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Problem Set 7
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Question 1

Suppose that $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$, and $F : \mathbb{R} \to \mathbb{R}$, suppose that f and g are differentiable, and suppose that

$$f' = F$$

and

$$g' = F.$$

Then there exists a $c \in \mathbb{R}$ such that

$$\forall x \in \mathbb{R} : \quad g(x) = f(x) + c.$$

Proof:
Suppose $f' = F$ and $g' = F$. Thus, we have $f' - g' = (f - g)' = F - F = 0$. Therefore, $f - g = c$. Then, we know $f(x) = g(x) + c$.

Question 2

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable, and suppose that

$$\forall x \in \mathbb{R} : f'(x) = 0.$$

Then prove that there exists a $c \in \mathbb{R}$ such that

$$\forall x \in \mathbb{R} : f(x) = c.$$

Proof:
Suppose $f'(x) = 0$. Then, $f'(x) = \lim_{h \to 0} \frac{f(b) - f(a)}{b - a} = 0$. Therefore, $f(b) - f(a) = 0$. Thus, $f(b) = f(a)$. This means $f(x)$ is constant. $f(x) = c$.

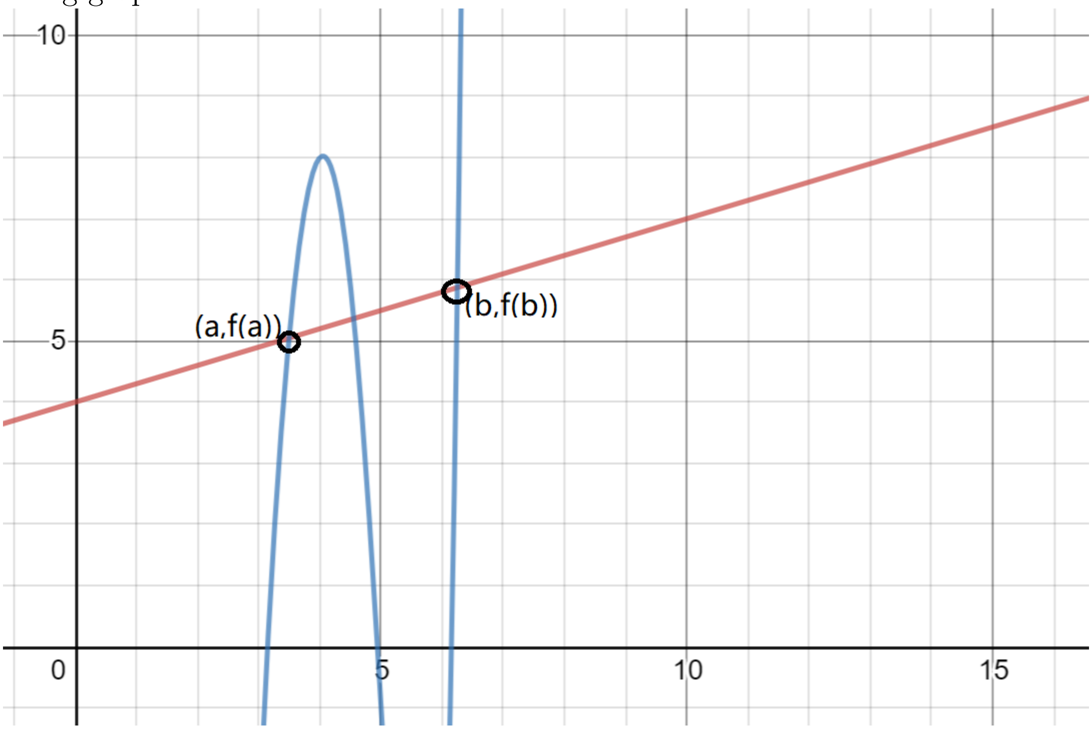
Question 3

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable, and suppose that $a, b \in \mathbb{R}$, with $a < b$. Then there exists a $c \in \mathbb{R}$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof:

Suppose $a, b \in \mathbb{R}$, and $f(x)$ is differentiable $\forall x, x \in \mathbb{R}$. Therefore, we could have the following graph:



