Eigenvalues, Eigenvectors, and Matrix Decompositions

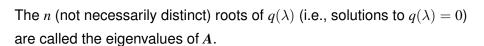
 $\mathbf{A} \Rightarrow q(\lambda) = |\mathbf{A} - \lambda \mathbf{I}|$ is a polynomial of degree n.

For example,

$$\mathbf{A}_{2\times 2} \Rightarrow |\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix}
= (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}a_{21}
= \lambda^2 + (-a_{11} - a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}.$$

The definition of $|A - \lambda I|$ implies that the coefficient on λ^n is $(-1)^n$.

$$\begin{vmatrix} a_{11} - \lambda & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} - \lambda & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} - \lambda \end{vmatrix}$$



The Fundamental Theorem of Algebra guarantees n roots, and the Polynomial Factorization Theorem implies that

$$q(\lambda) = |A - \lambda I| = \prod_{i=1}^{n} (\lambda_i - \lambda)$$

for the eigenvalues $\lambda_1, \ldots, \lambda_n$.

Example:

Suppose
$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$
, then

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{vmatrix}$$
$$= (2 - \lambda)(2 - \lambda) - 1$$
$$= \lambda^2 - 4\lambda + 3$$
$$= (3 - \lambda)(1 - \lambda).$$

Eigenvalues are

$$\lambda_1 = 3, \lambda_2 = 1 : (3 - \lambda)(1 - \lambda) = 0 \iff \lambda = 3 \text{ or } \lambda = 1.$$

Example:

Suppose
$$A = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$$
, then

$$|\mathbf{A} - \lambda \mathbf{I}| = \begin{vmatrix} 3 - \lambda & 0 \\ 0 & 3 - \lambda \end{vmatrix}$$
$$= (3 - \lambda)(3 - \lambda)$$
$$= \lambda^2 - 6\lambda + 9$$

Eigenvalues are $\lambda_1 = 3, \lambda_2 = 3$: $(3 - \lambda)(3 - \lambda) = 0 \iff \lambda = 3$.

In the previous example, the eigenvalue 3 is said to have algebraic multiplicity 2.

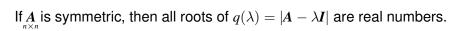
The algebraic multiplicity of eigenvalue λ^* is

$$\sum_{i=1}^n 1 [\lambda_i = \lambda^*].$$

Note that the roots of $q(\lambda)$ are not necessarily real numbers.

For example,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \Rightarrow \mathbf{A} - \lambda \mathbf{I} = \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}$$
$$\Rightarrow |\mathbf{A} - \lambda \mathbf{I}| = \lambda^2 + 1$$
$$= (\sqrt{-1} - \lambda)(-\sqrt{-1} - \lambda)$$
$$\therefore \lambda_1 = \sqrt{-1}, \lambda_2 = -\sqrt{-1}.$$



λ is an eigenvalue of A

$$\iff |A - \lambda I| = 0$$

$$\iff A - \lambda I \text{ is singular}$$

$$\iff rank(A - \lambda I) < n$$

$$\iff \exists x \neq 0 \ni (A - \lambda I)x = 0$$

$$\iff \exists x \neq 0 \ni Ax - \lambda x = 0$$

$$\iff \exists x \neq 0 \ni Ax = \lambda x.$$

Such a vector $x \neq 0$ satisfying $Ax = \lambda x$ is called an <u>eigenvector</u> of A corresponding to eigenvalue λ .

Note that if x is an eigenvector corresponding to eigenvalue λ , then cx is also an eigenvector $\forall c \in \mathbb{R}$.

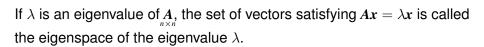
$$\therefore Ax = \lambda x \Rightarrow cAx = c\lambda x \Rightarrow A(cx) = \lambda(cx).$$

It is customary to take eigenvalues to be unit norm, i.e., we take

$$c = \frac{1}{\|\boldsymbol{x}\|}$$

so that the eigenvector cx satisfies

$$||c\mathbf{x}|| = 1.$$



Note that the eigenspace

$$\{x : Ax = \lambda x\} = \{x : (A - \lambda I)x = 0\} = \mathcal{N}(A - \lambda I).$$

The dimension of the eigenspace of λ is known as the geometric multiplicity of λ .

