University of Neuchâtel
Discrete Mathematics and Applications - Fall 2025
Problems - 3

Part A: Rules of Inference

1. Let $C(x)$ be the predicate "x has a cat", let $D(x)$ be the predicate "x has a dog", and let $F(x)$ be the predicate "x has a ferret". Express each of the following statements in terms of $C(x)$, $D(x)$, $F(x)$, quantifiers and logical operators. The domain is the set of all students in your class.
   (a) A student in your class has a cat, a dog and a ferret.
   (b) All students in your class have a cat, a dog or a ferret.
   (c) Some student in your class has a cat and a ferret but not a dog.
   (d) No student in your class has a cat, a dog and a ferret.
   (e) For each of the three animals, there is a student in your class who has one of these as a pet.
2. Let $F(x, y)$ be the predicate "x can fool y", where the domain consists of all the people in the world. Use quantifiers to express each of the following statements.
   (a) Everybody can fool Nancie.
   (b) Peter can fool everybody.
   (c) Everybody can fool somebody.
   (d) There is no one who can fool everybody.
   (e) Everyone can be fooled by somebody.
   (f) No one can fool both Ouny and Leyla.
   (g) Essil can fool exactly two people.
   (h) Ehlena cannot fool anyone.
   (i) There is exactly one person whom everybody can fool.
   (j) No one can fool themselves.
   (k) There is someone who can fool exactly one person, and that one person is not themselves.

   Disclaimer: *All names and characters used are fictitious! Any resemblance to actual persons, living or dead, is purely coincidental and should not be inferred.*
3. By definition, $\lim_{x\to a} f(x) = L$ if and only if:
$$\forall\varepsilon > 0, \exists\delta > 0 \text{ such that } \forall x, |x - a| < \delta \Longrightarrow |f(x) - L| < \varepsilon$$
   Write a similar expression for: $\lim_{x\to a} f(x) \neq L$
4. Consider the following propositions:
   p : It's sunny this afternoon.
   q : It's colder than yesterday.
   r : We will go swimming.
   s : We will take a canoe trip.

t : We will be home by sunset.

Suppose we have the premises:
a. "It's not sunny this afternoon and it's colder than yesterday."
b. "We will go swimming only if it's sunny this afternoon."
c. "If we won't go swimming, then we will take a canoe trip."
d. "If we will take a canoe trip, then we will be home by sunset."

Show using rules of inference that: "We will be home by sunset."

Part B: Methods of Proof

5. Prove *by induction* that for $n \in \mathbb{N}$, $n(n+1)(n+2)$ is divisible by 6. You can use the fact that $n(n+1)$ is even for all $n \in \mathbb{N}$.

6. Prove *by contradiction* that for $a \in \mathbb{Z}$, a^2 even $\implies a$ is even.

7. Prove *by contraposition* that if $n = ab$, where $a, b \in \mathbb{N}^*$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$ holds.

8. Show by *mathematical induction* that De Morgan's Laws hold for any number of variables.

$$\forall n \geq 1,\ \forall (p_1, \ldots, p_n) \in \{0,1\}^n : \begin{cases} \neg(p_1 \wedge \ldots \wedge p_n) = \neg p_1 \vee \ldots \vee \neg p_n \\ \neg(p_1 \vee \ldots \vee p_n) = \neg p_1 \wedge \ldots \wedge \neg p_n \end{cases}$$

9. For $n \in \mathbb{N}^*$, find a formula for:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{2^n}$$

Use mathematical induction to prove the conjecture.