Project 2: Image Recovery

Yufeng Yuan
yy208@duke.edu

Abstract

In this project, for a given image, we need to sample some pixel points in the image to recover the original image. The sample points should be segmented into training set and test set randomly and only the training set would be used before finding the optimal hyper-parameter. With Discrete Cosine Transformation and the sampled points, an under-determined linear system could be formed and it could be solved by Orthogonal Match Pursuit. Then cross validation could be applied to find the best hyper-parameter. With the optimal hyper-parameter, the sparse solution could obtained by Orthogonal Match Pursuit algorithms. Inverse Discrete Cosine Transform could be applied next to recovery the image. The recovery quality is measured by the mean square error.

1 Overview

First, I implemented a function for random sampling with a Matlab built-in function randomperm which generates random integer sequence which ensures that there's no repeated points in the samples. For OMP Solver function, part of it is to generate the under-determined linear system of Discrete Cosine Transform and the other part of it is to solve the linear system with given lambda by OMP algorithms. Also, this function is implemented with some matrix operation techniques to improve the running efficiency. For the implementation of cross validation, all the sample points will first be segmented into training set and test set. Then, the OMP solver and inverse DCT will be applied based on the training set to recover the image. By comparing the recovery quality, the optimal hyper-parameter could be found by doing cross validation for several iterations. The last function is Inverse Discrete Cosine Transform, with the optimal hyper-parameter, by applying IDCT, the image could be recovered. Note: The whole project was done in Octave environment, there might be some capability issues with Matlab.

2 Mathematical Formulation

This section consists two subsection, the first is how the linear system is obtained and the second is how to implement the Orthogonal Match Pursuit algorithms.

2.1 Linear System

A 2-D image can be mapped to frequency domain by Discrete Cosine Transform (DCT)

$$G(u,v) = \sum_{x=1}^{P}\sum_{y=1}^{Q} a_u \cdot b_v \cdot g(x,y) \cdot cos\frac{\pi(2x-1)(u-1)}{2\cdot P}$$

$$\cdot cos\frac{\pi(2y-1)(v-1)}{2\cdot Q}$$

$$x,u \in \{1,2,...P\} \quad y,v \in \{1,2,...Q\}$$

$$a_u = \begin{cases} \sqrt{\dfrac{1}{P}}\ (u=1) \\ \sqrt{\dfrac{2}{P}}\ (2 \le u \le P) \end{cases}$$

$$b_v = \begin{cases} \sqrt{\dfrac{1}{Q}}\ (v=1) \\ \sqrt{\dfrac{2}{Q}}\ (2 \le v \le Q) \end{cases}$$

Reversely, a 2-D image can also be uniquely determined by Inverse Discrete Cosine Transform (IDCT), if all DCT coefficients are known

$$g(x,y) = \sum_{u=1}^{P}\sum_{v=1}^{Q} a_u \cdot b_v \cdot G(u,v) \cdot cos\frac{\pi(2x-1)(u-1)}{2\cdot P}$$

$$\cdot cos\frac{\pi(2y-1)(v-1)}{2\cdot Q}$$

We can find the approximate solution of DCT coefficients by sampling a 2-D image at M spatial locations and form a set of M linear equations.

$$g(x_1,y_1) = \sum_{u=1}^{P}\sum_{v=1}^{Q} a_u \cdot b_v \cdot G(u,v) \cdot cos\frac{\pi(2x_1-1)(u-1)}{2\cdot P}$$

$$\cdot cos\frac{\pi(2y_1-1)(v-1)}{2\cdot Q}$$

$$g(x_2,y_2) = \sum_{u=1}^{P}\sum_{v=1}^{Q} a_u \cdot b_v \cdot G(u,v) \cdot cos\frac{\pi(2x_2-1)(u-1)}{2\cdot P}$$

$$\cdot cos \frac{\pi(2y_2 - 1)(v-1)}{2 \cdot Q}$$

$$\cdots$$

$$g(x_M, y_M) = \sum_{u=1}^{P} \sum_{v=1}^{Q} a_u \cdot b_v \cdot G(u,v) \cdot cos \frac{\pi(2x_M - 1)(u-1)}{2 \cdot P}$$

$$\cdot cos \frac{\pi(2y_M - 1)(v-1)}{2 \cdot Q}$$

Rewrite the linear equations in matrix form, which is under-determined linear equations. For simplicity, we use $C_{m,u,v}$ to denote $a_u \cdot b_v \cdot cos \frac{\pi(2x_m-1)(u-1)}{2 \cdot P} \cdot cos \frac{\pi(2y_m-1)(v-1)}{2 \cdot Q}$

$$\begin{bmatrix} g(x_1, y_1) \\ g(x_2, y_2) \\ \vdots \\ g(x_M, y_M) \end{bmatrix} =$$

$$\begin{bmatrix} C_{1,1,1} & C1,1,2 & \cdots & C_{1,P,Q-1} & C_{1,P,Q} \\ C_{2,1,1} & C2,1,2 & \cdots & C_{2,P,Q-1} & C_{2,P,Q} \\ & & \vdots & & \\ C_{M,1,1} & CM,1,2 & \cdots & C_{M,P,Q-1} & C_{M,P,Q} \end{bmatrix} \cdot \begin{bmatrix} G(1,1) \\ G(1,2) \\ \vdots \\ G(P,Q) \end{bmatrix}$$

This is an under-determined linear system which could be solved by OMP algorithms described below

