Mathématiques discrètes et applications
Discrete Mathematics and Applications

Problem Session 4

Peterson Yuhala

Instituts d'informatique et de mathématiques
Université de Neuchâtel

Automne 2025 / Fall 2025

Université de Neuchâtel
Faculté des sciences

Contact information

Peterson Yuhala
Office B119
peterson.yuhala@unine.ch

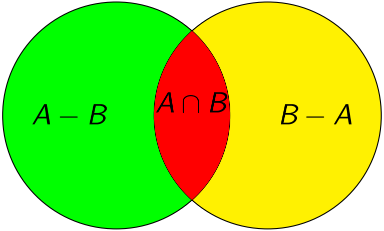
Ex-1a.

$$(A - B) - (B - C) \stackrel{?}{=} A - B$$

$$\begin{aligned}
L.H.S &= (A \cap \bar{B}) - (B \cap \bar{C}) && \textbf{Definition of -} \\
&= (A \cap \bar{B}) \cap \overline{(B \cap \bar{C})} && \textbf{Definition of -} \\
&= (A \cap \bar{B}) \cap (\bar{B} \cup C) && \textbf{De Morgan's law} \\
&= ((A \cap \bar{B}) \cap \bar{B}) \cup ((A \cap \bar{B}) \cap C) && \textbf{Distributive law} \\
&= (A \cap (\bar{B} \cap \bar{B})) \cup (A \cap (\bar{B} \cap C)) && \textbf{Associative law} \\
&= (A \cap \bar{B}) \cup (A \cap (\bar{B} \cap C)) && \textbf{Idempotent law} \\
&= A \cap (\bar{B} \cup (\bar{B} \cap C)) && \textbf{Distributive law} \\
&= A \cap \bar{B} && \textbf{Absorption law} \\
&= A - B \quad \square
\end{aligned}$$

Ex-1b.

$|A \cup B| \stackrel{?}{=} |A| + |B| - |A \cap B|$



From the picture, we have $A \cup B = (A - B) \cup (A \cap B) \cup (B - A)$
These sets are **mutually disjoint**, so we have:

$$\begin{aligned} |A \cup B| &= |A - B| + |A \cap B| + |B - A| \\ &= (|A| - |A \cap B|) + |A \cap B| + (|B| - |A \cap B|) \\ &= |A| + |B| - |A \cap B| \quad \square \end{aligned}$$

Ex-2

$$\overbrace{(B-A)\cup(C-A)}^{\text{L.H.S}} \stackrel{?}{=} \overbrace{(B\cup C)-A}^{\text{R.H.S}}$$

Let's show $L.H.S \subseteq R.H.S$

$$\begin{aligned}
x \in (B-A)\cup(C-A) &\implies x \in (B-A) \lor x \in (C-A) \\
&\implies (x \in B \land x \notin A) \lor (x \in C \land x \notin A) \\
&\implies x \notin A \land (x \in B \lor x \in C) \\
&\implies x \in (B \cup C) \land x \notin A \\
&\implies x \in (B \cup C) - A \\
&\implies L.H.S \subseteq R.H.S
\end{aligned}$$

Ex-2

Let's show $R.H.S \subseteq L.H.S$

$$\begin{aligned} x \in (B \cup C) - A &\implies x \in (B \cup C) \land x \notin A \\ &\implies (x \in B \lor x \in C) \land x \notin A \\ &\implies (x \in B \land x \notin A) \lor (x \in C \land x \notin A) \\ &\implies x \in (B - A) \lor x \in (C - A) \\ &\implies x \in (B - A) \cup (C - A) \\ &\implies R.H.S \subseteq L.H.S \end{aligned}$$

$L.H.S \subseteq R.H.S$ and $R.H.S \subseteq L.H.S \implies L.H.S = R.H.S \quad \square$

Ex-3a

· For a set X, $\mathcal{P}(X)$ is called the *power set* of X. It is the set containing all subsets of X.
· Note that the *empty set* ($\varnothing = \{\}$) is a subset of every set, including itself.

· $\mathcal{P}(A) = \{\varnothing\} = \{\{\}\}$
· $\mathcal{P}(B) = \{\varnothing, \{\varnothing\}\} = \{\{\}, \{\{\}\}\}$
· $\mathcal{P}(C) = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$

Ex-3b

Consider the statement:
$P(n)$: for every set X, if $|X| = n$, then $|\mathcal{P}(X)| = 2^n$.
Base step: $n = 0$
P(0):

$$|X| = 0 \iff X = \varnothing$$
$$\mathcal{P}(X) = \{\varnothing\} \implies |\mathcal{P}(X)| = 1 = 2^0$$
$$\implies P(0) \text{ is true}$$

· Let's assume that $P(k)$ is true and show that "$P(k)$ true $\implies$ $P(k+1)$ true".
Consider a set $S = \{e_1, e_2, ..., e_k\}$
$P(k)$: $|S| = k \implies |\mathcal{P}(S)| = 2^k$ (Inductive Hypothesis)
Let T be a set such that $T = \{e_1, e_2, ..., e_k, e_{k+1}\}$. We have:
$T = S \cup \{e_{k+1}\}$

Ex-3b

$T = S \cup \{e_{k+1}\} \implies$ every subset of S is also a subset of T. Moreover, the subsets of T can be divided into 2 disjoint sets: those that do contain e_{k+1} (T_1) and those that do not (T_2).

