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Problem Set 8
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Problem Set 8

Question 1
Suppose that $f : [a,b] \to \mathbb{R}$, is integrable, and suppose that $m = inf\{f(x) : x \in [a,b]\}$ and $M = sup\{f(x) : x \in [a,b]\}$. Then, we have $m(b-a) \leq \int_a^b f \leq M(b-a)$.
Proof:
Lemma: Suppose $f : [a,b] \to \mathbb{R}$ is integrable and suppose that P is any partition $[a,b]$. Then we have $L(f,P) \leq \int_a^b f \leq U(f,P)$.
Proof of Lemma:
We will prove this lemma from two parts:
First, we prove that $L(f,P) \leq \int_a^b f$.
Since f is integrable, then we know $sup\{L(f,P)|P, \text{a partition in range } [a,b]\} = \int_a^b f$. And let $A = \{L(f,P)|P, \text{a partition in range } [a,b]\}$. Then $supA$ is the least upper bound of A. Let $y = supA$, then $\forall x \in A, x \leq y$. Then, because we have $y = \int_a^b f$, $\forall x \in A, x \leq y = \int_a^b f$. Thus, $L(f,P) \leq \int_a^b f$.
Then, we prove that $\int_a^b f \leq U(f,P)$.
Since f is integrable, we know $inf\{U(f,P)|P, \text{a partition in range } [a,b]\} = \int_a^b f$. And let $B = \{U(f,P)|P, \text{a partition in range } [a,b]\}$. Let $z = infB$, then $\forall x \in B, z \leq x$. Therefore, since we have $U(f,P) \geq \int_a^b f$.
Thus, $L(f,P) \leq \int_a^b f \leq U(f,P)$.
From the definition, $L(f,P) = \sum_{i=1}^n m_i(t_i - t_{i-1})$ and $U(f,P) = \sum_{i=1}^n M_i(t_i - t_{i-1})$. In this case, we have the partition P as $[a,b]$ meaning there's only one pair of m and M, and $t_i = b$, $t_{i-1} = a$. Therefore, we have $L(f,P) = m(b-a)$ and $U(f,P) = M(b-a)$. Thus, from the lemma we know that $m(b-a) \leq \int_a^b f \leq M(b-a)$.

Question 2
Prove that the function $f : [-1,1] \to \mathbb{R}$, defined by

$$f(n) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases},$$

is integrable on $[-1,1]$.
Proof:

At first we choose the partition $P = \{-1, 0, 1\}$. Then, by definition, $L(f, P) = \sum_{i=1}^{n} m_i(t_i - t_{i-1})$ and $U(f, P) = \sum_{i=1}^{n} M_i(t_i - t_{i-1})$. Therefore, we have,

$$\begin{aligned} L(f, P) &= \sum_{i=1}^{2} m_i(t_i - t_{i-1}) \\ &= m_1 \cdot (t_1 - t_0) + m_2 \cdot (t_2 - t_1) \\ &= 0 \cdot (-0 + 1) + 1 \cdot (1 - 0) \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} U(f, P) &= \sum_{i=1}^{2} M_i(t_i - t_{i-1}) \\ &= M_1 \cdot (t_1 - t_0) + M_2 \cdot (t_2 - t_1) \\ &= 1 \cdot (-0 + 1) + 1 \cdot (1 - 0) \\ &= 2 \end{aligned}$$

Thus, we have $1 \leq \int_a^b f \leq 2$.
Then, suppose f is not integrable on $[-1, 1]$. Therefore, $\forall \epsilon \in \mathbb{R}, \epsilon > 0$ such that if $P = \{-1, -\epsilon, 1\}$, then, $L(f, P) < U(f, P)$. Therefore, we have

$$\begin{aligned} U(f, P) &= \sum_{i=1}^{2} M_i(t_i - t_{i-1}) \\ &= M_1 \cdot (t_1 - t_0) + M_2 \cdot (t_2 - t_1) \\ &= 0 \cdot (-\epsilon + 1) + 1 \cdot (1 + \epsilon) \\ &= 1 + \epsilon \end{aligned}$$

Then, $1 + \epsilon \geq \int_a^b f$. Also we have $1 \leq \int_a^b f$. Therefore, $0 \leq \int_a^b f - 1 \leq \epsilon$. Since f is not integrable, then we have $0 < \int_a^b f - 1$. Let $\delta = \int_a^b f - 1, \delta > 0$. Since $\epsilon > 0$, therefore, $\exists \epsilon, \delta > \epsilon > 0$. Thus, $\exists \epsilon, \int_a^b f - 1 > \epsilon > 0$. Then, we have $\int_a^b f > 1 + \epsilon$, which is a contradiction. Thus, f is integrable on $[-1, 1]$.

