Laplace-Fourier analysis for slow mode driving

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1 Setup

We drive slow mode in gyrokinetic system by adding $\delta B_{\parallel a}$ to Maxwell's equations. We begin with the Fourier transformed gyrokinetic equation:

$$\frac{\partial g_{\mathbf{k}s}}{\partial t} + ik_\parallel v_\parallel g_{\mathbf{k}s} + \frac{q_s}{T_s} v_\parallel F_{0s} ik_\parallel \tilde{\phi} = -\frac{q_s}{T_s} F_{0s} \frac{\tilde{A}}{\partial t} \tag{1}$$

where $\tilde{\phi}$ and $\tilde{A}$ are the source terms:

$$\begin{aligned} \tilde{\phi} &= j_0(k_\perp \rho_{\perp s})\phi_{\mathbf{k}} + \frac{J_1(k_\perp \rho_{\perp s})}{k_\perp \rho_{\perp s}} \frac{mv_\perp^2}{q_s} \frac{\delta B_{\parallel \mathbf{k}}}{B_0} && (2) \\ \tilde{A} &= J_0\left(k_\perp \rho_{\perp s}\right) \frac{v_\parallel A_{\parallel \mathbf{k}}}{c} && (3) \end{aligned}$$

where $\rho_{\perp s} = v_\perp / \Omega_{cs}$. Note that ϕ, $A_\parallel$ and $\delta B_\parallel$, g_s are short for the Fourier components $\phi_{\mathbf{k}}$, $A_{\parallel \mathbf{k}}$, $\delta B_{\parallel \mathbf{k}}$ and $g_{s\mathbf{k}}$.

The Poisson equation:

$$\begin{aligned} \sum_s \frac{q_s^2 n_s}{T_s}(1 - \Gamma_{0s}(\alpha_s))\phi_{\mathbf{k}} - \sum_s q_s n_s \Gamma_{1s}(\alpha_s) \left(\frac{\delta B_{\parallel \mathbf{k}}}{B_0} + \frac{\delta B_{\parallel \mathbf{k} a}}{B_0} \right) \\ = \sum_s q_s \int d^3\mathbf{v} J_0(k_\perp \rho_{\perp s}) g_{\mathbf{k}s} \end{aligned} \tag{4}$$

The parallel component of Ampere's law

$$\frac{ck_\perp^2}{4\pi} A_{\parallel \mathbf{k}} = \sum_s q_s \int d^3\mathbf{v} v_\parallel J_0(k_\perp \rho_{\perp s}) g_{\mathbf{k}s} \tag{5}$$

where $\alpha_s = k_\perp^2 \rho_s^2/2$

The perpendicular component of Ampere's law

$$\sum_s q_s n_s \Gamma_{1s}(\alpha_s)\phi_{\mathbf{k}} + \left(\frac{\delta B_{\|\mathbf{k}}}{B_0} + \frac{\delta B_{\|\mathbf{k}a}}{B_0}\right)\left(\frac{B_0^2}{4\pi} + \sum_s n_s T_s \Gamma_{2s}(\alpha_s)\right) = -\sum_s T_s \int d^3\mathbf{v} \frac{v_\perp^2}{v_{ts}^2} \frac{2J_1(k_\perp \rho_{\perp s})}{k_\perp \rho_{\perp s}} g_{\mathbf{k}s} \tag{6}$$

2 Laplace-Fourier solution with $A_\| = \phi = 0$

Consider antenna driving term

$$\delta B_{\|a} = \delta B_{\|0} e^{i(\mathbf{k}_0\cdot\mathbf{r} - \omega_0 t)} \tag{7}$$

Its Laplace transform is

$$\delta \hat{B}_{\|\mathbf{k}a} = \int_0^\infty \delta B_{\|\mathbf{k}_0} e^{-i\omega_0 t} e^{-pt} dt = \frac{\delta B_{\|\mathbf{k}_0}}{p + i\omega_0} \tag{8}$$

