Yufei Lin
Problem Set 1
Sep 10^{th} 2019

Problem Set 1

I. Propositions
Basic Properties of Equivalent:
(E0) If $a = b$, b can substitute a in any real formula
(E1) $\forall a$, $a = a$
(E2) $\forall a, b$, if $a = b$, then $b = a$
(E3) $\forall a, b, c$, if $a = b$, $b = c$, then $c = a$
Basic Properties of Numbers
(P1) $a + (b + c) = (a + b) + c$
(P2) $a + 0 = 0 + a = a$
(P3) $a + (-a) = (-a) + a = 0$
(P4) $a + b = b + a$
(P5) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$
(P6) $a \cdot 1 = 1 \cdot a = a, 1 \neq 0$
(P7) $a \cdot a^{-1} = a^{-1} \cdot a = 1$, for $a \neq 0$
(P8) $a \cdot b = b \cdot a$
(P9) $a \cdot (b + c) = a \cdot b + a \cdot c$
(P10)For every number a, one and only one of the following holds:
$(i) a = 0$
$(ii) a \in P$
$(ii) (-a) \in P$
(P11) If a and b are in P, then $a + b$ is in P
(P12) If a and b are in P, then $a \cdot b$ is in P
II. Solutions
Question 6 $\forall a, b$: if $a \cdot b = 0$, then either $a = 0$ or $b = 0$
Proof:
First, suppose $a \cdot b = 0$. Then, let's assume $a = 0$, from Question 5, we have proved that 0 multiplies any number is 0, then $a \cdot b = 0$.
Then, let's assume $a \neq 0$. From Question 2 we have proved that if we multiply the same thing on both side of a equation, then the equation is still valid. Therefore, we could obtain, by multiplying a^{-1} on both sides of the equation:

$$a^{-1} \cdot (a \cdot b) = a^{-1} \cdot 0 = 0$$

By (P5), we can reformat the equation:

$$(a^{-1} \cdot a) \cdot b = 0$$

By (P7), we have $a^{-1} \cdot a = 1$

$$\therefore \ (a^{-1} \cdot a) \cdot b = 1 \cdot b = 0$$

By (P6) we have $1 \cdot b = b$,

$$\therefore \ b = 0$$

Question 7 $\forall a, b$: $(a + b)^2 = a^2 + 2ab + b^2$
Proof:
By definition of the exponentials we have $\forall a, a^2 = a \cdot a$
$\therefore$ we have $(a + b)^2 = (a + b) \cdot (a + b)$

$$\begin{aligned}
(a + b) \cdot (a + b) &= (a + b) \cdot a + (a + b) \cdot b \qquad \text{By(P9)} \\
&= a \cdot (a + b) + b \cdot (a + b) \qquad \text{By(P8)} \\
&= a \cdot a + a \cdot b + b \cdot a + b \cdot b \qquad \text{By(P9)} \\
&= a^2 + a \cdot b + a \cdot b + b^2 \qquad \text{By(P8)} \\
&= a^2 + (a \cdot b) \cdot 1 + (a \cdot b) \cdot 1 + b^2 \qquad \text{By(P6)} \\
&= a^2 + (a \cdot b) \cdot (1 + 1) + b^2 \qquad \text{By(P9)} \\
&= a^2 + (a \cdot b) \cdot 2 + b^2 \qquad \text{By(P9)} \\
&= a^2 + 2 \cdot (a \cdot b) + b^2 \qquad \text{By(P8)}
\end{aligned}$$

Question 9 $\forall a$: $(-1) \cdot a = -a$
Proof:
If we do the following:

$$\begin{aligned}
a + (-1) \cdot a &= a \cdot 1 + (-1) \cdot a \qquad \text{By(P6)} \\
&= a \cdot 1 + (-1) \qquad \text{By(P9)} \\
&= a \cdot 0 \qquad \text{By(P3)} \\
&= 0 \qquad \text{By(Question 5)}
\end{aligned}$$

$$\therefore \ a + (-1) \cdot a = 0$$

For $-a$, we have:

$$-a + a = 0 \qquad \text{By(P3)}$$

$$\therefore\ a + (-1)\cdot a = -a + a$$

Then we add $(-a)$ on both sides, by Question 1, the equation should still be valid

