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Problem Set 1
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Problem Set 1

I. Propositions

Basic Properties of Equivalent:

(E0) If $a = b$, b can substitute a in any real formula

(E1) $\forall a$, $a = a$

(E2) $\forall a, b$, if $a = b$, then $b = a$

(E3) $\forall a, b, c$, if $a = b$, $b = c$, then $c = a$

Basic Properties of Numbers

(P1) $a + (b + c) = (a + b) + c$

(P2) $a + 0 = 0 + a = a$

(P3) $a + (-a) = (-a) + a = 0$

(P4) $a + b = b + a$

(P5) $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

(P6) $a \cdot 1 = 1 \cdot a = a, 1 \neq 0$

(P7) $a \cdot a^{-1} = a^{-1} \cdot a = 1$, for $a \neq 0$

(P8) $a \cdot b = b \cdot a$

(P9) $a \cdot (b + c) = a \cdot b + a \cdot c$

(P10)For every number a, one and only one of the following holds:

$(i) a = 0$

$(ii) a \in P$

$(ii) (-a) \in P$

(P11) If a and b are in P, then $a + b$ is in P

(P12) If a and b are in P, then $a \cdot b$ is in P

II. Solutions

Question 6 $\forall a, b$: if $a \cdot b = 0$, then either $a = 0$ or $b = 0$

Proof:

First, suppose $a \cdot b = 0$. Then, let's assume $a = 0$, from Question 5, we have proved that 0 multiplies any number is 0, then $a \cdot b = 0$.

Then, let's assume $a \neq 0$. From Question 2 we have proved that if we multiply the same thing on both side of a equation, then the equation is still valid. Therefore, we could obtain, by multiplying a^{-1} on both sides of the equation:

$$a^{-1} \cdot (a \cdot b) = a^{-1} \cdot 0 = 0$$

By (P5), we can reformat the equation:

$$(a^{-1} \cdot a) \cdot b = 0$$

By (P7), we have $a^{-1} \cdot a = 1$

$$\therefore \ (a^{-1} \cdot a) \cdot b = 1 \cdot b = 0$$

By (P6) we have $1 \cdot b = b$,

$$\therefore \ b = 0$$

Question 7 $\forall a, b$: $(a+b)^2 = a^2 + 2ab + b^2$

Proof:

By definition of the exponentials we have $\forall a, a^2 = a \cdot a$

$\therefore$ we have $(a+b)^2 = (a+b) \cdot (a+b)$

$$\begin{aligned}
(a+b) \cdot (a+b) &= (a+b) \cdot a + (a+b) \cdot b \qquad \text{By(P9)} \\
&= a \cdot (a+b) + b \cdot (a+b) \qquad \text{By(P8)} \\
&= a \cdot a + a \cdot b + b \cdot a + b \cdot b \qquad \text{By(P9)} \\
&= a^2 + a \cdot b + a \cdot b + b^2 \qquad \text{By(P8)} \\
&= a^2 + (a \cdot b) \cdot 1 + (a \cdot b) \cdot 1 + b^2 \qquad \text{By(P6)} \\
&= a^2 + (a \cdot b) \cdot (1+1) + b^2 \qquad \text{By(P9)} \\
&= a^2 + (a \cdot b) \cdot 2 + b^2 \qquad \text{By(P9)} \\
&= a^2 + 2 \cdot (a \cdot b) + b^2 \qquad \text{By(P8)}
\end{aligned}$$

Question 9 $\forall a$: $(-1) \cdot a = -a$

Proof:

If we do the following:

$$\begin{aligned}
a + (-1) \cdot a &= a \cdot 1 + (-1) \cdot a \qquad \text{By(P6)} \\
&= a \cdot 1 + (-1) \qquad \text{By(P9)} \\
&= a \cdot 0 \qquad \text{By(P3)} \\
&= 0 \qquad \text{By(Question 5)}
\end{aligned}$$

$$\therefore \ a + (-1) \cdot a = 0$$

For $-a$, we have:

$$-a + a = 0 \qquad \text{By(P3)}$$

$$\therefore \; a + (-1) \cdot a = -a + a$$

Then we add $(-a)$ on both sides, by Question 1, the equation should still be valid

$$a + (-a) + (-1) \cdot a = -a + a + (-a)$$

$$(-1) \cdot a = -a \qquad \text{By(P3)}$$

Question 10 $\forall a, b$: $(-a) \cdot (-b) = a \cdot b$
Proof:

$$\begin{aligned}(-a) \cdot (-b) &= ((-1) \cdot a) \cdot ((-1) \cdot b) \qquad \text{By(Question 9)} \\ &= (-1) \cdot a \cdot (-1) \cdot b \qquad \text{By(P5)} \\ &= (-1) \cdot (-1) \cdot a \cdot b \qquad \text{By(P8)} \\ &= a \cdot b\end{aligned}$$

$$\therefore (-a) \cdot (-b) = a \cdot b = a \cdot b \qquad \text{By(E1)}$$

Question 13 $\forall a, b, c$: if $a + c < b + c$, then $a < b$
Proof:
It's because $a + c < b + c$, therefore, $b + c - (a + c) \in P$

$$\begin{aligned}b + c - (a + c) &= b + c + (-1) \cdot (a + c) \qquad \text{By(Question 9)} \\ &= b + c + (-1) \cdot a + (-1) \cdot c \qquad \text{By(P9)} \\ &= b + c + (-a) + (-c) \qquad \text{By(Question 9)} \\ &= b + (-a) + c + (-c) \qquad \text{By(P1)} \\ &= b + (-a) + (c + (-c)) \qquad \text{By(P1)} \\ &= b + (-a) \qquad \text{By(P3)} \\ &= b - a\end{aligned}$$

Therefore, $b + c - (a + c) = b - a \in P$. Thus, $a < b$.
Question 14 $\forall a, b$: if $a < 0$, $b < 0$,then $a \cdot b > 0$
Proof:

Suppose $a < 0$,$b < 0$, then $0 - a, 0 - b \in P$ From (P2) we have

$$\begin{aligned} 0 - a &= 0 + (-a) \\ &= -a \end{aligned}$$

And similar for b, $0 - b = -b$

$$\therefore -a, -b \in P$$

From (P12), we can have because $-a, -b \in P$, then $-a \cdot (-b) \in P$ From Question 10 we know, $\forall a, b$: $(-a) \cdot (-b) = a \cdot b$

$$\therefore a \cdot b \in P$$

From (P2), we have $a \cdot b + (-0) = a \cdot b - 0 \in P$

$$\therefore a \cdot b > 0$$

Question 16 $\forall a, b$: $a \cdot b > 0$, then either $a > 0$ and $b > 0$ or $a < 0$ and $b < 0$
Proof:
Suppose $a = 0$, from Question 5, we have $\forall a$,$a \cdot 0 = 0$. And 0 cannot be greater than 0. Therefore, $a \neq 0$.
Assume $a < 0$.
Suppose $b > 0$. From Question 15, we have $\forall a, b$ if $a < 0, b > 0$, then $a \cdot b < 0$. Thus, $a < 0$ and $b < 0$.
Assume $a > 0$.
Suppose$b < 0$. Because we can use our symbols interchangeably, from Question 15, we can have $\forall a, b$ if $b < 0, a > 0$, then $a \cdot b < 0$. Thus, $a > 0$ and $b > 0$.
Question 22 $\forall a, b, c$: if $a < b$ and $c > 0$, then $a \cdot c < b \cdot c$
Proof:
It's because $a < b$, therefore, $b - a \in P$. Also, because $c > 0$, meaning $c - 0 = c \in P$. Therefore, $c, (b-a) \in P$. From (P12), if both c and $(b-a)$ belong to P, then $c \cdot (b - a) \in P$. Then, we have

$$\begin{aligned} c \cdot (b - a) &= c \cdot (b + (-a)) \\ &= c \cdot b + c \cdot (-a)) \qquad \text{By(P9)} \\ &= c \cdot b - c \cdot a \end{aligned}$$

Therefore, $c \cdot b - c \cdot a \in P$. So, $c \cdot a < c \cdot b$
Question 24 $\forall a, b, c$: if $a < b$ and $c < 0$, then $b \cdot c < b \cdot a$.

Proof:

It's because $a < b$, therefore, $b - a \in P$. Also, because $c < 0$, meaning $0 - c = -c \in P$. Therefore, $-c, (b - a) \in P$. From (P12), if both $-c$ and $(b - a)$ belong to P, then $-c \cdot (b - a) \in P$.

