2. Some Key Linear Models Results

A General Linear Model (GLM)

Suppose $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where

- $\boldsymbol{y} \in \mathbb{R}^n$ is the response vector,
- $\boldsymbol{X}$ is an $n \times p$ matrix of known constants,
- $\boldsymbol{\beta} \in \mathbb{R}^p$ is an unknown parameter vector, and
- $\boldsymbol{\epsilon}$ is a vector of random “errors” satisfying $E(\boldsymbol{\epsilon}) = \boldsymbol{0}$.

This GLM says simply that $\boldsymbol{y}$ is a random vector satisfying $E(\boldsymbol{y}) = \boldsymbol{X}\boldsymbol{\beta}$ for some $\boldsymbol{\beta} \in \mathbb{R}^p$.

The distribution of $\boldsymbol{y}$ is left unspecified.

We know only that $\boldsymbol{y}$ is random and its mean is in the column space of $\boldsymbol{X}$; i.e., $E(\boldsymbol{y}) \in \mathcal{C}(\boldsymbol{X})$.

Ordinary Least Squares (OLS) Estimation of $E(\boldsymbol{y})$

Because we know $E(\boldsymbol{y}) \in \mathcal{C}(\boldsymbol{X})$, a natural estimator of $E(\boldsymbol{y})$ is the *Ordinary Least Squares Estimator* (OLSE), which is the unique point in $\mathcal{C}(\boldsymbol{X})$ that is closest to $\boldsymbol{y}$ in terms of Euclidean distance.

The OLSE of $E(\boldsymbol{y})$ is given by $\hat{\boldsymbol{y}} \equiv \boldsymbol{P_X y}$, where

$$\boldsymbol{P_X} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}',$$

because $\boldsymbol{P_X y} \in \mathcal{C}(\boldsymbol{X})$ and

$$||\boldsymbol{y} - \boldsymbol{P_X y}||^2 < ||\boldsymbol{y} - \boldsymbol{z}||^2 \;\forall\, \boldsymbol{z} \in \mathcal{C}(\boldsymbol{X}) \setminus \{\boldsymbol{P_X y}\}.$$

The Orthogonal Projection Matrix

$\boldsymbol{P_X} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'$ is known as the *orthogonal-projection matrix* onto the column space of $\boldsymbol{X}$ and has the following properties:

- $\boldsymbol{P_X}$ is symmetric (i.e., $\boldsymbol{P_X} = \boldsymbol{P}'_{\boldsymbol{X}}$).
- $\boldsymbol{P_X}$ is idempotent (i.e., $\boldsymbol{P_X}\boldsymbol{P_X} = \boldsymbol{P_X}$).
- $\boldsymbol{P_X}\boldsymbol{X} = \boldsymbol{X}$ and $\boldsymbol{X}'\boldsymbol{P_X} = \boldsymbol{X}'$.
- $\text{rank}(\boldsymbol{X}) = \text{rank}(\boldsymbol{P_X}) = \text{tr}(\boldsymbol{P_X})$.
- $\boldsymbol{P_X} = \boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'$ is the same matrix for all generalized inverses $(\boldsymbol{X}'\boldsymbol{X})^{-}$ of $\boldsymbol{X}'\boldsymbol{X}$.

The OLSE of a Linear Function of $E(\boldsymbol{y})$

For any $q \times n$ matrix $\boldsymbol{A}$, $\boldsymbol{A}E(\boldsymbol{y})$ is a linear function of $E(\boldsymbol{y})$.

For any $q \times n$ matrix $\boldsymbol{A}$, the OLSE of $\boldsymbol{A}E(\boldsymbol{y}) = \boldsymbol{A}\boldsymbol{X}\boldsymbol{\beta}$ is

$$\boldsymbol{A}[\text{OLSE of } E(\boldsymbol{y})] = \boldsymbol{A}\hat{\boldsymbol{y}} = \boldsymbol{A}\boldsymbol{P}_X\boldsymbol{y} = \boldsymbol{A}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{X}'\boldsymbol{y}.$$

The OLSE of an estimable function of $\boldsymbol{\beta}$

For any $q \times n$ matrix $\boldsymbol{A}$, $\boldsymbol{A}E(\boldsymbol{y}) = \boldsymbol{A}\boldsymbol{X}\boldsymbol{\beta}$ is a linear function of $\boldsymbol{\beta}$ of the form $\boldsymbol{C}\boldsymbol{\beta}$, where $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{X}$.

