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Problem Set 2

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Problem Set 2

Chapter 2

1.(i) $1^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{6}$.

Proof:

Let $n = 1$, then we have on the left hand side:

$$1^2 = 1$$

Then, on the right hand side:

$$\begin{aligned} \frac{1 \times (1+1) \times (2 \cdot 1 + 1)}{6} &= \frac{1 \times 2 \times 3}{6} \\ &= \frac{6}{6} \\ &= 1 \end{aligned}$$

Therefore, left hand side equals to right hand side. This claim holds for 1.

Then, assume if $n = k$, and $1^2 + \dots + k^2 = \frac{k(k+1)(2k+1)}{6}$.

Let $n = k + 1$, on the left hand side, we would have:

$$\begin{aligned} 1^2 + \dots + k^2 + (k+1)^2 &= \frac{k(k+1)(2k+1)}{6} + (k+1)^2 \\ &= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6} \\ &= \frac{k(k+1)(2k+1)}{6} + \frac{6(k+1) \cdot (k+1)}{6} \\ &= \frac{((k+1)(2k^2+k) + (6k+6)(k+1))}{6} \\ &= \frac{(k+1)(2k^2+k+6k+6)}{6} \\ &= \frac{(k+1)(2k^2+7k+6)}{6} \\ &= \frac{(k+1)(2k+3)(k+2)}{6} \end{aligned}$$

$$\begin{aligned}
&= \frac{(k+1)(2(k+1)+1)((k+1)+1)}{6} \\
&= \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}
\end{aligned}$$

And on the right hand side, we would have:

$$\frac{(k+1)((k+1)+1)(2(k+1)+1)}{6}$$

\therefore Left hand side equals to right hand side

The claim holds.

1.(ii) $1^3 + \dots + n^3 = (1 + \dots + n)^2$.

Proof:

Let $n = 1$, then we would have on the left hand side:

$$1^3 = 1$$

On the right hand side:

$$1^2 = 1$$

Therefore, left hand side equals to right hand side this claim holds for 1.

Then, assume if $n = k$, and $1^3 + \dots + k^3 = (1 + \dots + k)^2$.

Let $n = k + 1$, on the left hand side, we would have:

$$\begin{aligned}
1^3 + \dots + k^3 + (k+1)^3 &= (1 + \dots + k)^2 + (k+1)^3 \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\
&= \left(\frac{k^2(k+1)^2}{4}\right) + (k+1) \cdot (k+1)^2 \\
&= \frac{k^2}{4} \cdot (k+1)^2 + (k+1) \cdot (k+1)^2 \\
&= (k+1)^2 \cdot \left(\frac{k^2}{4} + (k+1)\right) \\
&= (k+1)^2 \cdot \left(\frac{k^2}{4} + \frac{4(k+1)}{4}\right) \\
&= (k+1)^2 \cdot \left(\frac{k^2}{4} + \frac{4k+4}{4}\right)
\end{aligned}$$

$$\begin{aligned}
&= (k+1)^2 \cdot \left(\frac{k^2 + 4k + 4}{4}\right) \\
&= (k+1)^2 \cdot \left(\frac{(k+2)^2}{4}\right) \\
&= (k+1)^2 \cdot \left(\frac{(k+2)}{2}\right)^2 \\
&= (k+1)^2 \cdot \left(\frac{((k+1)+1)}{2}\right)^2 \\
&= \left(\frac{(k+1)((k+1)+1)}{2}\right)^2 \\
&= (1 + \dots + (k+1))^2
\end{aligned}$$

And on the right hand side, we would have:

$$(1 + \dots + (k+1))^2$$

\therefore Left hand side equals to right hand side

The claim holds.

