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Problem Set 2
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Problem Set 2

Chapter 2
1.(i) $1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$.
Proof:
Let $n = 1$, then we have on the left hand side:

$$1^2 = 1$$

Then, on the right hand side:

$$\begin{aligned}\frac{1 \times (1+1) \times (2 \cdot 1 + 1)}{6} &= \frac{1 \times 2 \times 3}{6} \\ &= \frac{6}{6} \\ &= 1\end{aligned}$$

Therefore, left hand side equals to right hand side. This claim holds for 1.
Then, assume if $n = k$, and $1^2 + \cdots + k^2 = \frac{k(k+1)(2k+1)}{6}$.

Let $n = k+1$, on the left hand side, we would have:

$$\begin{aligned}1^2 + \cdots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\ &= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)\cdot(k+1)}{6} \\ &= \frac{((k+1)(2k^2+k) + (6k+6)(k+1))}{6} \\ &= \frac{(k+1)(2k^2+k+6k+6)}{6} \\ &= \frac{(k+1)(2k^2+7k+6)}{6} \\ &= \frac{(k+1)(2k+3)(k+2)}{6}\end{aligned}$$

$$
\begin{aligned}
&= \frac{(k+1)(2(k+1)+1)((k+1)+1)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}
\end{aligned}
$$

And on the right hand side, we would have:

$$\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

$$\therefore \text{Left hand side equals to right hand side}$$

The claim holds.

1.(ii) $1^3 + \cdots + n^3 = (1 + \cdots + n)^2$.

Proof:

Let $n = 1$, then we would have on the left hand side:

$$1^3 = 1$$

On the right hand side:

$$1^2 = 1$$

Therefore, left hand side equals to right hand side this claim holds for 1.

Then, assume if $n = k$, and $1^3 + \cdots + k^3 = (1 + \cdots + k)^2$.

Let $n = k + 1$, on the left hand side, we would have:

$$
\begin{aligned}
1^3 + \cdots + k^3 + (k+1)^3 &= (1 + \cdots + k)^2 + (k+1)^3 \\
&= (\frac{k(k+1)}{2})^2 + (k+1)^3 \\
&= (\frac{k^2(k+1)^2}{4}) + (k+1) \cdot (k+1)^2 \\
&= \frac{k^2}{4} \cdot (k+1)^2 + (k+1) \cdot (k+1)^2 \\
&= (k+1)^2 \cdot (\frac{k^2}{4} + (k+1)) \\
&= (k+1)^2 \cdot (\frac{k^2}{4} + \frac{4(k+1)}{4}) \\
&= (k+1)^2 \cdot (\frac{k^2}{4} + \frac{4k+4}{4})
\end{aligned}
$$

$$
\begin{aligned}
&= (k+1)^2 \cdot (\frac{k^2+4k+4}{4} \\
&= (k+1)^2 \cdot (\frac{(k+2)^2}{4} \\
&= (k+1)^2 \cdot (\frac{(k+2)}{2})^2 \\
&= (k+1)^2 \cdot (\frac{((k+1)+1)}{2})^2 \\
&= (\frac{(k+1)((k+1)+1)}{2})^2 \\
&= (1+\cdots+(k+1))^2
\end{aligned}
$$

And on the right hand side, we would have:

$$(1+\cdots+(k+1))^2$$

$$\therefore \text{Left hand side equals to right hand side}$$

The claim holds.

2.(i)

$$\sum_{i=1}^{n}(2i-1) = n^2$$

2.(ii)

$$
\begin{aligned}
\sum_{i=1}^{n}(2i-1)^2 &= \sum_{i=1}^{2n}(i)^2 - \sum_{i=1}^{n}(2i)^2 \\
&= \sum_{i=1}^{2n}(i)^2 - 4\sum_{i=1}^{n}(i)^2 \\
&= \frac{(2n)((2n)+1)(2(2n)+1)}{6} \\
&= \frac{(2n)(2n+1)(4n+1) - 4n(n+1)(2n+1)}{6} \\
&= \frac{(2n)(2n+1)(4n+1-2n-2)}{6} \\
&= \frac{(2n)(2n+1)(2n-1)}{6}
\end{aligned}
$$