Question 3

Suppose that $f : [a, b] \to \mathbb{R}$, is bounded. Then f is integrable on $[a, b]$ if, and only if, for every $\epsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U(f, P) - L(f, P) < \epsilon.$$

Proof:

Suppose $\forall \epsilon > 0 U(f,P) - L(f,P) < \epsilon$, then we know by definition of $sup\{L(f,P)\}$ and $inf\{U(f,P)\}$ that $sup\{L(f,P\} \geq L(f,P)$, $inf\{U(f,P)\} \leq U(f,P)$ and $inf\{U(f,P)\} \geq sup\{L(f,P\}$. Thus, $0 \leq inf\{U(f,P)\} - sup\{L(f,P)\} \leq U(f,P) - L(f,P) < \epsilon$. Let $\delta = inf\{U(f,P)\} - sup\{L(f,P)\}$. Suppose $\delta > 0$. Since $\forall \epsilon > 0 \delta < \epsilon$. But we have $a = \delta - 0 = \delta$ such that if $\epsilon < a = \delta$, then, we have $0 < \epsilon < \delta$ and we reach a contradiction. Therefore, $\delta = 0$. Thus, $inf\{U(f,P)\} = sup\{L(f,P)\}$. Then, f is integrable on $[a,b]$, based on the definition.

Conversely, suppose f is integrable on $[a,b]$. Therefore, we know that $inf\{U(f,P)\} = sup\{L(f,P)\}$. Also, we know that $sup\{L(f,P)\} \geq L(f,P)$ and $inf\{U(f,P)\} \leq U(f,P)$. Therefore, $0 = inf\{U(f,P)\} - sup\{L(f,P)\} \leq U(f,P) - L(f,P)$. Suppose $\exists \epsilon > 0$ such that $\forall P, U(f,P) - L(f,P) > \epsilon$. Therefore, we would have $inf\{U(f,P)\} - sup\{L(f,P)\} = 0 > \epsilon$. Then, we have a contradiction. Therefore, we know that if f is integrable on $[a,b]$, then $\forall \epsilon > 0$ there exists a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \epsilon$.

Question 4

Use the theorem you proved in question #3 to solve question #2 again in a slightly different way. (It should be easier this way, but it is worth doing it both ways.)

Proof:

Suppose $P = \{t_0, t_1, ..., t_n\}$ is a partition of $[-1,1]$ with $t_j = 0, j \in \mathbb{Z}, j \in [0,n]$. Then, we have when $i < j$, $m_i = M_i = 0$ and if $i > j$, we have $m_i = M_i = 1$, $m_j = 0$ and $M_j = 1$. Since we have:

$$L(f,P_n) = \sum_{i=1}^{j-1} m_i(t_i - t_{i-1}) + m_j(t_j - t_{j-1}) + \sum_{i=j+1}^{n} m_i(t_i - t_{i-1})$$
$$U(f,P_n) = \sum_{i=1}^{j-1} M_i(t_i - t_{i-1}) + M_j(t_j - t_{j-1}) + \sum_{i=j+1}^{n} M_i(t_i - t_{i-1})$$

Then, $U(f,P) - L(f,P) = t_j - t_{j-1}$. Suppose $\epsilon > 0$ and assume P with an interval such that $t_i - t_{i-1} < \epsilon$, for each i. Thus, we have $t_j - t_{j-1} < \epsilon$ and therefore, f is integrable.

Chpater 13. #1

Prove that $\int_0^b x^3 dx = \frac{b^4}{4}$, by considering partitions into n equal intervals.

Proof:

Since we are going to have a partition with n intervals, then we would have $P = \{t_0, t_1, ..., t_n\}$

with $t_0 = 0, t_i = i \cdot \frac{b}{n}$. Then, we have

$$\begin{aligned}
L(f, P_n) &= \sum_{i=1}^{n} t_{i-1}^3 (t_i - t_{i-1}) \\
&= \sum_{i=1}^{n} (\frac{(i-1) \cdot b}{n})^3 \cdot \frac{b}{n} \\
&= (\frac{b}{n})^4 \cdot \sum_{i=1}^{n} (i-1)^3 \\
&= (\frac{b}{n})^4 \cdot \sum_{j=0}^{n-1} j^3
\end{aligned}$$

$$\begin{aligned}
U(f, P_n) &= \sum_{i=1}^{n} t_i^3 (t_i - t_{i-1}) \\
&= \sum_{i=1}^{n} (\frac{i \cdot b}{n})^3 \cdot \frac{b}{n} \\
&= (\frac{b}{n})^4 \cdot \sum_{i=1}^{n} i^3
\end{aligned}$$

