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Problem Set 3

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Question

Prove that if $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then $\lim_{x \to a} f(x) \cdot g(x) = L \cdot M$.

Proof:

We know $\forall x$ such that $0 < |x - a| < \delta_1, |f(x) - L| < \epsilon_1$, and such that $0 < |x - a| < \delta_2, |g(x) - M| < \epsilon_2$. In order for these two ineqaulities to hold at the same time, we need to have $0 < |x - a| < \delta = \text{Min}(\delta_1, \delta_2)$. Then, we would have $|g(x) - M| \cdot |f(x) - L| < \epsilon_1 \cdot \epsilon_2$, based on the theorem that if $0 < a, b$ and $a < c, b < d$ then $ab < cd$. Also, from the theorem that $|a| \cdot |b| = |ab|$. Then we would have the following:

$$|g(x) - M| \cdot |f(x) - L| < \epsilon_1 \epsilon_2$$

$$|f(x) \cdot g(x) - M \cdot f(x) - L \cdot g(x) + LM| < \epsilon_1 \epsilon_2$$

By another theorem that $|a| + |b| < |a + b|$, we could on both sides of the inequality add $|M \cdot f(x) + L \cdot g(x) - 2LM|$. Also, we know that $0 < \epsilon_1, \epsilon_2$. Thus, $\epsilon_1 \epsilon_2 = |\epsilon_1 \epsilon_2|$. Then, we would have for the inequality:

$$|f(x) \cdot g(x) - M \cdot f(x) - L \cdot g(x) + LM| + |M \cdot f(x) + L \cdot g(x) - 2LM| \tag{1}$$

$$|f(x) \cdot g(x) - M \cdot f(x) - L \cdot g(x) + LM + M \cdot f(x) + L \cdot g(x) - 2LM| \tag{2}$$

$$|\epsilon_1 \epsilon_2| + |M \cdot f(x) + L \cdot g(x) - 2LM| \tag{3}$$

Where $(2) < (1)$ and $(1) < (3)$. Thus, $(2) < (3)$.

Then we have:

$$(2) < |\epsilon_1 \epsilon_2| + |M \cdot f(x) + L \cdot g(x) - LM - LM|$$

$$(2) < |\epsilon_1 \epsilon_2| + |M \cdot f(x) - LM + L \cdot g(x) - LM|$$

$$(2) < |\epsilon_1 \epsilon_2| + |M \cdot (f(x) - L) + L \cdot (g(x) - M)|$$

$$(2) < |\epsilon_1 \epsilon_2| + |M(f(x) - L) + L(g(x) - M)|$$

Also, we know from the definition of a limit that $|f(x) - L| < \epsilon_1$, $|g(x) - M| < \epsilon_2$. We

would therefore have:

$$|M| \cdot |f(x) - L| < |M| \cdot \epsilon_1$$

$$|L| \cdot |g(x) - M| < |L| \cdot \epsilon_2$$

$$\therefore |M \cdot (f(x) - L)| < |M \cdot \epsilon_1|, |L \cdot (g(x) - M)| < |L \cdot \epsilon_2|$$

Thus, we would have a new inequality:

$$|\epsilon_1 \epsilon_2| + |M(f(x) - L)| + |L(g(x) - M)| < |\epsilon_1 \epsilon_2| + |M \cdot \epsilon_1 + L \cdot \epsilon_2|$$