From the previous slide, we know

OLSE of $\boldsymbol{C}\boldsymbol{\beta} = \boldsymbol{A}\boldsymbol{X}\boldsymbol{\beta} = \boldsymbol{A}E(\boldsymbol{y})$ is $\boldsymbol{A}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{y} = \boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{y}$.

Now if $\boldsymbol{C}$ is any $q \times p$ matrix, we say that the linear function of $\boldsymbol{\beta}$ given by $\boldsymbol{C}\boldsymbol{\beta}$ is *estimable* if and only if $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{X}$ for some matrix $q \times n$ matrix $\boldsymbol{A}$.

The OLSE of an estimable linear function $\boldsymbol{C}\boldsymbol{\beta}$ is $\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{y}$.

The OLSE of Estimable Functions of $\boldsymbol{\beta}$

An equivalent definition for the OLSE of an estimable linear function of $\boldsymbol{\beta}$, given by $\boldsymbol{C\beta} = \boldsymbol{AX\beta}$, can be stated in terms of solutions to the

$$\textit{Normal Equations}: \qquad \boldsymbol{X}'\boldsymbol{X}\boldsymbol{b} = \boldsymbol{X}'\boldsymbol{y}.$$

The OLSE of estimable $\boldsymbol{C\beta}$ is $\boldsymbol{C}\hat{\boldsymbol{\beta}}$, where $\hat{\boldsymbol{\beta}}$ is any solution for $\boldsymbol{b}$ in the Normal Equations.

Solutions to the Normal Equations

The Normal Equations

$$\boldsymbol{X}'\boldsymbol{X}\boldsymbol{b} = \boldsymbol{X}'\boldsymbol{y}$$

have $(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{X}'\boldsymbol{y}$ as the unique solution for $\boldsymbol{b}$ if $\text{rank}(\boldsymbol{X}) = p$.

The Normal Equations have infinitely many solutions for $\boldsymbol{b}$ if $\text{rank}(\boldsymbol{X}) < p$.

$\hat{\boldsymbol{\beta}} = (\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{y}$ is always a solution to the Normal Equations for any $(\boldsymbol{X}'\boldsymbol{X})^{-}$, a generalized inverse of $\boldsymbol{X}'\boldsymbol{X}$.

Uniqueness of the OLSE of an Estimable $\boldsymbol{C}\boldsymbol{\beta}$

If $\boldsymbol{C}\boldsymbol{\beta}$ is estimable, then $\boldsymbol{C}\hat{\boldsymbol{\beta}}$ is the same for all solutions $\hat{\boldsymbol{\beta}}$ to the Normal Equations.

In particular, the unique OLSE of $\boldsymbol{C}\boldsymbol{\beta}$ is

$$\boldsymbol{C}\hat{\boldsymbol{\beta}} = \boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{y} = \boldsymbol{A}\boldsymbol{X}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{y} = \boldsymbol{A}\boldsymbol{P}_{\boldsymbol{X}}\boldsymbol{y},$$

where $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{X}$.

The OLSE is a Linear Unbiased Estimator

If $\boldsymbol{C\beta}$ is estimable, then $\boldsymbol{C\hat{\beta}}$ is a *linear unbiased estimator* of $\boldsymbol{C\beta}$.