$$a + (-a) + (-1)\cdot a = -a + a + (-a)$$

$$(-1)\cdot a = -a \qquad \text{By(P3)}$$

Question 10 $\forall a, b$: $(-a)\cdot(-b) = a\cdot b$
Proof:

$$\begin{aligned}(-a)\cdot(-b) &= ((-1)\cdot a)\cdot((-1)\cdot b) \qquad \text{By(Question 9)}\\ &= (-1)\cdot a\cdot(-1)\cdot b \qquad \text{By(P5)}\\ &= (-1)\cdot(-1)\cdot a\cdot b \qquad \text{By(P8)}\\ &= a\cdot b\end{aligned}$$

$$\therefore (-a)\cdot(-b) = a\cdot b = a\cdot b \qquad \text{By(E1)}$$

Question 13 $\forall a, b, c$: if $a + c < b + c$, then $a < b$
Proof:
It's because $a + c < b + c$, therefore, $b + c - (a + c) \in P$

$$\begin{aligned}b + c - (a + c) &= b + c + (-1)\cdot(a + c) \qquad \text{By(Question 9)}\\ &= b + c + (-1)\cdot a + (-1)\cdot c \qquad \text{By(P9)}\\ &= b + c + (-a) + (-c) \qquad \text{By(Question 9)}\\ &= b + (-a) + c + (-c) \qquad \text{By(P1)}\\ &= b + (-a) + (c + (-c)) \qquad \text{By(P1)}\\ &= b + (-a) \qquad \text{By(P3)}\\ &= b - a\end{aligned}$$

Therefore, $b + c - (a + c) = b - a \in P$. Thus, $a < b$.
Question 14 $\forall a, b$: if $a < 0$, $b < 0$,then $a\cdot b > 0$
Proof:

Suppose $a < 0$,$b < 0$, then $0 - a, 0 - b \in P$ From (P2) we have

$$\begin{aligned} 0 - a &= 0 + (-a) \\ &= -a \end{aligned}$$

And similar for b, $0 - b = -b$

$$\therefore -a, -b \in P$$

From (P12), we can have because $-a, -b \in P$, then $-a \cdot (-b) \in P$ From Question 10 we know, $\forall a, b$: $(-a) \cdot (-b) = a \cdot b$

$$\therefore a \cdot b \in P$$

From (P2), we have $a \cdot b + (-0) = a \cdot b - 0 \in P$

$$\therefore a \cdot b > 0$$

Question 16 $\forall a, b$: $a \cdot b > 0$, then either $a > 0$ and $b > 0$ or $a < 0$ and $b < 0$
Proof:
Suppose $a = 0$, from Question 5, we have $\forall a$,$a \cdot 0 = 0$. And 0 cannot be greater than 0. Therefore, $a \neq 0$.
Assume $a < 0$.
Suppose $b > 0$. From Question 15, we have $\forall a, b$ if $a < 0, b > 0$, then $a \cdot b < 0$. Thus, $a < 0$ and $b < 0$.
Assume $a > 0$.
Suppose$b < 0$. Because we can use our symbols interchangeably, from Question 15, we can have $\forall a, b$ if $b < 0, a > 0$, then $a \cdot b < 0$. Thus, $a > 0$ and $b > 0$.
Question 22 $\forall a, b, c$: if $a < b$ and $c > 0$, then $a \cdot c < b \cdot c$
Proof:
It's because $a < b$, therefore, $b - a \in P$. Also, because $c > 0$, meaning $c - 0 = c \in P$. Therefore, $c, (b-a) \in P$. From (P12), if both c and $(b-a)$ belong to P, then $c \cdot (b - a) \in P$. Then, we have

$$\begin{aligned} c \cdot (b - a) &= c \cdot (b + (-a)) \\ &= c \cdot b + c \cdot (-a)) \qquad \text{By(P9)} \\ &= c \cdot b - c \cdot a \end{aligned}$$

Therefore, $c \cdot b - c \cdot a \in P$. So, $c \cdot a < c \cdot b$
Question 24 $\forall a, b, c$: if $a < b$ and $c < 0$, then $b \cdot c < c \cdot a$.