2.(i)

$$\sum_{i=1}^n (2i-1) = n^2$$

2.(ii)

$$\begin{aligned}
\sum_{i=1}^n (2i-1)^2 &= \sum_{i=1}^{2n} (i)^2 - \sum_{i=1}^n (2i)^2 \\
&= \sum_{i=1}^{2n} (i)^2 - 4 \sum_{i=1}^n (i)^2 \\
&= \frac{(2n)((2n)+1)(2(2n)+1)}{6} \\
&= \frac{(2n)(2n+1)(4n+1) - 4n(n+1)(2n+1)}{6} \\
&= \frac{(2n)(2n+1)(4n+1-2n-2)}{6} \\
&= \frac{(2n)(2n+1)(2n-1)}{6}
\end{aligned}$$

3.(a)

On the right hand side we have:

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-(k-1))!} + \frac{n!}{k!(n-k)!} \\
&= \frac{k \cdot n! + (n-k+1) \cdot n!}{k!(n-k+1)!} \\
&= \frac{(k+n-k+1)n!}{k!((n+1)-k)!} \\
&= \frac{(n+1)!}{k!((n+1)-k)!} \\
&= \binom{n+1}{k}
\end{aligned}$$

\therefore Left hand side is the same as the right hand side.

3.(b)

$\forall n, n \in \mathbb{Z}_{\geq 0}$, we would have by definition $\binom{n}{n} = \binom{n}{0} = 1$, which is a natural number.

Let $n = 0$, we would have $\binom{0}{0} = 1$, and it is a natural number. The claim holds for 0.

Then, assume $n = m$ and $\forall k, k \leq m$, $\binom{m}{k}$ is natural number.

Let $n = m + 1$, we would have $\forall k, k < m + 1$, $\binom{m+1}{k} = \binom{m}{k-1} + \binom{m}{k}$ from the theorem we prove in (a). It is because from the assumption above, we have both $\binom{m}{k-1}, \binom{m}{k}$ are natural numbers. Therefore, $\binom{m+1}{k}$ is a natural number. And $\binom{m+1}{m+1} = m + 10 = 1$ by definition of a binomial coefficient. Therefore, the claim holds for all $n \in \mathbb{Z}_{\geq 0}$.

3.(c)

When choosing k things out of n things, we are looking at the arrangement of first k things in an ordered list of n things without considering their order. Therefore, first we are arranging all n things, then we have $n!$ ways to do the arrangement. Then, we choose the first k things and find their arrangements. Therefore, we are overcounting all such arrangements by $(n-k)!$ times, since we do not care about the rest $n-k$ things' arrangements. Then we would have $\frac{n!}{(n-k)!}$ ways of arrangement left. However, we are still overcounting the arrangements $k!$ times because we do not need to know the order for the arrangements of these k things. Therefore, we have $\frac{n!}{k! \cdot (n-k)!}$ in total.

3.(d)

Let $n = 1$, then we have on the left hand side: $(a+b)^1 = a+b$ and on the right hand side: $\binom{1}{0}a^1 + \binom{1}{1}b^1 = a+b$. We have left hand side equals to the right hand side. The claim holds for $n = 1$.

Let $n = m$, such that $(a+b)^m = a^m + \binom{m}{1}a^{m-1}b + \binom{m}{2}a^{m-2}b^2 + \cdots + \binom{m}{m-1}ab^{m-1} + b^m$

Let $n = m + 1$, on the left hand side we would have:

$$\begin{aligned}
(a+b)^{m+1} &= (a+b)^m \cdot (a+b) \\
&= (a^m + \binom{m}{1}a^{m-1}b + \binom{m}{2}a^{m-2}b^2 + \cdots + \binom{m}{m-1}ab^{m-1} + b^m) \cdot (a+b) \\
&= a \cdot \left(\binom{m}{0}a^m + \binom{m}{1}a^{m-1}b + \binom{m}{2}a^{m-2}b^2 + \cdots + \binom{m}{m-1}ab^{m-1} + \binom{m}{m}b^m \right) \\
&\quad + b \cdot \left(\binom{m}{0}a^m + \binom{m}{1}a^{m-1}b + \binom{m}{2}a^{m-2}b^2 + \cdots + \binom{m}{m-1}ab^{m-1} + \binom{m}{m}b^m \right) \\
&= \binom{m}{0}a^{m+1} + \binom{m}{1}a^m b + \cdots + \binom{m}{m-1}a^2 b^{m-1} + \binom{m}{m}ab^m \\
&\quad + \binom{m}{0}a^m b + \binom{m}{1}a^m b^2 + \cdots + \binom{m}{m-1}ab^m + \binom{m}{m}b^{m+1} \\
&= \binom{m}{0}a^{m+1} + \left(\binom{m}{0} + \binom{m}{1} \right) a^m b + \cdots + \left(\binom{m}{m-1} + \binom{m}{m} \right) ab^m + \binom{m}{m}b^{m+1}
\end{aligned}$$