Performing Laplace transform to the gyrokinetic equation gives

$$p\hat{g}_{\mathbf{k}s} - g_{\mathbf{k}s}(t=0) + ik_\| v_\| \hat{g}_{\mathbf{k}s} + ik_\| v_\| \frac{mv_\perp^2 F_{0s}}{T_s B_0} \frac{J_1(k_\perp \rho_\perp s)}{k_\perp \rho_{\perp s}} \frac{\delta \hat{B}_{\|\mathbf{k}}}{B_0} = 0 \tag{9}$$

Choosing zero initial condition, i.e., $g_{\mathbf{k}s}(t=0) = 0$, the distribution function is solved to be:

$$\hat{g}_{\mathbf{k}s} = -\frac{ik_\| v_\|}{p + ik_\| v_\|} \frac{mv_\perp^2 F_{0s}}{T_s B_0} \frac{J_1(k_\perp \rho_{\perp s})}{k_\perp \rho_{\perp s}} \delta \hat{B}_{\|\mathbf{k}} \tag{10}$$

With $A_\| = \phi = 0$, the perpendicular Ampere's law takes the form

$$\left(\frac{\delta B_{\|\mathbf{k}}}{B_0} + \frac{\delta B_{\|\mathbf{k}a}}{B_0}\right)\left(\frac{B_0^2}{4\pi} + \sum_s n_s T_s \Gamma_{2s}(\alpha_s)\right) = -\sum_s T_s \int d^3\mathbf{v} \frac{v_\perp^2}{v_{ts}^2} \frac{2J_1(k_\perp \rho_{\perp s})}{k_\perp \rho_{\perp s}} g_{\mathbf{k}s} \tag{11}$$

Substitute Eq. (10) into the above equation and perform the integral:

$$
\begin{aligned}
\mathrm{RHS} = \sum_s \frac{T_s n_s}{\pi^{3/2} v_{ts}^3} \frac{\delta B_{\|\mathbf{k}}}{B_0} \\
\int 2\pi v_\perp dv_\perp \frac{4v_\perp^2}{v_{ts}^4} \frac{J_1^2(k_\perp \rho_{\perp s})}{(k_\perp \rho_{\perp s})^2} e^{-v_\perp^2/v_{ts}^2} \\
\int dv_\| \frac{ik_\| v_\|}{p + ik_\| v_\|} e^{-v_\|^2/v_{ts}^2}
\end{aligned}
\tag{12}
$$

Using relations A.1 and A.27 in AstroGK Manual (Howes et al. 2007), the integrals are expressed in terms of Bessel functions and plasma dispersion function:

$$
\mathrm{RHS} = \sum_s T_s n_s \Gamma_{2s}(\alpha_s) \left(1 + \xi_s Z(\xi_s)\right) \frac{\delta \hat{B}_{\|\mathbf{k}}}{B_0} \tag{13}
$$

where $\xi_s = \frac{p}{-ikv_{ts}}$ and $Z(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dt \frac{e^{-t^2}}{t-\xi}$ is the Plasma Dispersion Function. Equating LHS and RHS of Eq. (11), we obtain that

$$
\begin{aligned}
\delta \hat{B}_{\|\mathbf{k}} &= \frac{\frac{B_0^2}{4\pi} + \sum_s n_s T_s \Gamma_{2s}(\alpha_s)}{-\frac{B_0^2}{4\pi} + \sum_s n_s T_s \Gamma_{2s}(\alpha_s) \xi_s Z(\xi_s)} \frac{\delta B_{\|\mathbf{k}_0}}{p + i\omega_0} &(14)\\
&= \frac{\frac{1}{\beta_i} + \sum_s \frac{T_s}{T_i} \Gamma_{1s}(\alpha_s)}{-\frac{1}{\beta_i} + \sum_s \frac{T_s}{T_i} \Gamma_{1s}(\alpha_s) \xi_s Z(\xi_s)} \frac{\delta B_{\|\mathbf{k}_0}}{p + i\omega_0} &(15)
\end{aligned}
$$