· T_2 : If a subset of T does not contain e_{k+1}, then it must be a subset of S and there are 2^k of such subsets (I.H.). So $|T_2| = 2^k$.

· T_1 : If a subset of T contains e_{k+1}, then it must be of the form $Z \cup \{e_{k+1}\}$, where $Z \in \mathcal{P}(S)$.

However, there are 2^k of such subsets Z (I.H.) $\implies$ there are also 2^k subsets of the form $Z \cup \{e_{k+1}\}$. That is, $|T_1| = 2^k$. So we have:

$$\begin{aligned} |\mathcal{P}(T)| &= |T_1| + |T_2| \\ &= 2^k + 2^k \\ &= 2 \cdot 2^k \\ &= 2^{k+1} \end{aligned}$$

Thus $P(k) \implies P(k+1)$. Conclusion: $P(n)$ holds $\forall n \geq 0$.

Ex-4a

To show that g is injective, we take arbitrary $x, y \in A$, assume $g(x) = g(y)$, and show that $x = y$ follows from this hypothesis.

$$\begin{tikzcd}
A \arrow[d, "g"'] \arrow[dr, "f \circ g"] & \\
B \arrow[r, "f"'] & C
\end{tikzcd}$$

$$\begin{aligned}
g(x) = g(y) &\implies f(g(x)) = f(g(y)) && \textbf{because } g(x) \in B \\
&\implies (f \circ g)(x) = (f \circ g)(y) && \textbf{by def. of composition} \\
&\implies x = y && f \circ g \textbf{ is injective}
\end{aligned}$$

Hence, g is injective.

Ex-4b

$f \circ g$ surjective means $\forall c \in C,\ \exists a \in A \mid (f \circ g)(a) = c$.

For an arbitrary $c \in C$, let's find $b \in B \mid f(b) = c$.

$$c \in C \wedge f \circ g \text{ surjective} \implies \exists a \in A \mid (f \circ g)(a) = c$$

Let $a \in A$ such that $(f \circ g)(a) = c$, then $f(g(a)) = c$.

But $g(a) \in B$, so for $b = g(a)$, we have $f(b) = c$.

Hence, f is surjective.

Ex-5

$$\begin{tikzcd}
\{1,2,3\} \arrow[d, "f"'] \arrow[dr, "g \circ f"] & \\
\{a,b,c,d\} \arrow[r, "g"'] & \{1,2,3,4,5\}
\end{tikzcd}$$

· We have: $g \circ f : \{1,2,3\} \to \{1,2,3,4,5\}$

$$(g \circ f)(1) = g(f(1)) = g(a) = 2$$
$$(g \circ f)(2) = g(f(2)) = g(c) = 4$$
$$(g \circ f)(3) = g(f(3)) = g(d) = 5$$

Question: is $g \circ f$ surjective?

Ex-6a

- For an arbitrary $y \in \mathbb{Z}$, let's find $(a, b) \in \mathbb{Z} \mid f(a, b) = y$.
- $f(m, n) = y \iff 2m - n = y$. Take $m = 0 \in \mathbb{Z}$.

$$\begin{aligned} m = 0 \wedge 2m - n = y &\iff m = 0 \wedge -n = y \\ &\iff m = 0 \wedge n = -y \in \mathbb{Z}. \end{aligned}$$

- So for $(a, b) = (0, -y)$, we have $f(a, b) = y$. So f is surjective.

Ex-6b

$f : A \to B$ **not surjective** means $\exists b \in B \mid \forall a \in A,\ f(a) \neq b$

· For $f(m, n) = m^2 - n^2$, if we take $y = 2 \in \mathbb{Z}$, then:

$$\begin{aligned} f(m, n) = 2 &\Longrightarrow m^2 - n^2 = 2 \\ &\Longrightarrow (m + n)(m - n) = 2 \end{aligned}$$

· 2 is prime $\Longrightarrow$
$(m + n, m - n) \in \{(1, 2), (2, 1), (-1, -2), (-2, -1)\}$

We consider the first case, where $m + n = 1$ and $m - n = 2$.

$$\begin{aligned} m + n = 1 &\Longrightarrow m = 1 - n \\ &\Longrightarrow (1 - n) - n = m - n = 2 \end{aligned}$$

Ex-6b

$$
\begin{aligned}
(1 - n) - n = 2 &\implies -2n = 1 \\
&\implies n = -\frac{1}{2} \notin \mathbb{Z}
\end{aligned}
$$

The other three cases similarly lead to $n \notin \mathbb{Z}$ and thus to a contradiction. We conclude that there exists no couple $(m, n) \in \mathbb{Z} \times \mathbb{Z}$ such that $f(m, n) = 2$.

