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Problem Set 6

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Question 1

Prove that if $f(x)$ is differentiable at a, then it is continuous at a.

Proof:

Suppose $f(x)$ is differentiable at a. Then, $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = L$ and $f(a)$ exist. Since h is constant $\lim_{h \to 0} h$ exist. Then,

$$\begin{aligned} \lim_{h \to 0} h \cdot \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \to 0} h \cdot L \\ &= 0 \cdot L \\ &= 0 \\ &= \lim_{h \to 0} (f(a+h) - f(a)) \end{aligned}$$

Then we know $\lim_{h \to 0} f(a+h) = \lim_{h \to 0} f(a)$. Thus, $\lim_{x \to a} f(x) = f(a))$.

Question 2

Prove if $\forall x, f(x) = c, c \in \mathbb{R}$, then $\forall a : f'(a) = 0$.

Proof:

Suppose a is any number, then $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$.

$$\begin{aligned} \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \to 0} \frac{c - c}{h} \\ &= \lim_{h \to 0} 0 \\ &= 0 \end{aligned}$$

Therefore, $f'(a) = 0$.

Question 3

Prove if $f(x) = x$, then $\forall a : f'(a) = 1$.

Proof:

Suppose a is any number, then $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$.

$$\begin{aligned}
\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} &= \lim_{h \to 0} \frac{a + h - a}{h} \\
&= \lim_{h \to 0} \frac{h}{h} \\
&= \lim_{h \to 0} 1 \\
&= 1
\end{aligned}$$

Therefore, $f'(a) = 0$.

Question 4

Prove that if $f(x)$ and $g(x)$ are differentiable at a, then $f(x) + g(x)$ is differentiable at a and $(f+g)'(a) = f'(a) + g'(a)$.

Proof:

Suppose $f(x)$ and $g(x)$ are differentiable at a, then $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ and $g'(a) = \lim_{h \to 0} \dfrac{g(a+h) - g(a)}{h}$. Thus,

$$\begin{aligned}
f'(a) + g'(a) &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} + \lim_{h \to 0} \frac{g(a+h) - g(a)}{h} \\
&= \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} + \frac{g(a+h) - g(a)}{h} \right) \\
&= \lim_{h \to 0} \frac{f(a+h) - f(a) + g(a+h) - g(a)}{h} \\
&= \lim_{h \to 0} \frac{(f(a+h) + g(a+h)) - (f(a) + g(a))}{h} \\
&= \lim_{h \to 0} \frac{(f+g)(a+h) - (f+g)(a)}{h} \\
&= (f+g)'(a)
\end{aligned}$$

Thus, $(f+g)'(a) = f'(a) + g'(a)$.

Question 5

Prove that if $f(x)$ and $g(x)$ are differentiable at a, then $f(x) \cdot g(x)$ is differentiable at a and $(f \cdot g)'(a) = f'(a) \cdot g(a) + f(a) \cdot g'(a)$.

Proof:

Suppose $f(x)$ and $g(x)$ are differentiable at a, then $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ and $g'(a) =$

$\lim_{h\to 0}\dfrac{g(a+h)-g(a)}{h}$. Also, we know both $f(a)$ and $g(a)$ exist. Thus,

$$\begin{aligned}
(f\cdot g)'(a) &= \lim_{h\to 0}\frac{(f\cdot g)(a+h)-(f\cdot g)(a)}{h}\\
&= \lim_{h\to 0}\frac{(f(a+h)g(a+h)-f(a)g(a)}{h}\\
&= \lim_{h\to 0}\frac{(f(a+h)g(a+h)-f(a)g(a+h)+f(a)g(a+h)-f(a)g(a)}{h}\\
&= \lim_{h\to 0}\frac{((f(a+h)-f(a))g(a+h)+f(a)(g(a+h)-g(a))}{h}\\
&= \lim_{h\to 0}\frac{((f(a+h)-f(a))g(a+h)}{h}+\lim_{h\to 0}\frac{f(a)(g(a+h)-g(a))}{h}\\
&= \lim_{h\to 0}\frac{((f(a+h)-f(a))}{h}\cdot\lim_{h\to 0}g(a+h)+\lim_{h\to 0}\frac{(g(a+h)-g(a))}{h}\cdot\lim_{h\to 0}f(a)\\
&= f'(a)\cdot\lim_{h\to 0}g(a+h)+g'(a)\cdot\lim_{h\to 0}f(a)
\end{aligned}$$