From there, we could say that $|f(x) \cdot g(x) - LM| < |\epsilon_1 \epsilon_2| + |M \cdot \epsilon_1 + L \cdot \epsilon_2| \leq \epsilon$
Assume $\epsilon_1 \epsilon_2 \leq \frac{\epsilon}{2}$. From there we know, $\epsilon_1, \epsilon_2 \leq \sqrt{\frac{\epsilon}{2}}$. On the other hand, from $|M \cdot \epsilon_1 + L \cdot \epsilon_2| \leq \frac{\epsilon}{2}$. We know that $|M \cdot \epsilon_1 + L \cdot \epsilon_2| \leq |M \cdot \epsilon_1| + |L \cdot \epsilon_2|$. In order for the previous inequality to hold, we assign $|M \cdot \epsilon_1| < \epsilon/4$ and $|L \cdot \epsilon_2| < \epsilon/4$. Therefore, $\epsilon_2 < \frac{\epsilon}{4 \cdot |M|}$ and $\epsilon_1 < \frac{\epsilon}{4 \cdot |L|}$, where $L, M \neq 0$. If we have $\epsilon_1 < \text{Min}(\frac{\epsilon}{4 \cdot |L|}, \sqrt{\frac{\epsilon}{2}})$ and $\epsilon_1 < \text{Min}(\frac{\epsilon}{4 \cdot |M|}, \sqrt{\frac{\epsilon}{2}})$. Also, if we have L or M equal to 0. We can just say $\epsilon_1 < \sqrt{\frac{\epsilon}{2}}$ and $\epsilon_2 < \sqrt{\frac{\epsilon}{2}}$. Then, we have ϵ to be a very small number. We then have $\lim_{x \to a} f(x) \cdot g(x) = L \cdot M$.

Question
Suppose that $\lim_{x \to a} f(x)$ exists, and that $\lim_{x \to a} f(x) = L$. Suppose M is any number. Then prove that $\lim_{x \to a} (M f(x))$ exists, and $\lim_{x \to a} (M f(x)) = M \lim_{x \to a} f(x)$.

Proof:
Suppose $\epsilon_1 = \frac{\epsilon}{|M|}$ where $|M| \neq 0$ and $\forall x$ such that $0 < |x - a| < \delta, |f(x) - L| < \epsilon$. From the theorem that $|a| \cdot |b| = |ab|$. Thus, $|M| \cdot |f(x) - L| = |M(f(x) - L)|$. Then, we know that

$$M \cdot |f(x) - L| = |M(f(x) - L)| < |M| \cdot \epsilon$$

$$|M \cdot f(x) - LM| < |M \cdot \epsilon| = \epsilon$$

Then we could have $|M \cdot f(x) - LM|$ be a small number and therefore, $\lim_{x \to a} (M \cdot f(x)) = L \cdot M$. Also, because $\lim_{x \to a} f(x)$ is a number, then we know that $M \cdot \lim_{x \to a} f(x) = M \cdot L$. Therefore, $M \cdot \lim_{x \to a} f(x) = \lim_{x \to a} (M \cdot f(x))$.
If we have $M = 0$, then we know that on the left hand side, we are finding the limit of 0, which is 0. And on the right hand side, we have $0 \cdot L = 0$. Therefore, the theorem still holds.

Question
Show that a function cannot have two different limits at a. That is, if $\lim_{x \to a} f(x)$ exists, and $\lim_{x \to a} f(x) = L$, and $\lim_{x \to a} f(x) = M$, then we must have $L = M$.

Proof:

Suppose we have $L < M$ and we assign $0 < \epsilon \leq \frac{M-L}{2}$ such that $2\epsilon \leq M-L$ and $L+\epsilon \leq M-\epsilon$ therefore, $(M-\epsilon, M+\epsilon) \cap (L-\epsilon, L+\epsilon) = \emptyset$. Also, $\forall x$ such that $0 < |x-a| < \delta, |f(x)-L| < \epsilon$ and $|f(x) - M| < \epsilon$. From that we have $L - \epsilon < f(x) < L + \epsilon$ and $M - \epsilon < f(x) < M + \epsilon$. Then, we know that we need to have two different $f(x)$ in order to have $f(x)$ to be in two different ranges. This means $f(x)$ is not a one-to-one relationship, thus not a function. Therefore, for $f(x)$ there should be only one possible function. Then, if $\lim_{x \to a} f(x)$ exists, and $\lim_{x \to a} f(x) = L$, and $\lim_{x \to a} f(x) = M$, then we must have $L = M$.

Chapter 5. #8