Preliminary Linear Algebra

Notation

$\forall$ for all

$\exists$ there exists

$\ni$ such that

$\therefore$ therefore

$\because$ because

$\square$ end of proof (QED)

Notation

$A \implies B$ A implies B

$A \iff B$ A if and only if (iff) B

$a \in B$ a is an element of the set B

$A \subset B$ A is a proper subset of B

$A \subseteq B$ A is a subset of B

$\mathbb{R}^n$ Euclidean n-space

Matrix Notation

- $\underset{m \times n}{A} = \underset{m \times n}{[a_{ij}]} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ is a matrix with m rows and n columns.

- The entry in the ith row and j^{th} column of A is a_{ij}.

Square Matrices and Vectors

- Matrix $\underset{m \times n}{A}$ is said to be <u>square</u> if $m = n$.

- A matrix with one column is called a <u>vector</u>.

- A matrix with one row is called a <u>row vector</u>.

Notation

In these notes, bold uppercase letters are used to denote matrices, and bold lowercase letters are used to denote vectors.

Examples

- $\mathbf{0}$ or $\mathbf{0}_n$ is a vector of zeros.

- $\mathbf{1}$ or $\mathbf{1}_n$ is a vector of ones.

- $\boldsymbol{I}$ or $\boldsymbol{I}_n$ or $\boldsymbol{I}_{n \times n}$ is the identity matrix. For example,

$$
\boldsymbol{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = diag(1, 1, 1).
$$

Special Types of Square Matrices

- A square matrix $\underset{n \times n}{A}$ is <u>upper triangular</u> if $a_{ij} = 0, \forall\, i > j$.

- A square matrix $\underset{n \times n}{A}$ is <u>lower triangular</u> if $a_{ij} = 0, \forall\, i < j$.

- A square matrix $\underset{n \times n}{A}$ is <u>diagonal</u> if $a_{ij} = 0, \forall\, i \neq j$.

- Write one example for each of these types of matrices.

Examples

- Upper triangular $\begin{bmatrix} 1 & 0 & 4 & 6 \\ 0 & 2 & -3 & 5 \\ 0 & 0 & 8 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

- Lower triangular $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}$

- Diagonal $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & -13 \end{bmatrix}$.

Matrix Transpose

If $\underset{m \times n}{A} = [a_{ij}]$, the <u>transpose</u> of A, denoted A', is the matrix $\underset{n \times m}{B} = [b_{ij}]$, where $b_{ij} = a_{ji}, \forall \; i = 1, \ldots, m; \quad j = 1, \ldots, n$.

That is, $B = A'$ is the matrix whose columns are the rows of A and whose rows are the columns of A.

A Symmetric Matrix

A square matrix $\underset{n \times n}{A}$ is <u>symmetric</u> if $A = A'$.

Examples

- Find the transpose of

$$\begin{bmatrix} 4 & -2 \\ 3 & 7 \end{bmatrix}.$$

- Provide an example of a symmetric matrix.

Examples

- $\begin{bmatrix} 4 & -2 \\ 3 & 7 \end{bmatrix}' = \begin{bmatrix} 4 & 3 \\ -2 & 7 \end{bmatrix}.$

- The matrix

$$\begin{bmatrix} 4 & 2 & -1 \\ 2 & 0 & 3 \\ -1 & 3 & 5 \end{bmatrix}$$

  is symmetric.

Matrix Multiplication

- Suppose

$$\underset{m \times n}{\boldsymbol{A}} = [a_{ij}] = \begin{bmatrix} \boldsymbol{a}'_{(1)} \\ \vdots \\ \boldsymbol{a}'_{(m)} \end{bmatrix} = [\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n]$$

and

$$\underset{n \times k}{\boldsymbol{B}} = [b_{ij}] = \begin{bmatrix} \boldsymbol{b}'_{(1)} \\ \vdots \\ \boldsymbol{b}'_{(n)} \end{bmatrix} = [\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k].$$

