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Problem Set 2
Sep 17^{th} 2019

Problem Set 2

Chapter 2
1.(i) $1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.
Proof:
Let $n = 1$, then we have on the left hand side:

$$1^2 = 1$$

Then, on the right hand side:

$$\begin{aligned}\frac{1 \times (1+1) \times (2 \cdot 1 + 1)}{6} &= \frac{1 \times 2 \times 3}{6} \\ &= \frac{6}{6} \\ &= 1\end{aligned}$$

Therefore, left hand side equals to right hand side. This claim holds for 1.
Then, assume if $n = k$, and $1^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$.
Let $n = k+1$, on the left hand side, we would have:

$$\begin{aligned}1^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\ &= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)\cdot(k+1)}{6} \\ &= \frac{((k+1)(2k^2+k) + (6k+6)(k+1))}{6} \\ &= \frac{(k+1)(2k^2+k+6k+6)}{6} \\ &= \frac{(k+1)(2k^2+7k+6)}{6} \\ &= \frac{(k+1)(2k+3)(k+2)}{6}\end{aligned}$$

$$
\begin{aligned}
&= \frac{(k+1)(2(k+1)+1)((k+1)+1)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}
\end{aligned}
$$

And on the right hand side, we would have:

$$\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$\therefore \text{Left hand side equals to right hand side}$$

The claim holds.

1.(ii) $1^3 + \cdots + n^3 = (1 + \cdots + n)^2$.

Proof:

Let $n = 1$, then we would have on the left hand side:

$$1^3 = 1$$

On the right hand side:

$$1^2 = 1$$

Therefore, left hand side equals to right hand side this claim holds for 1.

Then, assume if $n = k$, and $1^3 + \cdots + k^3 = (1 + \cdots + k)^2$.

Let $n = k + 1$, on the left hand side, we would have:

$$
\begin{aligned}
1^3 + \cdots + k^3 + (k+1)^3 &= (1 + \cdots + k)^2 + (k+1)^3 \\
&= (\frac{k(k+1)}{2})^2 + (k+1)^3 \\
&= (\frac{k^2(k+1)^2}{4}) + (k+1) \cdot (k+1)^2 \\
&= \frac{k^2}{4} \cdot (k+1)^2 + (k+1) \cdot (k+1)^2 \\
&= (k+1)^2 \cdot (\frac{k^2}{4} + (k+1)) \\
&= (k+1)^2 \cdot (\frac{k^2}{4} + \frac{4(k+1)}{4}) \\
&= (k+1)^2 \cdot (\frac{k^2}{4} + \frac{4k+4}{4})
\end{aligned}
$$

$$
\begin{aligned}
&= (k+1)^2 \cdot (\frac{k^2+4k+4}{4} \\
&= (k+1)^2 \cdot (\frac{(k+2)^2}{4} \\
&= (k+1)^2 \cdot (\frac{(k+2)}{2})^2 \\
&= (k+1)^2 \cdot (\frac{((k+1)+1)}{2})^2 \\
&= (\frac{(k+1)((k+1)+1)}{2})^2 \\
&= (1+\cdots+(k+1))^2
\end{aligned}
$$

And on the right hand side, we would have:

$$(1+\cdots+(k+1))^2$$

$$\therefore \text{Left hand side equals to right hand side}$$

The claim holds.

2.(i)

$$\sum_{i=1}^{n}(2i-1) = n^2$$

2.(ii)

$$
\begin{aligned}
\sum_{i=1}^{n}(2i-1)^2 &= \sum_{i=1}^{2n}(i)^2 - \sum_{i=1}^{n}(2i)^2 \\
&= \sum_{i=1}^{2n}(i)^2 - 4\sum_{i=1}^{n}(i)^2 \\
&= \frac{(2n)((2n)+1)(2(2n)+1)}{6} \\
&= \frac{(2n)(2n+1)(4n+1) - 4n(n+1)(2n+1)}{6} \\
&= \frac{(2n)(2n+1)(4n+1-2n-2)}{6} \\
&= \frac{(2n)(2n+1)(2n-1)}{6}
\end{aligned}
$$

3.(i) On the right hand side we have:

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-(k-1))!} + \frac{n!}{k!(n-k)!} \\
&= \frac{k \cdot n! + (n-k+1) \cdot n!}{k!(n-k+1))!} \\
&= \frac{(k+n-k+1)n!}{k!((n+1)-k))!} \\
&= \frac{(n+1)!}{k!((n+1)-k))!} \\
&= \binom{n+1}{k}
\end{aligned}$$

$\therefore$ Left hand side is the same as the right hand side.