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# Globally attracting positive periodic solution of the $n$-dimensional periodic Ricker system

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ABSTRACT

In this paper, we provide a criterion to guarantee the existence of a globally attracting positive periodic solution for the nonautonomous Ricker system whose coefficients are all periodic functions of a common period, that is, under certain circumstances, all species will coexist in a limit cycle fashion.

## 1. Introduction

In this paper, we consider the periodic time-dependent Ricker system

$$\dot{x}_i(t) = x_i(t)\Big(\exp\big(r_i(t) - \sum_{j=1}^{n} a_{ij}(t)x_j(t)\big) - 1\Big), \quad i = 1, \dots, n, \tag{1}$$

which describes the evolution of $n$ competing species [1–3], where $r_i(t)$ and $a_{ij}(t)$ are assumed to be continuous positive periodic functions of time with a common period $\omega > 0$ for all $i$ and $j$. For the autonomous case, there have been many interesting results. Jiang and Niu [4] showed that the autonomous Ricker system possesses a carrying simplex [1,5], that is a globally attracting invariant hypersurface of codimension one. They then classified the dynamics of the three-dimensional case into 33 classes by using geometric analysis of nullclines, and interestingly, they found that heteroclinic cycle and limit cycle can coexist in the system. Hou [6] provided sufficient conditions by using a geometric method to guarantee the global stability of an equilibrium. Li and Jiang [7] recently proved that multiple limit cycles can coexist in three-dimensional Ricker system.

It is widely recognized that inherent biological and physical environmental parameters actually often oscillate periodically over time, which can have significant effects on population dynamics [3,8,9]. A more realistic model would certainly allow for the periodic oscillations of these parameters in consideration of these obvious periodic fluctuations that such ecological systems would naturally be subjected. This inspired us to study the asymptotic behaviour of the periodic time-dependent Ricker system (1).

Each $k$-dimensional coordinate subspace of $\mathbb{R}^n$ is positively invariant under system (1) ($1 \le k \le n$), and we adopt the tradition of restricting attention to the closed nonnegative cone $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_i \ge 0, i = 1, \dots, n\}$. Denote by $x(t, x_0)$ the unique solution of system (1) with initial point $x_0 \in \mathbb{R}^n_+$. Clearly, the domain of $x(\cdot, x_0)$ includes $[0, +\infty)$. We denote the open positive cone by $\dot{\mathbb{R}}^n_+ := \{x \in \mathbb{R}^n_+ : x_i > 0, i = 1, \dots, n\}$, and call a vector $x$ positive if $x \in \dot{\mathbb{R}}^n_+$. For a bounded function $h(t)$ on $\mathbb{R}$, we define

$$h^L = \inf_{t\in\mathbb{R}} h(t), \quad h^M = \sup_{t\in\mathbb{R}} h(t).$$

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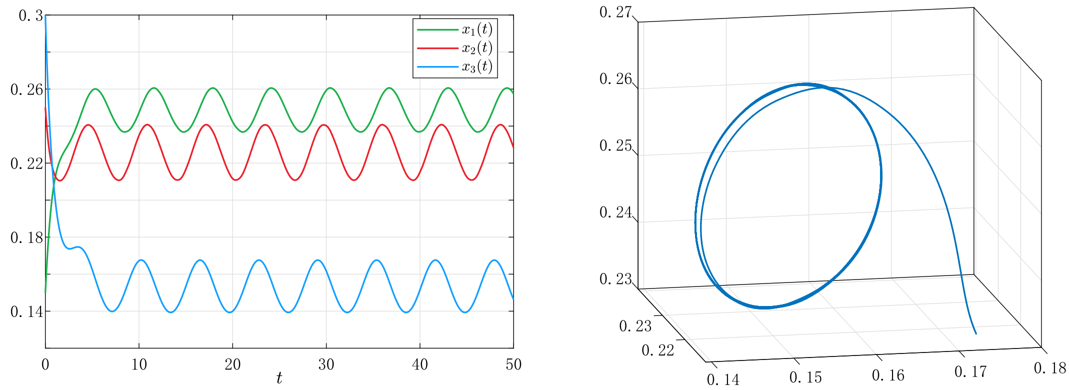


(a) The components of the solution (b) The trajectory of the solution

**Fig. 1.** The solution of system (3) through the initial point $x_0 = (0.15, 0.25, 0.3)$ is asymptotic to a positive periodic solution.