By the theorem from Chapter 5 Question 9, we know that $\lim_{h\to 0}g(a+h)=\lim_{x\to a}g(x)$. Also, because this function is differentiable at a. Thus, this function is continuous at a, meaning $\lim_{x\to a}g(x)=g(a)$. Therefore,

$$\begin{aligned}
&f'(a)\cdot\lim_{h\to 0}g(a+h)+g'(a)\cdot\lim_{h\to 0}f(a)\\
=&f'(a)\cdot g(a)+g'(a)\cdot f(a)
\end{aligned}$$

Thus, $(f\cdot g)'(a)=f'(a)\cdot g(a)+g'(a)\cdot f(a)$.

Question 6

Prove that if $f(x) : \mathbb{R} \to \mathbb{R}$ is differentiable at $a \in \mathbb{R}$ and $c \in \mathbb{R}$, then $g(x) = c \cdot f(x)$ is differentiable at a, and $g'(x) = c \cdot f'(x)$.

Proof:

Suppose $f(x)$ and $g(x)$ are differentiable at a, then $f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ and $g'(a) = \lim_{h\to 0} \frac{g(a+h) - g(a)}{h}$. Also, we know both $f(a)$ and $g(a)$ exist. Thus,

$$\begin{aligned} g'(a) &= \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} \\ &= \lim_{h\to 0} \frac{c \cdot f(a+h) - c \cdot f(a)}{h} \\ &= c \cdot \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\ &= c \cdot f'(a) \end{aligned}$$

Therefore, $g'(x) = c \cdot f'(x)$.

Question 7

Prove that if $\forall x \in \mathbb{R}, n \in \mathbb{N}, f(x) : \mathbb{R} \to \mathbb{R}$, $f(x) = x^n$, such that, if $f(x)$ is differentiable at $a \in \mathbb{R}$, then $f'(x) = na^{n-1}$.

Proof:

Suppose $n = 1$, then $f(x) = x$ and $f'(x) = 1 \cdot a^{1-1} = 1$, according to the assumption. Then we have,

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \\ &= \lim_{h\to 0} \frac{a+h-a}{h} \\ &= \lim_{h\to 0} \frac{h}{h} \\ &= \lim_{h\to 0} 1 \\ &= 1 \end{aligned}$$

Thus, we have $f'(x)$ as the assumption.

Suppose there is a $k, k \in \mathbb{N}$, such that $h(x) = x^k$ and $h'(k) = ka^{k-1}$.

For $k+1$, $f(x) = x^{k+1}$. Then, we have based on the definition of a derivative:

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \\
&= \lim_{h\to 0}\frac{(a+h)^{k+1}-a^{k+1}}{h} \\
&= \lim_{h\to 0}\frac{(a+h)^{k}\cdot(a+h)-a^{k}\cdot a}{h} \\
&= \lim_{h\to 0}\frac{(a+h)^{k}\cdot a+(a+h)^{k}\cdot h-a^{k}\cdot a}{h} \\
&= \lim_{h\to 0}\frac{((a+h)^{k}-a^{k})\cdot a+(a+h)^{k}\cdot h}{h} \\
&= \lim_{h\to 0}(\frac{((a+h)^{k}-a^{k})\cdot a}{h}+\frac{(a+h)^{k}\cdot h}{h}) \\
&= \lim_{h\to 0}\frac{((a+h)^{k}-a^{k})\cdot a}{h}+\lim_{h\to 0}\frac{(a+h)^{k}\cdot h}{h}) \\
&= a\cdot\lim_{h\to 0}\frac{((a+h)^{k}-a^{k})\cdot a}{h}+\lim_{h\to 0}(a+h)^{k} \\
&= a\cdot h'(x)+a^{k} \\
&= a\cdot ka^{k-1}+a^{k} \\
&= (k+1)\cdot a^{k}
\end{aligned}
$$

Therefore, $f'(a) = na^{n-1}$ when $f(a) = a^n$.

