
\end{aligned}
$$

It's because from (a) we have proved that $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$.Also, by definition of a binomial coefficient we know that $\forall m, m \in \mathbb{Z}_{\geq 0}\binom{m}{0} = \binom{m}{m} = 1$ Therefore, on the left hand side, we should have:

$$
\begin{aligned}
&\binom{m}{0}a^{m+1} + \left(\binom{m}{0} + \binom{m}{1}\right)a^m b + \cdots + \left(\binom{m}{m-1} + \binom{m}{m}\right)ab^m + \binom{m}{m}b^{m+1} \\
&= \binom{m+1}{0}a^{m+1} + \binom{m+1}{a}^m b + \cdots + \binom{m+1}{m}ab^m + b^{m+1} \\
&= \sum_{j=0}^{m+1}\binom{m+1}{j}a^{m+1-j}b^j
\end{aligned}
$$

On the right hand side, we would have when $n = m + 1$:

$$
\sum_{j=0}^{m+1}\binom{m+1}{j}a^{m+1-j}b^j
$$

$\therefore$ Left hand side equals to the right hand side. The claim holds.

3.(e)

(i)

Proof:

Let $n = 0$, then we would have on the left hand side $\binom{0}{0} = 1$, and on the right hand side, we would have $2^0 = 1$. Therefore, the claim holds for $n = 0$.

Let $n = m$ such that

$$\sum_{j=0}^{m} \binom{m}{j} = 2^m$$

Let $n = m + 1$, we would have on the left hand side:

$$\begin{aligned}
\sum_{j=0}^{m+1} \binom{m+1}{j} &= \binom{m+1}{0} + \binom{m+1}{1} + \cdots + \binom{m+1}{m+1} \\
&= \binom{m}{0} + \left(\binom{m}{0} + \binom{m}{1}\right) + \cdots + \left(\binom{m}{m-1} + \binom{m}{m}\right) + \binom{m}{m} \\
&= \binom{m}{0} + \binom{m}{0} + \binom{m}{1} + \binom{m}{1} + \cdots + \binom{m}{m} + \binom{m}{m} \\
&= 2 \cdot \left(\binom{m}{0} + \binom{m}{1} + \cdots + \binom{m}{m}\right) \\
&= 2 \cdot 2^n \\
&= 2^{n+1}
\end{aligned}$$

On the right hand side, we would have 2^{m+1}. Thus, the left hand side is equal to the right hand side. The claim holds.

(ii)

Proof:

This only holds for any number $n, n > 0$. It is because if $n = 0$, this equation would be 1 instead of 0.

Let $n = 1$, on the left hand side we would have $\binom{1}{0} - \binom{1}{1} = 0$. On the right hand side we have 0. Therefore, the claim holds for $n = 1$.

Let $n = m$, assume

$$\sum_{j=0}^{m} (-1)^j \cdot \binom{m}{j} = 0$$

Let $n = m + 1$, we would have on the left hand side:

$$\begin{aligned}
\sum_{j=0}^{m+1} (-1)^j \cdot \binom{m+1}{j} &= \binom{m+1}{0} - \binom{m+1}{1} + \cdots + (-1)^{m+1} \cdot \binom{m+1}{m+1} \\
&= \binom{m}{0} - \left(\binom{m}{0} + \binom{m}{1}\right) + \cdots + (-1)^m \cdot \left(\binom{m}{m-1} + \binom{m}{m}\right) + (-1)^m \cdot \binom{m}{m} \\
&= \binom{m}{0} - \binom{m}{0} - \binom{m}{1} + \cdots + (-1)^m \cdot \binom{m}{m-1} + (-1)^m \cdot \binom{m}{m} + (-1)^m \cdot \binom{m}{m} \\
&= 0
\end{aligned}$$

On the left hand side, we have 0.
Therefore, this claim holds for all number $n > 0$.

