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Problem Set 2

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## Problem Set 2

### Chapter 2

1.(i)  $1^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$ .

**Proof:**

Let  $n = 1$ , then we have on the left hand side:

$$1^2 = 1$$

Then, on the right hand side:

$$\begin{aligned} \frac{1 \times (1+1) \times (2 \cdot 1 + 1)}{6} &= \frac{1 \times 2 \times 3}{6} \\ &= \frac{6}{6} \\ &= 1 \end{aligned}$$

Therefore, left hand side equals to right hand side. This claim holds for 1.

Then, assume if  $n = k$ , and  $1^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$ .

Let  $n = k + 1$ , on the left hand side, we would have:

$$\begin{aligned} 1^2 + \dots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\ &= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1) \cdot (k+1)}{6} \\ &= \frac{((k+1)(2k^2+k) + (6k+6)(k+1))}{6} \\ &= \frac{(k+1)(2k^2+k+6k+6)}{6} \\ &= \frac{(k+1)(2k^2+7k+6)}{6} \\ &= \frac{(k+1)(2k+3)(k+2)}{6} \end{aligned}$$

$$\begin{aligned}
&= \frac{(k+1)(2(k+1)+1)((k+1)+1)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}
\end{aligned}$$

And on the right hand side, we would have:

$$\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$\therefore$  Left hand side equals to right hand side

The claim holds.

**1.(ii)**  $1^3 + \dots + n^3 = (1 + \dots + n)^2$ .

**Proof:**

Let  $n = 1$ , then we would have on the left hand side:

$$1^3 = 1$$

On the right hand side:

$$1^2 = 1$$

Therefore, left hand side equals to right hand side this claim holds for 1.

Then, assume if  $n = k$ , and  $1^3 + \dots + k^3 = (1 + \dots + k)^2$ .

Let  $n = k + 1$ , on the left hand side, we would have:

$$\begin{aligned}
1^3 + \dots + k^3 + (k+1)^3 &= (1 + \dots + k)^2 + (k+1)^3 \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\
&= \left(\frac{k^2(k+1)^2}{4}\right) + (k+1) \cdot (k+1)^2 \\
&= \frac{k^2}{4} \cdot (k+1)^2 + (k+1) \cdot (k+1)^2 \\
&= (k+1)^2 \cdot \left(\frac{k^2}{4} + (k+1)\right) \\
&= (k+1)^2 \cdot \left(\frac{k^2}{4} + \frac{4(k+1)}{4}\right) \\
&= (k+1)^2 \cdot \left(\frac{k^2}{4} + \frac{4k+4}{4}\right)
\end{aligned}$$

$$\begin{aligned}
&= (k+1)^2 \cdot \left(\frac{k^2 + 4k + 4}{4}\right) \\
&= (k+1)^2 \cdot \left(\frac{(k+2)^2}{4}\right) \\
&= (k+1)^2 \cdot \left(\frac{(k+2)}{2}\right)^2 \\
&= (k+1)^2 \cdot \left(\frac{((k+1)+1)}{2}\right)^2 \\
&= \left(\frac{(k+1)((k+1)+1)}{2}\right)^2 \\
&= (1 + \dots + (k+1))^2
\end{aligned}$$

And on the right hand side, we would have:

$$(1 + \dots + (k+1))^2$$

$\therefore$  Left hand side equals to right hand side

The claim holds.

**2.(i)**

$$\sum_{i=1}^n (2i-1) = n^2$$

**2.(ii)**

$$\begin{aligned}
\sum_{i=1}^n (2i-1)^2 &= \sum_{i=1}^{2n} (i)^2 - \sum_{i=1}^n (2i)^2 \\
&= \sum_{i=1}^{2n} (i)^2 - 4 \sum_{i=1}^n (i)^2 \\
&= \frac{(2n)((2n)+1)(2(2n)+1)}{6} \\
&= \frac{(2n)(2n+1)(4n+1) - 4n(n+1)(2n+1)}{6} \\
&= \frac{(2n)(2n+1)(4n+1-2n-2)}{6} \\
&= \frac{(2n)(2n+1)(2n-1)}{6}
\end{aligned}$$