It's because from (a) we have proved that $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$. Also, by definition of a binomial coefficient we know that $\forall m, m \in \mathbb{Z}_{\geq 0} \binom{m}{0} = \binom{m}{m} = 1$. Therefore, on the left hand side, we should have:

$$\begin{aligned}
&\binom{m}{0}a^{m+1} + \left(\binom{m}{0} + \binom{m}{1} \right) a^m b + \cdots + \left(\binom{m}{m-1} + \binom{m}{m} \right) ab^m + \binom{m}{m}b^{m+1} \\
&= \binom{m+1}{0}a^{m+1} + \binom{m+1}{1}a^m b + \cdots + \binom{m+1}{m}ab^m + b^{m+1} \\
&= \sum_{j=0}^{m+1} \binom{m+1}{j} a^{m+1-j} b^j
\end{aligned}$$

On the right hand side, we would have when $n = m + 1$:

$$\sum_{j=0}^{m+1} \binom{m+1}{j} a^{m+1-j} b^j$$

\therefore Left hand side equals to the right hand side. The claim holds.

3.(e)

(i)

Proof:

Let $n = 0$, then we would have on the left hand side $\binom{0}{0} = 1$, and on the right hand side, we would have $2^0 = 1$. Therefore, the claim holds for $n = 0$.

Let $n = m$ such that

$$\sum_{j=0}^m \binom{m}{j} = 2^m$$

Let $n = m + 1$, we would have on the left hand side:

$$\begin{aligned} \sum_{j=0}^{m+1} \binom{m+1}{j} &= \binom{m+1}{0} + \binom{m+1}{1} + \cdots + \binom{m+1}{m+1} \\ &= \binom{m}{0} + \left(\binom{m}{0} + \binom{m}{1} \right) + \cdots + \left(\binom{m}{m-1} + \binom{m}{m} \right) + \binom{m}{m} \\ &= \binom{m}{0} + \binom{m}{0} + \binom{m}{1} + \binom{m}{1} + \cdots + \binom{m}{m} + \binom{m}{m} \\ &= 2 \cdot \left(\binom{m}{0} + \binom{m}{1} + \cdots + \binom{m}{m} \right) \\ &= 2 \cdot 2^m \\ &= 2^{m+1} \end{aligned}$$

On the right hand side, we would have 2^{m+1} . Thus, the left hand side is equal to the right hand side. The claim holds.

(ii)

Proof:

This only holds for any number $n, n > 0$. It is because if $n = 0$, this equation would be 1 instead of 0.

Let $n = 1$, on the left hand side we would have $\binom{1}{0} - \binom{1}{1} = 0$. On the right hand side we have 0. Therefore, the claim holds for $n = 1$.

Let $n = m$, assume

$$\sum_{j=0}^m (-1)^j \cdot \binom{m}{j} = 0$$

Let $n = m + 1$, we would have on the left hand side:

$$\begin{aligned} \sum_{j=0}^{m+1} (-1)^j \cdot \binom{m+1}{j} &= \binom{m+1}{0} - \binom{m+1}{1} + \cdots + (-1)^{m+1} \cdot \binom{m+1}{m+1} \\ &= \binom{m}{0} - \left(\binom{m}{0} + \binom{m}{1} \right) + \cdots + (-1)^m \cdot \left(\binom{m}{m-1} + \binom{m}{m} \right) + (-1)^m \cdot \binom{m}{m} \\ &= \binom{m}{0} - \binom{m}{0} - \binom{m}{1} + \cdots + (-1)^m \cdot \binom{m}{m-1} + (-1)^m \cdot \binom{m}{m} + (-1)^m \cdot \binom{m}{m} \\ &= 0 \end{aligned}$$

On the left hand side, we have 0.