For symmetric matrices, it turns out that algebraic multiplicity of an eigenvalue is the same as the geometric multiplicity.

Show that eigenvectors corresponding to distinct eigenvalues of a symmetric matrix $\mathbf{A}_{n \times n}$ are orthogonal.

Proof:

Suppose
$$Ax_1 = \lambda_1 x_1, Ax_2 = \lambda_2 x_2$$
 and $\lambda_1 \neq \lambda_2$.

Then

$$\lambda_1 x_1' x_2 = (\lambda_1 x_1)' x_2 = (A x_1)' x_2 = x_1' A' x_2 = x_1' A x_2$$

= $x_1' (A x_2) = x_1' (\lambda_2 x_2) = \lambda_2 x_1' x_2$.

Now

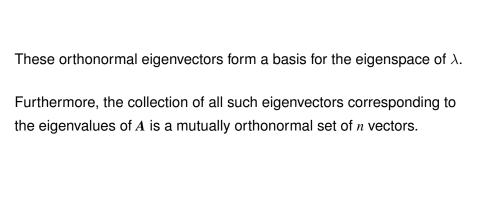
$$\lambda_1 \mathbf{x}_1' \mathbf{x}_2 = \lambda_2 \mathbf{x}_1' \mathbf{x}_2 \Rightarrow (\lambda_1 - \lambda_2) \mathbf{x}_1' \mathbf{x}_2 = 0 \Rightarrow \mathbf{x}_1' \mathbf{x}_2 = 0.$$

Fact V7:

If $S \subseteq \mathbb{R}^n$ is a vector space of dimension k > 0, then there exist a set of mutually orthonormal vectors that forms a basis for S.

Proof: Homework problem.

It follows that for each distinct eigenvalue λ of a matrix \mathbf{A} satisfying $\mathbf{A} = \mathbf{A}'$, there exists a set of orthonormal eigenvectors with $dim(\mathcal{N}(\mathbf{A} - \lambda \mathbf{I}))$ elements.



Let q_1, \ldots, q_n denote these n orthonormal eigenvectors corresponding to eigenvalues $\lambda_1, \ldots, \lambda_n$ respectively.

Let
$$Q = [q_1, ..., q_n]$$
.

Let
$$\Lambda = diag(\lambda_1, \ldots, \lambda_n)$$
.

 $\therefore q_1, \dots, q_n$ are mutually orthonormal, Q'Q = I.

Moreover, because Q is $n \times n$, it follows that Q' is Q^{-1} .

Thus,

$$QQ' = Q'Q = I.$$

Note that

$$Aq_i = \lambda_i q_i \ \forall \ i = 1, \dots, n$$
 $\iff AQ = Q\Lambda$
 $\iff AQQ' = Q\Lambda Q'$
 $\iff A = Q\Lambda Q' = \sum_{i=1}^n \lambda_i q_i q_i'.$

This result is known as the Spectral Decomposition Theorem:

If A is a symmetric matrix, then $A = Q\Lambda Q'$, where Q is an orthogonal matrix and Λ is a diagonal matrix.

Result A.19:

Let $A_{n \times n}$ be a symmetric matrix. Then rank(A) is equal to the number of nonzero eigenvalues of A.

Proof: Homework problem.

Suppose A has eigenvalues $\lambda_1,\dots,\lambda_n$. Then

(a)
$$trace(A) = \sum_{i=1}^{n} \lambda_i$$
, and

(b)
$$|A| = \prod_{i=1}^n \lambda_i$$
.

Proof: Homework problem.

Quadratic Forms

Suppose
$$\mathbf{A}_{n \times n} = [a_{ij}]$$
 and $\mathbf{x} = [x_1, \dots, x_n]'$.

$$x'Ax = \sum_{i=1}^{n} \sum_{j=1}^{n} x_i x_j a_{ij}$$
 is known as a quadratic form.

Nonnegative Definite and Positive Definite Matrices

A symmetric matrix A is nonnegative definite (NND) if and only if

$$x'Ax \geq 0 \ \forall \ x \in \mathbb{R}^n.$$

A symmetric matrix A is positive definite (PD) if and only if

$$x'Ax > 0 \ \forall \ x \in \mathbb{R}^n \setminus \{\mathbf{0}\}.$$

Suppose $A_{n \times n}$ is a symmetric matrix.

Prove that A is NND if and only if all eigenvalues of A are nonnegative.

Prove that *A* is PD if and only if all eigenvalues of *A* are positive.

Proof:

 (\Longrightarrow) Suppose A is a symmetric NND matrix. Then $A=Q\Lambda Q'$ and $q_i'Aq_i\geq 0\ \forall\ i=1,\ldots,n.$

Now note that $Q'q_i=e_i$ (the ith column of $I_{n\times n}$). Thus,

$$\lambda_i = \mathbf{e}_i' \mathbf{\Lambda} \mathbf{e}_i = \mathbf{q}_i' \mathbf{Q} \mathbf{\Lambda} \mathbf{Q}' \mathbf{q}_i = \mathbf{q}_i' \mathbf{A} \mathbf{q}_i \ge 0.$$

 (\longleftarrow) Suppose all eigenvalues of $A, \lambda_1, \ldots, \lambda_n$, are nonnegative.

Let $x \in \mathbb{R}^n$ be arbitrary.

Let y = Qx. Then

$$x'Ax = x'Q\Lambda Q'x$$

$$= y'\Lambda y$$

$$= \sum_{i=1}^{n} \lambda_i y_i^2 \ge 0.$$

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Suppose $A_{n \times n}$ is a symmetric NND matrix.

Show there exists a symmetric matrix B such that BB = A.

Such a matrix is called the symmetric square root of \boldsymbol{A} and is denoted by $\boldsymbol{A}^{1/2}$.

Proof:

Suppose $A_{n \times n}$ is a symmetric NND matrix.

By the Spectral Decomposition Theorem, $A = Q\Lambda Q'$ where Q is orthogonal and Λ is diagonal with eigenvalues of A on the diagonal.

Because A is NND, eigenvalues of A are nonnegative. Define

$$\mathbf{\Lambda}^{1/2} = diag(\lambda_1^{1/2}, \dots, \lambda_n^{1/2}),$$

where

$$\Lambda = diag(\lambda_1, \ldots, \lambda_n).$$

Now define $\mathbf{B} = \mathbf{Q} \mathbf{\Lambda}^{1/2} \mathbf{Q}'$. Note

$$B' = (Q\Lambda^{1/2}Q')'$$

$$= (Q')'(\Lambda^{1/2})'Q'$$

$$= Q\Lambda^{1/2}Q' = B.$$

and

$$BB = Q\Lambda^{1/2}Q'Q\Lambda^{1/2}Q'$$

$$= Q\Lambda^{1/2}\Lambda^{1/2}Q'$$

$$= Q\Lambda Q' = A.$$

Note that if A is PD, $A^{1/2} = Q\Lambda^{1/2}Q'$ is nonsingular

$$\therefore \mathbf{A}^{-1/2} = \mathbf{Q} \left[diag(1/\sqrt{\lambda_1}, \dots, 1/\sqrt{\lambda_1}) \right] \mathbf{Q}' = \mathbf{Q} \mathbf{\Lambda}^{-1/2} \mathbf{Q}'$$

is its inverse:

$$A^{1/2}A^{-1/2} = Q\Lambda^{1/2}Q'Q\Lambda^{-1/2}Q' = Q\Lambda^{1/2}\Lambda^{-1/2}Q' = QQ' = I.$$

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Result A.20 (Cholesky Decomposition):

Suppose A is a symmetric matrix. A is PD iff there exists a nonsingular lower triangular matrix L such that

A = LL'.