Then, we have

$$\begin{aligned} -c \cdot (b - a) &= -c \cdot (b + (-a)) \\ &= -c \cdot b + (-c) \cdot (-a)) \qquad \text{By(P9)} \\ &= -c \cdot b + c \cdot a \end{aligned}$$

Therefore, $c \cdot a - c \cdot b \in P$. So, $c \cdot b < c \cdot a$

Question 26 $\forall a, b$: $|a + b| \leq |a| + |b|$

Proof:

First, assume $a, b \geq 0$

$$\therefore |a + b| = a + b, |a| + |b| = a + b$$

$a + b = a + b$, meaning $|a + b| = |a| + |b|$ The assumption holds for $a, b \geq 0$

Then, assume $a, b < 0$

$|a + b| = -(a + b) = -a - b$ (By(P9)), $|a| + |b| = -a - b$ Therefore, we have $|a + b| = -a - b = |a| + |b|$. And this assumption holds when $a, b < 0$.

Assume, $a \geq 0, b \leq 0$, and $|a| \geq |b|$, it would be the same situation, when $b \geq 0, a \leq 0$, and $|b| \geq |a|$.

It is because $|a| \geq |b|$, then $a + b \geq 0$. It means $|a + b| = a + b$, and $|a| + |b| = a - b$. Both calculations are absolute values, meaning both of them are greater than 0.

Then, we have

$$\begin{aligned} a - b - (a + b) &= a - b - a - b \qquad \text{By(P9)} \\ &= a - a - b - b \qquad \text{By(P4)} \\ &= -b - b \qquad \text{By(P3)} \\ &= -b + (-b) \\ &= -1 \cdot b + (-1) \cdot b \\ &= (-1 + (-1)) \cdot b \qquad \text{By(P9)} \\ &= -2 \cdot b \qquad \text{By(P9)} \end{aligned}$$

$$\therefore a - b - (a + b) = -2b \geq 0$$

$$\therefore |a| + |b| = a - b \geq |a + b| = a + b$$

Assume, $a \geq 0, b \leq 0$, and $|a| \leq |b|$, it would be the same situation, when $b \geq 0, a \leq 0$, and $|b| \leq |a|$.
It is because $|a| \leq |b|$, then $a + b \leq 0$. It means $|a + b| = -(a + b)$, and $|a| + |b| = a - b$. Both calculations are absolute values, meaning both of them are greater than 0.

Then, we have

$$\begin{aligned} a - b - (-(a + b)) &= a - b + (a + b) \\ &= a + a + b - b \qquad \text{By(P4)} \\ &= a + a \qquad \text{By(P3)} \\ &= 1 \cdot a + 1 \cdot a \\ &= (1 + 1) \cdot a \qquad \text{By(P9)} \\ &= 2 \cdot a \end{aligned}$$

$$\therefore a - b - (-(a + b)) = 2a \geq 0$$

$$\therefore |a| + |b| = a - b \geq |a + b| = a + b$$

Question #1, 2, 3, 4, 5, 6, 7, 11, 12, 13, 14.

Chap 1, Q1

(i) If $ax = a$ for some number $a \neq 0$, then $x = 1$.

Proof:

Assume $ax = a$, then we can have $ax - a = 0$, meaning:

$$\begin{aligned} ax - a &= ax + a \cdot -1 \\ &= a \cdot (x-1) \qquad \text{By(P9)} \\ &= 0 \end{aligned}$$

$$\therefore a \cdot (x-1) = 0$$

It is because $a \neq 0$, from Question 6, $\forall a, b, if a \cdot b = 0$, either a or b is 0. Then, we know b in this equation is 0, which is $(x-1)$. $x - 1 = 0 \therefore x = 1$.

(ii) $x^2 - y^2 = (x-y)(x+y)$.

Proof: On the right hand side of the equation, we can have:

$$\begin{aligned} (x-y)(x+y) &= x(x-y) + y(x-y) \qquad \text{By(P9)} \\ &= x^2 - xy + yx - y^2 \qquad \text{By(P9)} \\ &= x^2 - y^2 \qquad \text{By(P3)} \end{aligned}$$

$$\therefore x^2 - y^2 = (x-y)(x+y)$$

(iii) If $x^2 = y^2$, then $x = y$ or $x = -y$.