**3.(a)**

On the right hand side we have:

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-(k-1))!} + \frac{n!}{k!(n-k)!} \\
&= \frac{k \cdot n! + (n-k+1) \cdot n!}{k!(n-k+1)!} \\
&= \frac{(k+n-k+1)n!}{k!((n+1)-k)!} \\
&= \frac{(n+1)!}{k!((n+1)-k)!} \\
&= \binom{n+1}{k}
\end{aligned}$$

$\therefore$  Left hand side is the same as the right hand side.

### 3.(b)

$\forall n, n \in \mathbb{Z}_{\geq 0}$ , we would have by definition  $\binom{n}{n} = \binom{n}{0} = 1$ , which is a natural number.

Let  $n = 0$ , we would have  $\binom{0}{0} = 1$ , and it is a natural number. The claim holds for 0.

Then, assume  $n = m$  and  $\forall k, k \leq m$ ,  $\binom{m}{k}$  is natural number.

Let  $n = m + 1$ , we would have  $\forall k, k < m + 1$ ,  $\binom{m+1}{k} = \binom{m}{k-1} + \binom{m}{k}$  from the theorem we prove in (a). It is because from the assumption above, we have both  $\binom{m}{k-1}, \binom{m}{k}$  are natural numbers. Therefore,  $\binom{m+1}{k}$  is a natural number. And  $\binom{m+1}{m+1} = m + 1 = 1$  by definition of a binomial coefficient. Therefore, the claim holds for all  $n \in \mathbb{Z}_{\geq 0}$ .

### 3.(c)

When choosing  $k$  things out of  $n$  things, we are looking at the arrangement of first  $k$  things in an ordered list of  $n$  things without considering their order. Therefore, first we are arranging all  $n$  things, then we have  $n!$  ways to do the arrangement. Then, we choose the first  $k$  things and find their arrangements. Therefore, we are overcounting all such arrangements by  $(n-k)!$  times, since we do not care about the rest  $n-k$  things' arrangements. Then we would have  $\frac{n!}{(n-k)!}$  ways of arrangement left. However, we are still overcounting the arrangements  $k!$  times because we do not need to know the order for the arrangements of these  $k$  things. Therefore, we have  $\frac{n!}{k!(n-k)!}$  in total.

### 3.(d)

Let  $n = 1$ , then we have on the left hand side:  $(a+b)^1 = a+b$  and on the right hand side:  $\binom{1}{0}a^1 + \binom{1}{1}b^1 = a+b$ . We have left hand side equals to the right hand side. The claim holds for  $n = 1$ .

Let  $n = m$ , such that  $(a+b)^m = a^m + \binom{m}{1}a^{m-1}b + \binom{m}{2}a^{m-2}b^2 + \cdots + \binom{m}{m-1}ab^{m-1} + b^m$

Let  $n = m + 1$ , on the left hand side we would have:

$$\begin{aligned}
(a+b)^{m+1} &= (a+b)^m \cdot (a+b) \\
&= (a^m + \binom{m}{1}a^{m-1}b + \binom{m}{2}a^{m-2}b^2 + \dots + \binom{m}{m-1}ab^{m-1} + b^m) \cdot (a+b) \\
&= a \cdot \left( \binom{m}{0}a^m + \binom{m}{1}a^{m-1}b + \binom{m}{2}a^{m-2}b^2 + \dots + \binom{m}{m-1}ab^{m-1} + \binom{m}{m}b^m \right) \\
&\quad + b \cdot \left( \binom{m}{0}a^m + \binom{m}{1}a^{m-1}b + \binom{m}{2}a^{m-2}b^2 + \dots + \binom{m}{m-1}ab^{m-1} + \binom{m}{m}b^m \right) \\
&= \binom{m}{0}a^{m+1} + \binom{m}{1}a^m b + \dots + \binom{m}{m-1}a^2 b^{m-1} + \binom{m}{m}ab^m \\
&\quad + \binom{m}{0}a^m b + \binom{m}{1}a^m b^2 + \dots + \binom{m}{m-1}ab^m + \binom{m}{m}b^{m+1} \\
&= \binom{m}{0}a^{m+1} + \left( \binom{m}{0} + \binom{m}{1} \right) a^m b + \dots + \left( \binom{m}{m-1} + \binom{m}{m} \right) ab^m + \binom{m}{m}b^{m+1}
\end{aligned}$$