Proof:
It's because $a < b$, therefore, $b - a \in P$. Also, because $c < 0$, meaning $0 - c = -c \in P$. Therefore, $-c, (b - a) \in P$. From (P12), if both $-c$ and $(b - a)$ belong to P, then $-c \cdot (b - a) \in P$.
Then, we have

$$\begin{aligned} -c \cdot (b - a) &= -c \cdot (b + (-a)) \\ &= -c \cdot b + (-c) \cdot (-a)) \qquad \text{By(P9)} \\ &= -c \cdot b + c \cdot a \end{aligned}$$

Therefore, $c \cdot a - c \cdot b \in P$. So, $c \cdot b < c \cdot a$
Question 26 $\forall a, b$: $|a + b| \le |a| + |b|$
Proof:
First, assume $a, b \ge 0$

$$\therefore |a + b| = a + b, |a| + |b| = a + b$$

$a + b = a + b$, meaning $|a + b| = |a| + |b|$ The assumption holds for $a, b \ge 0$
Then, assume $a, b < 0$
$|a + b| = -(a + b) = -a - b$ (By(P9)), $|a| + |b| = -a - b$ Therefore, we have $|a + b| = -a - b = |a| + |b|$. And this assumption holds when $a, b < 0$.
Assume, $a \ge 0, b \le 0$, and $|a| \ge |b|$, it would be the same situation, when $b \ge 0, a \le 0$, and $|b| \ge |a|$.
It is because $|a| \ge |b|$, then $a + b \ge 0$. It means $|a + b| = a + b$, and $|a| + |b| = a - b$. Both calculations are absolute values, meaning both of them are greater than 0.
Then, we have

$$\begin{aligned} a - b - (a + b) &= a - b - a - b \qquad \text{By(P9)} \\ &= a - a - b - b \qquad \text{By(P4)} \\ &= -b - b \qquad \text{By(P3)} \\ &= -b + (-b) \\ &= -1 \cdot b + (-1) \cdot b \\ &= (-1 + (-1)) \cdot b \qquad \text{By(P9)} \\ &= -2 \cdot b \qquad \text{By(P9)} \end{aligned}$$

$$\therefore a - b - (a + b) = -2b \ge 0$$

$$\therefore |a| + |b| = a - b \ge |a + b| = a + b$$

Assume, $a \geq 0, b \leq 0$, and $|a| \leq |b|$, it would be the same situation, when $b \geq 0, a \leq 0$, and $|b| \leq |a|$.
It is because $|a| \leq |b|$, then $a + b \leq 0$. It means $|a + b| = -(a + b)$, and $|a| + |b| = a - b$. Both calculations are absolute values, meaning both of them are greater than 0.

Then, we have

$$
\begin{aligned}
a - b - (-(a + b)) &= a - b + (a + b) \\
&= a + a + b - b \qquad \text{By(P4)} \\
&= a + a \qquad \text{By(P3)} \\
&= 1 \cdot a + 1 \cdot a \\
&= (1 + 1) \cdot a \qquad \text{By(P9)} \\
&= 2 \cdot a
\end{aligned}
$$

$$\therefore a - b - (-(a + b)) = 2a \geq 0$$

$$\therefore |a| + |b| = a - b \geq |a + b| = a + b$$

Chap 1, Q1

(i) If $ax = a$ for some number $a \neq 0$, then $x = 1$.

Proof:

Assume $ax = a$, then we can have $ax - a = 0$, meaning:

$$\begin{aligned} ax - a &= ax + a \cdot -1 \\ &= a \cdot (x-1) \qquad \text{By(P9)} \\ &= 0 \end{aligned}$$

$$\therefore a \cdot (x-1) = 0$$

It is because $a \neq 0$, from Question 6, $\forall a, b, if a \cdot b = 0$, either a or b is 0. Then, we know b in this equation is 0, which is $(x-1)$. $x - 1 = 0 \therefore x = 1$.

(ii) $x^2 - y^2 = (x-y)(x+y)$.

Proof:

On the right hand side of the equation, we can have:

$$\begin{aligned} (x-y)(x+y) &= x(x-y) + y(x-y) \qquad \text{By(P9)} \\ &= x^2 - xy + yx - y^2 \qquad \text{By(P9)} \\ &= x^2 - y^2 \qquad \text{By(P3)} \end{aligned}$$

$$\therefore x^2 - y^2 = (x-y)(x+y)$$

(iii) If $x^2 = y^2$, then $x = y$ or $x = -y$.