Our aim is to provide easily verifiable conditions which guarantee the existence of a globally attracting positive periodic solution for system (1). In other words, under certain conditions, all species can coexist in a limit cycle fashion when subjected to a periodic environment. Our result is stated in Theorem 1.1, which generalizes the results of Ahmad and Lazer in [10] for nonautonomous Lotka–Volterra system with periodic coefficients to the periodic time-dependent Ricker system.

**Theorem 1.1.** *Suppose that $r_i(t)$ and $a_{ij}(t)$, $i, j = 1, \dots, n$, are positive $\omega$-periodic functions. Let*

$$m_i = \left(\frac{r_i}{a_{ii}}\right)^M, \quad \varepsilon_i := \inf_{t\in\mathbb{R}}\left\{\frac{r_i(t) - \sum_{\substack{j=1\\ j\neq i}}^n a_{ij}(t)m_j}{a_{ii}(t)}\right\}, \quad i = 1, \dots, n.$$

*If system (1) satisfies that*

$$\beta_* r_i^L > \sum_{\substack{j=1\\ j\neq i}}^n \beta^* a_{ij}^M m_j, \quad i = 1, \dots, n, \tag{2}$$

*where*

$$\beta_* = \min_i \inf_{t\in\mathbb{R}}\left\{\exp\Big(r_i(t) - \sum_{j=1}^n a_{ij}(t)m_j\Big)\right\}, \quad \beta^* = \max_i \sup_{t\in\mathbb{R}}\left\{\exp\Big(r_i(t) - \sum_{j=1}^n a_{ij}(t)\varepsilon_j\Big)\right\},$$

*then it admits a globally attracting positive $\omega$-periodic solution $\hat{x}(t)$ with $\varepsilon_i \le \hat{x}_i(t) \le m_i$, $i = 1, \dots, n$, for all $t \ge 0$.*

**Example 1.1.** Consider the following three-dimensional Ricker system

$$\begin{cases} \dot{x}_1 = x_1 \exp\Big(2 - 7x_1 - (\frac{1}{2} + \frac{1}{2}\sin t)x_2 - x_3\Big) - x_1, \\ \dot{x}_2 = x_2 \exp\Big(2 - (\frac{1}{2} + \frac{1}{2}\sin t)x_1 - 8x_2 - (\frac{1}{2} + \frac{1}{2}\cos t)x_3\Big) - x_2, \\ \dot{x}_3 = x_3 \exp\Big(1 - \frac{1}{2}x_1 - (\frac{1}{2} + \frac{1}{2}\cos t)x_2 - 5x_3\Big) - x_3. \end{cases} \tag{3}$$

It is easy to check that

$$\beta_* r_i^L - \beta^* \left(a_{ij}^M m_j + a_{ik}^M m_k\right) > 0, \quad i = 1, 2, 3, \quad j \neq k \neq i,$$

that is conditions (2) hold. Therefore, it follows from Theorem 1.1 that system (3) has a globally attracting positive $2\pi$-periodic solution on $\dot{\mathbb{R}}_+^3$ (see Fig. 1).

## 2. Proof of the main result

**Lemma 2.1.** *Suppose that $\varepsilon_i > 0$, $i = 1, \dots, n$. Then for all $\varepsilon \in (0, \min_i\{\varepsilon_i\})$, there exists $0 < \sigma_\varepsilon < \varepsilon$ such that for all $0 < \sigma \le \sigma_\varepsilon$, the set*

$$\mathcal{A}_{\varepsilon,\sigma} := \{x \in \mathbb{R}_+^n : \varepsilon_i - \varepsilon \le x_i \le m_i + \sigma, \ i = 1, \dots, n\}$$

*is positively invariant and attracting on $\dot{\mathbb{R}}_+^n$, i.e., $x(t, \mathcal{A}_{\varepsilon,\sigma}) \subset \mathcal{A}_{\varepsilon,\sigma}$ for all $t \ge 0$, and for any $x_0 \in \dot{\mathbb{R}}_+^n$, there is $\tau \ge 0$ such that $x(t, x_0) \in \mathcal{A}_{\varepsilon,\sigma}$ for all $t \ge \tau$.*