Note that

$$
D(p) = -\frac{1}{\beta_i} + \sum_s \frac{T_s}{T_i} \Gamma_{1s}(\alpha_s) \xi_s Z(\xi_s) \tag{16}
$$

is exactly the dispersion relation, and is the same as Eq. (55) in [1]. We know that there $D(p) = 0$ has one solution $p_1 = -i\omega_1 = -\gamma_1$, corresponding to a non-propagating slow/ entropy mode. To obtain $\delta B_{\|\mathbf{k}}$, we apply the inverse Laplace transform via Bromwich integral:

$$
\begin{aligned}
\delta B_{\|\mathbf{k}}(t) &= \frac{1}{2\pi i} \int_{\beta - i\infty}^{\beta + i\infty} \frac{C_b e^{pt} dp}{D(p)(p - p0)} &(17)\\
&= \mathrm{Res}(p_1) + \mathrm{Res}(p_0) &(18)
\end{aligned}
$$

where

$$C_b = \left[\frac{1}{\beta_i} + \sum_s \frac{T_s}{T_i} \Gamma_{1s}(\alpha_s) \right] \delta B_{\|\mathbf{k}_0} \tag{19}$$

is a frequency-independent term. $p0 = -i\omega_0$. Res(p) denotes residue at simple pole p.

The Residues are evaluated as follows:

$$\mathrm{Res}(p_0) = \frac{C_b e^{p_0 t}}{D(p_0)} = \frac{C_b e^{-i\overline{\omega}_0 \bar{t}}}{D(p_0)} \tag{20}$$

where $\overline{\omega}_0 = \frac{p_0}{-ik_\| v_A} = \frac{\omega_0}{k_\| v_A}$ and $\bar{t} = tk_\| v_A$. And

$$\begin{aligned} \mathrm{Res}(p_1) &= \left[\frac{(p-p_1) C_b e^{pt}}{D(p)(p-p_0)} \right]_{p=p_1} && (21) \\ &= \frac{C_b e^{p_1 t}}{\left[\frac{dD}{dp} \right]_{p_1} (p_1 - p_0)} && (22) \end{aligned}$$

Define

$$\begin{aligned} G(p) &= \frac{dD}{dp}(p - p_0) && (23) \\ &= \sum_s \Gamma_{1s} \frac{T_s}{T_i} \frac{1}{\beta_i} \sqrt{\frac{T_s}{T_i} \frac{m_s}{m_i}} \left[(1 - 2\xi_s^2) Z_s - 2\xi_s \right] (\overline{\omega} - \overline{\omega}_0) && (24) \end{aligned}$$

Hence

$$\mathrm{Res}(p_1) = \frac{C_b e^{-i\overline{\omega}_1 \bar{t}}}{G(\overline{\omega}_1)} \tag{25}$$

To summarize, parallel magnetic field fluctuation is given by

$$\delta B_{\|\mathbf{k}}(t) = \frac{C_b e^{-i\overline{\omega}_0 \bar{t}}}{D(p_0)} + \frac{C_b e^{-i\overline{\omega}_1 \bar{t}}}{G(\overline{\omega}_1)} \tag{26}$$

3 Laplace-Fourier solution: general case

We take away the constraint on $\delta A_{\mathbf{k}\|}$ and ϕ. The process is the similar to the restricted case we considered in the above section. We obtain an expression for the Fourier-Laplace-transformed distribution function $\hat{g}_{\mathbf{k}s}(p)$ from the gyrokinetic equation. Substituting it into the three Maxwell's equations (Poisson, parallel and perpendicular components of Ampere's law), we obtain three equations for three independent field fluctuations, i.e., $\delta\phi$, $\delta A_\parallel$ and $\delta B_\parallel$. Doing inverse Laplace transform will give us the temporal evolution for the fields.