Therefore, this claim holds for all number $n > 0$.

(iii) and (iv)

Proof:

From the question above we have proved that $\sum_{j=0}^m (-1)^j \cdot \binom{m}{j} = 0$. This means that for even binomial coefficients, it is always positive in this equation, and for odd coefficients, it is always negative. Therefore, we can have $\sum_{j \text{ even}} \binom{n}{j} = \sum_{j \text{ odd}} \binom{n}{j}$. It is because from (i) we know $\sum_{j=0}^m \binom{m}{j} = 2^n$. Thus, $\sum_{j \text{ even}} \binom{n}{j} = \sum_{j \text{ odd}} \binom{n}{j} = 2^{n-1}$

5.(a)

Let $n = 0$, we would have on the left hand side: $r^0 = 1$.

On the right hand side, we would have:

$$\begin{aligned} \frac{1 - r^{0+1}}{1 - r} &= \frac{1 - r}{1 - r} \\ &= 1 \end{aligned}$$

\therefore Left hand side is the same as the right hand side.

The claim holds for $n = 0$.

Let $n = k$, assume $1 + \dots + r^k = \frac{1 - r^{k+1}}{1 - r}$

If $n = k + 1$, then on the left hand side we would have:

$$\begin{aligned} 1 + \dots + r^k + r^{k+1} &= \frac{1 - r^{k+1}}{1 - r} + r^{k+1} \\ &= \frac{1 - r^{k+1}}{1 - r} + \frac{(1 - r)r^{k+1}}{1 - r} \\ &= \frac{1 - r^{k+1}}{1 - r} + \frac{r^{k+1} - r^{k+2}}{1 - r} \\ &= \frac{1 - r^{k+1} + r^{k+1} - r^{k+2}}{1 - r} \\ &= \frac{1 - r^{k+2}}{1 - r} \\ &= \frac{1 - r^{(k+1)+1}}{1 - r} \end{aligned}$$

On the right hand side, we have: $\frac{1 - r^{(k+1)+1}}{1 - r}$.

\therefore Left hand side is the same as the right hand side.

The claim holds.

5.(b)

Let $S = 1 + \cdots + r^n$, by multiplying both sides with r , then we would have:

$$\begin{aligned} r \cdot S &= r \cdot 1 + \cdots + r^n \\ &= r + \cdots + r^{n+1} \end{aligned}$$

It is because we would like to know about S , then we could have:

$$\begin{aligned} r \cdot S - S &= (r - 1) \cdot S \\ &= r + \cdots + r^{n+1} - (1 + \cdots + r^n) \\ &= r^{n+1} - 1 \end{aligned}$$

$$\therefore (r - 1) \cdot S = r^{n+1} - 1$$

$$S = \frac{r^{n+1} - 1}{r - 1} = \frac{1 - r^{n+1}}{1 - r}$$

Chapter 3

1.(i)

$$\begin{aligned}f(f(x)) &= f\left(\frac{1}{1+x}\right) \\&= \frac{1}{1+\frac{1}{1+x}} \\&= \frac{1}{\frac{2+x}{1+x}} \\&= \frac{1+x}{2+x}\end{aligned}$$

$$\therefore x \neq -1, -2$$

1.(ii)

$$\begin{aligned}f\left(\frac{1}{x}\right) &= \frac{1}{1+\frac{1}{x}} \\&= \frac{1}{\frac{1+x}{x}} \\&= \frac{x}{1+x}\end{aligned}$$

$$\therefore x \neq -1$$

1.(iii)

$$f(cx) = \frac{1}{1+cx}$$

$$\therefore x \neq -\frac{1}{c}, \text{ if } c \neq 0$$

1.(iv)

$$f(x+y) = \frac{1}{1+x+y}$$

$$\therefore x+y \neq -1$$

1.(v)

$$\begin{aligned} f(x) + f(y) &= \frac{1}{1+x} + \frac{1}{1+y} \\ &= \frac{2+x+y}{(1+x)(1+y)} \end{aligned}$$

$$\therefore x, y \neq -1$$

1.(vi)

$f(cx) = f(x)$, $x \neq -1$, or $-c^{-1}$, then we would have $\frac{1}{1+x} = \frac{1}{1+cx}$.

$$\therefore 1+x = 1+cx$$

$$(c-1)x = 0$$

If we only want one x solution for such an equation, then we could have $(c-1) \cdot 0 = 0$. Therefore, $c \in \mathbb{R}$ the equation would always hold.