- Then

$$\underset{m \times n}{\boldsymbol{A}} \, \underset{n \times k}{\boldsymbol{B}} = \underset{m \times k}{\boldsymbol{C}} = [c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj}] = [c_{ij} = \boldsymbol{a}'_{(i)} \boldsymbol{b}_j]$$

$$= [\boldsymbol{A}\boldsymbol{b}_1, \ldots, \boldsymbol{A}\boldsymbol{b}_k] = \begin{bmatrix} \boldsymbol{a}'_{(1)}\boldsymbol{B} \\ \vdots \\ \boldsymbol{a}'_{(m)}\boldsymbol{B} \end{bmatrix} = \sum_{l=1}^{n} \boldsymbol{a}_{(l)} \boldsymbol{b}'_{(l)}.$$

Matrix Multiplication

- Suppose

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}.$$

- Work out AB using $AB = \sum_{l=1}^{n} a_l b'_{(l)}$.

Matrix Multiplication

$$
\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} 5 & 6 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 7 & 8 \end{bmatrix}
$$

$$
= \begin{bmatrix} 5 & 6 \\ 15 & 18 \end{bmatrix} + \begin{bmatrix} 14 & 16 \\ 28 & 32 \end{bmatrix}
$$

$$
= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}.
$$

Transpose of a Matrix Product

-
$$(AB)' = B'A'$$

- The transpose of a product is the product of the transposes in reverse order.

Scalar Multiplication of a Matrix

If $c \in \mathbb{R}$, then c times the matrix A is the matrix whose i,j^{th} element is c times the i,j^{th} element of A; i.e.,

$$c\underset{m \times n}{A} = c\underset{m \times n}{[a_{ij}]} = \underset{m \times n}{[ca_{ij}]} = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1n} \\ ca_{21} & ca_{22} & \cdots & ca_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m1} & ca_{m2} & \cdots & ca_{mn} \end{bmatrix}.$$

Linear Combination

If $x_1, \ldots, x_n \in \mathbb{R}^m$ and $c_1, \ldots, c_n \in \mathbb{R}$, then

$$\sum_{i=1}^{n} c_i x_i = c_1 x_1 + \cdots + c_n x_n$$

is a <u>linear combination</u> (LC) of $x_1, \ldots, x_n$.

The <u>coefficients</u> of the LC are $c_1, \ldots, c_n$.

Linear Independence and Linear Dependence

- A set of vectors $x_1, \ldots, x_n$ is <u>linearly independent</u> (LI) if

$$\sum_{i=1}^{n} c_i x_i = \mathbf{0} \iff c_1 = \cdots = c_n = 0.$$

- A set of vectors $x_1, \ldots, x_n$ is <u>linearly dependent</u> (LD) if

$$\exists\, c_1, \ldots, c_n \text{ not all } 0 \ni \sum_{i=1}^{n} c_i x_i = \mathbf{0}.$$

- Prove or disprove: any set of vectors that contains the vector $\mathbf{0}$ is LD.

- Prove or disprove: The following set of vectors is LI.

$$
\begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix}, \ \begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix}, \ \begin{bmatrix} 9 \\ -6 \\ 7 \end{bmatrix}.
$$

- Suppose $x_1 = 0$ and $x_2, \ldots, x_n$ are any other vectors of the same dimension as x_1.

- If we take $c_1 = 1, c_2 = \cdots = c_n = 0$, then $\sum_{i=1}^{n} c_i x_i = 0$ and $c_1, c_2, \ldots, c_n$ are not all zero.

- Thus, any set of vectors containing 0 is LD.

- If we take $c_1 = 2, c_2 = 1, c_3 = -1$ then

$$c_1 \begin{bmatrix} 1 \\ -5 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 7 \\ 4 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 9 \\ -6 \\ 7 \end{bmatrix} = \mathbf{0}.$$

Thus the set of vectors is LD.