Question 8

Prove that if $g(x)$ is differentiable at $a \in \mathbb{R}$, and $g(a) \neq 0$, then the function $\frac{1}{g(x)}$ is differentiable at a, and $\frac{1}{g(a)} = \frac{-g'(a)}{(g(a))^2}$.

Proof:

Suppose $g(x)$ is differentiable at a, then $g'(a) = \lim_{h\to 0}\frac{g(a+h)-g(a)}{h}$. Let $f(x) = \frac{1}{g(x)}$, since we only know that $g(a) \neq 0$, but we can find a δ such that $\delta > |h|$ and $g(a+h) \neq 0$.

Then we have,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} \\
&= \lim_{h\to 0}\frac{\frac{1}{g(a+h)}-\frac{1}{g'(a)}}{h} \\
&= \lim_{h\to 0}\frac{g(a)-g(a+h)}{h\cdot g(a)\cdot g(a+h)} \\
&= \lim_{h\to 0}(\frac{g(a)-g(a+h)}{h}\cdot\frac{1}{g(a)\cdot g(a+h)}) \\
&= \lim_{h\to 0}(\frac{g(a)-g(a+h)}{h}\cdot\lim_{h\to 0}\frac{1}{g(a)\cdot g(a+h)} \\
&= -g'(a)\cdot\frac{1}{g(a)\cdot g(a)} \\
&= -\frac{g'(a)}{(g(a))^2}
\end{aligned}$$

Therefore, $\frac{1}{g(x)}$ exist.

Question 9

Prove that if $f(x)$ and $g(x)$ are differentiable at a, then $\frac{f}{g}(x)$ is differentiable at a and

$$(\frac{f}{g})'(a) = \frac{f'(a)\cdot g(a)-f(a)\cdot g'(a)}{(g(a))^2}.$$

Suppose $f(x)$ and $g(x)$ are differentiable at a, then $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$ and $g'(a) = \lim_{h\to 0}\frac{g(a+h)-g(a)}{h}$. Then, let $h(x) = \frac{1}{g(x)}$. Therefore, $h'(x) = -\frac{g'(a)}{(g(a))^2}$
Thus, we have $\frac{f}{g}(x) = f(x)\cdot h(x)$. Then,

$$\begin{aligned}
(\frac{f}{g})'(a) &= (f\cdot h)'(a) \\
&= f(a)\cdot h'(a)+f'(a)\cdot h(a) \\
&= f(a)\cdot(-\frac{g'(a)}{(g(a))^2})+f'(a)\cdot\frac{1}{g(x)} \\
&= \frac{-f(a)g(a)}{(g(a))^2}+\frac{f'(a)}{g(a)} \\
&= \frac{f'(a)\cdot g(a)-f(a)\cdot g'(a)}{(g(a))^2}
\end{aligned}$$

Therefore, the derivative exist.

Question 10

Prove that if $g(x) : \mathbb{R} \to \mathbb{R}$ is differentiable at $a, a \in \mathbb{R}$, and if $f(x) : \mathbb{R} \to \mathbb{R}$ is differentiable at $g(a)$, then $f \circ g$ is also differentiable at a, and $(f \circ g)'(a) = f'(g(a)) \cdot g'(a)$.