Proof:

Assume $x^2 = y^2$, then we have:

$$x^2 - y^2 = 0$$

$$\text{From (ii): } (x+y)(x-y) = 0$$

$$\text{From (Question 6): either } (x+y) \text{ or } (x-y) = 0$$

When $(x+y) = 0$, subtract y on both sides, $(x+y) - y = x = 0 - y = -y$. Therefore, $x = -y$.

When $(x-y) = 0$, add y on both sides, $(x-y) + y = x = 0 + y = y$. Therefore, $x = y$.

(iv) $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$.

Proof:

(v) $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + ... + xy^{n-2} + y^{n-1})$.

Proof:

(vi) $x^3 + y^3 = (x+y)(x^2 - xy + y^2)$.

Proof:

Chap 1, Q2

Solution

This is incorrect because on the third step, when the person wants to divide both side of the equation by $(x-y)$, there is a possibility the person is dividing by 0. Therefore, this step is problematic and leading to the final conclusion that $2 = 1$.

Chap 1, Q3

(i) $\frac{a}{b} = \frac{ac}{bc}$, if $b, c \neq 0$.

Proof:

On the right hand side because $b, c \neq 0$, we have:

$$\begin{aligned}
\frac{ac}{bc} &= a \cdot c \cdot b^{-1} \cdot c^{-1} \\
&= a \cdot b^{-1} \cdot c \cdot c^{-1} \qquad \text{By(P8)} \\
&= a \cdot b^{-1} \qquad \text{By(P7)} \\
&= \frac{a}{b}
\end{aligned}$$

$$\therefore \frac{ac}{bc} = \frac{a}{b}$$

(ii) $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$, if $b, d \neq 0$.

Proof:

On the right hand side because $b, d \neq 0$, we have:

$$
\begin{aligned}
\frac{ad+bc}{bd} &= (ad+bc)\cdot b^{-1}\cdot d^{-1} \\
&= (ad+bc)\cdot (b^{-1}\cdot d^{-1}) \qquad \text{By(P5)} \\
&= ad\cdot (b^{-1}\cdot d^{-1}) + bc\cdot (b^{-1}\cdot d^{-1}) \qquad \text{By(P9)} \\
&= a\cdot b^{-1}\cdot d\cdot d^{-1} + c\cdot b\cdot b^{-1}\cdot d^{-1} \qquad \text{By(P5)} \\
&= a\cdot b^{-1} + c\cdot d^{-1} \qquad \text{By(P7)} \\
&= \frac{a}{b}+\frac{c}{d}
\end{aligned}
$$

$$\therefore \frac{a}{b}+\frac{c}{d} = \frac{ad+bc}{bd}$$

(iii)$(ab)^{-1} = a^{-1}b^{-1}$, if $a, b \neq 0$.

Proof:

On the left hand side, we can multiply by ab, because from P7, we have $a\cdot a^{-1} = 1$. Then we have $ab\cdot (ab)^{-1} = 1$.

On the right hand side, if we multiply by ab, we would have:

$$
\begin{aligned}
ab\cdot a^{-1}b^{-1} &= a\cdot a^{-1}\cdot b^{-1}\cdot b \qquad \text{By(P5)} \\
&= 1\cdot 1 \qquad \text{By(P7)} \\
&= 1
\end{aligned}
$$

$$\because \text{We multiply the same thing, and both of them give the same result}$$

$$\therefore \text{From Question 2, we know} (ab)^{-1} = a^{-1}b^{-1}.$$

(iv)$\frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd}$, if $b, d \neq 0$.

Proof:

On the left hand side, we have:

$$
\begin{aligned}
\frac{a}{b}\cdot\frac{c}{d} &= a\cdot b^{-1}\cdot c\cdot d^{-1} \\
&= a\cdot c\cdot b^{-1}\cdot d^{-1} \qquad \text{By(P5)} \\
&= a\cdot c\cdot (b\cdot d)^{-1} \\
&= \frac{ac}{bd}
\end{aligned}
$$

$$\therefore \frac{a}{b}\cdot\frac{c}{d} = \frac{ac}{bd}$$

(v) $\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$, if $b, c, d \neq 0$.

