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Problem Set 8
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# Problem Set 8

**Question 1**
Suppose that $f : [a, b] \to \mathbb{R}$, is integrable, and suppose that $m = inf\{f(x) : x \in [a, b]\}$ and $M = sup\{f(x) : x \in [a, b]\}$. Then, we have $m(b - a) \leq \int_a^b f \leq M(b - a)$.
**Proof:**
Lemma: Suppose $f : [a, b] \to \mathbb{R}$ is integrable and suppose that $P$ is any partition $[a, b]$. Then we have $L(f, P) \leq \int_a^b f \leq U(f, P)$.
Proof of Lemma:
We will prove this lemma from two parts:
First, we prove that $L(f, P) \leq \int_a^b f$.
Since $f$ is integrable, then we know $sup\{L(f, P)|P, \text{a partition in range } [a, b]\} = \int_a^b f$. And let $A = \{L(f, P)|P, \text{a partition in range } [a, b]\}$. Then $supA$ is the least upper bound of $A$. Let $y = supA$, then $\forall x \in A, x \leq y$. Then, because we have $y = \int_a^b f$, $\forall x \in A, x \leq y = \int_a^b f$. Thus, $L(f, P) \leq \int_a^b f$.
Then, we prove that $\int_a^b f \leq U(f, P)$.
Since $f$ is integrable, we know $inf\{U(f, P)|P, \text{a partition in range } [a, b]\} = \int_a^b f$. And let $B = \{U(f, P)|P, \text{a partition in range } [a, b]\}$. Let $z = infB$, then $\forall x \in B, z \leq x$. Therefore, since we have $U(f, P) \geq \int_a^b f$.
Thus, $L(f, P) \leq \int_a^b f \leq U(f, P)$.
From the definition, $L(f, P) = \sum_{i=1}^n m_i(t_i - t_{i-1})$ and $U(f, P) = \sum_{i=1}^n M_i(t_i - t_{i-1})$. In this case, we have the partition $P$ as $[a, b]$ meaning there's only one pair of $m$ and $M$, and $t_i = b$, $t_{i-1} = a$. Therefore, we have $L(f, P) = m(b - a)$ and $U(f, P) = M(b - a)$. Thus, from the lemma we know that $m(b - a) \leq \int_a^b f \leq M(b - a)$.

**Question 2**
Prove that the function $f : [-1, 1] \to \mathbb{R}$, defined by

$$f(n) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases},$$

is integrable on $[-1, 1]$.
**Proof:**

At first we choose the partition $P = \{-1, 0, 1\}$. Then, by definition, $L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1})$ and $U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1})$. Therefore, we have,

$$\begin{aligned} L(f, P) &= \sum_{i=1}^{2} m_i(t_i - t_{i-1}) \\ &= m_1 \cdot (t_1 - t_0) + m_2 \cdot (t_2 - t_1) \\ &= 0 \cdot (-0 + 1) + 1 \cdot (1 - 0) \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} U(f, P) &= \sum_{i=1}^{2} M_i(t_i - t_{i-1}) \\ &= M_1 \cdot (t_1 - t_0) + M_2 \cdot (t_2 - t_1) \\ &= 1 \cdot (-0 + 1) + 1 \cdot (1 - 0) \\ &= 2 \end{aligned}$$

Thus, we have $1 \leq \int_a^b f \leq 2$.
Then, suppose $f$ is not integrable on $[-1, 1]$. Therefore, $\forall \epsilon \in \mathbb{R}, \epsilon > 0$ such that if $P = \{-1, -\epsilon, 1\}$, then, $L(f, P) < U(f, P)$. Therefore, we have

$$\begin{aligned} U(f, P) &= \sum_{i=1}^{2} M_i(t_i - t_{i-1}) \\ &= M_1 \cdot (t_1 - t_0) + M_2 \cdot (t_2 - t_1) \\ &= 0 \cdot (-\epsilon + 1) + 1 \cdot (1 + \epsilon) \\ &= 1 + \epsilon \end{aligned}$$

Then, $1 + \epsilon \geq \int_a^b f$. Also we have $1 \leq \int_a^b f$. Therefore, $0 \leq \int_a^b f - 1 \leq \epsilon$. Since $f$ is not integrable, then we have $0 < \int_a^b f - 1$. Let $\delta = \int_a^b f - 1, \delta > 0$. Since $\epsilon > 0$, therefore, $\exists \epsilon, \delta > \epsilon > 0$. Thus, $\exists \epsilon, \int_a^b f - 1 > \epsilon > 0$. Then, we have $\int_a^b f > 1 + \epsilon$, which is a contradiction. Thus, $f$ is integrable on $[-1, 1]$.