Proof:

Suppose $g(x)$ is differentiable at a, and $f(x)$ is differentiable at $g(a)$. Then, we have $g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h}$. Then,

$$(f \circ g)(a) = \lim_{h \to 0} \frac{f(g(a+h)) - f(g(a))}{h}$$

Let $l = g(a)$, $m = g(a+h)$, $t = l - m$ and $l = m + t$, where $l - m$ is a very small increment. Thus, we have

$$\begin{aligned}
(f \circ g)(a) &= \lim_{t \to 0} \frac{f(m+t) - f(m)}{t} \\
&= \lim_{h \to 0} \frac{f(m+t) - f(m)}{t} \cdot \frac{t}{h} \\
&= \lim_{t \to 0} \frac{f(m+t) - f(m)}{t} \cdot \lim_{h \to 0} \frac{t}{h}
\end{aligned}$$

Chapter 10 Exercise

#1 Answers

(i)

$$f(x) = sin(x + x^2),$$

$$f'(x) = (2x+1)cos(x + x^2)$$

(ii)

$$f(x) = sinx + sinx^2,$$

$$f'(x) = 2x \cdot cos(x^2) + cos(x)$$

(iii)

$$f(x) = sin(cosx),$$

$$f'(x) = sinx \cdot (-cos(cos(x)))$$

(iv)

$$f(x) = sin(sinx),$$

$$f'(x) = cos(x) \cdot cos(sin(x))$$

(v)

$$f(x) = sin(\frac{cosx}{x}),$$

$$f'(x) = -\frac{cos(\frac{cos(x)}{x}) \cdot (x \cdot sin(x) + cos(x))}{x^2}$$

(vi)

$$f(x) = \frac{sin(cos(x))}{x},$$

$$f'(x) = -\frac{x \cdot sin(x)cos(cos(x)) + sin(cos(x))}{x^2}$$

(vii)

$$f(x) = sin(x + sinx),$$

$$f'(x) = (cosx + 1) \cdot cos(x + sin(x))$$

(viii)

$$f(x) = sin(cos(sinx)),$$

$$f'(x) = sin(sin(x))(-cos(x)cos(cos(sin(x))))$$

#2 Answers

(i)

$$f(x) = sin((x + 1)^2(x + 2)),$$

$$f'(x) = (3x^2 + 8x + 5) \cdot cos((x + 1)^2(x + 2))$$

(ii)

$$f(x) = sin^3(x^2 + sinx),$$

$$f'(x) = 3sin^2(x^2 + sin(x)) \cdot cos(x^2 + sin(x))x \cdot (2x + cosx)$$

(iii)

$$f(x) = sin^2((x + sinx)^2),$$

$$f'(x) = 4(x + sin(x))sin(x + sin(x))^2)(cos(x) + 1)cos(x + sin^2(x))$$

(iv)

$$f(x) = sin(\frac{x^3}{cos(x^3)}),$$

$$f'(x) = cos(\frac{x^3}{cos(x^3)}) \cdot \frac{cos(x^3)3x^2 + x^3sin(x^3) \cdot 3x^2}{cos^2(x^3)}$$

(v)

$$f(x) = sin(x \cdot sin(x)) + sin(sinx^2),$$

$$f'(x) = 2sin(x) \cdot cos(x)cos(sin^2(x)) + cos(x \cdot sin(x))(sin(x) + x \cdot cos(x))$$

(vi)

$$f(x) = cos(x)^{31^2},$$

$$f'(x) = 31^2(cosx)^{31^2-1} \cdot (-sin(x))$$

(vii)

$$f(x) = sin^2(x)sin(x^2)sin^2(x^2),$$

$$f'(x) = 2sin(x)sin^2(x^2) \cdot (3x \cdot sin(x)cos(x^2) + sin(x^2)cos(x))$$

(viii)

$$f(x) = sin^3(sin^2(sin(x))),$$

$$f'(x) = 3sin^2(sin^2(sin(x))) \cdot cos(sin^2(sin(x)) \cdot 2sin(sin(x)) \cdot cos(sin(x)) \cdot cos(x)$$

