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Problem Set 8

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Question 1

Suppose that $f : [a, b] \to \mathbb{R}$, is integrable, and suppose that $m = inf\{f(x) : x \in [a, b]\}$ and $M = sup\{f(x) : x \in [a, b]\}$. Then, we have $m(b - a) \leq \int_a^b f \leq M(b - a)$.

Proof:

Lemma: Suppose $f : [a, b] \to \mathbb{R}$ is integrable and suppose that P is any partition $[a, b]$. Then we have $L(f, P) \leq \int_a^b f \leq U(f, P)$.

Proof of Lemma:

We will prove this lemma from two parts:

First, we prove that $L(f, P) \leq \int_a^b f$.

Since f is integrable, then we know $sup\{L(f, P)|P, \text{a partition in range } [a, b]\} = \int_a^b f$. And let $A = \{L(f, P)|P, \text{a partition in range } [a, b]\}$. Then $supA$ is the least upper bound of A. Let $y = supA$, then $\forall x \in A, x \leq y$. Then, because we have $y = \int_a^b f$, $\forall x \in A, x \leq y = \int_a^b f$. Thus, $L(f, P) \leq \int_a^b f$.

Then, we prove that $\int_a^b f \leq U(f, P)$.

Since f is integrable, we know $inf\{U(f, P)|P, \text{a partition in range } [a, b]\} = \int_a^b f$. And let $B = \{U(f, P)|P, \text{a partition in range } [a, b]\}$. Let $z = infB$, then $\forall x \in B, z \leq x$. Therefore, since we have $U(f, P) \geq \int_a^b f$.

Thus, $L(f, P) \leq \int_a^b f \leq U(f, P)$.

From the definition, $L(f, P) = \sum_{i=1}^n m_i(t_i - t_{i-1})$ and $U(f, P) = \sum_{i=1}^n M_i(t_i - t_{i-1})$. In this case, we have the partition P as $[a, b]$ meaning there's only one pair of m and M, and $t_i = b$, $t_{i-1} = a$. Therefore, we have $L(f, P) = m(b - a)$ and $U(f, P) = M(b - a)$. Thus, from the lemma we know that $m(b - a) \leq \int_a^b f \leq M(b - a)$.

Question 2

Prove that the function $f : [-1, 1] \to \mathbb{R}$, defined by

$$f(n) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases},$$

is integrable on $[-1, 1]$.

Proof:

At first we choose the partition $P = \{-1, 0, 1\}$. Then, by definition, $L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1})$ and $U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1})$. Therefore, we have,

$$\begin{aligned}
L(f, P) &= \sum_{i=1}^{2} m_i(t_i - t_{i-1}) \\
&= m_1 \cdot (t_1 - t_0) + m_2 \cdot (t_2 - t_1) \\
&= 0 \cdot (-0 + 1) + 1 \cdot (1 - 0) \\
&= 1
\end{aligned}$$

and

$$\begin{aligned}
U(f, P) &= \sum_{i=1}^{2} M_i(t_i - t_{i-1}) \\
&= M_1 \cdot (t_1 - t_0) + M_2 \cdot (t_2 - t_1) \\
&= 1 \cdot (-0 + 1) + 1 \cdot (1 - 0) \\
&= 2
\end{aligned}$$

Thus, we have $1 \leq \int_a^b f \leq 2$.
Then, suppose f is not integrable on $[-1, 1]$. Therefore, $\forall \epsilon \in \mathbb{R}, \epsilon > 0$ such that if $P = \{-1, -\epsilon, 1\}$, then, $L(f, P) < U(f, P)$. Therefore, we have

$$\begin{aligned}
U(f, P) &= \sum_{i=1}^{2} M_i(t_i - t_{i-1}) \\
&= M_1 \cdot (t_1 - t_0) + M_2 \cdot (t_2 - t_1) \\
&= 0 \cdot (-\epsilon + 1) + 1 \cdot (1 + \epsilon) \\
&= 1 + \epsilon
\end{aligned}$$

Then, $1 + \epsilon \geq \int_a^b f$. Also we have $1 \leq \int_a^b f$. Therefore, $0 \leq \int_a^b f - 1 \leq \epsilon$. Since f is not integrable, then we have $0 < \int_a^b f - 1$. Let $\delta = \int_a^b f - 1, \delta > 0$. Since $\epsilon > 0$, therefore, $\exists \epsilon, \delta > \epsilon > 0$. Thus, $\exists \epsilon, \int_a^b f - 1 > \epsilon > 0$. Then, we have $\int_a^b f > 1 + \epsilon$, which is a contradiction. Thus, f is integrable on $[-1, 1]$.