Hence, f is not surjective.

Ex-6c

For an arbitrary $y \in \mathbb{Z}$, let's find $(m, n) \in \mathbb{Z} \times \mathbb{Z} \mid f(m, n) = y$.

$$f(m, n) = y \iff m + n - 4 = y$$

Take $m = 0$:

$$\begin{aligned} m = 0 \land m + n - 4 = y &\iff m = 0 \land n - 4 = y \\ &\iff m = 0 \land n = y + 4 \in \mathbb{Z} \end{aligned}$$

Therefore, for $(m, n) = (0, y + 4)$, we have that $f(m, n) = y$.

Hence, f is surjective.

Ex-7a

$$\overset{\text{L.H.S}}{f^{-1}(S \cup T)} \overset{?}{=} \overset{\text{R.H.S}}{f^{-1}(S) \cup f^{-1}(T)}$$

$$\begin{aligned} L.H.S \subseteq R.H.S: x \in f^{-1}(S \cup T) &\implies f(x) \in S \cup T \\ &\implies f(x) \in S \vee f(x) \in T \\ &\implies x \in f^{-1}(S) \vee x \in f^{-1}(T) \\ &\implies x \in f^{-1}(S) \cup f^{-1}(T) \end{aligned}$$

Therefore: $f^{-1}(S \cup T) \subseteq f^{-1}(S) \cup f^{-1}(T)$

A similar argument shows that $R.H.S \subseteq L.H.S$.

Conclusion: $L.H.S = R.H.S$

Ex-7b

$$\overset{\text{L.H.S}}{f^{-1}(S \cap T)} \stackrel{?}{=} \overset{\text{R.H.S}}{f^{-1}(S) \cap f^{-1}(T)}$$

Use the same reasoning as in Ex-7a.

Ex-7c

$$\overbrace{f^{-1}(\bar{S})}^{\text{L.H.S}} \stackrel{?}{=} \overbrace{\overline{f^{-1}(S)}}^{\text{R.H.S}}$$

We first prove the inclusion "$\subseteq$":

$$\begin{aligned} x \in f^{-1}(\bar{S}) &\implies f(x) \in \bar{S} \\ &\implies f(x) \notin S \\ &\implies x \notin f^{-1}(S) \\ &\implies x \in \overline{f^{-1}(S)} \end{aligned}$$

Therefore, $f^{-1}(\bar{S}) \subseteq \overline{f^{-1}(S)}$.

Ex-7c

We next prove the inclusion "$\supseteq$":

$$\begin{aligned} x \in \overline{f^{-1}(S)} &\Longrightarrow x \notin f^{-1}(S) \\ &\Longrightarrow f(x) \notin S \\ &\Longrightarrow f(x) \in \bar{S} \\ &\Longrightarrow x \in f^{-1}(\bar{S}) \end{aligned}$$

Therefore, $\overline{f^{-1}(S)} \subseteq f^{-1}(\bar{S})$.

$\overline{f^{-1}(S)} \subseteq f^{-1}(\bar{S})$ and $f^{-1}(\bar{S}) \subseteq \overline{f^{-1}(S)} \Longrightarrow f^{-1}(\bar{S}) = \overline{f^{-1}(S)}$.

Ex-8

A set is countable if it is either **finite** or **countably infinite**. A set A is countably infinite if it satisfies one of the following equivalent conditions:

· There exists a bijection between A and $\mathbb{N}^*$.

· A has the same cardinality as $\mathbb{N}^*$. That is: $A \sim \mathbb{N}^*$ or $|A| = |\mathbb{N}^*| = \aleph_0$.

· The set of positive odd integers $\mathbb{O}$ is not finite. So to show it is countable, we need to show it is countably infinite, i.e., we need to find a bijection between $\mathbb{O}$ and $\mathbb{N}^*$.

