

# IFT6390

## Fondements de l'apprentissage machine

**Bayes Classifier**  
**Naive Bayes Classifier**

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# Probabilistic approach to learning

- We assumed that the data is generated by an unknown process.
- $X, Y$  is seen as a pair of random variables, distributed according to an unknown probability law  $P(X, Y)$ .
- $X$  (a vector variable) is itself seen as a set of scalar random variables.

$$P(X, Y) = P(X_{[1]}, \dots, X_{[d]}, Y)$$

# Binary classifier (reminder)

- For **binary classification (2 classes)**, we often use a classifier **computing a real valued output  $g(x)$** .  
 $g$  is called a **discriminant function**.
- The **decision/classification function  $f(x)$**  is then obtained by **comparing the output  $g(x)$  with a threshold  $t$** . I.e. we predict the first class if  $g(x)$  is smaller than  $t$  and the second otherwise.
- One common choice for the *threshold  $t$*  is 0, which boils down to looking at the **signe** of  $g(x)$ .
- Another common choice is a *threshold* of 0.5 when the *output  $g(x)$*  is in  $[0, 1]$  and can be interpreted as the **probability** of one of the class given the input  $x$ . (observe that the probability of the other class is  $1-g(x)$ ; the 0.5 threshold corresponds to predicting the most likely class...)

# Multiclass classification

- When there are  $m$  classes, the output  $g(\mathbf{x})$  is an  $m$ -dimensional vector containing “scores” for each class.

*discriminant functions:*

$$g(\mathbf{x}) = (g_1(\mathbf{x}), g_2(\mathbf{x}), \dots, g_m(\mathbf{x}))$$

- The decision/classification  $f(\mathbf{x})$  is then the class with the highest score:

$$\text{prediction} = \arg \max(\text{output})$$

$$f(\mathbf{x}) = \arg \max(g(\mathbf{x}))$$

- When the outputs (scores) are positive and sum to 1, we can interpret the  $j$ -th output as the probability of the  $j$ -th class given the input.
- This works also for binary classification (2 classes: 2 outputs)!
- Note: with two classes, predicting the class with the highest score is equivalent to predicting the sign of the difference between the two scores.

# Probability

(what you should be familiar with)

- Discrete and continuous random variables
- Joint probability distribution
- Marginal distribution, marginalization
- Conditionnal probability
- Bayes Rule
- Independence

# Bayes Rule

$$P(A|B) = \frac{P(B|A) P(A)}{P(B)}$$



Thomas Bayes

1702 - 1761

# Bayes Classifier

*or how to build a classifier from density estimators*

- Split the training data into  $m$  subsets corresponding to the  $m$  each one containing the points from only one class.
- Estimate the density/distribution of points for each class  $c \in \{1, \dots, m\}$

$$\hat{p}_c(x) \simeq P(X = x | Y = c)$$

- Estimate *prior probabilities* for each class:  $\hat{P}_c = \frac{n_c}{n} \simeq P(Y = c)$   
(for ex. by counting the proportion of each class in the training data)

- Apply **Bayes rule** to compute the *posterior probability* of each class given  $x$ .

- Predict the most likely class.

$$\begin{aligned}
 \underbrace{P(Y = c | X = x)}_{\text{posterior}} &= \frac{\underbrace{P(X = x | Y = c)}_{\text{class-conditional density (likelihood)}} \underbrace{P(Y = c)}_{\text{prior}}}{P(X = x)} \\
 &= \frac{P(X = x | Y = c) P(Y = c)}{\sum_{c'=1}^m P(X = x | Y = c') P(Y = c')} \\
 &\simeq \frac{\hat{p}_c(x) \hat{P}_c}{\sum_{c'=1}^m \hat{p}_{c'}(x) \hat{P}_{c'}}
 \end{aligned}$$

# Naive Bayes

- For naive Bayes, we suppose that, for each class  $c \in \{1, \dots, m\}$ , **the components of  $X$  are independent given  $c$** :

$$P(X|Y = c) = P(X_{[1]}|Y = c)P(X_{[2]}|Y = c)\dots P(X_{[d]}|Y = c)$$

$$\hat{p}_c(x) = \hat{p}_{c,1}(x_{[1]})\hat{p}_{c,2}(x_{[2]})\dots\hat{p}_{c,d}(x_{[d]})$$

- So we just have to **estimate univariate densities**  $\hat{p}_{c,j}(x_{[j]})$  which is **an easy task** (univariate  $\Leftrightarrow$  dimension 1: no curse of dimensionality; histogram methods or Parzen often perform well).
- We then build a Bayes classifier using the estimate  $\hat{p}_c(x)$  for each class.



# Warning

- “Bayes Classifier”  $\neq$  “Naive Bayes”
- “Naive Bayes”  $\subset$  “Bayes Classifier”
- Naive Bayes is a very naive model, with very low capacity  
(it cannot capture/model interactions between the input components!)
- The Bayes classifier can have a very high capacity  
(depends on the density estimators used to model the class conditional densities)

# Other way to build a classifier:

## Use a good estimate of the joint distribution

- Estimate the joint distribution  $P(X,Y)$
- We can then compute the conditional probabilities for each class  $c$ :

$$\begin{aligned} P(Y = c | X = x) &= \frac{P(X = x, Y = c)}{P(X = x)} \\ &= \frac{P(X = x, Y = c)}{\sum_{c'=1}^m P(X = x, Y = c')} \end{aligned}$$

- Note that these class probabilities (given  $x$ ) are proportional to the joint probabilities. The denominator is a simple normalizer (so that they sum to one).