C2 S2: Extended Works of Kac's Idea: Heat Content

I. Mathematical Formula

A. Heat Equation

i. Fouier's heat equation

a. mathematical interpretation: deterministic model for the heat flow; how temperature chages over the space and time;

b. physical interpretation: conservation of heat per unit volume over an infinitesimally samll volumne lying in the interior of the flow domain;

ii. Connection with the Diffusion Equation

a. The diffusion equation describes the density fluctuations in a material undergoing diffusion.

b. When the diffusion coefficient is independent of the density (i.e. constant diffusion coefficient), the diffusion equation is also named the heat equation.

iii. Define an Initial-Boundary Value Problem (IBVP)

a. Domain: $\Omega$; homogeneous and isotropic;

b. Heat equation: $\frac{\partial u(\boldsymbol{s},t)}{\partial t} = \Delta u(\boldsymbol{s}, t)$

c. Initial condition: $u(\boldsymbol{s}, t) = f(\boldsymbol{s})$ for $t = 0$

d. Dirichlet boundary condition: $u(\boldsymbol{s}, t) = 0$ for $t > 0$ and $\boldsymbol{s} \in \partial\Omega$

B. General Solution to the IBVP

i. the forms

a. a series of error functions or related integrals (Laplace's transformation)

b. the trigonometric series or a series of Bessel functions (separation of variables)

ii. $u(\boldsymbol{s}, t) = \sum_{k=1}^{k=\infty} a_k u_k(\boldsymbol{s}) e^{-\lambda_k t}$

a. obtained by separation of variables

b. the $u_k$ form an orthonormal basis of $L^2(\Omega)$ of real valued eigenfunctions; the corresponding Dirichlet eigenvalues $\lambda_k \in \mathbb{R}^+$; $-\Delta u_k = \lambda_k u_k$

c. $h(t) = \sum_{k=1}^{k=\infty} e^{-\lambda_k t}$ is the heat trace, which is a smooth function and converges for every $t > 0$

d. $a_k = \int_\Omega f(\boldsymbol{s}) u_k(\boldsymbol{s}) d\boldsymbol{s}$

iii. $u(\boldsymbol{s}, t) = \int_\Omega G(\boldsymbol{s}, t, \boldsymbol{s_0}) \delta(\boldsymbol{s} - \boldsymbol{s_0}) d\boldsymbol{s_0}$

a. $\delta$ is the Dirac delta function.

b. the initial condition: $u(\boldsymbol{s}, t) = \delta(\boldsymbol{s} - \boldsymbol{s_0})$ for $t = 0$

c. $G(\boldsymbol{s}, t, \boldsymbol{s_0}) = \sum_{k=1}^{\infty} u_k(\boldsymbol{s}) u_k(\boldsymbol{s_0}) e^{-\lambda_k t}$

d. $u(\boldsymbol{s}, t)$ is the convolution of the initial condition with Green's function $G(\boldsymbol{s}, t, \boldsymbol{s_0})$

e. $G(\boldsymbol{s}, t, \boldsymbol{s_0})$ is the foundamental solution (heat kernel of $\Omega$) describing the distribution of the heat after time $t$ when there has a single heat source at $\boldsymbol{s_0} \in \Omega$;

C. Heat Content $Q_\Omega(t)$

i. $Q_\Omega(t) = \int_\Omega d\boldsymbol{s} u(\boldsymbol{s}, t)$

ii. As $t \to 0$, $Q_\Omega(t) = 1 + \sum_{k=1}^{\infty} \beta_k t^{\frac{k}{2}}$

iii. Obtain geometrical information of $\Omega$ from $\beta_k$

a. area

b. length

c. scalar Curvature

d. mass

e. etc.

II. Difficulties in Application

A. The pure analytical method for finding $u(\boldsymbol{s}, t)$:

i. It can apply strictly only to the linear form of the boundary conditions and to constant diffusion properties.

ii. Except in very few cases (i.e. rectangular, disk, certain triangles), $\lambda_k$ can not be calculated.

iii. Either irregular geometries or discontinuities lead to the complexities, so the explicit algebraic solutions $u(\boldsymbol{s}, t)$ are close to non-existed.

B. Difficulties in calculating the asympotic expansion of $Q_\Omega(t)$ as $t \to 0$ because of the complicated forms of the coeffients $\beta_k$.

C. The numerical evaluation of the analytical solutions $u(\boldsymbol{s}, t)$ and $Q_\Omega(t)$ is usually by no means trivial because they are in the form of infinite series.