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Problem Set 5

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Problem Set 5

Chapter 6 #3

(a) Proof

Suppose $\forall x, |f(x)| \leq |x|$. Thus, $0 \leq |f(x)| \leq |x|$. When, $x = 0$, $0 \leq |f(0)| \leq |0|$. Thus, $f(0) = 0$. Suppose $\exists \delta > 0$ such that, $\forall x, |x - 0| < \delta$. Then we have $|x| < \delta$. It's also because $|f(x)| \leq |x|$. Thus, $|f(x)| \leq |x| < \delta$. $|f(x)| = |f(x) - 0| = |f(x) - f(0)|$. Therefore, $|f(x) - f(0)|\delta$. Thus $\lim_{x \to 0} f(x) = 0$.

(b) Example

$$f(x) = \begin{cases} 0 & x \in \mathbb{R} - \mathbb{Q} \\ x & x \in \mathbb{Q} \end{cases}$$

(c) Proof

Suppose $\lim_{x \to 0} g(x) = 0$. Therefore, for any $\epsilon > 0$, $\exists \delta > 0$ such that if $|x - 0| < \delta$, then $|g(x) - g(0)| < \epsilon$. Also, we know that $g(0) = 0$, then if $|x| < \delta$, then $|g(x)| < \epsilon$. Since $0 \leq |f(x)| \leq |g(x)|$, then $0 \leq |f(0)| \leq |g(0)| = 0$. Thus, $f(0) = 0$. Then, since $|f(x) - f(0)| = |f(x)| \leq |g(x)| = |g(x) - 0| < \epsilon$, we know that $|f(x) - f(0)| < \epsilon$. Thus, $\lim_{x \to 0} f(x) = 0$.

Chapter 6 #4

Since we want $f(x)$ to be discontinuous, and $|f(x)|$ to be continuous, then we need to have this function with cases that is about irrational and rational and their value should be the contrary number. For instance:

$$f(x) = \begin{cases} 1 & x \in \mathbb{R} - \mathbb{Q} \\ -1 & x \in \mathbb{Q} \end{cases}$$

Chapter 6 #5

From the previous question in #3 (b), we know that if we want to have 0 to be the continuous point then we should have 0 for one case and x for the other case. Similarly, because we want to have it continuous at a and discontinuous everywhere else, we can just simply replace 0 with a.

$$f(x) = \begin{cases} a & x \in \mathbb{R} - \mathbb{Q} \\ x & x \in \mathbb{Q} \end{cases}$$

Chapter 6 #6

(a) Example

$$f(x) = \begin{cases} 0 & x = \frac{1}{n}n \in \mathbb{N} \\ x & x \neq \frac{1}{n}n \in \mathbb{N} \end{cases}$$

(b) Example

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x = \frac{1}{n}n \in \mathbb{N} \\ x & x \neq \frac{1}{n}n \in \mathbb{N} \text{ and } x \neq 0 \end{cases}$$

Chapter 6 #12

(a) Proof

Suppose $f(a)$ exists and $\lim\limits_{x \to a} f(x)$ exist and $\lim\limits_{x \to a} f(x) = f(a)$.

(b) Example

Suppose

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

such that $f(x)$ is not continuous at 0, but we could have $\lim\limits_{x \to 0} f(x) = 0$. Then, assume $g(x) = x$ such that $\lim\limits_{x \to 0} g(x) = 0$. Therefore, $\lim\limits_{x \to 0} f(g(x)) = \lim\limits_{x \to 0} f(0) = 0$ and $f(\lim\limits_{x \to 0} g(x)) = f(0) = 1$. Thus, $\lim\limits_{x \to a} f(g(x)) \neq f(\lim\limits_{x \to a} g(x))$.