- One way to arrive at such solution is to search for a solution to the system of the equations:

$$
\begin{aligned}
c_1 + 7c_2 + 9c_3 &= 0 \\
-5c_1 + 4c_2 - 6c_3 &= 0 \\
3c_1 + c_2 + 7c_3 &= 0.
\end{aligned}
$$

Fact V1:

The nonzero vectors $x_1, \ldots, x_n$ are LD $\iff$ x_j is a LC of $x_1, \ldots, x_{j-1}$ for some $j \in \{2, \ldots, n\}$.

Proof of Fact V1:

($\Longrightarrow$) Suppose there exist $c_1, \ldots, c_n$ such that $\sum_{i=1}^{n} c_i \boldsymbol{x}_i = \boldsymbol{0}$. Let

$$j = \max\{i : c_i \neq 0\}.$$

Since $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n$ are nonzero, $j > 1$. Then

$$\sum_{i=1}^{j} c_i \boldsymbol{x}_i = \boldsymbol{0} \implies \sum_{i=1}^{j-1} c_i \boldsymbol{x}_i = -c_j \boldsymbol{x}_j.$$

$$\implies \sum_{i=1}^{j-1} \frac{-c_i}{c_j} \boldsymbol{x}_i = \boldsymbol{x}_j.$$

($\Longleftarrow$) Suppose $\boldsymbol{x}_j = \sum_{i=1}^{j-1} c_i \boldsymbol{x}_i$, then $\sum_{i=1}^{n} d_i \boldsymbol{x}_i = \boldsymbol{0}$, where

$$d_i = \begin{cases} c_i & \text{if } i < j \\ -1 & \text{if } i = j \\ 0 & \text{if } i > j. \end{cases}$$

$\square$

Orthogonality

- The two vectors x, y are <u>orthogonal</u> to each other if their <u>inner product</u> is zero, i.e.,

$$x'y = y'x = \sum_{i=1}^{n} x_i y_i = 0.$$

- The <u>length</u> of a vector, also known as its <u>Euclidean norm</u>, is

$$\|x\| := \sqrt{x'x} = \sqrt{\sum_{i=1}^{n} x_i^2}.$$

- The vectors $x_1, \ldots, x_n$ are <u>mutually orthogonal</u> if

$$x_i' x_j = 0, \quad \forall \; i \neq j.$$

- The vectors $x_1, \ldots, x_n$ are <u>mutually orthonormal</u> if

$$x_i' x_j = 0 \quad \forall \; i \neq j, \text{ and } \|x_i\| = 1 \quad \forall \; i = 1, \ldots, n.$$

- Write down a set of mutually orthogonal but not mutually orthonormal vectors.

- Write down a set of mutually orthonormal vectors.

- $\begin{bmatrix} 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are mutually orthogonal but not mutually orthonormal.

- $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are mutually orthonormal.

- $\begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ are mutually orthonormal.

Orthogonal Matrix

- A square matrix with mutually orthonormal columns is called an
  <u>orthogonal matrix</u>.

- Show that if Q is orthogonal, then $Q'Q = I$.

- Show that if Q is orthogonal and x is any vector of appropriate dimension, then $\|Qx\| = \|x\|$.

- $Q'Q = [q_i'q_j]$, where $Q = [q_1, \ldots, q_n]$.

- By orthogonality of Q, $q_1, \ldots, q_n$ are mutually orthonormal.

- Thus,

$$q_i'q_j = 0 \quad \forall \ i \neq j$$

and

$$\|q_i\| = 1 \quad \forall \ i = 1, \ldots, n.$$

$\therefore Q'Q = I.$

$$
\begin{aligned}
\|\boldsymbol{Q}\boldsymbol{x}\| &= \sqrt{(\boldsymbol{Q}\boldsymbol{x})'\boldsymbol{Q}\boldsymbol{x}} \\
&= \sqrt{\boldsymbol{x}'\boldsymbol{Q}'\boldsymbol{Q}\boldsymbol{x}} \\
&= \sqrt{\boldsymbol{x}'\boldsymbol{I}\boldsymbol{x}} \\
&= \sqrt{\boldsymbol{x}'\boldsymbol{x}} \\
&= \|\boldsymbol{x}\|.
\end{aligned}
$$

An orthogonal matrix Q is sometimes called a <u>rotation matrix</u> because if a vector $\underset{n \times 1}{x}$ is premultiplied by $\underset{n \times n}{Q}$, the result (Qx) is the vector x rotated to a new position in $\mathbb{R}^n$.