Proof:

Assume $x^2 = y^2$, then we have:

$$x^2 - y^2 = 0$$

$$\text{From (ii): } (x+y)(x-y) = 0$$

$$\text{From (Question 6): either } (x+y) \text{ or } (x-y) = 0$$

When $(x+y) = 0$, subtract y on both sides, $(x+y) - y = x = 0 - y = -y$. Therefore, $x = -y$.

When $(x-y) = 0$, add y on both sides, $(x-y) + y = x = 0 + y = y$. Therefore, $x = y$.

(iv) $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$.

Proof:

On the right hand side of the equation, we can have:

$$\begin{aligned}(x-y)(x^2+xy+y^2) &= (x+(-y))(x^2+xy+y^2)\\ &= x(x^2+xy+y^2)+(-y)(x^2+xy+y^2) \qquad \text{By(P9)}\\ &= x^3+x^2y+xy^2+(-y)x^2+(-y)xy+(-y)y^2 \qquad \text{By(P9)}\\ &= x^3+x^2y+xy^2+(-x^2y)+(-xy^2)+(-y^3) \qquad \text{By(P8)}\\ &= x^3+(-y^3) \qquad \text{By(P3)}\\ &= x^3-y^3\end{aligned}$$

(v)$x^n - y^n = (x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+...+xy^{n-2}+y^{n-1})$.
Proof:
On the right hand side of the equation, we can have:
$(x-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+...+xy^{n-2}+y^{n-1})$

$$\begin{aligned}&= (x+(-y))(x^{n-1}+x^{n-2}y+x^{n-3}y^2+...+xy^{n-2}+y^{n-1})\\ &= x(x^{n-1}+x^{n-2}y+x^{n-3}y^2+...+xy^{n-2}+y^{n-1})\\ &\quad +(-y)(x^{n-1}+x^{n-2}y+x^{n-3}y^2+...+xy^{n-2}+y^{n-1}) \qquad \text{By(P9)}\\ &= x^n+x^{n-1}y+x^{n-2}y^2+...+x^2y^{n-2}+xy^{n-1}+(-y)x^{n-1}+...+(-y)y^{n-1} \qquad \text{By(P9)}\\ &= x^n+x^{n-1}y+...+xy^{n-1}+(-x^{n-1}y)+...+(-y^n) \qquad \text{By(P8)}\\ &= x^n+(-y^n) \qquad \text{By(P3)}\\ &= x^n-y^n\end{aligned}$$

(vi)$x^3+y^3 = (x+y)(x^2-xy+y^2)$.
Proof:

From (v), let $y = (-y)$ and $n = 3$, we would have $x^3+y^3 = x^3-(-y)^3$. Therefore, plugging these values into the equation and we could get: $x^3+y^3 = (x+y)(x^2-xy+y^2)$.

Chap 1, Q2
Solution
This is incorrect because on the third step, when the person wants to divide both side of the equation by $(x-y)$, there is a possibility the person is dividing by 0. Therefore, this step is problematic and leading to the final conclusion that $2 = 1$.

Chap 1, Q3
(i)$\frac{a}{b} = \frac{ac}{bc}$, if $b, c \neq 0$.
Proof:

On the right hand side because $b, c \neq 0$, we have:

$$\begin{aligned}
\frac{ac}{bc} &= a \cdot c \cdot b^{-1} \cdot c^{-1} \\
&= a \cdot b^{-1} \cdot c \cdot c^{-1} \qquad \text{By(P8)} \\
&= a \cdot b^{-1} \qquad \text{By(P7)} \\
&= \frac{a}{b}
\end{aligned}$$

$$\therefore \frac{ac}{bc} = \frac{a}{b}$$

(ii) $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$, if $b, d \neq 0$.