It's because from (a) we have proved that  $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ . Also, by definition of a binomial coefficient we know that  $\forall m, m \in \mathbb{Z}_{\geq 0} \binom{m}{0} = \binom{m}{m} = 1$ . Therefore, on the left hand side, we should have:

$$\begin{aligned}
&\binom{m}{0}a^{m+1} + \left( \binom{m}{0} + \binom{m}{1} \right) a^m b + \dots + \left( \binom{m}{m-1} + \binom{m}{m} \right) ab^m + \binom{m}{m}b^{m+1} \\
&= \binom{m+1}{0}a^{m+1} + \binom{m+1}{1}a^m b + \dots + \binom{m+1}{m}ab^m + b^{m+1} \\
&= \sum_{j=0}^{m+1} \binom{m+1}{j} a^{m+1-j} b^j
\end{aligned}$$

On the right hand side, we would have when  $n = m + 1$ :

$$\sum_{j=0}^{m+1} \binom{m+1}{j} a^{m+1-j} b^j$$

$\therefore$  Left hand side equals to the right hand side. The claim holds.

**3.(e)**

(i)

**Proof:**

Let  $n = 0$ , then we would have on the left hand side  $\binom{0}{0} = 1$ , and on the right hand side, we would have  $2^0 = 1$ . Therefore, the claim holds for  $n = 0$ .

Let  $n = m$  such that

$$\sum_{j=0}^m \binom{m}{j} = 2^m$$

Let  $n = m + 1$ , we would have on the left hand side:

$$\begin{aligned} \sum_{j=0}^{m+1} \binom{m+1}{j} &= \binom{m+1}{0} + \binom{m+1}{1} + \cdots + \binom{m+1}{m+1} \\ &= \binom{m}{0} + \left( \binom{m}{0} + \binom{m}{1} \right) + \cdots + \left( \binom{m}{m-1} + \binom{m}{m} \right) + \binom{m}{m} \\ &= \binom{m}{0} + \binom{m}{0} + \binom{m}{1} + \binom{m}{1} + \cdots + \binom{m}{m} + \binom{m}{m} \\ &= 2 \cdot \left( \binom{m}{0} + \binom{m}{1} + \cdots + \binom{m}{m} \right) \\ &= 2 \cdot 2^m \\ &= 2^{m+1} \end{aligned}$$

On the right hand side, we would have  $2^{m+1}$ . Thus, the left hand side is equal to the right hand side. The claim holds.

(ii)

**Proof:**

This only holds for any number  $n, n > 0$ . It is because if  $n = 0$ , this equation would be 1 instead of 0.

Let  $n = 1$ , on the left hand side we would have  $\binom{1}{0} - \binom{1}{1} = 0$ . On the right hand side we have 0. Therefore, the claim holds for  $n = 1$ .

Let  $n = m$ , assume

$$\sum_{j=0}^m (-1)^j \cdot \binom{m}{j} = 0$$

Let  $n = m + 1$ , we would have on the left hand side:

$$\begin{aligned} \sum_{j=0}^{m+1} (-1)^j \cdot \binom{m+1}{j} &= \binom{m+1}{0} - \binom{m+1}{1} + \cdots + (-1)^{m+1} \cdot \binom{m+1}{m+1} \\ &= \binom{m}{0} - \left( \binom{m}{0} + \binom{m}{1} \right) + \cdots + (-1)^m \cdot \left( \binom{m}{m-1} + \binom{m}{m} \right) + (-1)^m \cdot \binom{m}{m} \\ &= \binom{m}{0} - \binom{m}{0} - \binom{m}{1} + \cdots + (-1)^m \cdot \binom{m}{m-1} + (-1)^m \cdot \binom{m}{m} + (-1)^m \cdot \binom{m}{m} \\ &= 0 \end{aligned}$$

On the left hand side, we have 0.