The OLSE is a *linear estimator* because it's a linear function of $\boldsymbol{y}$:

$$\boldsymbol{C\hat{\beta}} = \boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{y} = \boldsymbol{My}, \text{ where } \boldsymbol{M} = \boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'.$$

The OLSE is *unbiased* because, for all $\boldsymbol{\beta} \in \mathbb{R}^p$,

$$\begin{aligned} E(\boldsymbol{C\hat{\beta}}) &= E(\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'\boldsymbol{y}) = \boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'E(\boldsymbol{y}) = \boldsymbol{AX}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{X}'E(\boldsymbol{y}) \\ &= \boldsymbol{AP_X}E(\boldsymbol{y}) = \boldsymbol{AP_X X\beta} = \boldsymbol{AX\beta} = \boldsymbol{C\beta}. \end{aligned}$$

The Guass-Markov Model (GMM)

Suppose $\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}$, where

- $\boldsymbol{y} \in \mathbb{R}^n$ is the response vector,
- $\boldsymbol{X}$ is an $n \times p$ matrix of known constants,
- $\boldsymbol{\beta} \in \mathbb{R}^p$ is an unknown parameter vector, and
- $\boldsymbol{\epsilon}$ is a vector of random “errors” satisfying $E(\boldsymbol{\epsilon}) = \mathbf{0}$ and $\text{Var}(\boldsymbol{\epsilon}) = \sigma^2\boldsymbol{I}$ for some unknown variance parameter $\sigma^2 \in \mathbb{R}^+$.

The GMM is a Special Case of the GLM

The GMM is a special case of the GLM presented previously.

We have added the assumption $\text{Var}(\boldsymbol{\epsilon}) = \sigma^2 \boldsymbol{I}$; i.e., we assume the errors are uncorrelated and have constant variance.

All the results presented for the GLM hold for the GMM.

The Guass-Markov Theorem

For the GMM, we have an additional result provided by the *Gauss-Markov Theorem*:

The OLSE of an estimable function $\boldsymbol{C\beta}$ is the

Best Linear Unbiased Estimator (BLUE) of $\boldsymbol{C\beta}$

in the sense that the OLSE $\boldsymbol{C}\hat{\boldsymbol{\beta}}$ has the smallest variance among all linear unbiased estimators of $\boldsymbol{C\beta}$.

Unbiased Estimation of σ^2

An unbiased estimator of σ^2 under the GMM is given by

$$\hat{\sigma}^2 \equiv \frac{\boldsymbol{y}'(\boldsymbol{I} - \boldsymbol{P_X})\boldsymbol{y}}{n - r}, \text{ where } r = \text{rank}(\boldsymbol{X}).$$

Because $\boldsymbol{I} - \boldsymbol{P_X} = (\boldsymbol{I} - \boldsymbol{P_X})(\boldsymbol{I} - \boldsymbol{P_X}) = (\boldsymbol{I} - \boldsymbol{P_X})'(\boldsymbol{I} - \boldsymbol{P_X})$,

$$\begin{aligned}
\boldsymbol{y}'(\boldsymbol{I} - \boldsymbol{P_X})\boldsymbol{y} &= \boldsymbol{y}'(\boldsymbol{I} - \boldsymbol{P_X})'(\boldsymbol{I} - \boldsymbol{P_X})\boldsymbol{y} = \{(\boldsymbol{I} - \boldsymbol{P_X})\boldsymbol{y}\}'\{(\boldsymbol{I} - \boldsymbol{P_X})\boldsymbol{y}\} \\
&= ||(\boldsymbol{I} - \boldsymbol{P_X})\boldsymbol{y}||^2 = ||\boldsymbol{y} - \boldsymbol{P_X}\boldsymbol{y}||^2 \\
&= ||\boldsymbol{y} - \hat{\boldsymbol{y}}||^2 = \text{“Sum of Squared Errors” (SSE).}
\end{aligned}$$

Gauss-Markov Model with Normal Errors (GMMNE)

Suppose

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon},$$

where

- $\boldsymbol{y} \in \mathbb{R}^n$ is the response vector,
- $\boldsymbol{X}$ is an $n \times p$ matrix of known constants,
- $\boldsymbol{\beta} \in \mathbb{R}^p$ is an unknown parameter vector, and
- $\boldsymbol{\epsilon} \sim N(\boldsymbol{0}, \sigma^2\boldsymbol{I})$ for some unknown variance parameter $\sigma^2 \in \mathbb{R}^+$.