Proof:

On the left hand side, because $b, c, d \neq 0$, we could have:

$$\begin{aligned}
\frac{a}{b} \div \frac{c}{d} &= \frac{a}{b} \cdot (\frac{c}{d})^{-1} \\
&= a \cdot b^{-1} \cdot (c \cdot d^{-1})^{-1} \\
&= a \cdot b^{-1} \cdot c^{-1} \cdot d \\
&= a \cdot d \cdot b^{-1} \cdot c^{-1} \qquad \text{By (P8)} \\
&= a \cdot d \cdot (b^{-1} \cdot c^{-1}) \qquad \text{By (P5)} \\
&= a \cdot d \cdot (b \cdot c)^{-1} \\
&= \frac{ac}{bd}
\end{aligned}$$

$$\therefore \frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc}$$

(vi) If $b, d \neq 0$, then $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$. Also, determine when $\frac{a}{b} = \frac{b}{a}$.

Proof:

It's because $\frac{a}{b} = \frac{c}{d}$, $b, d \neq 0$. Therefore, we could have both side of the equation multiply by bd:

$$\frac{a}{b} \cdot bd = \frac{c}{d} \cdot bd$$

$$a \cdot b^{-1} \cdot bd = c \cdot d^{-1} \cdot bd$$

$$a \cdot b \cdot b^{-1} \cdot d = b \cdot c \cdot d^{-1} \cdot d \qquad \text{By(P8)}$$

$$a \cdot d = b \cdot c \qquad \text{By(P7)}$$

$$\therefore \frac{a}{b} = \frac{c}{d} \text{ if and only if } ad = bc$$

It is because when $\frac{a}{b} = \frac{c}{d}$, $ad = bc$. We must have $a \cdot a = b \cdot b$, if we want to have $\frac{a}{b} = \frac{b}{a}$, $a, b \neq 0$.

Chap 1, Q4

(i) $4 - x < 3 - 2x$.

Solution:

$S = \{x \mid x < -1\}$

(ii) $5 - x^2 < 8$.

Solution:

$S = \{x \mid x \in \mathbb{R}\}$

(iii) $5 - x^2 < -2$.

Solution:

$S = \{x \mid x^2 > 7\}$

(iv) $(x-1)(x-3) > 0$.

Solution:

$S = \{x \mid x < 1, \text{ or } x > 3\}$

(v) $x^2 - 2x + 2 > 0$.

Solution:

$S = \{x \mid x \in \mathbb{R}\}$

(vi) $x^2 + x + 1 > 2$.

Solution:

$S = \{x \mid x < \frac{-1-\sqrt{5}}{2}, \text{ or } x > \frac{\sqrt{5}-1}{2}\}$

(vii) $x^2 - x + 10 > 16$.

Solution:

$S = \{x \mid x < -2, \text{ or } x > 3\}$

(viii) $x^2 + x + 1 > 0$.

Solution:

$S = \{x \mid x \in \mathbb{R}\}$

(ix) $(x-\pi)(x+5)(x-3) > 0$.

Solution:

$S = \{x \mid -5 < x < 3, \text{ or } x > \pi\}$

(x) $(x - \sqrt[3]{2})(x - \sqrt{2}) > 0$.

Solution:

$S = \{x \mid x < \sqrt[3]{2}, \text{ or } x > \sqrt{2}\}$

(xi) $2^x < 8$.

Solution:

$S = \{x \mid x < 3\}$

(xii) $x + 3^x < 4$.

Solution:

$S = \{x \mid x < 1\}$

(xiii) $\frac{1}{x} + \frac{1}{(1-x)} > 0$.

Solution:

$S = \{x \mid 1 > x > 0\}$

(xiv) $\frac{x-1}{x+1} > 0$.

Solution:

$S = \{x \mid 1 < x, \text{ or } -1 > x\}$

Chap 1, Q7

Proof:

It is because we have $0 < a < b$. Then, we know $b - a, b, a \in P$. From Question 16, we then know that $\sqrt{a}, \sqrt{b} \in P$. This means that $\sqrt{b} + \sqrt{a} \in P$. Therefore, $\sqrt{b} + \sqrt{a} > 0$.

Also, because $b - a \in P$, we know from Chap1, Q1(ii) that $(\sqrt{a} + \sqrt{b})(\sqrt{b} - \sqrt{a}) \in P$. From Question 16 again we know that for $(\sqrt{a} + \sqrt{b})$ and $(\sqrt{b} - \sqrt{a})$ either they are both positive or both negative. Since $\sqrt{b} + \sqrt{a} > 0$, then $\sqrt{b} - \sqrt{a} > 0$. Therefore, $\sqrt{b} - \sqrt{a} \in P$ and $\sqrt{b} > \sqrt{a}$. Also, by (P12), we know that $\sqrt{a} \cdot (\sqrt{b} - \sqrt{a}) = \sqrt{ab} - a \in P$. Thus, $\sqrt{ab} > a$.

