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Problem Set 3
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Question
Prove that if $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a} f(x)\cdot g(x) = L\cdot M$.
Proof:
We know $\forall x$ such that $0 < |x-a| < \delta_1, |f(x)-L| < \epsilon_1$, and such that $0 < |x-a| < \delta_2, |g(x)-M| < \epsilon_2$. In order for these two ineqaulities to hold at the same time, we need to have $0 < |x-a| < \delta = \text{Min}(\delta_1, \delta_2)$. Then, we would have $|g(x)-M|\cdot|f(x)-L| < \epsilon_1\cdot\epsilon_2$, based on the theorem that if $0 < a, b$ and $a < c, b < d$ then $ab < cd$. Also, from the theorem that $|a|\cdot|b| = |ab|$. Then we would have the following:

$$|g(x)-M|\cdot|f(x)-L| < \epsilon_1\epsilon_2$$

$$|f(x)\cdot g(x) - M\cdot f(x) - L\cdot g(x) + LM| < \epsilon_1\epsilon_2$$

By another theorem that $|a|+|b| < |a+b|$, we could on both sides of the inequality add $|M\cdot f(x) + L\cdot g(x) - 2LM|$. Also, we know that $0 < \epsilon_1, \epsilon_2$. Thus, $\epsilon_1\epsilon_2 = |\epsilon_1\epsilon_2|$. Then, we would have for the inequality:

$$|f(x)\cdot g(x) - M\cdot f(x) - L\cdot g(x) + LM| + |M\cdot f(x) + L\cdot g(x) - 2LM| \tag{1}$$

$$|f(x)\cdot g(x) - M\cdot f(x) - L\cdot g(x) + LM + M\cdot f(x) + L\cdot g(x) - 2LM| \tag{2}$$

$$|\epsilon_1\epsilon_2| + |M\cdot f(x) + L\cdot g(x) - 2LM| \tag{3}$$

Where (2) < (1) and (1) < (3). Thus, (2) < (3).
Then we have:

$$(2) < |\epsilon_1\epsilon_2| + |M\cdot f(x) + L\cdot g(x) - LM - LM|$$

$$(2) < |\epsilon_1\epsilon_2| + |M\cdot f(x) - LM + L\cdot g(x) - LM|$$

$$(2) < |\epsilon_1\epsilon_2| + |M\cdot(f(x)-L) + L\cdot(g(x)-M)|$$

$$(2) < |\epsilon_1\epsilon_2| + |M(f(x)-L) + L(g(x)-M)|$$

Also, we know from the definition of a limit that $|f(x)-L| < \epsilon_1$, $|g(x)-M| < \epsilon_2$. We would therefore have:

$$|M| \cdot |f(x) - L| < |M| \cdot \epsilon_1$$
$$|L| \cdot |g(x) - M| < |L| \cdot \epsilon_2$$
$$\therefore |M \cdot (f(x) - L)| < |M \cdot \epsilon_1|, |L \cdot (g(x) - M)| < |L \cdot \epsilon_2|$$

Thus, we would have a new inequality:

$$|\epsilon_1\epsilon_2| + |M(f(x) - L)| + |L(g(x) - M)| < |\epsilon_1\epsilon_2| + |M \cdot \epsilon_1 + L \cdot \epsilon_2|$$

From there, we could say that $|f(x) \cdot g(x) - LM| < |\epsilon_1\epsilon_2| + |M \cdot \epsilon_1 + L \cdot \epsilon_2| \leq \epsilon$
Assume $\epsilon_1\epsilon_2 \leq \epsilon/2$. From there we know, $\epsilon_1, \epsilon_2 \leq \sqrt{\frac{\epsilon}{2}}$. On the other hand from $|M \cdot \epsilon_1 + L \cdot \epsilon_2| \leq \epsilon/2$. We know that $|M \cdot \epsilon_1 + L \cdot \epsilon_2| \leq |M \cdot \epsilon_1| + |L \cdot \epsilon_2|$. In order for the previous inequality to hold, we assign $|M \cdot \epsilon_1| < \epsilon/4$ and $|L \cdot \epsilon_2| < \epsilon/4$. Therefore, $\epsilon_2 < \frac{\epsilon}{4 \cdot |M|}$ and $\epsilon_1 < \frac{\epsilon}{4 \cdot |L|}$. If we have $\epsilon_1 < \text{Min}(\frac{\epsilon}{4 \cdot |L|}, \sqrt{\frac{\epsilon}{2}})$ and $\epsilon_1 < \text{Min}(\frac{\epsilon}{4 \cdot |M|}, \sqrt{\frac{\epsilon}{2}})$. Then, we have ϵ to be a very small number. We then have $\lim_{x\to a} f(x) \cdot g(x) = L \cdot M$.

Question
Suppose that $\lim\limits_{x\to a} f(x)$ exists, and that $\lim\limits_{x\to a} f(x) = L$. Suppose M is any number. Then prove that $\lim\limits_{x\to a}(Mf(x))$ exists, and $\lim\limits_{x\to a}(Mf(x)) = M\lim\limits_{x\to a} f(x)$.
Proof:
$\forall x$ such that $0 < |x - a| < \delta, |f(x) - L| < \epsilon$. From the theorem that $|a| \cdot |b| = |ab|$. Thus, $|M| \cdot |f(x) - L| = |M(f(x) - L)|$. Then, we know that

$$M \cdot |f(x) - L| = |M(f(x) - L)| < |M| \cdot \epsilon$$

$$|M \cdot f(x) - LM| < |M \cdot \epsilon| = \epsilon_{final}$$

Let $\epsilon = \frac{\epsilon_{final}}{|M|}$, then we could have $|M \cdot f(x) - LM|$ be a small number and therefore, $\lim_{x\to a}(M \cdot f(x)) = L \cdot M$. Also, because $\lim_{x\to a} f(x)$ is a number, then we know that $M \cdot \lim_{x\to a} f(x) = M \cdot L$. Therefore, $M \cdot \lim_{x\to a} f(x) = \lim_{x\to a}(M \cdot f(x))$.

Question
Show that a function cannot have two different limits at a. That is, if $\lim\limits_{x\to a} f(x)$ exists, and $\lim\limits_{x\to a} f(x) = L$, and $\lim\limits_{x\to a} f(x) = M$, then we must have $L = M$.
Proof:
$\forall x$ such that $0 < |x - a| < \delta, |f(x) - L| < \epsilon_1$ and $|f(x) - M| < \epsilon_2$ where $L \neq M$. Therefore, $\epsilon_1 \neq \epsilon_2$. If we have $\epsilon_1 < \epsilon_2$, then we know that from the definition of a limit

$|f(x)-L| < |f(x)-M|$. It is because $|f(x)-L|$ is smaller, then $f(x)$ is closer to L instead of M. Thus, $\lim_{x\to a} f(x) = L$. Vice versa, if we have $\epsilon_1 > \epsilon_2$, then $|f(x)-L| > |f(x)-M|$ and $\lim_{x\to a} f(x) = M$. In conclusion if $\lim_{x\to a} f(x)$ exists, and $\lim_{x\to a} f(x) = L$, and $\lim_{x\to a} f(x) = M$, then we must have $L = M$.