Driven linear collisionless gyrokinetics with $\delta B_\parallel = 0$

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1 Set up

We start fro the Laplace-Fourier solution given in the AstroGK manual, i.e., Eq. (6.124):

$$\hat{A}_{\parallel\mathbf{k}}(p) = \frac{-Q^2 A_{\parallel\mathbf{k}_0}}{(p^2+Q^2)\,(p+i\omega_0)} \tag{1}$$

where

$$Q^2(p) = \frac{\alpha_i A k_\parallel^2 v_A^2}{AB - B^2} \tag{2}$$

and

$$A = \sum_s \frac{T_i}{T_s}(1+\Gamma_{0s}\xi_s Z_s) \tag{3}$$

$$B = \sum_s \frac{T_i}{T_s}(1-\Gamma_{0s}) \tag{4}$$

$\alpha_s = k_\perp^2 \rho_s^2/2$ and $\xi_s = ip/kv_{ts}$

p^2+Q^2 is just the dispersion relation. The system with $\delta B_\parallel = 0$ has two solutions, both Alfvén waves. We denote the two solutions $p_1 = -i\omega_1$ and $p_2 = p_1^* = -i\omega_2 = i\omega_1^*$. AstroGK manual approximates p^2+Q^2 by $(p-p_1)(p-p_2)$.

2 Inverse Laplace transform

We take a different path here and directly evaluate the inverse Laplace transform via Bromwich integral.

$$\begin{aligned} A_{\parallel\mathbf{k}} &= \mathrm{ILT}\left(\hat{A}_{\parallel\mathbf{k}}\Big(p)\right) && (5)\\ &= \frac{1}{2\pi i}\int_{\beta-i\infty}^{\beta+i\infty} \frac{-Q^2 e^{pt}}{(p^2+Q^2)(p+i\omega_0)}dp && (6)\\ &= \mathrm{Res}(p_1)+\mathrm{Res}(p_2)+\mathrm{Res}(p=-i\omega_0) && (7)\end{aligned}$$

where Res denotes residue.

The residue at $p_0 = -i\omega_0$ is trivially

$$\mathrm{Res}(p = -i\omega_0) = \left[\frac{-Q^2}{p^2 + Q^2}\right]_{p=-i\omega_0} \tag{8}$$

The residues at p_1 and p_2 are evaluated by

$$\begin{aligned}
\mathrm{Res}(p_1) &= \left[(p - p_1)\frac{-Q^2 e^{pt}}{(p^2 + Q^2)(p + i\omega_0)}\right]_{p=p_1} && (9)\\
&= \frac{-Q^2(p_1)e^{p_1 t}}{\left[\frac{d(p^2+Q^2)}{dp} + \sum_{n=2}^{\infty}\frac{1}{n!}\frac{d^n(p^2+Q^2)}{dp^n}(p - p_1)^n\right]_{p=p_1}(p_1 + i\omega_0)} && (10)\\
&= \frac{-Q^2(p_1)e^{p_1 t}}{\left[\frac{d(p^2+Q^2)}{dp}\right]_{p=p_1}(p_1 + i\omega_0)} && (11)\\
& && (12)
\end{aligned}$$

and

$$p_1 \leftrightarrow p_2 \tag{13}$$

For convenience to compare with simulation, we introduce the dimensionless frequency

$$\begin{aligned}
\overline{\omega} &= \frac{\omega}{k_\parallel v_A} = \frac{p}{-ik_\parallel v_A} && (14)\\
\bar{t} &= k_\parallel v_A t && (15)
\end{aligned}$$

Therefore

$$\xi_s = \sqrt{\frac{T_i}{T_s}\frac{m_s}{m_i}}\frac{\overline{\omega}}{\sqrt{\beta_I}} \tag{16}$$

The dimensionless dispersion relation takes the form:

$$\overline{\omega}^2 - \overline{Q}^2 = 0 \tag{17}$$

where

$$\overline{Q}^2 = \frac{Q^2}{-k_\parallel^2 v_A^2} = \frac{\alpha_i A}{(A - B)B} \tag{18}$$

Hence the residue $\mathrm{Res}(p = -i\omega_0)$ becomes

$$\mathrm{Res}(\overline{\omega}_0) = \frac{\overline{Q}^2(\overline{\omega}_0)e^{-i\overline{\omega}_0\bar{t}}}{\overline{\omega}_0^2 - \overline{Q}^2(\overline{\omega}_0)} \tag{19}$$

The residues of the other two poles are

$$\mathrm{Res}(\overline{\omega}_j) = \frac{\overline{Q}^2(\overline{\omega}_j)e^{-i\overline{\omega}_j\bar{t}}}{(G(\overline{\omega}_j) + 2\overline{\omega}_j)(\overline{\omega}_j - \overline{\omega}_0)} \tag{20}$$

where $j = 1, 2$ and

$$G = \frac{\alpha_i}{\sqrt{\beta_i}(A-B)^2} \sum_s \left(\frac{T_i}{T_s}\right)^{3/2} \sqrt{\frac{m_s}{m_i}} \Gamma_{0s} \left[(1 - 2\xi_s^2)Z_s - 2\xi_s\right] \tag{21}$$