Chapter 2: An Alternatively Mathematical Method for Shape Description

I. Kac's Idea [5]: Can one hear the shape of a drum?

   A. Interpretations

      i. How much can shape attributes be inferred from hearing all the pure tones produced by a drum?
      ii. Or, if you can obtain complete eigenvalues of the Dirichlet problem for the Laplacian precisely, will you determine the shape of a planar?

   B. Summarize Kac's idea

      i. Consider a simply connected membrane Ω in the Euclidean space bounded by a smooth convex curve Γ (i.e. a drum).
      ii. As $t \to 0$, establishing the leading terms of the asymptotic expansion of the spectral function (heat trace) $\sum_{n=1}^{\infty} e^{-\lambda_n t}$.
      iii. Without using rulers, some geometrical features of Ω can be deduced from the asymptotic expansions of spectral functions involving eigenvalues of Laplacian-type operators.

   C. Conclusion

      i. Except in very few cases (i.e. rectangular, disk, and certain triangles), it is impossible to obtain the λ_n, which results in the difficulties in practice.
      ii. The negative answers to Kac's original mathematical question in the realm of Riemannian manifolds have been known in Kac's time. Moreover, the counterexample in the plane proposed in 1992 [3].

II. Extended Works of Kac's Idea: Heat Content

   Finding the unique solution u of the heat equation defined in D

   A. Asymptotic expansion of $Q_D(t) := \int_D u$ as $t \to 0^+$

      i. Interpretation: the total amount of heat contained in D at the moment $t \geq 0$
      ii. A compact smooth Riemannian manifold of dimension m with C^∞ boundary ∂D [1].

         a. Volume
         b. Length
         c. Scalar Curvature
         d. Mass
         e. etc.

   B. Limitations

      It is not numerically implementable for shape description by

i. obtaining complete eigenvalues.
ii. calculating explicit solutions for heat (diffusion) equations in complex systems (i.e. plant root systems).
iii. $t \to 0^+$.
iv. acknowledging the infinite number of coefficients of asymptotic series .

III. Heat Content on an Annulus[1]

A. Analytical Results

i. Solving Diffusion Equation

a. Mathematical Expression

- Initial condition: uniform distribution
- Boundary conditions
  - D.B.C (absorbing) for target (interior) boundary
  - N.B.C (reflecting) for escape (exterior) boundary

b. Methods

- Dimensional analysis
- Separation of variables

c. Solutions: $u(\hat{r}, \theta, \tau)$

ii. Survival Probability $S(\tau)$

iii. Mean First-Passage Time $< \tau >$

B. Numerical Analysis

i. Eigenvalues (Newton-Raphson in Scipy) [15]

ii. Infinite Series Approximation (Algorithms)

a. $S(\tau)$

b. $< \tau >$

iii. Errors

a. Truncation

b. Round-off

IV. Monte Carlo Simulations

A. Background

i. Probabilistic Interpretation of the Diffusion Process

a. Diffusion Equations [4]

b. First-Passage Process [12]

- Occupation Probability: $P(\overrightarrow{r}, t)$
- First-Passage Probability: $F(\overrightarrow{r}, t)$
- Survival Probability: $S(t) = -\int F(t)dt$
- Mean First-Passage Time: $< t >= \int_0^\infty S(t)dt$

ii. Some deterministic numerical schemes can replace the original continuous problems by a set of linear equations (e.g. finite differences, finite elements, etc.). However, their accuracy and efficiency highly rely on the discretization [8].

iii. Monte Carlo Techniques for Continuous Diffusion Process [13][14]

a. Random Walks and Markov Process

b. Continuum Limit for Diffusion Equations [7]

c. Continuous Space and Discrete Time Isotropic Random Walks

- Karl Pearson's Question [9]
- Rayleigh's Random Flights [10] [11]

d. Discrete Space and Discrete Time Symmetric Hopping Process

- Simple Random Walks on the Integer Lattice $\mathbb{Z}^d$ [12]

B. Output Analysis

i. Kaplan-Meier Estimator [6]

C. Algorithm Description

i. Lattice Random Walks (LRWs)

ii. Pearson's Random Walks (PRWs)

V. Two-sample Statistical Tests

A. General Methods

i. Kolmogorov-Smirnov Test

ii. Anderson-Darling Test

B. Nonparametric Tests for Survival Distributions [2]

i. Mantel-Haenszel Test

ii. Fleming-Harrington Test

iii. Gehan-Breslow-Wilcoxon Test

iv. Tarone-Ware Test

VI. Research Design

A. Methodology

Without analytical calculation, the behaviours of the asymptotic expansion of heat content can be mimicked approximately by the survival distribution of particles' diffusing times in the random walks.

B. Idea

Instead of measuring with rulers, the innovative mathematical tool, random walks, can describe and compare the geometrical features of plant root shapes in the 2D images.

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