Proof:

On the right hand side because $b, d \neq 0$, we have:

$$\begin{aligned}
\frac{ad+bc}{bd} &= (ad+bc) \cdot b^{-1} \cdot d^{-1} \\
&= (ad+bc) \cdot (b^{-1} \cdot d^{-1}) \qquad \text{By(P5)} \\
&= ad \cdot (b^{-1} \cdot d^{-1}) + bc \cdot (b^{-1} \cdot d^{-1}) \qquad \text{By(P9)} \\
&= a \cdot b^{-1} \cdot d \cdot d^{-1} + c \cdot b \cdot b^{-1} \cdot d^{-1} \qquad \text{By(P5)} \\
&= a \cdot b^{-1} + c \cdot d^{-1} \qquad \text{By(P7)} \\
&= \frac{a}{b} + \frac{c}{d}
\end{aligned}$$

$$\therefore \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

(iii) $(ab)^{-1} = a^{-1}b^{-1}$, if $a, b \neq 0$.

Proof:

On the left hand side, we can multiply by ab, because from P7, we have $a \cdot a^{-1} = 1$. Then we have $ab \cdot (ab)^{-1} = 1$.

On the right hand side, if we multiply by ab, we would have:

$$\begin{aligned}
ab \cdot a^{-1}b^{-1} &= a \cdot a^{-1} \cdot b^{-1} \cdot b \qquad \text{By(P5)} \\
&= 1 \cdot 1 \qquad \text{By(P7)} \\
&= 1
\end{aligned}$$

$\because$ We multiply the same thing, and both of them give the same result

$$\therefore \text{From Question 2, we know} (ab)^{-1} = a^{-1}b^{-1}.$$

(iv) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$, if $b, d \neq 0$.
Proof:
On the left hand side, we have:

$$\begin{aligned} \frac{a}{b} \cdot \frac{c}{d} &= a \cdot b^{-1} \cdot c \cdot d^{-1} \\ &= a \cdot c \cdot b^{-1} \cdot d^{-1} \qquad \text{By(P5)} \\ &= a \cdot c \cdot (b \cdot d)^{-1} \\ &= \frac{ac}{bd} \end{aligned}$$

$$\therefore \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$$

(v) $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$, if $b, c, d \neq 0$.
Proof:
On the left hand side, because $b, c, d \neq 0$, we could have:

$$\begin{aligned} \frac{a}{b} \div \frac{c}{d} &= \frac{a}{b} \cdot (\frac{c}{d})^{-1} \\ &= a \cdot b^{-1} \cdot (c \cdot d^{-1})^{-1} \\ &= a \cdot b^{-1} \cdot c^{-1} \cdot d \\ &= a \cdot d \cdot b^{-1} \cdot c^{-1} \qquad \text{By (P8)} \\ &= a \cdot d \cdot (b^{-1} \cdot c^{-1}) \qquad \text{By (P5)} \\ &= a \cdot d \cdot (b \cdot c)^{-1} \\ &= \frac{ac}{bd} \end{aligned}$$

$$\therefore \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$$

(vi) If $b, d \neq 0$, then $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$. Also, determine when $\frac{a}{b} = \frac{b}{a}$.
Proof:
It's because $\frac{a}{b} = \frac{c}{d}$, $b, d \neq 0$. Therefore, we could have both side of the equation multiply by bd:

$$\frac{a}{b} \cdot bd = \frac{c}{d} \cdot bd$$

$$a \cdot b^{-1} \cdot bd = c \cdot d^{-1} \cdot bd$$

$$a \cdot b \cdot b^{-1} \cdot d = b \cdot c \cdot d^{-1} \cdot d \qquad \text{By(P8)}$$

$$a \cdot d = b \cdot c \qquad \text{By(P7)}$$

$$\therefore \frac{a}{b} = \frac{c}{d} \text{ if and only if } ad = bc$$

It is because when $\frac{a}{b} = \frac{c}{d}$, $ad = bc$. We must have $a \cdot a = b \cdot b$, if we want to have $\frac{a}{b} = \frac{b}{a}$, $a, b \neq 0$.

Chap 1, Q4

(i)$4 - x < 3 - 2x$.

Solution:

$S = \{x \mid x < -1\}$

(ii)$5 - x^2 < 8$.

Solution:

$S = \{x \mid x \in \mathbb{R}\}$

(iii)$5 - x^2 < -2$.

Solution:

$S = \{x \mid x^2 > 7\}$

(iv)$(x-1)(x-3) > 0$.

Solution:

$S = \{x \mid x < 1, \text{ or } x > 3\}$

(v)$x^2 - 2x + 2 > 0$.