Question 3

Suppose that $f : [a, b] \to \mathbb{R}$, is bounded. Then f is integrable on $[a, b]$ if, and only if, for every $\epsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon.$$

Proof:

Suppose f is integrable on $[a, b]$.

Question 4

Use the theorem you proved in question #?? to solve question #?? again in a slightly different way. (It should be easier this way, but it is worth doing it both ways.)

Proof:

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Chpater 13. #1

Prove that $\int_0^b x^3 dx = \frac{b^4}{4}$, by considering partitions into n equal intervals.

Proof:

Since we are going to have a partition with n intervals, then we would have $P = \{t_0, t_1, ..., t_n\}$ with $t_0 = 0, t_i = i \cdot \frac{b}{n}$. Then, we have

$$\begin{aligned} L(f, P_n) &= \sum_{i=1}^{n} t_{i-1}^3 (t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} (\frac{(i-1) \cdot b}{n})^3 \cdot \frac{b}{n} \\ &= (\frac{b}{n})^4 \cdot \sum_{i=1}^{n} (i-1)^3 \\ &= (\frac{b}{n})^4 \cdot \sum_{j=0}^{n-1} j^3 \end{aligned}$$

$$\begin{aligned} U(f, P_n) &= \sum_{i=1}^{n} t_i^3 (t_i - t_{i-1}) \\ &= \sum_{i=1}^{n} (\frac{i \cdot b}{n})^3 \cdot \frac{b}{n} \\ &= (\frac{b}{n})^4 \cdot \sum_{i=1}^{n} i^3 \end{aligned}$$

From the previous question Chapter 2 #6, we know that $\sum_{i=1}^{n} i^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{2}$, and the equation could be written:

$$
\begin{aligned}
L(f,P_n) &= (\frac{b}{n})^4 \cdot (\frac{(n-1)^4}{4} + \frac{(n-1)^3}{2} + \frac{(n-1)^2}{4}) \\
&= (\frac{b}{n})^4 \cdot \frac{1}{4}((n-1)^4 + 2(n-1)^3 + (n-1)^2) \\
&= \frac{b^4}{4} \cdot (\frac{(n-1)^4}{n^4} + \frac{2(n-1)^3}{n^4} + \frac{(n-1)^2}{n^4}) \\
U(f,P_n) &= (\frac{b}{n})^4 \cdot (\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}) \\
&= (\frac{b}{n})^4 \cdot \frac{1}{4}(n^4 + 2n^3 + n^2) \\
&= \frac{b^4}{4} \cdot (1 + \frac{2}{n} + \frac{1}{n^2})
\end{aligned}
$$

Since $n \geq 1, n \in \mathbb{N}$, therefore, we know when n gets very large both $U(f,P_n)$ and $L(f,P_n)$ are close to $\frac{b^4}{4}$. At the same time, we find that

$$U(f,P_n) - L(f,P_n) = \frac{b^4}{4}(\frac{2n^3-1}{n^4})$$

which is a positive number. And we can make this difference as small as possible, by theorem 2, this function is integrable. Therefore, we have $U(f,P_n) \geq \frac{b^4}{4} \geq L(f,P_n)$. Thus, $\int_0^b x^3 dx = \frac{b^4}{4}$.

Chpater 13. #13

(a) Prove that if f is integrable on $[a,b]$ and $f(x) \geq 0$ for all x in $[a,b]$, then $\int_a^b f \geq 0$.

Proof:

Since f is integrable on $[a,b]$, then we have $U(f,P_n) \geq \int_a^b f \geq L(f,P_n)$. Also, based on the definition, we have:

$$
\begin{aligned}
L(f,P_n) &= \sum_{i=1}^{n} f(t_{i-1})(t_i - t_{i-1}) \\
U(f,P_n) &= \sum_{i=1}^{n} f(t_i)(t_i - t_{i-1})
\end{aligned}
$$

