Implementation of Airy function using Graphics Processing Unit (GPU)

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Outline

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Abstract

- Fractional Calculus (FC) is a field of mathematics that extends the order of derivatives, Integrals and Differential equations to an arbitrary non-integer order from its integer order variant.

$$\frac{d^{1/2}y}{dx^{1/2}} = ?, \frac{d^{0.59}y}{dx^{0.59}} = ?, \frac{d^{\pi}y}{dx^{\pi}} = ?, \frac{d^{2-4j}y}{dx^{2-4j}} = ?$$

- This sometimes lead to Special Mathematical Functions (like in our case, Airy function) which are non-elementary functions and a gruelling task for computation.
- To overcome this barrier researchers have used Hardware Accelerators like Graphics Processing Unit (GPU) to decrease the computational time required to evaluate these functions.

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Outline

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Problem

The Airy function is a solution to the Airy differential equation and we have to implement the equation on a GPU i.e write a program that will evaluate the function on its domain and also reduce the computational time required to do so.

Solution

Write a program for Airy function according to its definition and use MATLAB's inbuilt Parallel Computing Toolbox (PCT) to program the GPU. Compare the computational time required on CPU and GPU for same input parameters.

A solution to the Schrödinger equation...

- Airy function is named after British astronomer and physicist George Biddell Airy, who stumbled upon the following differential equation during his study of caustics in optics domain.

$$W'' = -\frac{\pi^2}{12} mW \tag{1}$$

- Today, Airy function is popularly know for being a solution to the One Dimensional Time Independent Schrödinger equation when solved for a triangular potential well.

$$\frac{-\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + q\varepsilon x\psi(x) = En\psi(x) \tag{2}$$

- But, in general it is solution to differential equations of the following form called as Airy differential equation.

$$\frac{d^2y}{dx^2} = xy \tag{3}$$

Definition of Airy function

- As one would notice from equation (3) it is a non-linear second order ordinary differential equation, which means it can have two unique linearly independent solutions.
- Those 2 solutions are know as Airy function of first kind $Ai(x)$ and Airy function of second kind $Bi(x)$.

Airy function of First Kind

$$Ai(x) = \frac{1}{\pi}\int_0^{\infty} cos(\frac{t^3}{2} + xt)dt, x \in \mathbb{R} \tag{4}$$

Airy function of Second Kind

$$Bi(x) = \frac{1}{\pi}\int_0^{\infty} e^{\frac{-t^3}{3+xt}} + sin(\frac{t^3}{3} + xt)dt, x \in \mathbb{R} \tag{5}$$

- One can easily see that for computing any of the two functions we have to implement a numerical integration method!
- In literature, the Runge-Kutta-Nyström method (or RKN method) is being applied.
- But again, even if we use a numerical method we will not be able to compute the function "parallelly" because the numerical methods usually have dependence on the previous computed value and we have cannot evaluate the function to an arbitrary precision.

- To overcome this problem, we would use an alternate definition from B.K Agarwal & Hari Prakash (1996)'Quantum Mechanics', p.207-210, PHI.

Airy function of First Kind using Bessel functions

$$Ai(x) = \frac{1}{\pi}\sqrt{\frac{x}{3}}K_{\frac{1}{3}}(\frac{2}{3}x^{\frac{3}{2}}), x > 0 \quad (6)$$

$$Ai(0) = \frac{3^{\frac{2}{3}}}{\Gamma(\frac{2}{3})}, x = 0 \quad (7)$$

$$Ai(x) = \sqrt{\frac{x}{9}}(J_{\frac{1}{3}}(\frac{2}{3}x^{\frac{3}{2}}) + J_{-\frac{1}{3}}(\frac{2}{3}x^{\frac{3}{2}})), x < 0 \quad (8)$$

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Airy function of Second Kind using Bessel functions

$$Bi(x) = \sqrt{\frac{x}{3}}\left(I_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) + I_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\right), x > 0 \tag{9}$$

$$Bi(0) = \frac{3^{\frac{-1}{6}}}{\Gamma(\frac{2}{3})}, x = 0 \tag{10}$$

$$Bi(x) = \sqrt{\frac{x}{3}}\left(J_{\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right) - J_{-\frac{1}{3}}\left(\frac{2}{3}x^{\frac{3}{2}}\right)\right), x < 0 \tag{11}$$

- The functions $I_\alpha(\cdot), K_\alpha(\cdot)$ are the non-integer order Modified Bessel function of first and second kind of order α respectively, $J_\alpha(\cdot)$ is the Bessel function of first kind of order α. $\Gamma(\cdot)$ is the Gamma function.