1.(vii)

If we would like to have at least two different answers for x in the equation $(c-1)x = 0$, then we need to have $c-1 = 0$. Thus, $c = 1$.

2.(i)

It is because for any y is rational, $h(y) = 0$, and for any y that is irrational, $h(y) = 1$. If we would like to have $h(y) \geq y$, then we need $y \leq 0, y \in \mathbb{Q}$ or $y < 1, y \in \mathbb{R} - \mathbb{Q}$.

2.(ii)

It's because we want to have $h(y) \leq g(y)$. Thus meaning for $y \in \mathbb{Q}$, $y^2 \geq 0$, and for $y \in \mathbb{R} - \mathbb{Q}$, $y^2 \geq 1$. Therefore, $|y| \geq 0, y \in \mathbb{Q}$ or $|y| > 1, y \in \mathbb{R} - \mathbb{Q}$

2.(iii)

Suppose z is a rational number, then we have $h(z) = 0$, and $h(g(z)) = 0^2 = 0$. Therefore, $h(g(z)) - h(z) = 0$. Suppose z is a irrational number, we would have $h(z) = 1$, and $h(g(z)) = 1^2 = 1$. Therefore, $h(g(z)) - h(z) = 0$. In conclusion, $h(g(z)) - h(z) = 0$.

2.(iv)

For $g(w) \leq w$, we would have $w - w^2 = w \cdot (1 - w) \geq 0$. Thus, w and $1 - w$ should be the same sign or one of them is 0.

Both of them are positive:

$$w > 0$$

$$1 - w > 0$$

Therefore, $1 > w > 0$.

Both of the are negative:

$$w < 0$$

$$1 - w < 0$$

This is not possible, because we need $w > 1$ and $w < 0$.

If one of them is 0. Say w , then $w = 0$. If $w - 1 = 0$, then $w = 1$.

$$\therefore 1 \geq w \geq 0$$

2.(v)

Suppose $g(g(e)) = g(e)$, then we have $g(e^2) = e^4 = e^2$. Then we have:

$$e^4 - e^2 = 0$$

$$e^2 \cdot (e^2 - 1) = 0$$

$$e^2 = 0 \text{ or } e^2 - 1 = 0$$

$$e_1 = 0 \text{ and } e_{2,3} = \pm 1$$

3.(i)

$1 - x^2 \geq 0$. Thus, $1 \geq x^2$. Therefore, $-1 \leq x \leq 1$.

3.(ii)

$1 - \sqrt{1 - x^2} \geq 0$, and we know $1 + \sqrt{1 - x^2}$. Thus, we can have:

$$(1 - \sqrt{1 - x^2}) \cdot 1 + \sqrt{1 - x^2} \geq (1 + \sqrt{1 - x^2}) \cdot 0$$

$$1 - 1 + x^2 \geq 0$$

$$x^2 \geq 0$$

$$\therefore x \in \mathbb{R}$$

3.(iii)

$$x \neq 1, 2$$

3.(iv)

$1 - x^2 \geq 0$ and $x^2 - 1 \geq 0$. Thus, we have $1 \geq x^2$ and $1 \leq x^2$. Therefore, $x^2 = 1$ and $x_{1,2} = \pm 1$.

3.(v)

$1 - x \geq 0, x - 2 \geq 0$. Therefore, $1 \geq x$ and $x \leq 2$. It is not possible. Therefore, this function cannot happen in real number domain.