From (P12), we also know $(\sqrt{b} - \sqrt{a})^2 \in P$. Thus we know $b - 2\sqrt{ab} + a \in P$. Furthermore, $\frac{1}{2} \in P$ and leads to $\frac{1}{2} \cdot (b - 2\sqrt{ab} + a) \in P$ by (P12). We now have $\frac{1}{2} \cdot (a + b) - \sqrt{ab} \in P$. Then we now $\frac{(a+b)}{2} > \sqrt{ab}$.

At last, we have $b - \frac{(a+b)}{2} = \frac{b}{2} - \frac{a}{2} = \frac{(b-a)}{2}$. We know $b - a, \frac{1}{2} \in P$, then $\frac{(b-a)}{2} \in P$. Thus, $b > \frac{(a+b)}{2}$.

$$\therefore b > \frac{(a+b)}{2} > \sqrt{ab} > a$$

Chap 1, Q11

(i)$|x - 3| = 8$.

Solution:

$x_1 = -5, x_2 = 11$

(ii)$|x - 3| < 8$.

Solution:

$S = \{x \mid -5 < x < 11\}$

(iii)$|x + 4| < 2$.

Solution:

$S = \{x \mid -6 < x < -2\}$

(iv)$|x - 1| + |x - 2| > 1$.

Solution:

$S = \{x \mid x \in \mathbb{R}\}$

(v)$|x - 1| + |x + 1| < 2$.

Solution:

$S = \{\emptyset\}$

(vi)$|x - 1| + |x + 1| < 1$.

Solution:

$S = \{\emptyset\}$

(vii)$|x-1| \cdot |x+1| = 0$.
Solution:
$x_{1,2} = \pm 1$
(viii)$|x-1| \cdot |x+2| = 3$.
Solution:
$x_{1,2} = \frac{-1 \pm \sqrt{21}}{2}$

Chap 1, Q12
(i)$|xy| = |x| \cdot |y|$
Proof:
Assume $x, y \geq 0$. Therefore, $xy \geq 0$. We would have $|xy| = xy$ and $|x| \cdot |y| = x \cdot y = xy$. Then $|xy| = |x| \cdot |y|$.
Then, assume $x, y < 0$. Therefore, $xy \geq 0$. We would have $|xy| = xy$ and $|x| \cdot |y| = -x \cdot (-y) = xy$. Then $|xy| = |x| \cdot |y|$.
Then, assume $x \geq 0, y < 0$. It would be the same for $x < 0, y \geq 0$. In this case, we would have $xy \leq 0$ and therefore $|xy| = -xy$. Then, we would have $|x| \cdot |y| = x \cdot (-y) = -xy$. $|xy| = |x| \cdot |y|$.
(ii)$|\frac{1}{x}| = \frac{1}{|x|}$ if $x \neq 0$.
Proof:
Assume $x > 0$, then we would have $|\frac{1}{x}| = \frac{1}{x}$ and $\frac{1}{|x|} = \frac{1}{x}$. Then, $|\frac{1}{x}| = \frac{1}{|x|}$.
Assume $x < 0$, then we would have $|\frac{1}{x}| = -1 \cdot \frac{1}{x} = \frac{1}{-x}$ and $\frac{1}{|x|} = \frac{1}{-x}$. Then, $|\frac{1}{x}| = \frac{1}{|x|}$.
(iii)$\frac{|x|}{|y|} = |\frac{x}{y}|$ if $y \neq 0$.
Proof:
Assume $x, y \geq 0, y \neq 0$, then $\frac{|x|}{|y|} = \frac{x}{y} \geq 0$. Also, $\frac{x}{y} \geq 0$. Then we would have $|\frac{x}{y}| = \frac{x}{y}$. Therefore, $\frac{|x|}{|y|} = |\frac{x}{y}|$.
Assume $x, y < 0$, then $\frac{|x|}{|y|} = \frac{-x}{-y} \geq 0$. Also, $\frac{x}{y} \geq 0$. Then we would have $|\frac{x}{y}| = \frac{x}{y}$. Therefore, $\frac{|x|}{|y|} = |\frac{x}{y}|$.
Assume $x \geq 0, y < 0$. Then $\frac{|x|}{|y|} = \frac{x}{-y} \leq 0$. Therefore, $\frac{|x|}{|y|} = \frac{x}{-y} = -\frac{x}{y}$. Also, $\frac{x}{y} \leq 0$. Then we would have $|\frac{x}{y}| = -\frac{x}{y}$. Therefore, $\frac{|x|}{|y|} = |\frac{x}{y}|$.
Assume $x < 0, y > 0$. Then $\frac{|x|}{|y|} = \frac{-x}{y} \leq 0$. Therefore, $\frac{|x|}{|y|} = \frac{-x}{y} = -\frac{x}{y}$. Also, $\frac{x}{y} \leq 0$. Then we would have $|\frac{x}{y}| = -\frac{x}{y}$. Therefore, $\frac{|x|}{|y|} = |\frac{x}{y}|$.