Thus, we can show that if we can find a straight line function such that $(a, f(a))$ and $(b, f(b))$ is on the graph we can thus proof if the slope exist then, there must be a $f'(c)$ exist such that $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Let the straight line's function be $h(x)$, then $h(x) = \frac{f(b)-f(a)}{b-a} \cdot x + t, t \in \mathbb{R}$. Since $h(a) = f(a), h(b) = f(b)$ then, $h(x) = \frac{f(b)-f(a)}{b-a} \cdot x + \frac{b(f(a))-a(f(b))}{b-a}$. Then we have $g(x) = f(x) - h(x)$. $g(a) = g(b) = 0$. Thus, $g'(a) = g'(b) = 0$. Then we have $g'(c) = f'(c) - h'(c) = f'(c) - \frac{f(b)-f(a)}{b-a} = 0$. Therefore, $f'(c) = \frac{f(b)-f(a)}{b-a}$.

Question 4

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable, and suppose that $a, b \in \mathbb{R}$, with $a < b$. Suppose also that $f(a) = f(b)$. Then there exists a $c \in \mathbb{R}$ such that

$$f'(c) = 0.$$

Proof:

Suppose $a < b$, and $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then there must be a $c \in [a, b]$ such that $f(c) \geq f(x), \forall x, x \in [a, b]$. Thus, we have a local maximum in range $[a, b]$. Based on Question 5, we know that if we have a local maximum at c, then $f'(c) = 0$.

Question 5

Suppose that $f : \mathbb{R} \to \mathbb{R}$ has a local maximum at $c \in \mathbb{R}$, and suppose that f is differentiable at c. Then we have

$$f'(c) = 0.$$

Proof:

Suppose f is differentiable at c and $f(c)$ is a local maximum. Therefore, we should be looking at the two sides of this local maximum. Such that we will have $\lim\limits_{h \to 0+} \dfrac{f(c+h) - f(c)}{h}$. The denominator is negative because $f(c)$ is local maximum, and h is positive. Then, we know $\lim\limits_{h \to 0+} \dfrac{f(c+h) - f(c)}{h} \leq 0$. On the other hand, $\lim\limits_{h \to 0-} \dfrac{f(c+h) - f(c)}{h}$. The denominator is negative because $f(c)$ is local maximum, and h is negative. Then, we have $\lim\limits_{h \to 0-} \dfrac{f(c+h) - f(c)}{h} \geq 0$. Thus, $\lim\limits_{h \to 0} \dfrac{f(c+h) - f(c)}{h} = 0$.

Question 6

Suppose that $f : \mathbb{R} \to \mathbb{R}$, and $a \in \mathbb{R}$. Then the limit of f at a exists and equals L if, and only if, both the right- and left-handed limits of f at a exist and they both equal L.

Proof:

Suppose $\lim\limits_{x \to a} f(x) = L$. Then, $\forall \epsilon > 0, \exists \delta > 0$, such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

(I) Look at the right-handed limit first:

If $a < x < a + \delta$, then we have $0 < |x - a| < \delta$. Therefore, $|f(x) - L| < \epsilon$.

(II) Look at the left-handed limit:

If $a > x > a - \delta$, then we have $0 < |x - a| < \delta$. Therefore, $|f(x) - L| < \epsilon$.

Therefore, if $\lim\limits_{x \to a} f(x) = L$, then both the right- and left-handed limits of f at a exist and they both equal L.

Conversely, suppose $\lim\limits_{x \to a+} f(x) = \lim\limits_{x \to a-} f(x) = L$. Therefore, we can assume for the right-handed limit if $a < x < a + \delta$, then $|f(x) - L| < \epsilon$. Also, for the left-handed limit we have if $a > x > a - \delta$, then $|f(x) - L| < \epsilon$. Thus, we know if both $0 < x - a < \delta$ and $0 > x - a > -\delta$, then $|f(x) - L| < \epsilon$. Therefore, if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$. Thus, $\lim_{x \to a} f(x) = L$ if and only if both right-handed and left-handed limit has the same value.