Vector Space

A vector space $\mathcal{S}$ is a set of vectors that is closed under addition (i.e., if $x_1 \in \mathcal{S}, x_2 \in \mathcal{S}$, then $x_1 + x_2 \in \mathcal{S}$) and closed under scalar multiplication (i.e., if $c \in \mathbb{R}, x \in \mathcal{S}$, then $cx \in \mathcal{S}$).

In other words,

$$c_1 x_1 + c_2 x_2 \in \mathcal{S} \quad \forall \, c_1, c_2 \in \mathbb{R}; \; x_1, x_2 \in \mathcal{S}.$$

- Is $\{x \in \mathbb{R}^n : \|x\| = 1\}$ a vector space?

- Is $\{x \in \mathbb{R}^n : \mathbf{1}'x = 0\}$ a vector space?

- Is $\{\underset{n \times m}{A}x : x \in \mathbb{R}^m\}$ a vector space?

- Suppose $y \in \mathbb{R}^n, c \in \mathbb{R}$ and $\|y\| = 1$, then

$$
\begin{aligned}
\|cy\| &= \sqrt{(cy)'cy} \\
&= \sqrt{c^2 y' y} \\
&= |c| \|y\| = |c|.
\end{aligned}
$$

- Thus $y \in \{x \in \mathbb{R}^n : \|x\| = 1\}$ does not imply that
  $cy \in \{x \in \mathbb{R}^n : \|x\| = 1\}$. Therefore, this set is not a vector space.

- Let
$$\mathcal{S} = \{x \in \mathbb{R}^n : \mathbf{1}'x = 0\}.$$

- Suppose $c_1, c_2 \in \mathbb{R}$ and $x_1, x_2 \in \mathcal{S}$, then
$$\mathbf{1}'(c_1 x_1 + c_2 x_2) = c_1 \mathbf{1}'x_1 + c_2 \mathbf{1}'x_2 = 0.$$

- Thus $c_1 x_1 + c_2 x_2 \in \mathcal{S}$ and it follows that $\mathcal{S}$ is a vector space.

- Let

$$\mathcal{S} = \left\{ \underset{n \times m}{A} x : x \in \mathbb{R}^m \right\}.$$

- Suppose $c_1, c_2 \in \mathbb{R}$ and $y_1, y_2 \in \mathcal{S}$.

- $y_1, y_2 \in \mathcal{S} \implies \exists\, x_1, x_2 \in \mathbb{R}^m \ni$

$$y_1 = A x_1 \text{ and } y_2 = A x_2.$$

- Thus,

$$c_1 y_1 + c_2 y_2 = c_1 A x_1 + c_2 A x_2 = A(c_1 x_1 + c_2 x_2).$$

$\because c_1 x_1 + c_2 x_2 \in \mathbb{R}^m, c_1 y_1 + c_2 y_2 \in \mathcal{S}$. It follows that $\mathcal{S}$ is a vector space.

Generators of a Vector Space

A vector space $\mathcal{S}$ is said to be <u>generated</u> by a set of vectors $x_1, \ldots, x_n$ if

$$x \in \mathcal{S} \implies x = \sum_{i=1}^{n} c_i x_i \text{ for some } c_1, \ldots, c_n \in \mathbb{R}.$$

Span of a Set of Vectors

- The <u>span</u> of vectors $x_1, \ldots, x_n$ is the set of all LC of $x_1, \ldots, x_n$, i.e.,

$$span\{x_1, \ldots, x_n\} = \left\{ \sum_{i=1}^{n} c_i x_i : c_1, \ldots, c_n \in \mathbb{R} \right\}.$$

- $span\{x_1, \ldots, x_n\}$ is the vector space generated by $x_1, \ldots, x_n$.