Therefore, this claim holds for all number  $n > 0$ .

(iii) and (iv)

**Proof:**

From the question above we have proved that  $\sum_{j=0}^m (-1)^j \cdot \binom{m}{j} = 0$ . This means that for even binomial coefficients, it is always positive in this equation, and for odd coefficients, it is always negative. Therefore, we can have  $\sum_{j \text{ even}} \binom{n}{j} = \sum_{j \text{ odd}} \binom{n}{j}$ . It is because from (i) we know  $\sum_{j=0}^m \binom{m}{j} = 2^n$ . Thus,  $\sum_{j \text{ even}} \binom{n}{j} = \sum_{j \text{ odd}} \binom{n}{j} = 2^{n-1}$

**5.(a)**

Let  $n = 0$ , we would have on the left hand side:  $r^0 = 1$ .

On the right hand side, we would have:

$$\begin{aligned} \frac{1 - r^{0+1}}{1 - r} &= \frac{1 - r}{1 - r} \\ &= 1 \end{aligned}$$

$\therefore$  Left hand side is the same as the right hand side.

The claim holds for  $n = 0$ .

Let  $n = k$ , assume  $1 + \dots + r^k = \frac{1 - r^{k+1}}{1 - r}$

If  $n = k + 1$ , then on the left hand side we would have:

$$\begin{aligned} 1 + \dots + r^k + r^{k+1} &= \frac{1 - r^{k+1}}{1 - r} + r^{k+1} \\ &= \frac{1 - r^{k+1}}{1 - r} + \frac{(1 - r)r^{k+1}}{1 - r} \\ &= \frac{1 - r^{k+1}}{1 - r} + \frac{r^{k+1} - r^{k+2}}{1 - r} \\ &= \frac{1 - r^{k+1} + r^{k+1} - r^{k+2}}{1 - r} \\ &= \frac{1 - r^{k+2}}{1 - r} \\ &= \frac{1 - r^{(k+1)+1}}{1 - r} \end{aligned}$$

On the right hand side, we have:  $\frac{1 - r^{(k+1)+1}}{1 - r}$ .

$\therefore$  Left hand side is the same as the right hand side.

The claim holds.

**5.(b)**

Let  $S = 1 + \cdots + r^n$ , by multiplying both sides with  $r$ , then we would have:

$$\begin{aligned} r \cdot S &= r \cdot 1 + \cdots + r^n \\ &= r + \cdots + r^{n+1} \end{aligned}$$

It is because we would like to know about  $S$ , then we could have:

$$\begin{aligned} r \cdot S - S &= (r - 1) \cdot S \\ &= r + \cdots + r^{n+1} - (1 + \cdots + r^n) \\ &= r^{n+1} - 1 \end{aligned}$$

$$\therefore (r - 1) \cdot S = r^{n+1} - 1$$

$$S = \frac{r^{n+1} - 1}{r - 1} = \frac{1 - r^{n+1}}{1 - r}$$



## Chapter 3

1.(i)

$$\begin{aligned}f(f(x)) &= f\left(\frac{1}{1+x}\right) \\&= \frac{1}{1+\frac{1}{1+x}} \\&= \frac{1}{\frac{2+x}{1+x}} \\&= \frac{1+x}{2+x}\end{aligned}$$

$$\therefore x \neq -1, -2$$

1.(ii)

$$\begin{aligned}f\left(\frac{1}{x}\right) &= \frac{1}{1+\frac{1}{x}} \\&= \frac{1}{\frac{1+x}{x}} \\&= \frac{x}{1+x}\end{aligned}$$

$$\therefore x \neq -1$$

1.(iii)