Apply Laplace transform to Eq. (1) and set initial conditions to zero gives:

$$\hat{g}_{\mathbf{k}s} = -\frac{q_s F_{0s}}{T_s}\frac{ik_\parallel v_\parallel}{p+ik_\parallel v_\parallel}\left(J_{0s}\hat{\phi}_{\mathbf{k}} + \frac{J_{1s}}{k_\perp \rho_{\perp s}}\frac{mv_\perp^2}{q_s}\frac{\delta\hat{B}_{\parallel\mathbf{k}}}{B_0} + J_{0s}\frac{p\hat{A}_{\parallel\mathbf{k}}}{ik_\parallel c}\right) \tag{27}$$

Substituting Eq. (27) into Poisson's equation and performing integration over velocity yield

$$\begin{aligned}\sum_s \frac{q_s^2 n_s}{T_s}\left[(1+\Gamma_{0s}\xi_s Z_s)\left(\hat{\phi}_{\mathbf{k}} - \frac{ip\hat{A}_{\parallel\mathbf{k}}}{k_\parallel c}\right) + (1-\Gamma_{0s})\frac{ip\hat{A}_{\parallel\mathbf{k}}}{k_\parallel c}\right] \\ + \sum_s q_s n_s \Gamma_{1s}\left(\xi_s Z_s \frac{\delta\hat{B}_{\parallel\mathbf{k}}}{B_0} - \frac{\delta\hat{B}_{\parallel\mathbf{k}a}}{B_0}\right) = 0\end{aligned} \tag{28}$$

For convenience, define the following dimensionless quantities:

$$A = \sum_s \frac{T_i}{T_s}(1+\Gamma_{0s}\xi_s Z_s) \tag{29}$$

$$B = \sum_s \frac{T_i}{T_s}(1-\Gamma_{0s}) \tag{30}$$

$$C = \sum_s \frac{q_i}{q_s}\Gamma_{1s}\xi_s Z_s \tag{31}$$

$$D = \sum_s \frac{2T_s}{T_i}\Gamma_{1s}\xi_s Z_s \tag{32}$$

$$E = \sum_s \frac{q_i}{q_s}\Gamma_{1s} \tag{33}$$

$$F = \sum_s \frac{2T_s}{T_i}\Gamma_{1s} \tag{34}$$

and

$$
\begin{aligned}
X &= \hat{\phi}_{\mathbf{k}} - \frac{ip\hat{A}_{\|\mathbf{k}}}{k_\| c} = \hat{\phi}_{\mathbf{k}} - \frac{\omega \hat{A}_{\|\mathbf{k}}}{k_\| c} && (35)\\
Y &= \frac{ip\hat{A}_{\|\mathbf{k}}}{k_\| c} = \frac{\omega \hat{A}_{\|\mathbf{k}}}{k_\| c} && (36)\\
Z &= \frac{T_i}{q_i}\frac{\delta \hat{B}_{\|\mathbf{k}}}{B_0} && (37)\\
Z_a &= \frac{T_i}{q_i}\frac{\delta \hat{B}_{\|\mathbf{k}a}}{B_0} && (38)\\
\overline{\omega} &= \frac{\omega}{k_\| v_A} = \frac{p}{-ik_\| v_A} && (39)
\end{aligned}
$$

Assuming hydrogen plasma: $q_i = -q_e$, $n_{0i} = n_{0e}$, and using the above definition, the Poisson equation becomes:

$$AX + BY + CZ = EZ_a \tag{40}$$

Similarly, the parallel component of the Ampere's law yields

$$(A - B)X + \frac{\alpha_i}{\overline{\omega}^2}Y + (C + E)Z = 0 \tag{41}$$

The perpendicular component of the Ampere's law yields:

$$CX - EY + \left(D - \frac{2}{\beta_i}\right) Z = \left(F + \frac{2}{\beta_i}\right) Z_a \tag{42}$$

Combining the above three equations in the matrix form gives

$$\begin{pmatrix} A & B & C \\ A - B & \alpha_i/\overline{\omega}^2 & C + E \\ C & -E & D - 2/\beta_i \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} EZ_a \\ 0 \\ (F + 2/\beta_i)Z_a \end{pmatrix} \tag{43}$$

Notice that the 3×3 matrix in the above expression is exactly the dispersion tensor for linear collisionless gyrokinetics with spatial homogeneity (c.f. Eq. (6.24) in AstroGK manual (Howes, 2007) and Eq. (C15) in [1]).