Solution:

$S = \{x \mid x \in \mathbb{R}\}$

(vi)$x^2 + x + 1 > 2$.

Solution:

$S = \{x \mid x < \frac{-1-\sqrt{5}}{2}, \text{ or } x > \frac{\sqrt{5}-1}{2}\}$

(vii)$x^2 - x + 10 > 16$.

Solution:

$S = \{x \mid x < -2, \text{ or } x > 3\}$

(viii)$x^2 + x + 1 > 0$.

Solution:

$S = \{x \mid x \in \mathbb{R}\}$

(ix)$(x - \pi)(x + 5)(x - 3) > 0$.

Solution:

$S = \{x \mid -5 < x < 3, \text{ or } x > \pi\}$

(x)$(x - \sqrt[3]{2})(x - \sqrt{2}) > 0$.

Solution:

$S = \{x \mid x < \sqrt[3]{2}, \text{ or } x > \sqrt{2}\}$

(xi)$2^x < 8$.

Solution:
$S = \{x \mid x < 3\}$
(xii)$x + 3^x < 4.$
Solution:
$S = \{x \mid x < 1\}$
(xiii)$\frac{1}{x} + \frac{1}{(1-x)} > 0.$
Solution:
$S = \{x \mid 1 > x > 0\}$
(xiv)$\frac{x-1}{x+1} > 0.$
Solution:
$S = \{x \mid 1 < x, \text{ or } -1 > x\}$

Chap 1, Q7
Proof:

It is because we have $0 < a < b$. Then, we know $b - a, b, a \in P$. From Question 16, we then know that $\sqrt{a}, \sqrt{b} \in P$. This means that $\sqrt{b} + \sqrt{a} \in P$. Therefore, $\sqrt{b} + \sqrt{a} > 0$.

Also, because $b - a \in P$, we know from Chap1, Q1(ii) that $(\sqrt{a} + \sqrt{b})(\sqrt{b} - \sqrt{a}) \in P$. From Question 16 again we know that for $(\sqrt{a} + \sqrt{b})$ and $(\sqrt{b} - \sqrt{a})$ either they are both positive or both negative. Since $\sqrt{b} + \sqrt{a} > 0$, then $\sqrt{b} - \sqrt{a} > 0$. Therefore, $\sqrt{b} - \sqrt{a} \in P$ and $\sqrt{b} > \sqrt{a}$. Also, by (P12), we know that $\sqrt{a} \cdot (\sqrt{b} - \sqrt{a}) = \sqrt{ab} - a \in P$. Thus, $\sqrt{ab} > a$.

From (P12), we also know $(\sqrt{b} - \sqrt{a})^2 \in P$. Thus we know $b - 2\sqrt{ab} + a \in P$. Furthermore, $\frac{1}{2} \in P$ and leads to $\frac{1}{2} \cdot (b - 2\sqrt{ab} + a) \in P$ by (P12). We now have $\frac{1}{2} \cdot (a + b) - \sqrt{ab} \in P$. Then we now $\frac{(a+b)}{2} > \sqrt{ab}$.

At last, we have $b - \frac{(a+b)}{2} = \frac{b}{2} - \frac{a}{2} = \frac{(b-a)}{2}$. We know $b - a, \frac{1}{2} \in P$, then $\frac{(b-a)}{2} \in P$. Thus, $b > \frac{(a+b)}{2}$.

$$\therefore b > \frac{(a+b)}{2} > \sqrt{ab} > a$$

Chap 1, Q11
(i)$|x - 3| = 8.$
Solution:
$x_1 = -5, x_2 = 11$
(ii)$|x - 3| < 8.$
Solution:
$S = \{x \mid -5 < x < 11\}$
(iii)$|x + 4| < 2.$

Solution:

$S = \{x \mid -6 < x < -2\}$

(iv) $|x-1| + |x-2| > 1.$

Solution:

$S = \{x \mid x \in \mathbb{R}\}$

(v) $|x-1| + |x+1| < 2.$

Solution:

$S = \{\emptyset\}$

(vi) $|x-1| + |x+1| < 1.$

Solution:

$S = \{\emptyset\}$

(vii)$|x-1|\cdot|x+1|=0$.