$$f(cx) = \frac{1}{1+cx}$$

$$\therefore x \neq -\frac{1}{c}, \text{ if } c \neq 0$$

1.(iv)

$$f(x+y) = \frac{1}{1+x+y}$$

$$\therefore x+y \neq -1$$

**1.(v)**

$$\begin{aligned} f(x) + f(y) &= \frac{1}{1+x} + \frac{1}{1+y} \\ &= \frac{2+x+y}{(1+x)(1+y)} \end{aligned}$$

$$\therefore x, y \neq -1$$

**1.(vi)**

$f(cx) = f(x)$ ,  $x \neq -1$ , or  $-c^{-1}$ , then we would have  $\frac{1}{1+x} = \frac{1}{1+cx}$ .

$$\therefore 1+x = 1+cx$$

$$(c-1)x = 0$$

If we only want one  $x$  solution for such an equation, then we could have  $(c-1) \cdot 0 = 0$ . Therefore,  $c \in \mathbb{R}$  the equation would always hold.

**1.(vii)**

If we would like to have at least two different answers for  $x$  in the equation  $(c-1)x = 0$ , then we need to have  $c-1 = 0$ . Thus,  $c = 1$ .

**2.(i)**

It is because for any  $y$  is rational,  $h(y) = 0$ , and for any  $y$  that is irrational,  $h(y) = 1$ . If we would like to have  $h(y) \geq y$ , then we need  $y \leq 0, y \in \mathbb{Q}$  or  $y < 1, y \in \mathbb{R} - \mathbb{Q}$ .

**2.(ii)**

It's because we want to have  $h(y) \leq g(y)$ . Thus meaning for  $y \in \mathbb{Q}$ ,  $y^2 \geq 0$ , and for  $y \in \mathbb{R} - \mathbb{Q}$ ,  $y^2 \geq 1$ . Therefore,  $|y| \geq 0, y \in \mathbb{Q}$  or  $|y| > 1, y \in \mathbb{R} - \mathbb{Q}$

**2.(iii)**

Suppose  $z$  is a rational number, then we have  $h(z) = 0$ , and  $h(g(z)) = 0^2 = 0$ . Therefore,  $h(g(z)) - h(z) = 0$ . Suppose  $z$  is a irrational number, we would have  $h(z) = 1$ , and  $h(g(z)) = 1^2 = 1$ . Therefore,  $h(g(z)) - h(z) = 0$ . In conclusion,  $h(g(z)) - h(z) = 0$ .

**2.(iv)**

For  $g(w) \leq w$ , we would have  $w - w^2 = w \cdot (1 - w) \geq 0$ . Thus,  $w$  and  $1 - w$  should be the same sign or one of them is 0.

Both of them are positive:

$$w > 0$$

$$1 - w > 0$$

Therefore,  $1 > w > 0$ .

Both of the are negative:

$$w < 0$$

$$1 - w < 0$$

This is not possible, because we need  $w > 1$  and  $w < 0$ .

If one of them is 0. Say  $w$ , then  $w = 0$ . If  $w - 1 = 0$ , then  $w = 1$ .

$$\therefore 1 \geq w \geq 0$$

**2.(v)**

Suppose  $g(g(e)) = g(e)$ , then we have  $g(e^2) = e^4 = e^2$ . Then we have:

$$e^4 - e^2 = 0$$

$$e^2 \cdot (e^2 - 1) = 0$$

$$e^2 = 0 \text{ or } e^2 - 1 = 0$$

$$e_1 = 0 \text{ and } e_{2,3} = \pm 1$$

**3.(i)**

$1 - x^2 \geq 0$ . Thus,  $1 \geq x^2$ . Therefore,  $-1 \leq x \leq 1$ .