(ix)

$$f(x) = (x + sin^5(x))^6$$

$$f'(x) = 6(x + sin^5(x))^5(5sin^4(x)cos(x) + 1)$$

(x)

$$f(x) = sin(sin(sin(sin(sin(x))))),$$

$$f'(x) = cos(sin(sin(sin(sin(x))))) \cdot cos(sin(sin(sin(x)))) \cdot cos(sin(sin(x))) \cdot cos(sin(x)) \cdot cos(x)$$

(xi)

$$f(x) = sin((sin^7(x^7) + 1)^7),$$

$$f'(x) = 343x^6 \cdot sin^6(x^7)(sin^7(x^7) + 1)^6cos(x^7)cos((sin^7(x^7) + 1)^7$$

(xii)

$$f(x) = (((x^2 + x)^3 + x)^4 + x)^5,$$

$$f'(x) = 5(((x^2 + x)^3 + x)^4 + x^4) \cdot (1 + 4((x^2 + x)^3 + x)^3 + x)^3(1 + 3(x^2 + x)^2(1 + 2x)$$

(xiii)

$$f(x) = sin(x^2 + sin(x^2 + sin(x^2))),$$

$$f'(x) = 2x((cos(x^2) + 1)cos(x^2 + sin(x^2)) + 1)cos(x^2 + sin(x^2 + sin(x^2)))$$

(xiv)

$$f(x) = sin(6cos(6sin(6cos(6sin(x))))),$$

$$f'(x) = 1296cos(6cos(6sin(6cos(6x))))\cdot sin(6sin(6cos(6x))) \cdot cos(6cos(6x))) \cdot (sin(6x)$$

(xv)

$$f(x) = \frac{sin(x^2)sin^2(x)}{1 + sin(x)},$$

$$f'(x) = \frac{sin(x)(2x(1 + sin(x))sin(x)cos(x^2) + (sin(x) + 2)sin(x^2)cos(x)}{(1 + sin(x))^2}$$

(xvi)

$$f(x) = \frac{1}{x - \frac{2}{x+sinx}},$$

$$f'(x) = \frac{-1 - \frac{2(1+cos(x)}{(x+sin(x))^2}}{(x - \frac{2}{x+sin(x)})^2}$$

(xvii)

$$f(x) = sin(\frac{x^3}{sin(\frac{x^3}{sin(x)})}),$$

$$f'(x) = \frac{x^2}{sin(\frac{x^3}{sin(x)})} \cdot cos(\frac{x^3}{sin(\frac{x^3}{sin(x)}}) \cdot (\frac{x^3}{sin(x)} \cdot (\frac{x \cdot cos(x)}{sin(x)} - 3) \cdot \frac{cos(\frac{x^3}{sin(x)})}{sin(\frac{x^3}{sin(x)})} + 3)$$

(xviii)

$$f(x) = sin(\frac{x}{x - sin(\frac{x}{x-sin(x)})},$$

$$f'(x) = cos(\frac{x}{x - sin(\frac{x}{x-sin(x)})}) \cdot \frac{1}{(x - sin(\frac{x}{x-sin(x)}))^2}$$
$$\cdot x - sin(\frac{x}{x - sin(x)}) - x \cdot (1 - cos(\frac{x}{x - sin(x)}) \cdot \frac{x - sin(x) - x \cdot (1 - cos(x))}{(x - sin(x))^2})$$

#6 Answers

(i)

$$f(x) = g(x + g(a)),$$

$$f'(x) = g'(x + g(a))$$

(ii)

$$f(x) = g(x \cdot g(a)),$$

$$f'(x) = g'(x \cdot g(a)) \cdot g(a)$$

(iii)

$$f(x) = g(x + g(x)),$$

$$f'(x) = g'(x + g(x)) \cdot (g'(x) + 1)$$

(iv)

$$f(x) = g(x)(x - a),$$

$$f'(x) = g(x) + (x - a)g'(x)$$

(v)

$$f(x) = g(a)(x - a),$$

$$f'(x) = g(a)$$

(vi)

$$f(x + 3) = g(x^2) \to f(x) = g((x - 3)^2),$$

$$f'(x) = 2g'((x - 3)^2) \cdot (x - 3)$$