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Problem Set 3

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### Problem Set 3

#### Question

Prove that if  $\lim_{x \rightarrow a} f(x) = L$  and  $\lim_{x \rightarrow a} g(x) = M$ , then  $\lim_{x \rightarrow a} f(x) \cdot g(x) = L \cdot M$ .

#### Proof:

We know  $\forall x$  such that  $0 < |x - a| < \delta_1, |f(x) - L| < \epsilon_1$ , and such that  $0 < |x - a| < \delta_2, |g(x) - M| < \epsilon_2$ . In order for these two inequalities to hold at the same time, we need to have  $0 < |x - a| < \delta = \min(\delta_1, \delta_2)$ . Then, we would have  $|g(x) - M| \cdot |f(x) - L| < \epsilon_1 \cdot \epsilon_2$ , based on the theorem that if  $0 < a, b$  and  $a < c, b < d$  then  $ab < cd$ . Also, from the theorem that  $|a| \cdot |b| = |ab|$ . Then we would have the following:

$$|g(x) - M| \cdot |f(x) - L| < \epsilon_1 \epsilon_2$$

$$|f(x) \cdot g(x) - M \cdot f(x) - L \cdot g(x) + LM| < \epsilon_1 \epsilon_2$$

By another theorem that  $|a| + |b| < |a + b|$ , we could on both sides of the inequality add  $|M \cdot f(x) + L \cdot g(x) - 2LM|$ . Also, we know that  $0 < \epsilon_1, \epsilon_2$ . Thus,  $\epsilon_1 \epsilon_2 = |\epsilon_1 \epsilon_2|$ . Then, we would have for the inequality:

$$|f(x) \cdot g(x) - M \cdot f(x) - L \cdot g(x) + LM| + |M \cdot f(x) + L \cdot g(x) - 2LM| \quad (1)$$

$$|f(x) \cdot g(x) - M \cdot f(x) - L \cdot g(x) + LM + M \cdot f(x) + L \cdot g(x) - 2LM| \quad (2)$$

$$|\epsilon_1 \epsilon_2| + |M \cdot f(x) + L \cdot g(x) - 2LM| \quad (3)$$

Where  $(2) < (1)$  and  $(1) < (3)$ . Thus,  $(2) < (3)$ .

Then we have:

$$(2) < |\epsilon_1 \epsilon_2| + |M \cdot f(x) + L \cdot g(x) - LM - LM|$$

$$(2) < |\epsilon_1 \epsilon_2| + |M \cdot f(x) - LM + L \cdot g(x) - LM|$$

$$(2) < |\epsilon_1 \epsilon_2| + |M \cdot (f(x) - L) + L \cdot (g(x) - M)|$$

$$(2) < |\epsilon_1 \epsilon_2| + |M(f(x) - L) + L(g(x) - M)|$$

Also, we know from the definition of a limit that  $|f(x) - L| < \epsilon_1, |g(x) - M| < \epsilon_2$ . We

would therefore have:

$$\begin{aligned} |M| \cdot |f(x) - L| &< |M| \cdot \epsilon_1 \\ |L| \cdot |g(x) - M| &< |L| \cdot \epsilon_2 \\ \therefore |M \cdot (f(x) - L)| &< |M \cdot \epsilon_1|, |L \cdot (g(x) - M)| < |L \cdot \epsilon_2| \end{aligned}$$

Thus, we would have a new inequality:

$$|\epsilon_1 \epsilon_2| + |M(f(x) - L)| + |L(g(x) - M)| < |\epsilon_1 \epsilon_2| + |M \cdot \epsilon_1 + L \cdot \epsilon_2|$$

From there, we could say that  $|f(x) \cdot g(x) - LM| < |\epsilon_1 \epsilon_2| + |M \cdot \epsilon_1 + L \cdot \epsilon_2| \leq \epsilon$ . Assume  $\epsilon_1 \epsilon_2 \leq \frac{\epsilon}{2}$ . From there we know,  $\epsilon_1, \epsilon_2 \leq \sqrt{\frac{\epsilon}{2}}$ . On the other hand, from  $|M \cdot \epsilon_1 + L \cdot \epsilon_2| \leq \frac{\epsilon}{2}$ . We know that  $|M \cdot \epsilon_1 + L \cdot \epsilon_2| \leq |M \cdot \epsilon_1| + |L \cdot \epsilon_2|$ . In order for the previous inequality to hold, we assign  $|M \cdot \epsilon_1| < \epsilon/4$  and  $|L \cdot \epsilon_2| < \epsilon/4$ . Therefore,  $\epsilon_2 < \frac{\epsilon}{4 \cdot |M|}$  and  $\epsilon_1 < \frac{\epsilon}{4 \cdot |L|}$ , where  $L, M \neq 0$ . If we have  $\epsilon_1 < \min(\frac{\epsilon}{4 \cdot |L|}, \sqrt{\frac{\epsilon}{2}})$  and  $\epsilon_2 < \min(\frac{\epsilon}{4 \cdot |M|}, \sqrt{\frac{\epsilon}{2}})$ . Also, if we have  $L$  or  $M$  equal to 0. We can just say  $\epsilon_1 < \sqrt{\frac{\epsilon}{2}}$  and  $\epsilon_2 < \sqrt{\frac{\epsilon}{2}}$ . Then, we have  $\epsilon$  to be a very small number. We then have  $\lim_{x \rightarrow a} f(x) \cdot g(x) = L \cdot M$ .

### Question

Suppose that  $\lim_{x \rightarrow a} f(x)$  exists, and that  $\lim_{x \rightarrow a} f(x) = L$ . Suppose  $M$  is any number. Then prove that  $\lim_{x \rightarrow a} (Mf(x))$  exists, and  $\lim_{x \rightarrow a} (Mf(x)) = M \lim_{x \rightarrow a} f(x)$ .