**Proof.** Let $x(t) = x(t, x_0)$. Note that for any given $\sigma > 0$ whenever $x_i(t) \geq m_i + \sigma$,

$$\dot{x}_i(t) \leq x_i(t)\left(\exp\left(r_i(t) - a_{ii}(t)(m_i + \sigma)\right) - 1\right) \leq l_i x_i(t) < 0 \tag{4}$$

by the definition of $m_i$, where $l_i = e^{-\sigma a_{ii}^L} - 1 < 0$. It follows that

$$\mathcal{B}_\sigma := \{x \in \mathbb{R}_+^n : 0 < x_i \leq m_i + \sigma,\ i = 1, \dots, n\}$$

is positively invariant and the interval $(0, m_i + \sigma]$ is an attracting set for $x_i(t)$. In fact, if $x_i(t) > m_i + \sigma$ for all $t \geq 0$, then $x_i(t) \leq x_i(0)e^{l_i t} \to 0$ as $t \to +\infty$, a contradiction. Thus, there exists $\tau_\sigma \geq 0$ such that $0 < x_i(t) \leq m_i + \sigma$, $i = 1, \dots, n$, i.e., $x(t) \in \mathcal{B}_\sigma$ for all $t \geq \tau_\sigma$. Now fix a $\varepsilon \in (0, \min_i\{\varepsilon_i\})$. There exists $0 < \sigma_\varepsilon < \varepsilon$ such that

$$u_i := \exp(\varepsilon a_{ii}^L - \sigma_\varepsilon \sum_{\substack{j=1\\ j\neq i}}^n a_{ij}^M) - 1 > 0.$$

By the definition of $\varepsilon_i$, it is clear that $\varepsilon_i \leq m_i$, and

$$r_i(t) - \sum_{\substack{j=1\\ j\neq i}}^n a_{ij}(t)m_j \geq a_{ii}(t)\varepsilon_i, \quad \forall t \in \mathbb{R}. \tag{5}$$

Then by (5), for any $\sigma \leq \sigma_\varepsilon$ whenever $x_i(t) \leq \varepsilon_i - \varepsilon$ and $x_j(t) \leq m_j + \sigma$ for all $j \neq i$,

$$\dot{x}_i(t) \geq x_i(t)\left(\exp\left(r_i(t) - \sum_{\substack{j=1\\ j\neq i}}^n a_{ij}(t)(m_j + \sigma) - a_{ii}(t)(\varepsilon_i - \varepsilon)\right) - 1\right) \geq u_i x_i(t) > 0. \tag{6}$$

Together with (4), (6) implies that $\mathcal{A}_{\varepsilon,\sigma}$ is positively invariant. Recall that there exists $\tau_\sigma \geq 0$ such that $x(t) \in \mathcal{B}_\sigma$ for all $t \geq \tau_\sigma$. Therefore, by the similar arguments as above one can prove that the interval $[\varepsilon_i - \varepsilon, m_i + \sigma]$ is an attracting set for $x_i(t)$, $i = 1, \dots, n$, which implies that $\mathcal{A}_{\varepsilon,\sigma}$ is attracting on $\dot{\mathbb{R}}_+^n$. □

**Remark 2.1.** Lemma 2.1 implies that system (1) is permanent [2,11] if $\varepsilon_i > 0$, and in particular, every solution $x(t, x_0)$ with $x_0 \in \dot{\mathbb{R}}_+^n$ is bounded on $[0, +\infty)$.

**Lemma 2.2.** *Suppose that $\varepsilon_i > 0$, $i = 1, \dots, n$. Then for any $\mathcal{A}_{\varepsilon,\sigma}$ defined in Lemma 2.1, the system (1) has a positive $\omega$-periodic solution $\hat{x}^{\varepsilon,\sigma}(t) \in \mathcal{A}_{\varepsilon,\sigma}$ for all $t \geq 0$.*

**Proof.** To prove the lemma, we consider the Poincaré map $\mathcal{P}$ associated with system (1) on $\mathbb{R}_+^n$, which is given by

$$\mathcal{P}(x_0) = x(\omega, x_0), \quad x_0 \in \mathbb{R}_+^n.$$

By Lemma 2.1, $\mathcal{P}(\mathcal{A}_{\varepsilon,\sigma}) \subset \mathcal{A}_{\varepsilon,\sigma}$. Since $\mathcal{A}_{\varepsilon,\sigma}$ is a convex compact set, the Brouwer fixed point theorem implies that $\mathcal{P}$ has a fixed point $\hat{x}^{\varepsilon,\sigma} \in \mathcal{A}_{\varepsilon,\sigma}$, that is

$$\mathcal{P}(\hat{x}^{\varepsilon,\sigma}) = \hat{x}^{\varepsilon,\sigma}.$$

Then the solution $\hat{x}^{\varepsilon,\sigma}(t) := x(t, \hat{x}^{\varepsilon,\sigma})$ is a positive $\omega$-periodic solution of system (1) because the system is $\omega$-periodic. It is clear that $\hat{x}^{\varepsilon,\sigma}(t) \in \mathcal{A}_{\varepsilon,\sigma}$ for all $t \geq 0$. □

