Project 1: 2-D Thermal Analysis

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Abstract

In this project, we are given some 2-D thermal distributions and required to solve the steady states of those thermal distributions. Originally, the heat equations are partial differential equations. By applying finite difference methods, we simplify it as a linear system which can be solved by Cholesky Decomposition and back-substitution.

1.Mathematical Formulation

This is the original heat equation, which is a 2nd-order linear PDE

$$\rho \cdot C_p \cdot \frac{\partial T(x,y,t)}{\partial t} = \kappa \cdot \bigtriangledown^2 T(x,y,t) + f(x,y,t)$$

For the steady state, the equation can be simplified as

$$\kappa \cdot \bigtriangledown^2 T(x,y) + f(x,y) = 0$$

Explicitly write out the laplace operator

$$\kappa \cdot [\frac{\partial^2 T(i,j)}{\partial x^2} + \frac{\partial^2 T(i,j)}{\partial y^2}] = -f(i,j)$$

The 2nd-order term can be written as the following with finite difference method

$$\frac{\partial^2 T(i,j)}{\partial x^2} = \frac{T_{i+1,j} + T_{i-1,j} - 2T_{i,j}}{\Delta x^2}$$
$$\frac{\partial^2 T(i,j)}{\partial y^2} = \frac{T_{i,j+1} + T_{i,j-1} - 2T_{i,j}}{\Delta y^2}$$

Combine all linear equations

$$\kappa \cdot [\frac{T_{i+1,j} + T_{i-1,j} - 2T_{i,j}}{\Delta x^2} + \frac{T_{i,j+1} + T_{i,j-1} - 2T_{i,j}}{\Delta y^2}] = -f(i,j)$$
$$(1 \leq i \leq N, 1 \leq j \leq M)$$

Rewrite the equation and add T_c as the term for boundary condition

$$2(\Delta x^2 + \Delta y^2)T_{i,j} - \Delta x^2(T_{i,j+1} + T_{i,j-1}) - \Delta y^2(T_{i+1,j} + T_{i-1,j})$$
$$= \frac{\Delta x^2 \Delta y^2 f(i,j)}{\kappa} + T_{c_{i,j}}$$
$$(1 \leq i \leq N, 1 \leq j \leq M)$$

We get s system of linear equations

$$A \cdot X = B$$
$$X = [T_{1,1}, T_{1,2}, \ldots, T_{M,N}]^T$$

When M = N = 2, the simplest linear equations are shown below

$$
\begin{bmatrix} 2(\Delta x^2+\Delta y^2) & -\Delta x^2 & -\Delta y^2 & 0 \\ -\Delta x^2 & 2(\Delta x^2+\Delta y^2) & 0 & -\Delta y^2 \\ -\Delta y^2 & 0 & 2(\Delta x^2+\Delta y^2) & -\Delta x^2 \\ 0 & -\Delta y^2 & -\Delta x^2 & 2(\Delta x^2+\Delta y^2) \end{bmatrix} * \begin{bmatrix} T_{1,1} \\ T_{1,2} \\ T_{2,1} \\ T_{2,2} \end{bmatrix} = \begin{bmatrix} \frac{\Delta x^2 \Delta y^2 f(1,1)}{\kappa} + T_{c_{1,1}} \\ \frac{\Delta x^2 \Delta y^2 f(1,2)}{\kappa} + T_{c_{1,2}} \\ \frac{\Delta x^2 \Delta y^2 f(2,1)}{\kappa} + T_{c_{2,1}} \\ \frac{\Delta x^2 \Delta y^2 f(2,2)}{\kappa} + T_{c_{2,2}} \end{bmatrix}
$$

2.Linear System Solver

For the linear system$A \cdot X = B$, A is a positive definite matrix which we can apply Cholesky Factorization.

$$A = C \cdot C'$$

So the original equation can be written as

$$C \cdot C' \cdot X = B$$

This is my implementation of Cholesky Decomposition

```
function [S] = choleskyDecom(A)
  [n,n] = size(A);
  S = zeros(n,n);
  for j = 1 : n
    S(j,j) = sqrt(A(j,j) - S(j,1:j-1)*S'(1:j-1,j));
    S(j+1:n,j)=(A(j+1:n,j) - S(j+1:n,1:j-1) * S'(1:j-1,j))/S(j,j);
  end
end
```