**3.(ii)**

$1 - \sqrt{1 - x^2} \geq 0$ , and we know  $1 + \sqrt{1 - x^2}$ . Thus, we can have:

$$(1 - \sqrt{1 - x^2}) \cdot 1 + \sqrt{1 - x^2} \geq (1 + \sqrt{1 - x^2}) \cdot 0$$

$$1 - 1 + x^2 \geq 0$$

$$x^2 \geq 0$$

$$\therefore x \in \mathbb{R}$$

**3.(iii)**

$$x \neq 1, 2$$

**3.(iv)**

$1 - x^2 \geq 0$  and  $x^2 - 1 \geq 0$ . Thus, we have  $1 \geq x^2$  and  $1 \leq x^2$ . Therefore,  $x^2 = 1$  and  $x_{1,2} = \pm 1$ .

**3.(v)**

$1 - x \geq 0, x - 2 \geq 0$ . Therefore,  $1 \geq x$  and  $x \leq 2$ . It is not possible. Therefore, this function cannot happen in real number domain.

**4.(i)**

We should have the following:

$$\begin{aligned}
 (S \circ P)(y) &= S(P(y)) \\
 &= S(2^y) \\
 &= (2^y)^2 \\
 &= 2^{2y}
 \end{aligned}$$

**4.(ii)**

We should have the following:

$$\begin{aligned}
 (S \circ s)(y) &= S(P(y)) \\
 &= S(\sin(y)) \\
 &= (\sin(y))^2 \\
 &= \sin^2(y)
 \end{aligned}$$

**4.(iii)**

We should have the following:

$$\begin{aligned}
 (S \circ P \circ s)(t) + (s \circ P)(t) &= S(P(s(t))) + s(P(t)) \\
 &= S(P(\sin(t))) + s(2^t) \\
 &= S(2^{\sin(t)}) + \sin(2^t) \\
 &= (2^{\sin(t)})^2 + \sin(2^t)
 \end{aligned}$$

**4.(iv)**

We should have the following:

$$s(t^3) = \sin(t^3)$$

**5.**

$$S(x) = x^2, P(x) = 2^x, s(x) = \sin x$$

**5.(i)**

We should have the following:

$$\begin{aligned}
 f(x) &= 2^{\sin x} &= 2^{(s(x))} \\
 &= P(s(x)) \\
 &= (P \circ s)(x)
 \end{aligned}$$

**5.(ii)**

We should have the following:

$$\begin{aligned}
 f(x) &= \sin(2^x) \\
 &= \sin(P(x)) \\
 &= s(P(x)) \\
 &= (s \circ P)(x)
 \end{aligned}$$

**5.(iii)**

We should have the following:

$$\begin{aligned}
 f(x) &= \sin(x^2) \\
 &= \sin(S(x)) \\
 &= s(S(x)) \\
 &= (s \circ S)(x)
 \end{aligned}$$

**5.(iv)**

We should have the following:

$$\begin{aligned}
 f(x) &= \sin^2(x) \\
 &= (\sin(x))^2 \\
 &= (s(x))^2 \\
 &= S(s(x)) \\
 &= (S \circ s)(x)
 \end{aligned}$$

**5.(v)**

We should have the following:

$$\begin{aligned}
 f(t) &= 2^{2^t} \\
 &= 2^{P(t)} \\
 &= P(P(t)) \\
 &= (P \circ P)(x)
 \end{aligned}$$

**5.(vi)**

We should have the following:

$$\begin{aligned}
f(u) &= \sin(2^u + 2^{u^2}) \\
&= \sin(P(u) + 2^S(u)) \\
&= \sin(P(u) + P(S(u))) \\
&= s(P(u) + (P \circ S)(u)) \\
&= s(P + P \circ S)(u)
\end{aligned}$$

**5.(vii)**

We should have the following:

$$\begin{aligned}
f(y) &= \sin(\sin(\sin(2^{2^{siny}}))) \\
&= \sin(\sin(\sin(2^{2^{s(y)}}))) \\
&= \sin(\sin(\sin(2^{P(s(y))}))) \\
&= \sin(\sin(\sin(P(P(s(y))))) \\
&= s(s(s(P(P(s(y))))) \\
&= (s \circ s \circ s \circ P \circ P \circ s)(y)
\end{aligned}$$