Find a set of vectors that generates the space

$$\{\boldsymbol{x} \in \mathbb{R}^3 : \boldsymbol{1}'\boldsymbol{x} = 0\};$$

i.e., find a set of vectors whose span is

$$\mathcal{S} = \{\boldsymbol{x} \in \mathbb{R}^3 : \boldsymbol{1}'\boldsymbol{x} = 0\}.$$

- Let $x_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, x_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. Note that $\mathbf{1}'x_1 = 0$ and $\mathbf{1}'x_2 = 0$.

  Thus, $x_1, \ x_2 \in \mathcal{S}$ so that $span\{x_1, x_2\} \subseteq \mathcal{S}$.

- Now suppose $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \in \mathcal{S}$.

  Then $0 = \mathbf{1}'y = y_1 + y_2 + y_3 \Longrightarrow y_3 = -y_1 - y_2$ so that

  $$y_1 x_1 + y_2 x_2 = \begin{bmatrix} y_1 \\ 0 \\ -y_1 \end{bmatrix} + \begin{bmatrix} 0 \\ y_2 \\ -y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ -y_1 - y_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = y.$$

  $\therefore \mathcal{S} \subseteq span\{x_1, x_2\}$, and $\mathcal{S} = span\{x_1, x_2\}$.

Basis of a Vector Space

If a vector space $\mathcal{S}$ is generated by LI vectors $x_1, \ldots, x_n$, then $x_1, \ldots, x_n$ form a <u>basis</u> for $\mathcal{S}$.

Fact V2:

Suppose $a_1, \ldots, a_n$ form a basis for a vector space $\mathcal{S}$. If $b_1, \ldots, b_k$ are LI vectors in $\mathcal{S}$, then $k \leq n$.

Proof of Fact V2:

- Because $a_1, \ldots, a_n$ form a basis for $\mathcal{S}$ and $b_1 \in \mathcal{S}$, $b_1 = \sum_{i=1}^{n} c_i a_i$ for some $c_1, \ldots, c_n \in \mathbb{R}$. Thus, $a_1, \ldots, a_n, b_1$ are LD by Fact V1.

- Again, using V1, we have a_j a LC of $b_1, a_1, \ldots, a_{j-1}$ for some $j \in \{1, 2, \ldots, n\}$.

- Thus, $b_1, a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n$ generate $\mathcal{S}$. It follows that $b_1, a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n, b_2$ is a LD set of vectors by V1.

- Again by V1, one of the vectors $b_1, b_2, a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_n$ is a LC of the preceding vectors. It is not b_2 $\because$ $b_1, \ldots, b_k$ are LI.

- Thus $\boldsymbol{b}_1, \boldsymbol{b}_2$ and $n - 2$ of $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ generate $\mathcal{S}$.

- If $k > n$, we can continue adding $\boldsymbol{b}$ vectors and deleting $\boldsymbol{a}$ vectors to get $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_n$ generates $\mathcal{S}$. However, then V1 would imply $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_{n+1}$ are LD. This contradicts LI of $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k$ $\therefore k \leq n$. $\qquad \square$

Fact V3:

If $\{a_1, \ldots, a_n\}$ and $\{b_1, \ldots, b_k\}$ each provide a basis for a vector space $\mathcal{S}$, then $n = k$.

Proof: By V2, we have $k \leq n$ and $n \leq k$. $\therefore k = n$. $\qquad\square$

Dimension of a Vector Space

A basis for a vector space is not unique, but the number of vectors in the basis, known as <u>dimension</u> of the vector space, is unique.

Find $dim(\mathcal{S})$, the dimension of vector space $\mathcal{S}$, for

$$\mathcal{S} = \{\boldsymbol{x} \in \mathbb{R}^3 : \boldsymbol{1}'\boldsymbol{x} = 0\}.$$

- As demonstrated previously,

$$span\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\} = \mathcal{S}.$$

- Because $\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}$ is a LI set of vectors, $\left\{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}\right\}$ forms a basis for $\mathcal{S}$.