**Proof:**

Suppose  $\epsilon_1 = \frac{\epsilon}{|M|}$  where  $|M| \neq 0$  and  $\forall x$  such that  $0 < |x - a| < \delta, |f(x) - L| < \epsilon$ . From the theorem that  $|a| \cdot |b| = |ab|$ . Thus,  $|M| \cdot |f(x) - L| = |M(f(x) - L)|$ . Then, we know that

$$M \cdot |f(x) - L| = |M(f(x) - L)| < |M| \cdot \epsilon$$

$$|M \cdot f(x) - LM| < |M \cdot \epsilon| = \epsilon$$

Then we could have  $|M \cdot f(x) - LM|$  be a small number and therefore,  $\lim_{x \rightarrow a} (M \cdot f(x)) = L \cdot M$ . Also, because  $\lim_{x \rightarrow a} f(x)$  is a number, then we know that  $M \cdot \lim_{x \rightarrow a} f(x) = M \cdot L$ . Therefore,  $M \cdot \lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} (M \cdot f(x))$ .

If we have  $M = 0$ , then we know that on the left hand side, we are finding the limit of 0, which is 0. And on the right hand side, we have  $0 \cdot L = 0$ . Therefore, the theorem still holds.

### Question

Show that a function cannot have two different limits at  $a$ . That is, if  $\lim_{x \rightarrow a} f(x)$  exists, and  $\lim_{x \rightarrow a} f(x) = L$ , and  $\lim_{x \rightarrow a} f(x) = M$ , then we must have  $L = M$ .

**Proof:**

Suppose we have  $L < M$  and we assign  $0 < \epsilon \leq \frac{M-L}{2}$  such that  $2\epsilon \leq M-L$  and  $L+\epsilon \leq M-\epsilon$  therefore,  $(M-\epsilon, M+\epsilon) \cap (L-\epsilon, L+\epsilon) = \emptyset$ . Also,  $\forall x$  such that  $0 < |x-a| < \delta$ ,  $|f(x)-L| < \epsilon$  and  $|f(x)-M| < \epsilon$ . From that we have  $L-\epsilon < f(x) < L+\epsilon$  and  $M-\epsilon < f(x) < M+\epsilon$ . Then, we know that we need to have two different  $f(x)$  in order to have  $f(x)$  to be in two different ranges. This means  $f(x)$  is not a one-to-one relationship, thus not a function. Therefore, for  $f(x)$  there should be only one possible function. Then, if  $\lim_{x \rightarrow a} f(x)$  exists, and  $\lim_{x \rightarrow a} f(x) = L$ , and  $\lim_{x \rightarrow a} f(x) = M$ , then we must have  $L = M$ .

## Chapter 5. #8

### (i) Counter Example

For instance, if we let  $f(x) = \frac{1}{x^2}$  and  $g(x) = -\frac{1}{x^2}$ , then we have  $f(x) + g(x) = \frac{1}{x^2} + (-\frac{1}{x^2}) = 0$ . And if  $x \rightarrow 0$ , then both  $\lim_{x \rightarrow 0} f(x)$  and  $\lim_{x \rightarrow 0} g(x) = L$  do not exist, but  $\lim_{x \rightarrow 0} (f(x) + g(x)) = \lim_{x \rightarrow 0} 0 = 0$  which do exist. Therefore, a counter example.

If we were to generalize this situation, for any polynomial  $f(x)$  that does not contain a constant, let  $g(x) = \frac{1}{f(x)}$  and  $h(x) = -\frac{1}{f(x)}$ . As  $x \rightarrow 0$ ,  $\lim_{x \rightarrow 0} f(x)$  and  $\lim_{x \rightarrow 0} g(x) = L$  do not exist, but  $\lim_{x \rightarrow 0} (f(x) + g(x)) = \lim_{x \rightarrow 0} 0 = 0$  which do exist.

### (ii) Proof

Assume,  $\lim_{x \rightarrow 0} g(x)$  does not exist. From the theorem I, we have proved that  $\lim_{x \rightarrow 0} (f(x) + g(x)) = \lim_{x \rightarrow 0} f(x) + \lim_{x \rightarrow 0} g(x)$ . It is because  $\lim_{x \rightarrow 0} (f(x) + g(x))$  exist, and  $\lim_{x \rightarrow 0} f(x)$  exist. By definition of a real number such that the difference between any two real number is a real number.

Thus,  $\lim_{x \rightarrow 0} (f(x) + g(x)) - \lim_{x \rightarrow 0} f(x)$  exist, which contradicts with our assumption. Therefore,  $\lim_{x \rightarrow 0} g(x)$  does not exist.

### (iii) Proof

Let  $\lim_{x \rightarrow 0} g(x)$  does not exist, and we have  $\lim_{x \rightarrow 0} f(x) = L$ . We have  $\lim_{x \rightarrow 0} (f(x) + g(x)) = \lim_{x \rightarrow 0} f(x) + \lim_{x \rightarrow 0} g(x) = Undefined + L$ . It is because when we add an undefined number to a real number, it is not defined. Thus  $\lim_{x \rightarrow 0} (f(x) + g(x))$  does not exist.

### (iv) Counter Example

Let  $f(x) = x$  and  $g(x) = \sqrt{x}$ , such that for  $x \rightarrow -1$ ,  $f(x)$  has a limit and  $g(x)$  does not exist. However, for  $f(x) \cdot g(x) = x \cdot \sqrt{x} = 1$ ,  $\lim_{x \rightarrow -1} (f(x) \cdot g(x)) = \lim_{x \rightarrow -1} 1 = 1$ , which means the limit of  $f(x)g(x)$  exist.