Let $y(t), z(t)$ be two solutions of (1) with initial values $y(0), z(0) \in \dot{\mathbb{R}}_+^n$, respectively. Motivated by [10], we define a nonnegative function $v$ on $[0, +\infty)$ by

$$v(t) = v(y(t), z(t)) = \sum_{i=1}^n c_i \left|\ln \frac{y_i(t)}{z_i(t)}\right|. \tag{7}$$

**Lemma 2.3.** *Suppose that $\varepsilon_i > 0$ and there exist $c_1, \dots, c_n > 0$ such that*

$$\eta := \min_j \inf_{t\in\mathbb{R}} \left\{ \beta_* c_j a_{jj}(t) - \beta^* \sum_{\substack{i=1\\ i\neq j}}^n c_i a_{ij}(t) \right\} > 0. \tag{8}$$

*Then there exist $\gamma > 0$ and $t_0 > 0$ such that for all $t \geq t_0$,*

$$v(t) \leq v(t_0) - \gamma \int_{t_0}^t \sum_{i=1}^n |y_i(s) - z_i(s)|\, ds. \tag{9}$$

**Proof.** For $\varepsilon > 0$ and $\sigma > 0$, define

$$\beta_*^\sigma = \min_i \inf_{t\in\mathbb{R}} \left\{ \exp\left(r_i(t) - \sum_{j=1}^n a_{ij}(t)(m_j + \sigma)\right) \right\},$$

$$\beta_\varepsilon^* = \max_i \sup_{t\in\mathbb{R}} \left\{ \exp\Big(r_i(t) - \sum_{j=1}^n a_{ij}(t)(\varepsilon_j - \varepsilon)\Big) \right\}$$

and

$$\gamma_\varepsilon^\sigma = \min_j \inf_{t\in\mathbb{R}} \left\{ \beta_*^\sigma c_j a_{jj}(t) - \beta_\varepsilon^* \sum_{\substack{i=1\\ i\neq j}}^n c_i a_{ij}(t) \right\}.$$

Then $\beta_*^\sigma \geq \beta_*(g_\sigma + 1)$ and $\beta_\varepsilon^* \leq \beta^*(g_\varepsilon + 1)$, where $g_\sigma = \min_i \exp\big(-\sum_{j=1}^n a_{ij}^M \sigma\big) - 1$ and $g_\varepsilon = \max_i \exp\big(\sum_{j=1}^n a_{ij}^M \varepsilon\big) - 1$. Let

$$H_\sigma^\varepsilon = \min_j \Big\{ \beta_* g_\sigma c_j a_{jj}^M - \beta^* g_\varepsilon \sum_{\substack{i=1\\ i\neq j}}^n c_i a_{ij}^M \Big\}.$$

Note that $g_\sigma \leq 0$ and $g_\varepsilon \geq 0$. It then follows that

$$\gamma_\varepsilon^\sigma \geq \min_j \inf_{t\in\mathbb{R}} \Big\{ \beta_* c_j a_{jj}(t) - \beta^* \sum_{\substack{i=1\\ i\neq j}}^n c_i a_{ij}(t) \Big\} + H_\sigma^\varepsilon = \eta + H_\sigma^\varepsilon. \tag{10}$$

Since $\varepsilon_i > 0$ and $\eta > 0$, there exist $\varepsilon \in (0, \min_i\{\varepsilon_i\})$ and $\sigma \in (0, \varepsilon)$ such that $\mathcal{A}_{\varepsilon,\sigma}$ is positively invariant and attracting on $\dot{\mathbb{R}}_+^n$ by Lemma 2.1 and

$$\gamma := \gamma_\varepsilon^\sigma \geq \frac{\eta}{2} > 0$$

by (10) because $H_\sigma^\varepsilon \to 0$ as $(\varepsilon, \sigma) \to (0, 0)$. Therefore, there exists $t_0 \geq 0$ such that for all $t \geq t_0$,

$$\varepsilon_i - \varepsilon \leq y_i(t), z_i(t) \leq m_i + \sigma, \quad i = 1, \ldots, n. \tag{11}$$