(iv)$|x-y| \leq |x|+|y|$

Proof:

Assume $x, y \geq 0, x \geq y$, $|x-y| = x-y$, $|x|+|y| = x+y$.

$$\begin{aligned} |x|+|y|-|x-y| &= x+y-(x-y) \\ &= x+y-x+y \\ &= 2y \geq 0 \end{aligned}$$

$$\therefore |x|+|y|-|x-y| \geq 0, \text{ meaning, } |x-y| \leq |x|+|y|.$$

Assume $x, y \geq 0, x < y$, $|x-y| = y-x$, $|x|+|y| = x+y$.

$$\begin{aligned} |x|+|y|-|x-y| &= x+y-(y-x) \\ &= x+y-y+x \\ &= 2x \geq 0 \end{aligned}$$

$$\therefore |x|+|y|-|x-y| \geq 0, \text{ meaning, } |x-y| \leq |x|+|y|.$$

Assume $x, y \leq 0, x \geq y$, $|x-y| = x-y$, $|x|+|y| = -x-y$.

$$\begin{aligned} |x|+|y|-|x-y| &= -x-y-(x-y) \\ &= -x-y-x+y \\ &= -2x \geq 0 \end{aligned}$$

$$\therefore |x|+|y|-|x-y| \geq 0, \text{ meaning, } |x-y| \leq |x|+|y|.$$

Assume $x, y \leq 0, x \leq y$, $|x-y| = y-x$, $|x|+|y| = -x-y$.

$$\begin{aligned} |x|+|y|-|x-y| &= -x-y-(y-x) \\ &= -x-y-y+x \\ &= -2y \geq 0 \end{aligned}$$

$$\therefore |x|+|y|-|x-y| \geq 0, \text{ meaning, } |x-y| \leq |x|+|y|.$$

Assume $x \geq 0, y < 0$, $|x-y| = x-y$, $|x|+|y| = x-y$.

$$\begin{aligned} |x|+|y|-|x-y| &= x-y-(x-y) \\ &= 0 \end{aligned}$$

$$\therefore |x| + |y| - |x - y| = 0, \text{ meaning, } |x - y| = |x| + |y|.$$

Assume $x < 0, y \geq 0$, $|x - y| = y - x$, $|x| + |y| = y - x$.

$$\begin{aligned} |x| + |y| - |x - y| &= y - x - (y - x) \\ &= 0 \end{aligned}$$

$$\therefore |x| + |y| - |x - y| = 0, \text{ meaning, } |x - y| = |x| + |y|.$$

(v)$|x| - |y| \leq |x - y|$
Proof:
If $|x| \geq |y|$ and x, y are both negative or both positive, then $|x| - |y| = |x - y|$. Else if $x \geq 0, y < 0$, we would have $|x - y| - |x| + |y| = x - y - x - y = -2y \geq 0$ meaning $|x| - |y| \leq |x - y|$. Or if we have $x < 0, y \geq 0$, then $|x - y| - |x| + |y| = y - x + x + y = 2y \geq 0$ meaning $|x| - |y| \leq |x - y|$.
If $|x| < |y|$, $|x| - |y| < 0$. It is because $|x - y| > 0$, then $|x| - |y| \leq |x - y|$.

(vi)$|(|x| - |y|)| \leq |x - y|$
Proof:
(vii)$|x + y + z| \leq |x| + |y| + |z|$
Proof: