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Problem Set 1
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I. Propositions

Basic Properties of Equivalent:

(E0) If $a = b$, b can substitute a in any real formula
(E1) $\forall a$, $a = a$
(E2) $\forall a, b$, if $a = b$, then $b = a$
(E3) $\forall a, b, c$, if $a = b$, $b = c$, then $c = a$

Basic Properties of Numbers

(P1) $a + (b + c) = (a + b) + c$
(P2) $a + 0 = 0 + a = a$
(P3) $a + (-a) = (-a) + a = 0$
(P4) $a + b = b + a$
(P5) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
(P6) $a \cdot 1 = 1 \cdot a = a, 1 \neq 0$
(P7) $a \cdot a^{-1} = a^{-1} \cdot a = 1$, for $a \neq 0$
(P8) $a \cdot b = b \cdot a$
(P9) $a \cdot (b + c) = a \cdot b + a \cdot c$

II. Solutions

Questions 6, 7, 9, 10, 13, 14, 16, 22, 24, 26

Question 6 $\forall a, b$: if $a \cdot b = 0$, then either $a = 0$ or $b = 0$

Proof:

First, let's assume $a = 0$, from Question 5, we have proved that 0 multiplies any number is 0, then $a \cdot b = 0$.

Then, let's assume $a \neq 0$. From Question 2 we have proved that if we multiply the same thing on both side of a equation, then the equation is still valid. Therefore, we could obtain, by multiplying a^{-1} on both sides of the equation:

$$a^{-1} \cdot (a \cdot b) = a^{-1} \cdot 0 = 0$$

By (P5), we can reformat the equation:

$$(a^{-1} \cdot a) \cdot b = 0$$

By (P7), we have $a^{-1} \cdot a = 1$

$$\therefore \ (a^{-1} \cdot a) \cdot b = 1 \cdot b = 0$$

By (P6) we have $1 \cdot b = b$,

$$\therefore \ b = 0$$

Question 7 $\forall a, b$: $(a+b)^2 = a^2 + 2ab + b^2$
Proof:
By definition of the exponentials we have $\forall a, a^2 = a \cdot a$
$\therefore$ we have $(a+b)^2 = (a+b) \cdot (a+b)$

$$\begin{aligned}
(a+b) \cdot (a+b) &= (a+b) \cdot a + (a+b) \cdot b \text{ ——By(P9)} \\
&= a \cdot (a+b) + b \cdot (a+b) \text{ ——By(P8)} \\
&= a \cdot a + a \cdot b + b \cdot a + b \cdot b \text{ ——By(P9)} \\
&= a^2 + a \cdot b + a \cdot b + b^2 \text{ ——By(P8)} \\
&= a^2 + (a \cdot b) \cdot 1 + (a \cdot b) \cdot 1 + b^2 \text{ ——By(P6)} \\
&= a^2 + (a \cdot b) \cdot (1+1) + b^2 \text{ ——By(P9)} \\
&= a^2 + (a \cdot b) \cdot 2 + b^2 \text{ ——By(P9)} \\
&= a^2 + 2 \cdot (a \cdot b) + b^2 \text{ ——By(P8)}
\end{aligned}$$

Question 9 $\forall a$: $(-1) \cdot a = -a$
Proof:
If we do the following:

$$\begin{aligned}
a + (-1) \cdot a &= a \cdot 1 + (-1) \cdot a \text{ ——By(P6)} \\
&= a \cdot 1 + (-1) \text{ ——By(P9)} \\
&= a \cdot 0 \text{ ——By(P3)} \\
&= 0 \text{ ——By(Question 5)}
\end{aligned}$$

$$\therefore \ a + (-1) \cdot a = 0$$

For $-a$, we have:

$$-a + a = 0 \text{ ——By(P3)}$$

$$\therefore\ a + (-1) \cdot a = -a + a$$

Then we add $(-a)$ on both sides, by Question 1, the equation should still be valid

$$a + (-a) + (-1) \cdot a = -a + a + (-a)$$

$$(-1) \cdot a = -a \text{ ——By(P3)}$$