Let $f_{1i}(t) = r_i(t) - \sum_{j=1}^n a_{ij}(t) y_j(t)$ and $f_{2i}(t) = r_i(t) - \sum_{j=1}^n a_{ij}(t) z_j(t)$, $i = 1, \ldots, n$. According to the Mean Value Theorem, for each $t$ there exists

$$\xi_i(t) \in \left(\min\left\{f_{1i}(t), f_{2i}(t)\right\}, \max\left\{f_{1i}(t), f_{2i}(t)\right\}\right)$$

such that

$$\exp(f_{1i}(t)) - \exp(f_{2i}(t)) = \exp(\xi_i(t)) \left(f_{1i}(t) - f_{2i}(t)\right).$$

By (11), $\beta_*^\sigma \leq \exp(\xi_i(t)) \leq \beta_\varepsilon^*$ for all $t \geq t_0$, $i = 1, \ldots, n$. Since $c_i |\ln \frac{y_i(t)}{z_i(t)}|$ is absolutely continuous, one has for almost all $t \geq t_0$,

$$\begin{aligned} \frac{d}{dt} c_i \left| \ln \frac{y_i(t)}{z_i(t)} \right| &= c_i \left(\exp(f_{1i}(t)) - \exp(f_{2i}(t))\right) \operatorname{sgn}\left(y_i(t) - z_i(t)\right) \\ &= c_i \exp\left(\xi_i(t)\right) \Big(\sum_{j=1}^n a_{ij}(t)\left(z_j(t) - y_j(t)\right)\Big) \operatorname{sgn}\left(y_i(t) - z_i(t)\right) \\ &\leq -c_i \beta_*^\sigma a_{ii}(t) \left|y_i(t) - z_i(t)\right| + c_i \beta_\varepsilon^* \sum_{\substack{j=1\\ j\neq i}}^n a_{ij}(t) \left|y_j(t) - z_j(t)\right|. \end{aligned} \tag{12}$$

Applying (12) to each summand in $\frac{dv(t)}{dt}$ and rearranging terms we obtain

$$\begin{aligned} \frac{dv(t)}{dt} &\leq \sum_{j=1}^n \Big(-c_j \beta_*^\sigma a_{jj}(t) + \sum_{\substack{i=1\\ i\neq j}}^n c_i \beta_\varepsilon^* a_{ij}(t)\Big) \left|y_j(t) - z_j(t)\right| \\ &\leq -\gamma \sum_{j=1}^n \left|y_j(t) - z_j(t)\right| \end{aligned}$$

for almost all $t \geq t_0$, which yields (9) by the Fundamental Theorem of Calculus for Lebesgue Integral. □

**Lemma 2.4** (*Barbălat's Lemma [12]*). *Suppose that $g : [a, +\infty) \to \mathbb{R}$ is uniformly continuous and $\lim_{t\to+\infty} \int_a^t g(s)\mathrm{d}s$ exists and is finite. Then $\lim_{t\to+\infty} g(t) = 0$ holds.*

**Lemma 2.5.** *Suppose that the conditions in Lemma 2.3 hold. Then for any two solutions $y(t), z(t)$ with initial values $y(0), z(0) \in \dot{\mathbb{R}}_+^n$, one has*

$$\lim_{t\to+\infty} \big(y(t) - z(t)\big) = 0.$$

**Proof.** By Lemma 2.3 and $v(t) \geq 0$ for all $t \geq 0$, one has

$$\int_{t_0}^{+\infty} \left|y_i(s) - z_i(s)\right| ds \leq \frac{v\left(t_0\right)}{\gamma} < +\infty, \quad i = 1, \dots, n.$$

Since $\left|y_i(t) - z_i(t)\right|$ is absolutely continuous, there exists $K_i > 0$ such that

$$\left|\frac{d}{dt}\left|y_i(t) - z_i(t)\right|\right| \leq \left|y_i(t)\exp(f_{1i}(t)) - z_i(t)\exp(f_{2i}(t))\right| \leq K_i \tag{13}$$

for almost all $t \geq 0$, because $y(t)$ and $z(t)$ are bounded by Lemma 2.1. Then by the Fundamental Theorem of Calculus for Lebesgue Integral and (13), one has that $\left|y_i(t) - z_i(t)\right|$ is uniformly continuous on $[0, +\infty)$. Thus, Lemma 2.4 implies that

$$\lim_{t\to+\infty} |y_i(t) - z_i(t)| = 0, \quad i = 1, \dots, n. \quad \square$$