**5.(viii)**

We should have the following:

$$\begin{aligned}
f(a) &= 2^{\sin^2(a)} + \sin(a^2) + 2^{\sin(a^2 + \sin(a))} \\
&= 2^{s(a)^2} + \sin(S(a)) + 2^{\sin(S(a) + s(a))} \\
&= 2^{S(s(a))} + s(S(a)) + 2^{s(S(a) + s(a))} \\
&= P(S(s(a))) + s(S(a)) + P(s(S(a) + s(a))) \\
&= P(S(s(a))) + s(S(a)) + P(s(S(a) + s(a))) \\
&= (P \circ S \circ s)(a) + (s \circ S)(a) + (P \circ s(S + s))(a)
\end{aligned}$$

**21.(a)**

This is not equivalent. Let  $f(x) = x^2$  and  $g(x) = x, h(x) = 1$ . Thus,  $f \circ (g + h)(x) = f(x + 1) = (x + 1)^2 = x^2 + 2x + 1$  and  $f \circ g(x) + f \circ h(x) = f(x) + f(1) = x^2 + 1^2 = x^2 + 1$ . Therefore, they are not the same.

**21.(b)** We know that for any two given functions  $a(x) + b(x) = (a + b)(x)$ . Therefore, in

this equation, we have on the left hand side:

$$\begin{aligned}
 ((g + h) \circ f)(x) &= (g + h)(f(x)) \\
 &= g(f(x)) + h(f(x)) \\
 &= (g \circ f)(x) + (h \circ f)(x) \\
 &= RHS
 \end{aligned}$$

**21.(c)**

Let  $h(x) = 1/x$ , then on the left hand side, we would have  $1/(f \circ g) = (h \circ (f \circ g))(x)$ . On the right hand side, we would have  $(1/f) \circ g = ((h \circ f) \circ g)(x)$ . From the theorem stating that composition of functions are associative, we know that  $(h \circ (f \circ g))(x) = ((h \circ f) \circ g)(x)$ . Thus, LHS=RHS.

**21.(d)**

This is not equivalent. For instance, let  $f(x) = 2, g(x) = x$ . Then, on the left hand side we would have  $\frac{1}{(f \circ g)(x)} = \frac{1}{f(x)} = \frac{1}{2}$ . On the right hand side, we would have  $(f \circ \frac{1}{g})(x) = f(1/x) = 2$ . Therefore, left hand side does not equal to right hand side.

**22.(a)**

It is because  $g(x) = (h \circ f)(x) = h(f(x))$ , we would have  $g(x) = h(f(x))$  and  $g(y) = h(f(y))$ . If  $f(x) = f(y)$ , then  $h(f(x)) = h(f(y)) \rightarrow g(x) = g(y)$ . Therefore, if  $f(x) = f(y)$ , then  $g(x) = g(y)$ .

**22.(b)**

Suppose whenever  $f(x) = f(y)$ ,  $g(x) = g(y)$ . Let  $h(z)$  be a function that takes in  $z = f(x)$ . If  $f(x) = f(y)$ , we would have  $h(f(x)) = h(f(y))$  because we know if the values of the variable is the same then the value of the function is the same. We also know that  $g(x) = g(y)$ , then we can say that  $g(x) = h(f(x)) = g(y) = h(f(y))$ . Thus,  $g(x) = (h \circ f)(x)$ .

**23.(a)**

Want to prove that if  $x \neq y$ , then  $g(x) \neq g(y)$ . Thus, showing if  $g(x) = g(y)$ , then  $x = y$  still holds. Thus, suppose  $g(x) = g(y)$ , we would have  $f(g(x)) = f(g(y)) = I(x) = I(y) = x = y$ . Therefore,  $x = y$ .

**23. (b)**

Suppose  $b = f(a)$ , and let  $a = g(x)$ . From the identity function we know that all  $f(g(x)) = x$ , meaning for all  $a = g(x)$  we can have  $b = f(a) = f(g(x)) = x$ . Also, because the domain of  $I = x$  is all real numbers. Therefore, for any number  $b$  there exist an  $a$  such that  $b = f(a)$ .