Proof of Result A.20:

$$(\longleftarrow) \ \forall \ x \neq 0, L'x \neq 0 \therefore L'$$
 is nonsingular and thus full column rank.

Thus, $\forall x \neq 0$,

$$x'Ax = x'LL'x = (L'x)'(L'x) > 0.$$

$$(\Longrightarrow)$$
 If $n=1, \underline{A}=[a]$. Moreover,

$$\mathbf{A}_{1\times 1} \text{ is PD} \Rightarrow x'ax > 0 \ \forall \ x \neq 0$$

$$\Rightarrow ax^2 > 0 \ \forall \ x \neq 0$$

$$\Rightarrow a > 0.$$

If we take $L = [\sqrt{a}]$, then LL' = A, L is a nonsingular lower triangular matrix, and the result holds.

Now suppose the result holds for n = 1, ..., k for some integer $k \ge 1$.

Suppose A is any $(k+1) \times (k+1)$ matrix. Partition A as

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_k & \boldsymbol{a}_k \\ \boldsymbol{a}'_k & a \end{bmatrix}$$

$$A \text{ is PD } \Rightarrow A_k \text{ is PD } :$$

$$x'Ax > 0 \ \forall \ x \neq \mathbf{0}$$

 $\Rightarrow x'Ax > 0 \ \forall \ x \neq \mathbf{0}$ with the last component 0
 $\Rightarrow [y'\ 0]A \begin{bmatrix} y \\ 0 \end{bmatrix} > 0 \ \forall \ y \neq \mathbf{0}$
 $\Rightarrow y'A_k y > 0 \ \forall \ y \neq \mathbf{0}$.

 $\therefore \exists L_k \text{ nonsingular and lower triangular } \ni A_k = L_k L'_k.$

Now let $\boldsymbol{l}_k = \boldsymbol{L}_k^{-1} \boldsymbol{a}_k$.

 $\therefore A \text{ is PD,}$

$$\begin{bmatrix} \boldsymbol{l}_k' \boldsymbol{L}_k^{-1} & -1 \end{bmatrix} \begin{bmatrix} \boldsymbol{A}_k & \boldsymbol{a}_k \\ \boldsymbol{a}_k' & a \end{bmatrix} \begin{bmatrix} \boldsymbol{L}_k'^{-1} \boldsymbol{l}_k \\ -1 \end{bmatrix} > 0 \Rightarrow$$

$$\Rightarrow \mathbf{l}'_k \mathbf{L}_k^{-1} \mathbf{A}_k \mathbf{L}'_k^{-1} \mathbf{l}_k - 2\mathbf{l}'_k \mathbf{L}_k^{-1} \mathbf{a}_k + a > 0$$

$$\Rightarrow \mathbf{l}'_k \mathbf{L}_k^{-1} \mathbf{L}_k \mathbf{L}'_k \mathbf{L}'_k^{-1} \mathbf{l}_k - 2\mathbf{l}'_k \mathbf{l}_k + a > 0$$

$$\Rightarrow -\mathbf{l}'_k \mathbf{l}_k + a > 0.$$

Now let
$$l = \sqrt{-l'_k l_k + a}$$
.

Then

$$m{L} = egin{bmatrix} m{L}_k & 0 \ m{l}_k' & l \end{bmatrix}$$

is lower triangular and nonsingular ($\because l > 0 \Rightarrow L$ is full rank) and

$$LL' = \begin{bmatrix} L_k & 0 \\ l'_k & l \end{bmatrix} \begin{bmatrix} L'_k & l_k \\ 0 & l \end{bmatrix}$$

$$= \begin{bmatrix} L_k L'_k & L_k l_k \\ l'_k L'_k & l'_k l_k + l^2 \end{bmatrix}$$

$$= \begin{bmatrix} A_k & L_k L_k^{-1} a_k \\ a'_k L'_k^{-1} L'_k & l'_k l_k + a - l'_k l_k \end{bmatrix}$$

$$= \begin{bmatrix} A_k & a_k \\ a'_k & a \end{bmatrix}$$

$$= A.$$