**Lemma 2.6.** *If the conditions in Theorem 1.1 hold, then $\varepsilon_i > 0$, $i = 1, \dots, n$, and there exist $c_1, \dots, c_n > 0$ such that $\eta > 0$ which is defined by (8).*

**Proof.** We use a modification of the method of Ahmad and Lazer [10, Lemma 4] to prove the lemma. Since $\varepsilon_i \leq m_i$ for all $i$, one has $0 < \beta_* \leq \beta^*$. If conditions (2) hold, then one has $r_i^L > \sum_{\substack{j=1\\ j\neq i}}^n a_{ij}^M m_j$ for all $i$, which implies that $\varepsilon_i > 0$, $i = 1, \dots, n$, and

$$\beta_* a_{ii}^L m_i \geq \beta_* r_i^L > \sum_{\substack{j=1\\ j\neq i}}^n \beta^* a_{ij}^M m_j, \quad i = 1, \dots, n. \tag{14}$$

Let $e_{ii} = 0$ and $e_{ij} = a_{ij}^M / a_{ii}^L$ for $j \neq i$. Then by (14), one has $\sum_{j=1}^n \beta^* e_{ij} m_j < \beta_* m_i$, $i = 1, \dots, n$. Let $W$ be a matrix with positive components $w_{ij} > e_{ij}$ such that

$$\sum_{j=1}^n \beta^* w_{ij} m_j < \beta_* m_i, \quad i = 1, \dots, n. \tag{15}$$

Then by the Perron–Frobenius theorem, $W$ has a positive left eigenvector $q = (q_1, \dots, q_n)$ associated with a positive eigenvalue $\lambda > 0$, that is,

$$\sum_{i=1}^n w_{ij} q_i = \lambda q_j, \quad j = 1, \dots, n. \tag{16}$$

It follows from (15) and (16) that

$$\lambda \sum_{j=1}^n \beta^* m_j q_j = \sum_{j=1}^n \sum_{i=1}^n \beta^* w_{ij} m_j q_i = \sum_{i=1}^n \left(\sum_{j=1}^n \beta^* w_{ij} m_j\right) q_i < \sum_{i=1}^n \beta_* m_i q_i.$$

which implies that $\lambda < \frac{\beta_*}{\beta^*}$. Then by the definition of $e_{ij}$ and (16), one has

$$\sum_{\substack{i=1\\ i\neq j}}^n \beta^* \frac{a_{ij}^M}{a_{ii}^L} q_i = \sum_{i=1}^n \beta^* e_{ij} q_i < \sum_{i=1}^n \beta^* w_{ij} q_i < \beta_* q_j, \quad j = 1, \dots, n. \tag{17}$$

Thus, one has $\eta > 0$ by (17) if we choose $c_i = \frac{q_i}{a_{ii}^L}$, $i = 1, \dots, n$. $\square$

**Proof of Theorem 1.1.** By Lemma 2.6, one has $\varepsilon_i > 0$ for all $i$ and there exist $c_1, \dots, c_n > 0$ such that $\eta > 0$. Then Lemma 2.2 implies the existence of a positive $\omega$-periodic solution $\hat{x}(t) = x(t, \hat{x})$. Assume that there is another positive $\omega$-periodic solution which is denoted by $\hat{y}(t) = x(t, \hat{y})$. It then follows from Lemma 2.5 that

$$\hat{x} - \hat{y} = \lim_{k\to+\infty} \left(\hat{x}(k\omega) - \hat{y}(k\omega)\right) = 0,$$

and hence $\hat{x}(t) = \hat{y}(t)$ by uniqueness of solution of system (1). Therefore, by Lemma 2.2,

$$\hat{x}(t) = \hat{x}^{\varepsilon,\sigma}(t) \in \mathcal{A}_{\varepsilon,\sigma}, \quad \forall t \geq 0,$$

for any $\mathcal{A}_{\varepsilon,\sigma}$ defined in Lemma 2.1. Then let $\varepsilon \to 0$, one obtains that $\varepsilon_i \leq \hat{x}_i(t) \leq m_i$, $i = 1, \dots, n$, for all $t \geq 0$. By Lemma 2.5, one also has

$$\lim_{t\to+\infty} \left(x(t, x_0) - \hat{x}(t)\right) = 0$$

for any $x_0 \in \dot{\mathbb{R}}_+^n$. This completes the proof.

## Data availability

No data was used for the research described in the article.

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