(iii) and (iv)

Proof:

From the question above we have proved that $\sum_{j=0}^{m}(-1)^j \cdot \binom{m}{j} = 0$. This means that for even binomial coefficients, it is always posivtive in this equation, and for odd coefficients, it is always negative. Therefore, we can have $\sum_{j \text{ even}} \binom{n}{j} = \sum_{j \text{ odd}} \binom{n}{j}$. It is because from (i) we know $\sum_{j=0}^{m} \binom{m}{j} = 2^n$. Thus, $\sum_{j \text{ even}} \binom{n}{j} = \sum_{j \text{ odd}} \binom{n}{j} = 2^{n-1}$

5.(a)

Let $n = 0$, we would have on the left hand side: $r^0 = 1$.
On the right hand side, we would have:

$$\begin{aligned}\frac{1-r^{0+1}}{1-r} &= \frac{1-r}{1-r} \\ &= 1\end{aligned}$$

$\therefore$ Left hand side is the same as the right hand side.

The claim holds for $n = 0$.
Let $n = k$, assume $1 + \cdots + r^k = \frac{1-r^{k+1}}{1-r}$
If $n = k + 1$, then on the left hand side we would have:

$$\begin{aligned}1 + \cdots + r^k + r^{k+1} &= \frac{1-r^{k+1}}{1-r} + r^{k+1} \\ &= \frac{1-r^{k+1}}{1-r} + \frac{(1-r)r^{k+1}}{1-r} \\ &= \frac{1-r^{k+1}}{1-r} + \frac{r^{k+1}-r^{k+2}}{1-r} \\ &= \frac{1-r^{k+1}+r^{k+1}-r^{k+2}}{1-r} \\ &= \frac{1-r^{k+2}}{1-r} \\ &= \frac{1-r^{(k+1)+1}}{1-r}\end{aligned}$$

On the right hand side, we have: $\frac{1-r^{(k+1)+1}}{1-r}$.

$\therefore$ Left hand side is the same as the right hand side.

The claim holds.

5.(b)

Let $S = 1 + \cdots + r^n$, by multiplying both sides with r, then we would have:

$$\begin{aligned} r \cdot S &= r \cdot 1 + \cdots + r^n \\ &= r + \cdots + r^{n+1} \end{aligned}$$

It is because we would like to know about S, then we could have:

$$\begin{aligned} r \cdot S - S &= (r-1) \cdot S \\ &= r + \cdots + r^{n+1} - (1 + \cdots + r^n) \\ &= r^{n+1} - 1 \end{aligned}$$

$$\begin{aligned} \therefore (r-1) \cdot S &= r^{n+1} - 1 \\ S &= \frac{r^{n+1} - 1}{r - 1} = \frac{1 - r^{n+1}}{1 - r} \end{aligned}$$

Chapter 3

1.(i)

$$\begin{aligned} f(f(x)) &= f(\frac{1}{1+x}) \\ &= \frac{1}{1+\frac{1}{1+x}} \\ &= \frac{1}{\frac{2+x}{1+x}} \\ &= \frac{1+x}{2+x} \end{aligned}$$

$$\therefore x \neq -1, -2$$

1.(ii)

$$\begin{aligned} f(\frac{1}{x}) &= \frac{1}{1+\frac{1}{x}} \\ &= \frac{1}{\frac{1+x}{x}} \\ &= \frac{x}{1+x} \end{aligned}$$

$$\therefore x \neq -1$$

1.(iii)

$$f(cx) = \frac{1}{1+cx}$$

$$\therefore x \neq -\frac{1}{c}, if c \neq 0$$

1.(iv)

$$f(x+y) = \frac{1}{1+x+y}$$

$$\therefore x+y \neq -1$$

1.(v)

$$f(x) + f(y) = \frac{1}{1+x} + \frac{1}{1+y}$$
$$= \frac{2+x+y}{(1+x)(1+y)}$$

$$\therefore x, y \neq -1$$

1.(vi)
$f(cx) = f(x), x \neq -1, or - c^{-1}$, then we would have $\frac{1}{1+x} = \frac{1}{1+cx}$.

$$\therefore 1 + x = 1 + cx$$

$$(c-1)x = 0$$

If we only want one x solution for such an equation, then we could have $(c-1) \cdot 0 = 0$. Therefore, $c \in \mathbb{R}$ the equation would always hold.
1.(vii)
If we would like to have at least two different answers for x in the equation $(c-1)x = 0$, then we need to have $c - 1 = 0$. Thus, $c = 1$.