3.(a)

On the right hand side we have:

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-(k-1))!} + \frac{n!}{k!(n-k)!} \\
&= \frac{k \cdot n! + (n-k+1) \cdot n!}{k!(n-k+1))!} \\
&= \frac{(k+n-k+1)n!}{k!((n+1)-k))!} \\
&= \frac{(n+1)!}{k!((n+1)-k))!} \\
&= \binom{n+1}{k}
\end{aligned}$$

$\therefore$ Left hand side is the same as the right hand side.

3.(b)

5.(a)

Let n=0, we would have on the left hand side: $r^0 = 1$.

On the right hand side, we would have:

$$\begin{aligned}
\frac{1-r^{0+1}}{1-r} &= \frac{1-r}{1-r} \\
&= 1
\end{aligned}$$

$\therefore$ Left hand side is the same as the right hand side.

The claim holds for $n = 0$.

Let $n = k$, assume $1 + \cdots + r^k = \frac{1-r^{k+1}}{1-r}$

If $n = k + 1$, then on the left hand side we would have:

$$\begin{aligned}
1 + \cdots + r^k + r^{k+1} &= \frac{1 - r^{k+1}}{1 - r} + r^{k+1} \\
&= \frac{1 - r^{k+1}}{1 - r} + \frac{(1 - r)r^{k+1}}{1 - r} \\
&= \frac{1 - r^{k+1}}{1 - r} + \frac{r^{k+1} - r^{k+2}}{1 - r} \\
&= \frac{1 - r^{k+1} + r^{k+1} - r^{k+2}}{1 - r} \\
&= \frac{1 - r^{k+2}}{1 - r} \\
&= \frac{1 - r^{(k+1)+1}}{1 - r}
\end{aligned}$$

On the right hand side, we have: $\frac{1-r^{(k+1)+1}}{1-r}$.

$\therefore$ Left hand side is the same as the right hand side.

The claim holds.

5.(b)

Let $S = 1 + \cdots + r^n$, by multiplying both sides with r, then we would have:

$$\begin{aligned}
r \cdot S &= r \cdot 1 + \cdots + r^n \\
&= r + \cdots + r^{n+1}
\end{aligned}$$

It is because we would like to know about S, then we could have:

$$\begin{aligned}
r \cdot S - S &= (r - 1) \cdot S \\
&= r + \cdots + r^{n+1} - (1 + \cdots + r^n) \\
&= r^{n+1} - 1
\end{aligned}$$

$$\therefore (r - 1) \cdot S = r^{n+1} - 1$$

$$S = \frac{r^{n+1} - 1}{r - 1} = \frac{1 - r^{n+1}}{1 - r}$$

Chapter 3

1.(i)

$$\begin{aligned} f(f(x)) &= f(\frac{1}{1+x}) \\ &= \frac{1}{1+\frac{1}{1+x}} \\ &= \frac{1}{\frac{2+x}{1+x}} \\ &= \frac{1+x}{2+x} \end{aligned}$$

$$\therefore x \neq -1, -2$$

1.(ii)

$$\begin{aligned} f(\frac{1}{x}) &= \frac{1}{1+\frac{1}{x}} \\ &= \frac{1}{\frac{1+x}{x}} \\ &= \frac{x}{1+x} \end{aligned}$$

$$\therefore x \neq -1$$

1.(iii)

$$f(cx) = \frac{1}{1+cx}$$

$$\therefore x \neq -\frac{1}{c}, if c \neq 0$$

1.(iv)

$$f(x+y) = \frac{1}{1+x+y}$$

$$\therefore x + y \neq -1$$

1.(v)

$$\begin{aligned} f(x) + f(y) &= \frac{1}{1+x} + \frac{1}{1+y} \\ &= \frac{2+x+y}{(1+x)(1+y)} \end{aligned}$$

$$\therefore x, y \neq -1$$