4 Driving gyrokinetics with $A_\parallel = 0$

To simplify comparison with AstroGK simulation of linearly driving slow mode, we set $A_\parallel = 0$. We choose the Poisson's equation and the perpendicular component of the Ampere's law to solve for fields:

$$\begin{pmatrix} A & C \\ C & D - 2/\beta_i \end{pmatrix} \begin{pmatrix} X \\ Z \end{pmatrix} = \begin{pmatrix} E \\ F + 2/\beta_i \end{pmatrix} Z_a \tag{44}$$

The Fourier-Laplace transformed fields X and Z are obtained by multiplying the matrix inverse:

$$\begin{pmatrix} \hat{\phi}(p) \\ \frac{T_i}{q_i}\frac{\delta\hat{B}_{\parallel\mathbf{k}}(p)}{B_0} \end{pmatrix} = \begin{pmatrix} X \\ Z \end{pmatrix} = \frac{1}{\mathrm{Det}M} \begin{pmatrix} D - 2/\beta_i & -C \\ -C & A \end{pmatrix} \begin{pmatrix} E \\ F + 2/\beta_i \end{pmatrix} Z_a \tag{45}$$

Using the same antenna driving term $\delta B_{\parallel a}$ as in Eq. (7) and its Laplace transform Eq. (8), we obtain

$$\begin{aligned} \begin{pmatrix} \phi(t) \\ \frac{T_i}{q_i}\frac{\delta B_{\parallel\mathbf{k}}(t)}{B_0} \end{pmatrix} &= \\ & ILT\left[\frac{1}{\mathrm{Det}M} \begin{pmatrix} D - 2/\beta_i & -C \\ -C & A \end{pmatrix} \begin{pmatrix} E \\ F + 2/\beta_i \end{pmatrix} \frac{T_i}{q_i}\frac{\delta B_{\parallel\mathbf{k}0}}{B_0}\frac{1}{p - p_0}\right] \\ &= \sum_{i=0}^{n} \mathrm{Res}(p_i) \end{aligned} \tag{46}$$

where the sum is over all the simple poles p_i. In particular, $p_0 = -i\omega_0$ results from antenna driving.

To evaluate residues at p_i for $i \neq 0$, we expand $\mathrm{Det}M$ in Taylor series near p_i in the same way as we did in Section 2:

$$\mathrm{Res}(p_i) = \left[\frac{1}{\frac{d\mathrm{Det}M}{dp}} \begin{pmatrix} D - 2/\beta_i & -C \\ -C & A \end{pmatrix} \begin{pmatrix} E \\ F + 2/\beta_i \end{pmatrix} \frac{T_i}{q_i}\frac{\delta B_{\parallel\mathbf{k}0}}{B_0}\frac{e^{pt}}{p - p_0}\right]_{p=p_i} \tag{47}$$

where

$$\frac{d\mathrm{Det}M}{dp} = A'\left(D - \frac{2}{\beta_i}\right) + AD' - 2CC' \tag{48}$$

A', C', D' are derivatives w.r.t. p:

$$A' = \frac{dA}{dp} = \sum_s \frac{T_i}{T_s} \Gamma_{0s} G_s \tag{49}$$

$$C' = \frac{dC}{dp} = \sum_s \frac{q_i}{q_s} \Gamma_{1s} G_s \tag{50}$$

$$D' = \frac{dD}{dp} = \sum_s \frac{2T_s}{T_i} \Gamma_{1s} G_s \tag{51}$$

where

$$\begin{aligned} G_s &= \frac{(1 - 2\xi_s^2)\, Z_s - 2\xi_s}{-ik_\| v_{ts}} \\ &= \frac{(1 - 2\xi_s^2)\, Z_s - 2\xi_s}{-ik_\| v_A} \frac{1}{\sqrt{\beta_i}} \sqrt{\frac{m_s}{m_i} \frac{T_i}{T_s}} \end{aligned} \tag{52}$$

References

[1] G. G. Howes, S. C. Cowley, W. Dorland, G. W. Hammett, E. Quataert, and A. A. Schekochihin. Astrophysical Gyrokinetics: Basic Equations and Linear Theory. ApJ, 651:590–614, November 2006.