Solution:

$x_{1,2}=\pm 1$

(viii)$|x-1|\cdot|x+2|=3$.

Solution:

$x_{1,2}=\frac{-1\pm\sqrt{21}}{2}$

Chap 1, Q12

(i)$|xy|=|x|\cdot|y|$

Proof:

Assume $x,y\geq 0$. Therefore, $xy\geq 0$. We would have $|xy|=xy$ and $|x|\cdot|y|=x\cdot y=xy$. Then $|xy|=|x|\cdot|y|$.

Then, assume $x,y<0$. Therefore, $xy\geq 0$. We would have $|xy|=xy$ and $|x|\cdot|y|=-x\cdot(-y)=xy$. Then $|xy|=|x|\cdot|y|$.

Then, assume $x\geq 0, y<0$. It would be the same for $x<0, y\geq 0$. In this case, we would have $xy\leq 0$ and therefore $|xy|=-xy$. Then, we would have $|x|\cdot|y|=x\cdot(-y)=-xy$. $|xy|=|x|\cdot|y|$.

(ii)$|\frac{1}{x}|=\frac{1}{|x|}$ if $x\neq 0$.

Proof:

Assume $x>0$, then we would have $|\frac{1}{x}|=\frac{1}{x}$ and $\frac{1}{|x|}=\frac{1}{x}$. Then, $|\frac{1}{x}|=\frac{1}{|x|}$.

Assume $x<0$, then we would have $|\frac{1}{x}|=-1\cdot\frac{1}{x}=\frac{1}{-x}$ and $\frac{1}{|x|}=\frac{1}{-x}$. Then, $|\frac{1}{x}|=\frac{1}{|x|}$.

(iii)$\frac{|x|}{|y|}=|\frac{x}{y}|$ if $y\neq 0$.

Proof:

Assume $x,y\geq 0, y\neq 0$, then $\frac{|x|}{|y|}=\frac{x}{y}\geq 0$. Also, $\frac{x}{y}\geq 0$. Then we would have $|\frac{x}{y}|=\frac{x}{y}$. Therefore, $\frac{|x|}{|y|}=|\frac{x}{y}|$.

Assume $x,y<0$, then $\frac{|x|}{|y|}=\frac{-x}{-y}\geq 0$. Also, $\frac{x}{y}\geq 0$. Then we would have $|\frac{x}{y}|=\frac{x}{y}$. Therefore, $\frac{|x|}{|y|}=|\frac{x}{y}|$.

Assume $x\geq 0, y<0$. Then $\frac{|x|}{|y|}=\frac{x}{-y}\leq 0$. Therefore, $\frac{|x|}{|y|}=\frac{x}{-y}=-\frac{x}{y}$. Also, $\frac{x}{y}\leq 0$. Then we would have $|\frac{x}{y}|=-\frac{x}{y}$. Therefore, $\frac{|x|}{|y|}=|\frac{x}{y}|$.

Assume $x<0, y>0$. Then $\frac{|x|}{|y|}=\frac{-x}{y}\leq 0$. Therefore, $\frac{|x|}{|y|}=\frac{-x}{y}=-\frac{x}{y}$. Also, $\frac{x}{y}\leq 0$. Then we would have $|\frac{x}{y}|=-\frac{x}{y}$. Therefore, $\frac{|x|}{|y|}=|\frac{x}{y}|$.

(iv) $|x-y| \leq |x|+|y|$

Proof:

Assume $x, y \geq 0, x \geq y$, $|x-y| = x-y$, $|x|+|y| = x+y$.

$$\begin{aligned}|x|+|y|-|x-y| &= x+y-(x-y)\\ &= x+y-x+y\\ &= 2y \geq 0\end{aligned}$$

$$\therefore |x|+|y|-|x-y| \geq 0, \text{ meaning, } |x-y| \leq |x|+|y|.$$

Assume $x, y \geq 0, x < y$, $|x-y| = y-x$, $|x|+|y| = x+y$.

$$\begin{aligned}|x|+|y|-|x-y| &= x+y-(y-x)\\ &= x+y-y+x\\ &= 2x \geq 0\end{aligned}$$

$$\therefore |x|+|y|-|x-y| \geq 0, \text{ meaning, } |x-y| \leq |x|+|y|.$$

Assume $x, y \leq 0, x \geq y$, $|x-y| = x-y$, $|x|+|y| = -x-y$.