The GMMNE is a special case of the GMM.

We have added the assumption $\boldsymbol{\epsilon}$ is multivariate normal.

The GMMNE implies $\boldsymbol{y} \sim N(\boldsymbol{X}\boldsymbol{\beta}, \sigma^2\boldsymbol{I})$.

The GMMNE is useful for drawing statistical inferences regarding estimable $\boldsymbol{C}\boldsymbol{\beta}$.

Throughout the remainder of these slides we will assume

- the GMMNE model holds,
- $\boldsymbol{C}$ is a $q \times p$ matrix such that $\boldsymbol{C\beta}$ is estimable,
- $\text{rank}(\boldsymbol{C}) = q$, and
- $\boldsymbol{d}$ is a known $q \times 1$ vector.

These assumptions imply $H_0 : \boldsymbol{C\beta} = \boldsymbol{d}$ is a *testable hypothesis*.

The Distribution of $\boldsymbol{C}\hat{\boldsymbol{\beta}}$ and $\hat{\sigma}^2$

- $\boldsymbol{C}\hat{\boldsymbol{\beta}} \sim N(\boldsymbol{C}\boldsymbol{\beta}, \sigma^2\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{C}')$.
- $\frac{(n-r)\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-r} \iff \frac{\hat{\sigma}^2}{\sigma^2} \sim \frac{\chi^2_{n-r}}{n-r} \iff \hat{\sigma}^2 \sim \frac{\sigma^2}{n-r}\chi^2_{n-r}$.
- $\boldsymbol{C}\hat{\boldsymbol{\beta}}$ and $\hat{\sigma}^2$ are independent.

The F Test of $H_0 : \boldsymbol{C}\boldsymbol{\beta} = \boldsymbol{d}$

The test statistic

$$\begin{aligned} F &\equiv (\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{d})'[\widehat{\text{Var}}(\boldsymbol{C}\hat{\boldsymbol{\beta}})]^{-1}(\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{d})/q \\ &= (\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{d})'[\hat{\sigma}^2\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{C}']^{-1}(\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{d})/q \\ &= \frac{(\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{d})'[\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{C}']^{-1}(\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{d})/q}{\hat{\sigma}^2} \end{aligned}$$

has a non-central F distribution with non-centrality parameter

$$\frac{(\boldsymbol{C}\boldsymbol{\beta} - \boldsymbol{d})'[\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{C}']^{-1}(\boldsymbol{C}\boldsymbol{\beta} - \boldsymbol{d})}{2\sigma^2}$$

and degrees of freedom q and $n - r$.

The F Test of $H_0 : \boldsymbol{C\beta} = \boldsymbol{d}$ (continued)

The non-negative non-centrality parameter

$$\frac{(\boldsymbol{C\beta} - \boldsymbol{d})'[\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{C}']^{-1}(\boldsymbol{C\beta} - \boldsymbol{d})}{2\sigma^2}$$

is equal to zero if and only if $H_0 : \boldsymbol{C\beta} = \boldsymbol{d}$ is true.

If $H_0 : \boldsymbol{C\beta} = \boldsymbol{d}$ is true, the statistic F has a central F distribution with q and $n - r$ degrees of freedom ($F_{q,n-r}$).

The F Test of $H_0 : \boldsymbol{C\beta} = \boldsymbol{d}$ (continued)

Thus, to test $H_0 : \boldsymbol{C\beta} = \boldsymbol{d}$, we compute the test statistic F and compare the observed value of F to the $F_{q,n-r}$ distribution.

If F is so large that it seems unlikely to have been a draw from the $F_{q,n-r}$ distribution, we reject H_0 and conclude $\boldsymbol{C\beta} \neq \boldsymbol{d}$.

The p-value of the test is the probability that a random variable with distribution $F_{q,n-r}$ matches or exceeds the observed value of the test statistic F.

