Practical Part: Linear Regression

October 16, 2018

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In [35]: #!/usr/bin/env python
         import sys, os
         import numpy as np
         from random import uniform
         import math
         get_ipython().run_line_magic('pylab', 'inline')
         import matplotlib.pyplot as plt

         class regression_gradient:
             def __init__(self, lamb=100, step_size=1, n_steps=100):
                 self.lamb = lamb
                 self.step_size = step_size
                 self.n_steps = n_steps

             def train(self, X, y):
                 """Question 1: ridge regression with gradient descent"""

                 # if X is a vector, add dummy dimension
                 if np.ndim(X) == 1:
                     X = np.expand_dims(X, axis=1)

                 stopping_tolerance = 1e-15

                 self.n = np.shape(X)[0]
                 self.d = np.shape(X)[1]
                 # includes b at position 0
                 self.w = np.random.uniform(low=-0.01, high=0.01, size=self.d + 1)

                 # include column of 1s for b
                 X = np.hstack((np.ones((self.n, 1)), X))

                 empirical_risk = np.zeros((self.d + 1))
                 for i in range(self.n_steps):
                     empirical_risk = X.T.dot(X).dot(self.w) - X.T.dot(y)
                     regularization = self.lamb * 2 * self.w
                     gradient = (empirical_risk + regularization)
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            # stopping criteria
            if np.sum(np.abs(gradient)) < stopping_tolerance:
                break

            self.w -= self.step_size  * gradient


    def predict(self, X):
        n = X.shape[0]
        y = np.zeros(n)
        w = self.w[1:]
        b = self.w[0]

        for j in range(n):
            y[j] = w.T.dot(X[j]) + b

        return(y)


def sample_h(n):
    """
    Question 2.
    h(x) = sin(x) + 0.3x -1
    Returns dataset D (x, h(x)) with n points. x in [-5, 5].
    """
    D = np.zeros((n,2))
    for i in range(n):
        D[i,0] = uniform(-5,5)
        D[i,1] = math.sin(D[i,0]) + 0.3*(D[i,0]) - 1
    return D


# plottng options
n_bins = 100
axes_min = -10
axes_max = 10
alpha = 0.75
x = np.atleast_2d(np.linspace(axes_min, axes_max, n_bins)).T

fig, axs = plt.subplots(2, 2, figsize=(9, 9))

# question 3
# plot data
D = sample_h(15)
axs[0][0].scatter(D[:,0], D[:,1]) # raw data

# plot h(x)
y1 = np.zeros(n_bins)
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for i in range(n_bins):
    y1[i] = math.sin(x[i]) + 0.3*(x[i]) - 1
axs[0][0].plot(x, y1, color='black', alpha=alpha)

# training settings
step_size = 5e-5
n_steps = 10000

# plot regression_gradient --lambda 0
mdl_1 = regression_gradient(lamb=0, step_size=step_size, n_steps=n_steps)
mdl_1.train(D[:,0], D[:,1])
axs[0][0].plot(x, mdl_1.predict(x), color='red', alpha=alpha)

# question 4
# plot regression_gradient - lambda intermediate
mdl_2 = regression_gradient(lamb=5, step_size=step_size, n_steps=n_steps)
mdl_2.train(D[:,0], D[:,1])
axs[0][0].plot(x, mdl_2.predict(x), color='green', alpha=alpha)

# plot regression_gradient - lambda large
mdl_3 = regression_gradient(lamb=25, step_size=step_size, n_steps=n_steps)
mdl_3.train(D[:,0], D[:,1])
axs[0][0].plot(x, mdl_3.predict(x), color='blue', alpha=alpha)

axs[0][0].legend(['h(x)',
                  'no lambda',
                  'intermediate lambda',
                  'large lambda',
                  'raw data'])

axs[0][0].set_ylabel('Predicted value')
axs[0][0].set_title('D_n 15 samples, lambda comparison')

# question 5
D_test = sample_h(100)
lambs = [0.0001, 0.001, 0.01, 0.1, 1, 10, 100]

avg_loss = np.zeros(len(lambs))
for i, lamb in enumerate(lambs):
    mdl = regression_gradient(lamb=lamb , step_size=step_size, n_steps=n_steps)
    mdl.train(D[:,0], D[:,1])

    b= mdl.w[0]
    w= mdl.w[1:]

    loss = 0
    for j in range(len(D_test)):
        loss += (np.take(((w.T.dot(D_test[j,0]) + b) - D_test[j,1]), 0))**2
    avg_loss[i] = loss/len(D_test)

# bar plot of average quadratic loss per lambda
width = 1.0
axs[0][1].scatter(lambs, avg_loss, c='black' )
axs[0][1].set_ylabel('Average quadratic loss')
axs[0][1].set_xlabel('Lambda')
axs[0][1].set_xscale('log')
axs[0][1].set_xlim([lambs[0], lambs[-1]])
axs[0][1].set_title('Average quadratic loss per lambda')

# question 6
# training settings
step_size = 5e-6
n_steps = 50000

orders = [1, 2, 3]

loss_test = np.zeros((100, len(orders)))
loss_train = np.zeros((15, len(orders)))

for i, l in enumerate(orders):

    # polynomial preprocessing
    poly = np.zeros((np.shape(D)[0], i+1))
    for j in range(np.shape(D)[0]):
        poly[j] = np.array([D[j, 0]**exp for exp in orders[0:i+1]])

    poly_test = np.zeros((np.shape(D_test)[0], i+1))
    for j in range(np.shape(D_test)[0]):
        poly_test[j] = np.array([D_test[j, 0]**exp for exp in orders[0:i+1]])

    # ridge regression
    mdl = regression_gradient(lamb=0.01, step_size=step_size, n_steps=n_steps)
    mdl.train(poly, D[:,1])
    b = mdl.w[0]
    w = mdl.w[1:]

    # ploting f learned with ridge regression
    y = np.zeros(n_bins)
    for z in range(n_bins):
        processed_x = np.array([x[z]**exp for exp in orders[:i+1]])
        y[z] = np.dot(w.T, processed_x) + b
    axs[1][0].plot(x, y, alpha=alpha)

    # collect empirical risk, true risk
    train_predictions = mdl.predict(poly)
    test_predictions = mdl.predict(poly_test)
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        for j in range(len(D_test)):
            loss_test[j, i] = (np.take(((w.T.dot(poly_test[j, :]) + b) - D_test[j,1]), 0)

        for j in range(len(D)):
            loss_train[j, i] = (np.take(((w.T.dot(poly[j, :]) + b) - D[j,1]), 0))**2

    # print models
    axs[1][0].scatter(D[:,0], D[:,1])
    axs[1][0].legend(['poly=1', 'poly=2', 'poly=3', 'raw data'])
    axs[1][0].set_ylabel('Predicted value')
    axs[1][0].set_ylim([-10, 10])
    axs[1][0].set_title('Polynomial regression')

    # print losses
    axs[1][1].scatter(np.array([1,2,3,4,5,6]), np.array([
        np.mean(loss_train[:, 0]), np.mean(loss_test[:, 0]),
        np.mean(loss_train[:, 1]), np.mean(loss_test[:, 1]),
        np.mean(loss_train[:, 2]), np.mean(loss_test[:, 2])
    ]), c='black')


    axs[1][1].set_xticklabels(['',
                               'l=1, train', 'l=1, test',
                               'l=2, train', 'l=2, test',
                               'l=3, train', 'l=3, test'])
    for tick in axs[1][1].get_xticklabels():
        tick.set_rotation(65)
    axs[1][1].set_title('Test and train loss over model orders')
    axs[1][1].set_ylabel('Average quadratic loss')

    plt.show()