Now our task becomes to solve the following two equations, but it's much easier

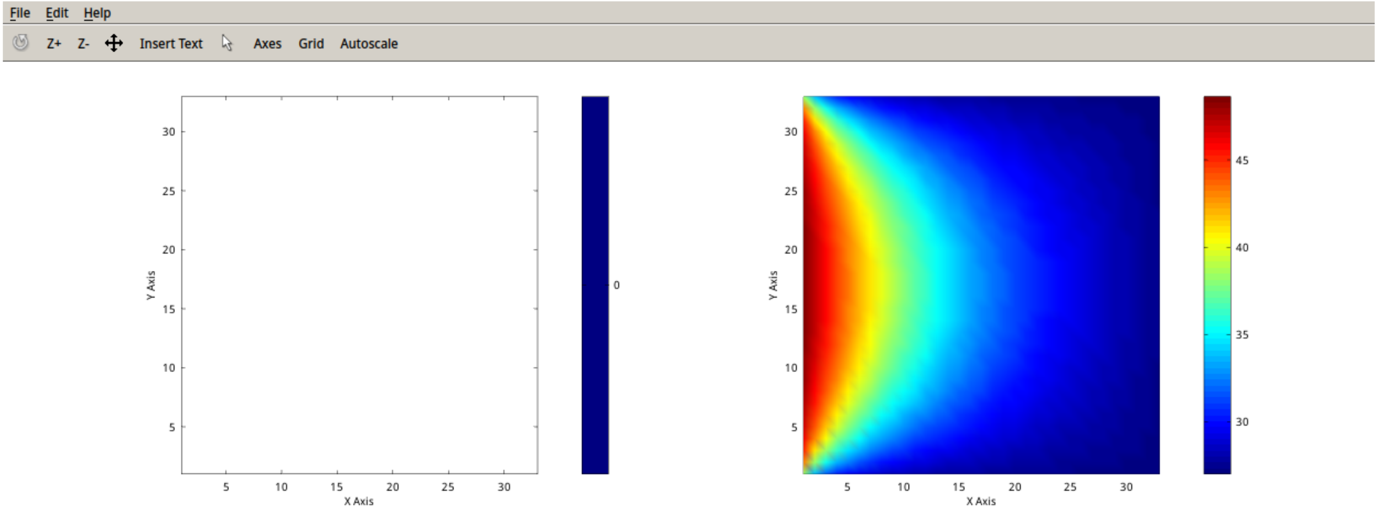
$$\begin{cases} CV = B \\ C'X = V \end{cases}$$

C and C' are lower-triangular and upper-triangular matrix, the equations can be easily solved by backward substitution

```
function [X] = choleskySolver(A, B)
  %solve the outer loop of the linear equations
  [n, n] = size(A);
  V = zeros(n,1);
  for i = 1 : n
    V(i) = (B(i) - A(i,1:i-1) * V(1:i-1)) / A(i,i);
  end
  %solve the inner loop of the linear equations
  A = A';
  X = zeros(n,1);
  for i = n : -1 : 1
    X(i) = (V(i) - A(i,i+1:n) * X(i+1:n)) / A(i,i);
  end
end
```

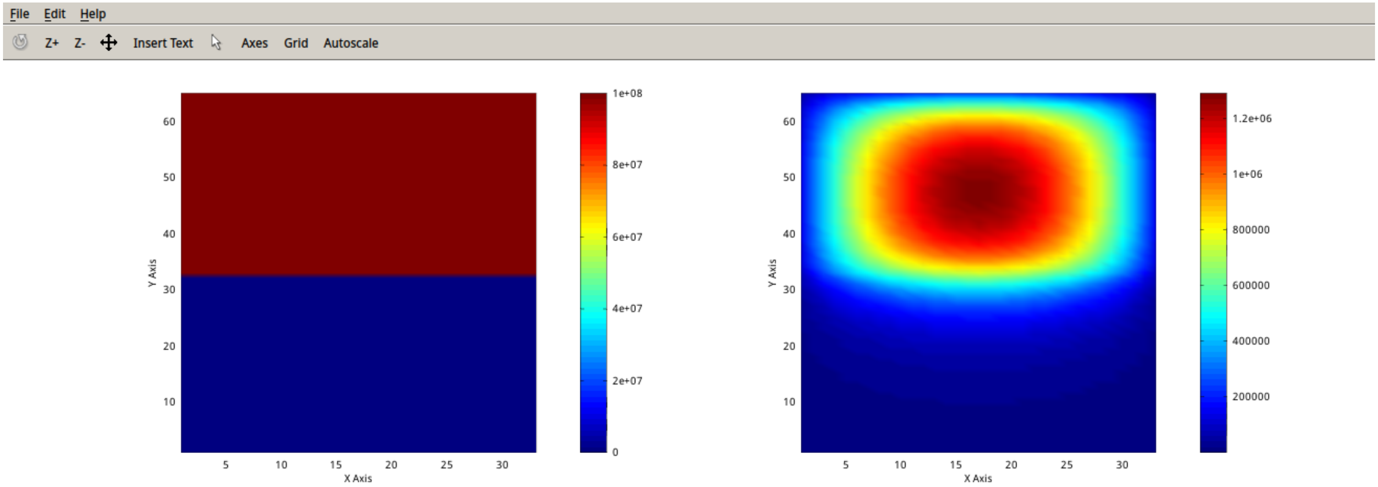
3.Experimental Results

The left side are original thermal distributions and the right side are the thermal distributions of steady state
Case 1
Runtime = 13.0667s