From the previous question Chapter 2 #6, we know that $\sum_{i=1}^{n} i^3 = \frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{2}$, and the equation could be written:

$$\begin{aligned}
L(f, P_n) &= (\frac{b}{n})^4 \cdot (\frac{(n-1)^4}{4} + \frac{(n-1)^3}{2} + \frac{(n-1)^2}{4}) \\
&= (\frac{b}{n})^4 \cdot \frac{1}{4}((n-1)^4 + 2(n-1)^3 + (n-1)^2) \\
&= \frac{b^4}{4} \cdot (\frac{(n-1)^4}{n^4} + \frac{2(n-1)^3}{n^4} + \frac{(n-1)^2}{n^4}) \\
U(f, P_n) &= (\frac{b}{n})^4 \cdot (\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4}) \\
&= (\frac{b}{n})^4 \cdot \frac{1}{4}(n^4 + 2n^3 + n^2) \\
&= \frac{b^4}{4} \cdot (1 + \frac{2}{n} + \frac{1}{n^2})
\end{aligned}$$

Since $n \geq 1, n \in \mathbb{N}$, therefore, we know when n gets very large both $U(f, P_n)$ and $L(f, P_n)$ are close to $\frac{b^4}{4}$. At the same time, we find that

$$U(f, P_n) - L(f, P_n) = \frac{b^4}{4}(\frac{2n^3 - 1}{n^4})$$

which is a positive number. And we can make this difference as small as possible, by theorem 2, this function is integrable. Therefore, we have $U(f,P_n) \geq \frac{b^4}{4} \geq L(f,P_n)$. Thus, $\int_0^b x^3 dx = \frac{b^4}{4}$.

Chpater 13. #13
(a) Prove that if f is integrable on $[a,b]$ and $f(x) \geq 0$ for all x in $[a,b]$, then $\int_a^b f \geq 0$.
Proof:
Since f is integrable on $[a,b]$, then we have $U(f,P_n) \geq \int_a^b f \geq L(f,P_n)$. Also, based on the definition, we have:

$$L(f,P_n) = \sum_{i=1}^{n} f(t_{i-1})(t_i - t_{i-1})$$
$$U(f,P_n) = \sum_{i=1}^{n} f(t_i)(t_i - t_{i-1})$$

Also, because $f(x) \geq 0, \forall x \in [a,b]$ and $t_{i-1} \in [a,b], \forall i \in \mathbb{N}, i \leq n$, then $f(t_{i-1}) \geq 0$. Also, because the definition of t_i guarantees, $t_i > t_{i-1}$, then $t_i - t_{i-1} > 0$ Therefore, let $q_i = f(t_{i-1})(t_i - t_{i-1})$, since both $f(t_{i-1})$ and$(t_i - t_{i-1})$ are greater than or equal to 0. We have $q_i \geq 0$. Similarly, because $t_i \in [a,b], \forall i \in \mathbb{N}, i \leq n$, then $f(t_i) \geq 0$. Let $p_i = f(t_i)(t_i - t_{i-1})$. Because both $f(t_{i-1})$ and$(t_i - t_{i-1})$ are both greater than or equal to 0. Then, $p_i \geq 0$. Thus, we have $U(f,P_n) \geq \int_a^b f \geq L(f,P_n) \geq 0$. Thus, $\int_a^b f \geq 0$.

(b) Prove that if f and g are both integrable on $[a,b]$ and $f(x) \geq g(x), \forall x \in [a,b]$, then $\int_a^b f \geq \int_a^b g$.
Proof:
Suppose f and g are both integrable on $[a,b]$ and $f(x) \geq g(x), \forall x \in [a,b]$. Then, we know that $f(x) - g(x) \geq 0$. Thus, $(f-g)(x) \geq 0, \forall x \in [a,b]$. Furthermore, from theorem 5, we know that for any two functions that are integrable at the same range, we have $\int_a^b f + \int_a^b g = \int_a^b (f+g)$. Furthermore, from theorem 6 we know that $\int_a^b cg = c \cdot \int_a^b g$. Thus, $\int_a^b -g = -\int_a^b g$. Then, we have $\int_a^b f - \int_a^b g = \int_a^b (f-g)$. Let $L(x) = (f-g)(x)$, then $\int_a^b (f-g) = \int_a^b L$ and from the previous theorem that if f is integrable on $[a,b]$ and $f(x) \geq 0$ for all x in $[a,b]$, then $\int_a^b f \geq 0$. We know that $\int_a^b L \geq 0$. Thus $\int_a^b f - \int_a^b g = \int_a^b (f-g) = \int_a^b L \geq 0$. Therefore, $\int_a^b f \geq \int_a^b g$.