2.2 The Implementation of OMP Algorithms

These are the goal function and constraint for this problem

$$\min_{\alpha} \|A\alpha - B\|_2^2$$

$$S.T. \|\alpha\|_0 \leq \lambda$$

Step 1: Set F = B, $\Omega = \{\}$ and p = 1

Step 2: Calculate the inner product values $\theta_i =< A_i, F >$

Step 3: Identify the index s for which $|\theta_s|$ takes the largest number

Step 4: Update Ω by $\Omega = \Omega \cup \{s\}$

Step5: Approximate F by the linear combination of $\{A_i, i \in \Omega\}$

$$\min_{a_i, j \in \Omega} \| \sum_{i \in \Omega} \alpha_i \cdot A - B\|_2^2$$

Step 6: Update F

$$F = B - \sum_{i \in \Omega} \alpha_i \cdot A$$

Step 7: if p $\leq \lambda$, p = p + 1 and go to Step 2. Otherwise, stop.

$\alpha_i = 0 (i \notin \Omega)$

3 Experimental Results

3.1 Small Test Image

The block size of this image is 8 by 8

Sample Size = 10, Mean Square Error = 1563.0

Sample Size = 20, Mean Square Error = 1374.4

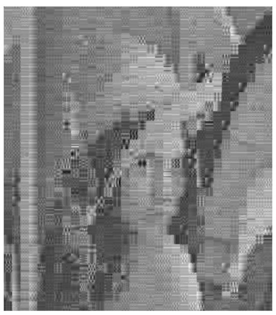
Sample Size = 30, Mean Square Error = 1340.2

Sample Size = 40, Mean Square Error = 1157.7

Sample Size = 50, Mean Square Error = 1090.7

3.2 Large Test Image

The block size of this image is 16 by 16



Sample Size = 10, Mean Square Error = 742.9

Sample Size = 30, Mean Square Error = 638.9

Sample Size = 50, Mean Square Error = 598.9

Sample Size = 100, Mean Square Error = 540.1

Sample Size = 150, Mean Square Error = 544.4

3.3 Cross Validation

In cross validation, we need to find the optimal lambda which is the hyper-parameter in OMP algorithms. We use lambda ranging from 5 to 50. More specifically, lambda = [5, 10, 15, 20, 30, 50]. For better comparison, we use the same lambda setting for both images. The following diagrams are the optimal lambda found by cross validation, the first figure is the small one and the second figure is the large one. In general, the optimal lambda increases with the sample number.

Samples	Lambda
10	5
20	20
30	10
40	30
50	50

Samples	Lambda
10	5
30	5
50	10
100	50
150	50

3.4 Recovery Error

The first diagram is for the small test image whose block sizes are 10, 20, 30, 40 and 50. The second diagram is for the large test image whose block sizes are 10, 30, 50, 100 and 150. Error 1 is the mean square error after median filtering, while error 2 is the mean square error before median filtering.

Samples	Error 1	Error 2
10	1563.0	2320.8
20	1374.4	2168.2
30	1340.2	1520.8
40	1157.7	3007.0
50	1090.7	3374.9

Samples	Error 1	Error 2
10	742.9	1049.8
30	638.9	790.1
50	598.9	652.8
100	540.1	984.0
150	544.4	666.9

4 Discussion

4.1 Factors that Impact the Recovery Quality

Sample Number

With the increase of sample number, the improvement of the recovery quality could be observed here. In our settings, the quality improvement is more obvious at first several times of increasing sample points, after that, the improvement becomes less obvious. However, the cost of more sample points is longer sampling process and computation. It hard to find the optimal sample to balance the computation time and recovery quality because it's very problem-dependent.

Lambda for OMP Algorithms

According to the figure above, lambda chosen by cross validation is different according to the sample size. For different sample number, there are different lambdas that fit the problem optimally. However, the cross validation process is very computational-expensive and the how to choose the range of lambdas is also experience-dependent.

Block Size

Though the block size is given in this project, an appropriate block size is crucial to image recovery which is why we have two different settings for two test image. For each block, we need to find the sparse solution by OMP algorithms. If the block size is too large, the solution may not be sparse. If the block size is too small, the recovery quality may not be that good.

4.2 Limits of This approach

Computational Complexity

This algorithms consists many loops inside, the whole algorithms has high computational complexity though some of the computation could be accelerated by matrix operation. The time required for computation increases rapidly with the increase of problem size, which makes this approach unable to handle high-resolution images.

Hyper-Parameters

This approach consists of a lot of hyper-parameters like block size, and lambda for OMP algorithms. Those parameters should be carefully tuned because these parameters are critical to the recovery quality. However, this is very time-consuming and experience-dependent .

4.3 Improvement of Running Efficiency

The bottleneck of this approach is that there are so many loop in it, and part of the loop could be rewritten as matrix operation form. For example, the generation of DCT coefficients linear system, I implemented it without any loop which greatly accelerated the computation. The code is shown below.

```
[b,n,m] = size(points);
block = 1:1:blockSize;
x = points(1,:,2)'; y = points(1,:,3)';
matrix1 = cos(pi*(2*x-1)*(block -1)/(2*blockSize));
matrix2 = cos(pi*(2*y-1)*(block -1)/(2*blockSize));
matrix1 = repmat(matrix1, 1, blockSize);
matrix2 = kron(matrix2, ones(1, blockSize));
matrix = matrix1 .* matrix2;
matrix /= blockSize; matrix *= 2;
matrix(1,:) /= sqrt(2);
matrix(:,1) /= sqrt(2);
```