The t Test of $H_0 : \boldsymbol{c}'\boldsymbol{\beta} = d$ for Estimable $\boldsymbol{c}'\boldsymbol{\beta}$

The test statistic

$$\begin{aligned} t &\equiv \frac{\boldsymbol{c}'\hat{\boldsymbol{\beta}} - d}{\sqrt{\widehat{\mathrm{Var}}(\boldsymbol{c}'\hat{\boldsymbol{\beta}})}} \\ &= \frac{\boldsymbol{c}'\hat{\boldsymbol{\beta}} - d}{\sqrt{\hat{\sigma}^2\boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{c}}} \end{aligned}$$

has a non-central t distribution with non-centrality parameter

$$\frac{\boldsymbol{c}'\boldsymbol{\beta} - d}{\sqrt{\sigma^2\boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{c}}}$$

and degrees of freedom $n - r$.

The t Test (continued)

The non-centrality parameter

$$\frac{\boldsymbol{c}'\boldsymbol{\beta} - d}{\sqrt{\sigma^2 \boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^{-}\boldsymbol{c}}}$$

is equal to zero if and only if $H_0 : \boldsymbol{c}'\boldsymbol{\beta} = d$ is true.

If $H_0 : \boldsymbol{c}'\boldsymbol{\beta} = d$ is true, the statistic t has a central t distribution with $n - r$ degrees of freedom (t_{n-r}).

The t Test (continued)

Thus, to test $H_0 : \boldsymbol{c}'\boldsymbol{\beta} = d$, we compute the test statistic t and compare the observed value of t to the t_{n-r} distribution.

If t is so far from zero that it seems unlikely to have been a draw from the t_{n-r} distribution, we reject H_0 and conclude $\boldsymbol{c}'\boldsymbol{\beta} \neq d$.

The p-value of the test is the probability that a random variable with distribution t_{n-r} would be as far or farther from 0 than the observed value of the t test statistic.

A $100(1-\alpha)\%$ Confidence Interval for Estimable $\boldsymbol{c}'\boldsymbol{\beta}$

$$\left(\boldsymbol{c}'\hat{\boldsymbol{\beta}} - t_{n-r,1-\alpha/2}\sqrt{\hat{\sigma}^2\boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{c}},\ \ \boldsymbol{c}'\hat{\boldsymbol{\beta}} + t_{n-r,1-\alpha/2}\sqrt{\hat{\sigma}^2\boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{c}}\right)$$

$$\boldsymbol{c}'\hat{\boldsymbol{\beta}} \pm t_{n-r,1-\alpha/2}\sqrt{\hat{\sigma}^2\boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{c}}$$

estimate $\pm$ (distribution quantile) $\times$ (standard error of estimate)

Form of the t Statistic for Testing $H_0 : \boldsymbol{c}'\boldsymbol{\beta} = d$

$$t = \frac{\text{estimate} - d}{\text{standard error of estimate}} = \frac{\text{estimate} - d}{\sqrt{\widehat{\text{Var}}(\text{estimator})}}$$

$$t^2 = \frac{(\text{estimate} - d)^2}{\widehat{\text{Var}}(\text{estimator})}$$

$$= (\text{estimate} - d)\left[\widehat{\text{Var}}(\text{estimator})\right]^{-1}(\text{estimate} - d)/1$$

Revisiting the F Statistic for Testing $H_0 : \boldsymbol{C\beta} = \boldsymbol{d}$

$$
\begin{aligned}
F &= (\textbf{estimate} - \boldsymbol{d})'\left[\widehat{\text{Var}}(\textbf{estimator})\right]^{-1}(\textbf{estimate} - \boldsymbol{d})/q \\
&= (\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{d})'[\widehat{\text{Var}}(\boldsymbol{C}\hat{\boldsymbol{\beta}})]^{-1}(\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{d})/q \\
&= (\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{d})'[\hat{\sigma}^2\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{C}']^{-1}(\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{d})/q \\
&= \frac{(\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{d})'[\boldsymbol{C}(\boldsymbol{X}'\boldsymbol{X})^-\boldsymbol{C}']^{-1}(\boldsymbol{C}\hat{\boldsymbol{\beta}} - \boldsymbol{d})/q}{\hat{\sigma}^2}
\end{aligned}
$$