Ex-8

Consider the following function:

$$
\begin{aligned}
f : \mathbb{N}^* &\to \mathbb{O} \\
n &\mapsto 2n - 1
\end{aligned}
$$

The idea behind the choice of such a function f is to have the following correspondence under f. NB: f is not the only possible choice.

$$
\begin{array}{lcccc}
\mathbb{N}^* : & 1 & 2 & 3 & \cdots \\
 & \updownarrow & \updownarrow & \updownarrow & \\
\mathbb{O} : & 1 & 3 & 5 & \cdots
\end{array}
$$

Ex-8

That is, f should map $\mathbb{N}^*$ bijectively to the set in question ($\mathbb{O}$ in this case). Let's show that f is bijective.

· **Injectivity:**

For $a, b \in \mathbb{N}^*$, assume that $f(a) = f(b)$.

$$\begin{aligned} f(a) = f(b) &\implies 2a - 1 = 2b - 1 \\ &\implies 2a = 2b \\ &\implies a = b \end{aligned}$$

Therefore, f is injective.

Ex-8

· **Surjectivity:**

For an arbitrary $y \in \mathbb{O}$, we want to find $n_y \in \mathbb{N}^* \mid f(n_y) = y$.

· $y \in \mathbb{O}$ (i.e. y odd) $\Longrightarrow \exists k \in \mathbb{Z} \mid y = 2k - 1$ (by the def. of odd integers). Since $y \geq 1$, we have that $k \geq 1$, so $k \in \mathbb{N}^*$. So we can take $n_y = k$.

· f injective and surjective $\Longrightarrow$ f is bijective. Therefore $\mathbb{O}$ is countable (countably infinite to be more specific).

Ex-9

· Let A and B be disjoint and countably infinite (the case where at least one of A, B is finite can be treated similarly): $A = \{a_1, a_2, ...\}$ and $B = \{b_1, b_2, ...\}$. This means there are bijections $f : \mathbb{N}^* \to A$ and $g : \mathbb{N}^* \to B$.

· To show that $A \cup B$ is countable, let's define a function $h : \mathbb{N}^* \to A \cup B$ such that:

$$h(n) = \begin{cases} f(\frac{n+1}{2}) & \text{if } n \text{ is odd} \\ g(\frac{n}{2}) & \text{if } n \text{ is even} \end{cases}$$

· Let's show that h is bijective. You can verify that h maps $\mathbb{N}^*$ to $A \cup B = \{a_1, b_1, a_2, b_2, ...\}$

Ex-9

· **Injectivity:**

Suppose we have $a, b \in \mathbb{N}^*$, such that $h(a) = h(b)$. Since A and B are disjoint, it follows that either a and b are both odd or a and b are both even.

If a and b are odd, then:

$$f\left(\frac{a+1}{2}\right) = f\left(\frac{b+1}{2}\right)$$
$$\implies \frac{a+1}{2} = \frac{b+1}{2} \implies a = b \text{ (because } f \text{ is injective)}$$

On the other hand, if a and b are even, then:

$$g\left(\frac{a}{2}\right) = g\left(\frac{b}{2}\right)$$
$$\implies \frac{a}{2} = \frac{b}{2} \implies a = b \text{ (because } g \text{ is injective)}$$

So h is injective.

Ex-9

· **Surjectivity:**
For an arbitrary element $e \in A \cup B$, let's find $n_e \in \mathbb{N}^*$ such that $h(n_e) = e$.
· Case 1: $e \in A$. Since f is surjective, $\exists k_1 \in \mathbb{N}^*$ such that $f(k_1) = e$. We can thus take n_e such that $k_1 = \frac{n_e+1}{2}$. That is, $n_e = 2k_1 - 1 \in \mathbb{N}^*$.
· Case 2: $e \in B$. Since g is surjective, $\exists k_2 \in \mathbb{N}^*$ such that $g(k_2) = e$. We can thus take n_e such that $k_2 = \frac{n_e}{2}$. That is, $n_e = 2k_2 \in \mathbb{N}^*$.
Thus h is surjective.
· **Conclusion**: h is both injective and surjective, thus bijective. So $A \cup B$ is countably infinite.

Ex-10

B countable $\implies B$ is either finite or countably infinite.
$\cdot$ Case 1: B **finite**: B finite $\implies A$ finite (because $A \subseteq B$) $\implies A$ is countable.
$\cdot$ Case 2: B **countably infinite**: B countably infinite $\implies \exists$ a bijection $f : B \to \mathbb{N}^* \implies f(A) \subset \mathbb{N}^*$. From the lemma, we have that $f(A)$ is countable.
$\cdot$ f bijective $\implies |A| = |f(A)|$ (i.e. A has the same cardinality as $f(A)$).
$\cdot$ $|A| = |f(A)|$ and $f(A)$ countable $\implies A$ countable. (NB: Any set that has the same cardinality as a subset of $\mathbb{N}^*$ is countable.)
$\cdot$ Conclusion: B countable $\implies A$ countable. $\square$

End

For more exercises and notes, see Rosen, 7th Edition, Chap. 2.