$$\begin{aligned}|x|+|y|-|x-y| &= -x-y-(x-y)\\ &= -x-y-x+y\\ &= -2x \geq 0\end{aligned}$$

$$\therefore |x|+|y|-|x-y| \geq 0, \text{ meaning, } |x-y| \leq |x|+|y|.$$

Assume $x, y \leq 0, x \leq y$, $|x-y| = y-x$, $|x|+|y| = -x-y$.

$$\begin{aligned}|x|+|y|-|x-y| &= -x-y-(y-x)\\ &= -x-y-y+x\\ &= -2y \geq 0\end{aligned}$$

$$\therefore |x|+|y|-|x-y| \geq 0, \text{ meaning, } |x-y| \leq |x|+|y|.$$

Assume $x \geq 0, y < 0$, $|x-y| = x-y$, $|x|+|y| = x-y$.

$$\begin{aligned}|x|+|y|-|x-y| &= x-y-(x-y)\\ &= 0\end{aligned}$$

$$\therefore |x| + |y| - |x - y| = 0, \text{ meaning, } |x - y| = |x| + |y|.$$

Assume $x < 0, y \geq 0$, $|x - y| = y - x$, $|x| + |y| = y - x$.

$$\begin{aligned} |x| + |y| - |x - y| &= y - x - (y - x) \\ &= 0 \end{aligned}$$

$$\therefore |x| + |y| - |x - y| = 0, \text{ meaning, } |x - y| = |x| + |y|.$$

(v)$|x| - |y| \leq |x - y|$
Proof:
If $|x| \geq |y|$ and x, y are both negative or both positive, then $|x| - |y| = |x - y|$. Else if $x \geq 0, y < 0$, we would have $|x - y| - |x| + |y| = x - y - x - y = -2y \geq 0$ meaning $|x| - |y| \leq |x - y|$. Or if we have $x < 0, y \geq 0$, then $|x - y| - |x| + |y| = y - x + x + y = 2y \geq 0$ meaning $|x| - |y| \leq |x - y|$.
If $|x| < |y|$, $|x| - |y| < 0$. It is because $|x - y| > 0$, then $|x| - |y| \leq |x - y|$.

(vi)$|(|x| - |y|)| \leq |x - y|$
Proof:
It is because both sides of this inequality are absolute values, meaning both of them are greater than or equal to 0. If $x, y \geq 0$ then we would have $|(|x| - |y|)| = |x - y|$, and the claim holds
If $x, y < 0$, then on the left hand side, we have $|(|x| - |y|)| = | - x - (-y)| = |y - x|$. If $x \geq y$, we would have $|y - x| = x - y$ and $|x - y| = x - y$. Thus, $|(|x| - |y|)| = |x - y|$. On the other hand, if $y \geq x$, it is the exactly same result. Therefore, the calim holds.
Futhermore, if $x \geq 0, y < 0$, then we know, $|(|x| - |y|)| = |x + y|$. If $|x| \geq |y|$, then $|x - y| = x - y$ and $|x + y| = x + y$. $x - y - (x + y) = x - y - x - y = -2y > 0$ Therefore, $|(|x| - |y|)| \leq |x - y|$. This would be the exact same proof for $y \geq 0, x < 0$.

(vii)$|x + y + z| \leq |x| + |y| + |z|$
Proof:
From Question 26 we know that $\forall x, y$: $|x + y| \leq |x| + |y|$. Therefore, if add $|c|$ on both sides of the inequality, by (P11) we would have $|x + y| + |z| \leq |x| + |y| + |z|$. Then, for this question, we only need to prove that $|x + y + z| \leq |x + y| + |z|$. Let, $a = x + y, b = z$, then we could have $|a + b| \leq |a| + |b|$. Therefore, $|x + y + z| \leq |x + y| + |z| \leq |x| + |y| + |z|$.
From Question 26, we know that if $|x + y| = |x| + |y|$ if x, y are the same sign or either x or y is 0. Therefore, in this case, either x, y, z are the same sign, or 2 of them ($x + y$ or $x + z$, or $y + z = 0$) cancels each other.