Definition of Bessel functions

Bessel function of First Kind of α order

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\alpha+1)}\left(\frac{x}{2}\right)^{2m+\alpha}, \alpha \in \mathbb{R} \tag{12}$$

Modified Bessel function of First Kind of α order

$$I_\alpha(x) = \left(\frac{x}{2}\right)^{\alpha} \sum_{k=0}^{\infty} \frac{\left(\frac{x^2}{4}\right)^k}{k!\Gamma(k+\alpha+1)}, \alpha \in \mathbb{R} \tag{13}$$

Modified Bessel function of Second Kind of α order

$$K_\alpha(x) = \frac{\pi}{2} \frac{I_{-\alpha}(x) - I_\alpha(x)}{\sin(\alpha\pi)}, \alpha \in \mathbb{R} \tag{14}$$

Outline

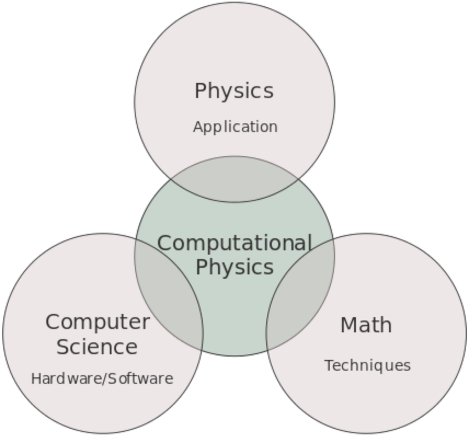

Figure: Interdisciplinary nature of the this experiment.

Image reference: https://www.wikiwand.com/en/Computational-physics

CPU v/s GPU

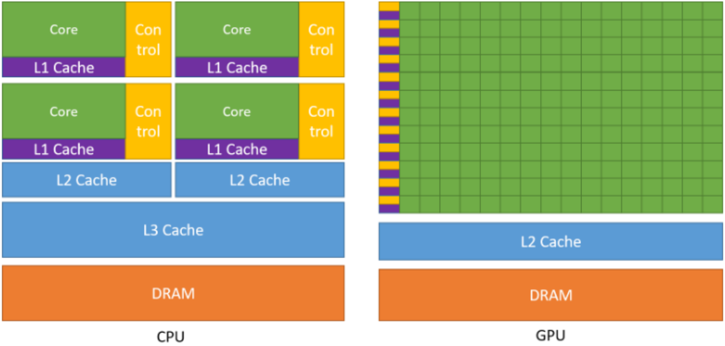

Figure: The GPU Devotes More Transistors to Data Processing.

Image reference: https://docs.nvidia.com/cuda/cuda-c-programming-guide/index.htmlfrom-graphics-processing-to-general-purpose-parallel-computing

Programming model of GPU

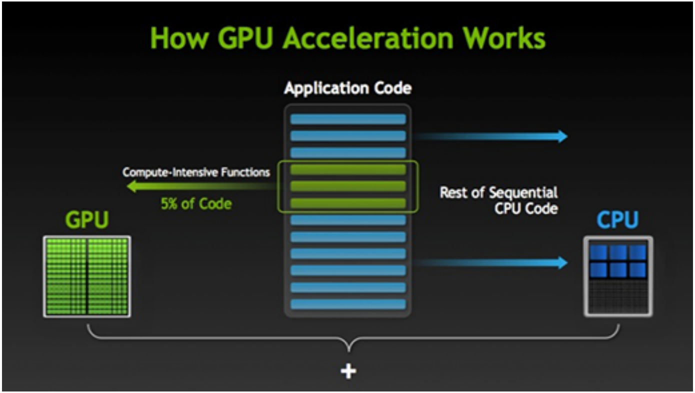

Figure: Programming on a heterogeneous system.

Image reference: https://www.analyticsvidhya.com/blog/2018/04/sequence-modelling-an-introduction-with-practical-use-cases/how-gpu-acceleration-works/

Speedup Parameter

The Speedup parameter was calculated by

$$Speedup = \frac{CPUTime(T_c)}{GPUTime(T_g)} \tag{15}$$

Speedup for $Ai(x)$

x	CPU Time (T_c in ms)	GPU Time (T_g in ms)	Speedup
1	0.494	0.186	2.655913978
2	0.475	0.184	2.581521739
3	0.474	0.183	2.590163934
4	0.473	0.182	2.598901099
5	0.473	0.181	2.613259669
10	0.467	0.181	2.580110497
50	0.455	0.177	2.570621469
100	0.447	0.174	2.568965517
1000	0.448	0.168	2.666666667

- Note: All time parameters are a mean of 10,000 observations for computational time.

Speedup for $Bi(x)$

x	CPU Time (T_c in ms)	GPU Time (T_g in ms)	Speedup
1	0.468	0.191	2.450261783
2	0.463	0.190	2.436842105
3	0.461	0.189	2.439153439
4	0.460	0.185	2.486486486
5	0.457	0.184	2.483695652
10	0.453	0.180	2.516666667
50	0.450	0.176	2.556818182
100	0.447	0.173	2.583815029
1000	0.437	0.168	2.601190476

- Note: All time parameters are a mean of 10,000 observations for computational time.

Conclusion

After the calculation of Speedup parameter for both kinds of Airy function, it can be concluded that the computational time required for execution on CPU is 2 to 3 fold than GPU when implemented on NVIDIA Tesla K80 system.

Any Questions?

Thank You!

Get Fractionalized!!!