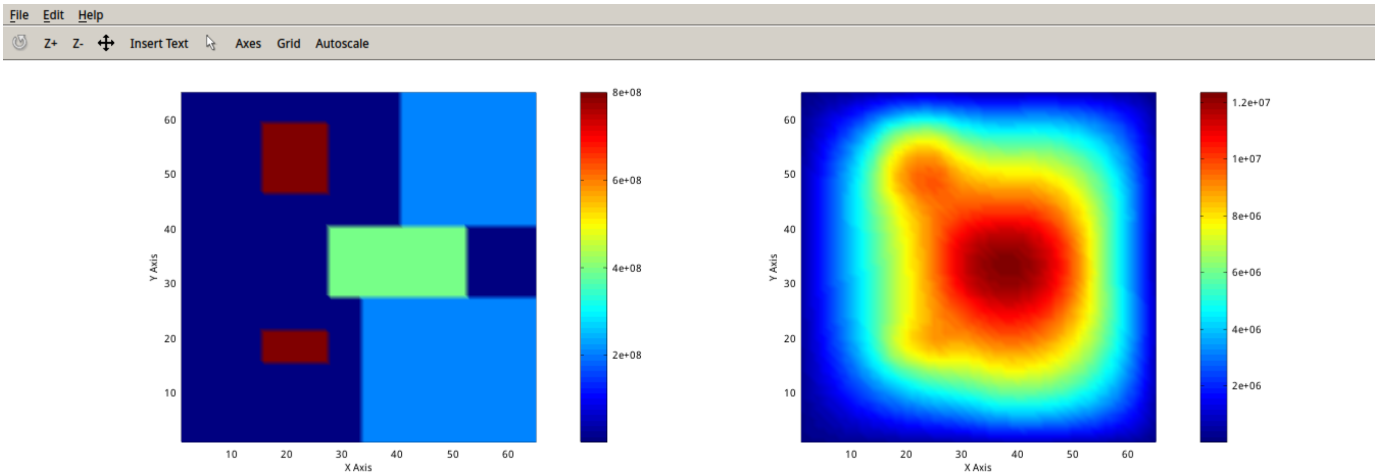
Case 2

Runtime = 169.576s



Case 3

Runtime = 1074.8s



4.Discussion

Using vector operations can greatly accelerate the computation. Originally, I implemented the CholeskyDecomposition and BackSubstitution in a very naive way which means a lot of 'for' loops. It took 3032.15 seconds to solve case 1, which means I wouldn't get the solution for case 3 with such naive implementation. The original code are shown below.

```
%Cholesky Decomposition naive version
for j = 1 : n
    for i = j : n
      tempSum = S(i,1:j-1)*S'(1:j-1,j);
```

```
      if i == j
        S(i,j) = sqrt(A(i,j) - tempSum);
      else
        S(i,j) = (A(i,j) - tempSum) / S(j,j);
      end
    end
  end

%Back Substitution naive version
for i = 1 : n
    tempSum = 0;
    for j = 1 : i-1
      tempSum += A(i,j)*V(j);
    end
    V(i) = (B(i) - tempSum) / A(i,i);
  end
```

I realised part of the loops can easily be replaced by vector operation which makes the code more elegant. The time needed to solve case 1 reduced from 3032 seconds to 13 seconds.

```
%Cholesky Decomposition fast version
for j = 1 : n
    S(j,j) = sqrt(A(j,j) - S(j,1:j-1)*S'(1:j-1,j));
    S(j+1:n,j)=(A(j+1:n,j) - S(j+1:n,1:j-1) * S'(1:j-1,j))/S(j,j);
  end
%Back Susbtitution fast version
for i = 1 : n
    V(i) = (B(i) - A(i,1:i-1) * V(1:i-1)) / A(i,i);
  end
```

It looks like a lot of progress, but I compared the built-in chol() function and the Cholesky Decomposition I implemented and it shows the built-in chol() is much faster.

Matrix Size	Built-in chol()	My Implementation
50 * 50	0.00136s	0.01781s
200 * 200	0.00212s	0.08110s
500 * 500	0.00980s	1.0816s
1000 * 1000	0.01855s	13.1817s

Someone also asked this question why the built-in chol() is so fast on Matlab forum, but there’s no exact answer to it. And I think the key is to get rid of the last loop in my function, but I can’t figure out how to do that without using backslash. One of my hypothesis is that they might use a totally different algorithms for chol() which makes it so fast.