4.(i)

We should have the following:

$$\begin{aligned}
 (S \circ P)(y) &= S(P(y)) \\
 &= S(2^y) \\
 &= (2^y)^2 \\
 &= 2^{2y}
 \end{aligned}$$

4.(ii)

We should have the following:

$$\begin{aligned}
 (S \circ s)(y) &= S(P(y)) \\
 &= S(\sin(y)) \\
 &= (\sin(y))^2 \\
 &= \sin^2(y)
 \end{aligned}$$

4.(iii)

We should have the following:

$$\begin{aligned}
 (S \circ P \circ s)(t) + (s \circ P)(t) &= S(P(s(t))) + s(P(t)) \\
 &= S(P(\sin(t))) + s(2^t) \\
 &= S(2^{\sin(t)}) + \sin(2^t) \\
 &= (2^{\sin(t)})^2 + \sin(2^t)
 \end{aligned}$$

4.(iv)

We should have the following:

$$s(t^3) = \sin(t^3)$$

5.

$$S(x) = x^2, P(x) = 2^x, s(x) = \sin x$$

5.(i)

We should have the following:

$$\begin{aligned}
 f(x) &= 2^{\sin x} &= 2^{(s(x))} \\
 &= P(s(x)) \\
 &= (P \circ s)(x)
 \end{aligned}$$

5.(ii)

We should have the following:

$$\begin{aligned}
 f(x) &= \sin(2^x) \\
 &= \sin(P(x)) \\
 &= s(P(x)) \\
 &= (s \circ P)(x)
 \end{aligned}$$

5.(iii)

We should have the following:

$$\begin{aligned}
 f(x) &= \sin(x^2) \\
 &= \sin(S(x)) \\
 &= s(S(x)) \\
 &= (s \circ S)(x)
 \end{aligned}$$

5.(iv)

We should have the following:

$$\begin{aligned}
 f(x) &= \sin^2(x) \\
 &= (\sin(x))^2 \\
 &= (s(x))^2 \\
 &= S(s(x)) \\
 &= (S \circ s)(x)
 \end{aligned}$$

5.(v)

We should have the following:

$$\begin{aligned}
 f(t) &= 2^{2^t} \\
 &= 2^{P(t)} \\
 &= P(P(t)) \\
 &= (P \circ P)(x)
 \end{aligned}$$

5.(vi)

We should have the following:

$$\begin{aligned}
 f(u) &= \sin(2^u + 2^{u^2}) \\
 &= \sin(P(u) + 2^S(u)) \\
 &= \sin(P(u) + P(S(u))) \\
 &= s(P(u) + (P \circ S)(u)) \\
 &= s(P + P \circ S)(u)
 \end{aligned}$$

5.(vii)

We should have the following:

$$\begin{aligned}
 f(y) &= \sin(\sin(\sin(2^{2^{siny}}))) \\
 &= \sin(\sin(\sin(2^{2^{s(y)}}))) \\
 &= \sin(\sin(\sin(2^{P(s(y))}))) \\
 &= \sin(\sin(\sin(P(P(s(y))))) \\
 &= s(s(s(P(P(s(y))))) \\
 &= (s \circ s \circ s \circ P \circ P \circ s)(y)
 \end{aligned}$$

5.(viii)

We should have the following:

$$\begin{aligned}
 f(a) &= 2^{\sin^2(a)} + \sin(a^2) + 2^{\sin(a^2 + \sin(a))} \\
 &= 2^{s(a)^2} + \sin(S(a)) + 2^{\sin(S(a) + s(a))} \\
 &= 2^{S(s(a))} + s(S(a)) + 2^{s(S(a) + s(a))} \\
 &= P(S(s(a))) + s(S(a)) + P(s(S(a) + s(a))) \\
 &= P(S(s(a))) + s(S(a)) + P(s(S(a) + s(a))) \\
 &= (P \circ S \circ s)(a) + (s \circ S)(a) + (P \circ s(S + s))(a)
 \end{aligned}$$

21.(i)

Want to prove that if $x \neq y$, then $g(x) \neq g(y)$. Thus, showing if $g(x) = g(y)$, then $x = y$ still holds.