Overview of 3D Transformations

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Rotations-1

Rotations around z-axis (z NO Change)

- **Rz(θ) =** $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Rotations-2

Rotations around x-axis (x NO Change)

- **Rx(θ) =** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$

Rotations-3

Rotations around y-axis (y NO Change)

- **Ry(θ) =** $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$

Translations

Move X by ΔX, Y by ΔY, and Z by ΔZ

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} = \begin{bmatrix} X + \Delta X \\ Y + \Delta Y \\ Z + \Delta Z \end{bmatrix}$$

Scaling

Scale X by S_X, Y by S_Y, and Z by S_Z

Can be represented by a single matrix:

$$\begin{bmatrix} X_s \\ Y_s \\ Z_s \end{bmatrix} = \begin{bmatrix} S_X & 0 & 0 \\ 0 & S_Y & 0 \\ 0 & 0 & S_Z \end{bmatrix} * \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

Combination of transforms

– **A sequence of rotations and scalings can be represented as a single 3x3 matrix which is the product of the individual transforms**

– **However there is no way to combine translations with rotations & translations into a single 3x3 matrix; translations need to be represented as vector additions**

– **Combination of matrix multiplication & vector additions makes it impossible to compute the inverse transformation in ONE step**

Combination of transforms-2

- **Rotation around X by 30 degrees, followed by translation by T1; followed by scaling by 0.5 on X, 0.75 on Y & 1 on Z; followed by translation by (10, -20, 10); followed by rotation around Z by 90 degrees**

- **Only the translation component prevents a compact (easily invertible) representation**

Homogeneous transforms-1

- **Want to represent rotations, scalings and translations all within the same matrix format, so that inverse of any sequence of transforms can be computed in one step & we can have one compact representation for all transforms**

- **To allow one format for all transforms with matrix multiplications, we extend the transformation to 4 x 4 matrices**

Homogeneous transforms-2

- **Rotations & scaling: simply add a 4th row & column corresponding to the identity matrix and add a 1 after (X, Y, Z). E.g., rotation around Z can be redefined as:**

$$\begin{bmatrix} X_r \\ Y_r \\ Z_r \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Homogeneous transforms-3

- **However, translations can now be redefined as a matrix multiplication:**

$$\begin{bmatrix} X_t \\ Y_t \\ Z_t \\ 1 \end{bmatrix} + \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \Delta X \\ 0 & 1 & 0 & \Delta Y \\ 0 & 0 & 1 & \Delta Z \\ 0 & 0 & 0 & 1 \end{bmatrix} * \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Arbitrary *Rotations*

- Rotating from one direction to another, e.g. change vector V0 to V1
  - Angle between two vectors V0 & V1 = ?
  - Direction normal to the two vectors = ?

Arbitrary *Rotations*

- Angle between two vectors V0 & V1
  - Normalized Inner product of two vectors
  V0 = (x0, y0, z0), V1 = (x1, y1, z1):
  (V0.V1)/(||V0||*||V1||) =
  (x0*x1 + y0*y1 + z0*z1)/(||V0||*||V1||)
  - Cosine (angle) = normalized inner product
  - angle = ?

Arbitrary *Rotations*

- Direction normal to two vectors
  - Cross product of the two vectors

- Detailed equations can be found in:

  arbitrarytransforms.pdf