Chpater 13. #20
Suppose that f is nondecreasing on $[a,b]$. Notice that f is automatically bounded on $[a,b]$,

because $f(a) \geq f(x) \geq f(b), \forall x \in [a, b]$.

(a) If $P = \{t_0, t_1, ..., t_n\}$ is a partition of $[a, b]$, then what is $L(f, P)$ and $U(f, P)$

Answer:

By definition of $L(f, P)$ and $U(f, P)$, we have the following:

$$L(f, P_n) = \sum_{i=1}^{n} f(t_{i-1})(t_i - t_{i-1})$$
$$U(f, P_n) = \sum_{i=1}^{n} f(t_i)(t_i - t_{i-1})$$

(b) Suppose that $t_i - t_{i-1} = \delta$ for each i. Prove that $U(f, P_n) - L(f, P_n) = \delta \cdot (f(b) - f(a))$.

Proof:

Suppose $t_i - t_{i-1} = \delta$ for each i. Therefore, we know that

$$\begin{aligned}
U(f, P_n) - L(f, P_n) &= \sum_{i=1}^{n} f(t_i)(t_i - t_{i-1}) - \sum_{i=1}^{n} f(t_{i-1})(t_i - t_{i-1}) \\
&= \sum_{i=1}^{n}(f(t_i) \cdot \delta) - \sum_{i=1}^{n}(f(t_{i-1}) \cdot \delta) \\
&= \delta \cdot (\sum_{i=1}^{n}(f(t_i) - \sum_{i=1}^{n}(f(t_{i-1}))) \\
&= \delta \cdot (\sum_{i=1}^{n}(f(t_i) - f(t_{i-1})) \\
&= \delta \cdot ((f(t_1) - f(t_0)) + (f(t_2) - f(t_1)) + ... + (f(t_n) - f(t_{n-1}))) \\
&= \delta \cdot (f(t_n) - f(t_0)) \\
&= \delta \cdot (f(b) - f(a))
\end{aligned}$$

(c) Prove f is integrable.

Proof:

Since we have $U(f, P_n) - L(f, P_n) = \delta \cdot (f(b) - f(a))$ and δ is arbitrary and $f(b) - f(a)$ is given because we know f and both $f(a)$ and $f(b)$ exist. Therefore, we could have $\forall \epsilon > 0, \exists \delta < \frac{\epsilon}{f(b)-f(a)}$. Then, we have $\delta \cdot (f(b) - f(a)) < \epsilon$ and $U(f, P_n) - L(f, P_n) < \epsilon$.

(d) Give an example of a nondecreasing function on $[0, 1]$ which is discontinuous at infinitely many points.

Example:

$$y = \begin{cases} 0 & x = 0 \\ \frac{1}{\left\lfloor \frac{1}{x} \right\rfloor} & 0 < x < 1 \\ 1 & x = 1 \end{cases}$$

Chapter 13 #23

(a) Prove that if f is integrable on $[a, b]$ and $m \leq f(x) \leq M$ for all x in $[a, b]$, then $\int_a^b f(x)dx = (b - a)\mu$, for some number μ with $m \leq \mu \leq M$.

Proof:

From Theorem 7, we have if f is integrable on $[a, b]$ and $m \leq f(x) \leq M$, then $m(b - a) \leq \int_a^b f \leq M(b - a)$. Let $q = \int_a^b f$. Then, $m(b - a) \leq q \leq M(b - a)$.

Assume $b - a = 0$, then we have $0 \leq q \leq 0$, meaning $q = 0$. We also know that every number is a factor of 0. Therefore, the assumption holds that if f is integrable on $[a, b]$ and $m \leq f(x) \leq M$ for all x in $[a, b]$, then $\int_a^b f(x)dx = (b - a)\mu$, for some number μ with $m \leq \mu \leq M$.

Let $(b - a) > 0$, we have $m \leq \frac{q}{(b-a)} \leq M$. Thus, let $\mu = \frac{q}{(b-a)}$, then $m \leq \mu \leq M$ and $q = \mu \cdot (b - a) = \int_a^b f$. Thus, if f is integrable on $[a, b]$ and $m \leq f(x) \leq M$ for all x in $[a, b]$, then $\int_a^b f(x)dx = (b - a)\mu$, for some number μ with $m \leq \mu \leq M$.