Also, because $f(x) \geq 0, \forall x \in [a,b]$ and $t_{i-1} \in [a,b], \forall i \in \mathbb{N}, i \leq n$, then $f(t_{i-1}) \geq 0$. Also, because the definition of t_i guarantees, $t_i > t_{i-1}$, then $t_i - t_{i-1} > 0$ Therefore, let $q_i = f(t_{i-1})(t_i - t_{i-1})$, since both $f(t_{i-1})$ and$(t_i - t_{i-1})$ are greater than or equal to 0. We have $q_i \geq 0$. Similarly, because $t_i \in [a,b], \forall i \in \mathbb{N}, i \leq n$, then $f(t_i) \geq 0$. Let

$p_i = f(t_i)(t_i - t_{i-1})$. Because both $f(t_{i-1})$ and$(t_i - t_{i-1})$ are both greater than or equal to 0. Then, $p_i \geq 0$. Thus, we have $U(f, P_n) \geq \int_a^b f \geq L(f, P_n) \geq 0$. Thus, $\int_a^b f \geq 0$.

(b) Prove that if f and g are both integrable on $[a, b]$ and $f(x) \geq g(x), \forall x \in [a, b]$, then $\int_a^b f \geq \int_a^b g$.

Proof:

Suppose f and g are both integrable on $[a, b]$ and $f(x) \geq g(x), \forall x \in [a, b]$. Then, we know that $f(x) - g(x) \geq 0$. Thus, $(f - g)(x) \geq 0, \forall x \in [a, b]$. Furthermore, from theorem 5, we know that for any two functions that are integrable at the same range, we have $\int_a^b f + \int_a^b g = \int_a^b (f + g)$. Furthermore, from theorem 6 we know that $\int_a^b cg = c \cdot \int_a^b g$. Thus, $\int_a^b -g = -\int_a^b g$. Then, we have $\int_a^b f - \int_a^b g = \int_a^b (f - g)$. Let $L(x) = (f - g)(x)$, then $\int_a^b (f - g) = \int_a^b L$ and from the previous theorem that if f is integrable on $[a, b]$ and $f(x) \geq 0$ for all x in $[a, b]$, then $\int_a^b f \geq 0$. We know that $\int_a^b L \geq 0$. Thus $\int_a^b f - \int_a^b g = \int_a^b (f - g) = \int_a^b L \geq 0$. Therefore, $\int_a^b f \geq \int_a^b g$.

Chpater 13. #20

Suppose that f is nondecreasing on $[a, b]$. Notice that f is automatically bounded on $[a, b]$, because $f(a) \geq f(x) \geq f(b), \forall x \in [a, b]$.

(a) If $P = \{t_0, t_1, ..., t_n\}$ is a partition of $[a, b]$, then what is $L(f, P)$ and $U(f, P)$

Answer:

By definition of $L(f, P)$ and $U(f, P)$, we have the following:

$$L(f, P_n) = \sum_{i=1}^{n} f(t_{i-1})(t_i - t_{i-1})$$
$$U(f, P_n) = \sum_{i=1}^{n} f(t_i)(t_i - t_{i-1})$$

(b) Suppose that $t_i - t_i = \delta$ for each i. Prove that $U(f, P_n) - L(f, P_n) = \delta \cdot (f(b) - f(a))$.

Proof:

Suppose $t_i - t_i = \delta$ for each i. Therefore, we know that

$$\begin{aligned}
U(f, P_n) - L(f, P_n) &= \sum_{i=1}^{n} f(t_i)(t_i - t_{i-1}) - \sum_{i=1}^{n} f(t_{i-1})(t_i - t_{i-1}) \\
&= \sum_{i=1}^{n} (f(t_i) \cdot \delta) - \sum_{i=1}^{n} (f(t_{i-1}) \cdot \delta) \\
&= \delta \cdot (\sum_{i=1}^{n} (f(t_i) - \sum_{i=1}^{n} (f(t_{i-1}))) \\
&= \delta \cdot (\sum_{i=1}^{n} (f(t_i) - f(t_{i-1})) \\
&= \delta \cdot (f(b) - f(a))
\end{aligned}$$

Based on the idea that this equations is a nondecreasing equation, then we know that $f(a) \geq f(x) \geq f(b), \forall x \in [a, b]$. Meaning, the largest possible value of $f(t_i) \geq f(b)$ for each i.

(d) Give an example of a nondecreasing function on $[0, 1]$ which is discontinuous at infinitely many points.

Example:

$$y = \begin{cases} 0 & x = 0 \\ \frac{1}{\lfloor \frac{1}{x} \rfloor} & 0 < x < 1 \\ 1 & x = 1 \end{cases}$$