    plt.savefig('report.jpg')
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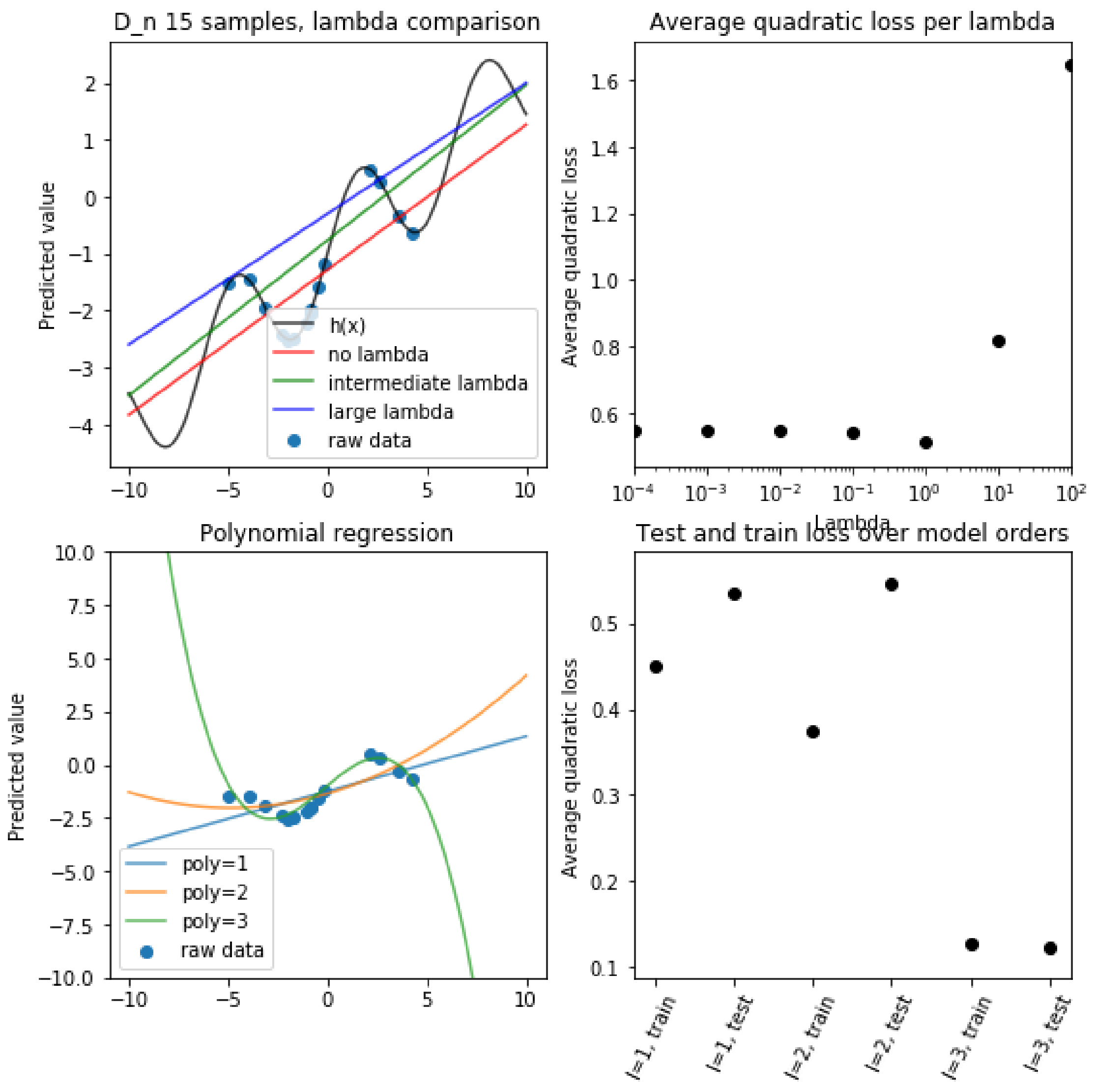
Populating the interactive namespace from numpy and matplotlib


/opt/python/sci_36/lib/python3.6/site-packages/IPython/core/magics/pylab.py:160: UserWarning: p
`%matplotlib` prevents importing * from pylab and numpy
  "\n`%matplotlib` prevents importing * from pylab and numpy"
```



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<matplotlib.figure.Figure at 0x7f3c6edc41d0>
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As l increases, the empirical risk stays steady for l=1 and l=2, and drops dramatically for l=3, since this particular polynomial fits the function generating the data fairly well. Increasing model order does not help the test performance for l=1 and l=2. Since l=3 is a good model for this particular dataset, the test performance is also good for this model. We predict that as one adds more orders to the polynomial expansion, empirical risk will stay low, however, the true risk (performance on test set) will become worse. This is because our model will become overfit to the training data for l > 3.