(b) Prove that if f is continuous on $[a, b]$, then $\int_a^b f(x)dx = (b - a)f(\xi)$, for some number ξ in $[a, b]$. and show by an example that continuity is essential.

Proof:

From Chpater 7, Theorem 4, we know that if f is continuous on $[a, b]$, and $f(a) < c < f(b)$, then there is some x in $[a, b]$ such that $f(x) = c$. Therefore, from the previous problem we have shown that $\int_a^b f(x)dx = (b-a)\mu$, for some number μ with $m \leq \mu \leq M$, then we know that from this theorem, since f is continuous, we have there exists $\xi \in [a, b]$ such that $f(\xi) = \mu$. This continuous is essential because if we have a $g(x) = \begin{cases} f(x) & x \neq \xi \\ k, k \in \mathbb{R}, k \neq f(\xi) & x = \xi \end{cases}$, then this assumption does not hold since there is no $f(x) = \mu$.

(c) More generally suppose that f is continuous on $[a, b]$ and that g is integrable and nonnegative on $[a, b]$. Prove that $\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx$ for some number ξ in $[a, b]$. This is called the Mean Value Theorem in Integrals.

Proof:

From the assumption $m \leq f(x) \leq M$, we know $mg(x) \leq f(x)g(x) \leq Mg(x)$. Therefore, we have $\int_a^b mg(x)dx \leq \int_a^b f(x)g(x)dx \leq \int_a^b Mg(x)dx$. And from theorem 6, we have this inequality rewritten as $m\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq M\int_a^b g(x)dx$. Let $q = \int_a^b f(x)g(x)dx$,

then we have $m\int_a^b g(x)dx \leq q \leq M\int_a^b g(x)dx$.
Assume $\int_a^b g(x)dx = 0$, then we have $0 \leq q \leq 0$ and therefore, $q = 0$. Thus, $\forall x \in [a,b], f(x)\cdot\int_a^b g(x)dx = 0$ and therefore holds the claim that $\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx$ for some number ξ in $[a,b]$.
Furthermore, if $\int_a^b g(x)dx > 0$, then we have $m \leq \frac{q}{\int_a^b g(x)dx} \leq M$. Let $\mu = \frac{q}{\int_a^b g(x)dx}$. Thus, we know from part (b) that $\exists\xi$ such that $f(\xi) = \mu$ and $f(\xi)\int_a^b g(x)dx = \int_a^b f(x)g(x)dx$.

(d) Deduce the same result if g is integrable and nonpositive on $[a,b]$.
Proof:
Since we have $g(x)$ as a nonpositive number, then we multiply the original inequality $m \leq f(x) \leq M$ by $-g(x)$. Then, we have $-mg(x) \leq -f(x)g(x) \leq -Mg(x)$. Therefore, we know that $\int_a^b -mg(x)dx \leq \int_a^b -f(x)g(x)dx \leq \int_a^b -Mg(x)dx$. Then, we know $m\int_a^b -g(x)dx \leq \int_a^b -f(x)g(x)dx \leq M\int_a^b -g(x)dx$.
Similarly from previous proof, if $\int_a^b -g(x)dx = 0$, then we know that $\int_a^b -f(x)g(x)dx = 0$. Thus, we also have $\forall x, x \in [a,b], f(x)\int_a^b g(x)dx = 0$. And therefore, the statement holds.
Furthermore, we have $\int_a^b -g(x)dx > 0$. Then, let $\mu = \frac{\int_a^b -f(x)g(x)dx}{\int_a^b -g(x)dx}$. We know that $m < \mu < M$. From Chapter 7 theorem 4 we know that there exists ξ such that $f(\xi) = \mu$ and therefore, the statement holds.

(e) Show that one of these two hypotheses for g is essential.
Answer:
If $g(x) = x^3$ on $[-1,1]$ and $f(x) = x$, then we have

$$\begin{aligned}\int_{-1}^1 f(x)g(x)dx &= \int_{-1}^1 x^4 dx \\ &= \frac{x^5}{4}]_{-1}^1 \\ &= \frac{1}{2}\end{aligned}$$

Then, we have

$$\begin{aligned} int_{-1}^1 f(x)dx &= \frac{x^2}{2}]_{-1}^1 \\ &= 0 \\ int_{-1}^1 g(x)dx &= \frac{x^4}{3}]_{-1}^1 \\ &= \frac{2}{3}\end{aligned}$$

Therefore, we have $\mu = 0$ and $\mu \cdot int^1_{-1} g(x)dx = 0 \neq \frac{1}{2}$, and the statement does not hold.