- Thus,

$$dim(\mathcal{S}) = 2 \quad \text{(even though dimension of vectors in } \mathcal{S} \text{ is 3).}$$

Consider the set $\left\{\underset{n \times 1}{\mathbf{0}}\right\}$. Is this a vector space? If so, what is its dimension?

- $\{\mathbf{0}\}$ is a vector space because

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 = \mathbf{0} \in \{\mathbf{0}\} \quad \forall\, c_1, c_2 \in \mathbb{R} \text{ and } \forall\, \mathbf{x}_1, \mathbf{x}_2 \in \{\mathbf{0}\}.$$

- The set $\{\mathbf{0}\}$ generates the vector space $\{\mathbf{0}\}$. However, $\{\mathbf{0}\}$ is not a LI set of vectors. $dim(\{\mathbf{0}\}) = 0$.

Fact V4:

Suppose $a_1, \ldots, a_n$ are LI vectors in a vector space $\mathcal{S}$ with dimension n. Then $a_1, \ldots, a_n$ form a basis for $\mathcal{S}$.

Proof of Fact V4:

- It suffices to show that $a_1, \ldots, a_n$ generate $\mathcal{S}$.

- Let a denote an arbitrary vector in $\mathcal{S}$.

- By V2, $a_1, \ldots, a_n, a$ are LD. By V1, $a = \sum_{i=1}^{n} c_i a_i$ for some $c_1, \ldots, c_n \in \mathbb{R}$.

- Thus
$$\mathcal{S} = \left\{ \sum_{i=1}^{n} c_i x_i : c_1, \ldots, c_n \in \mathbb{R} \right\},$$

and the result follows. $\qquad \square$

Fact V5:

If $a_1, \ldots, a_k$ are LI vectors in an n-dimensional vector space $\mathcal{S}$, then there exists a basis for $\mathcal{S}$ that contains $a_1, \ldots, a_k$.

Proof of Fact V5:

- $k \leq n$ by V2.

- If $k = n$, the result follows from V4.

- Suppose $k < n$. Then, there exist $\boldsymbol{a}_{k+1} \in \mathcal{S}$ such that $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{k+1}$ are LI. Because if not, $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k$ would generate $\mathcal{S}$ (by V1), and thus be a basis of dimension $k < n$, which is impossible by V3. Similarly, we can continue to add vectors to $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{k+1}\}$ until we have $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$ LI vectors. The result follows from V4. $\qquad \square$

Fact V6:

If $a_1, \ldots, a_k$ are LI and orthonormal vectors in $\mathbb{R}^n$, then there exist $a_{k+1}, \ldots, a_n$ such that $a_1, \ldots, a_n$ are LI and orthornormal.

Proof: The proof is left as an exercise for the reader. $\qquad\square$

Rank of a Matrix

It can be shown that

- the (maximum) number of LI rows of a matrix $\underset{m \times n}{A}$ is the same as the (maximum) number of LI columns of $\underset{m \times n}{A}$.

- This number of LI rows or columns is known as the <u>rank</u> of $\underset{m \times n}{A}$ and is denoted $rank(\underset{m \times n}{A})$ or $r(\underset{m \times n}{A})$.

- If $r(\underset{m \times n}{A}) = m$, $\underset{m \times n}{A}$ is said to have <u>full row rank</u>.

- If $r(\underset{m \times n}{A}) = n$, $\underset{m \times n}{A}$ is said to have <u>full column rank</u>.

Inverse of a Matrix

- If $r(\underset{n \times n}{A}) = n$, there exists a matrix $\underset{n \times n}{B}$ such that $\underset{n \times n}{A} \underset{n \times n}{B} = \underset{n \times n}{I}$.

- Such a matrix B is called the <u>inverse</u> of A and is denoted A^{-1}.

- Prove that $r(\underset{n \times n}{A}) = n \Longleftrightarrow \exists \underset{n \times n}{B} \ni \underset{n \times n}{A} \underset{n \times n}{B} = \underset{n \times n}{I}$.

- Prove that $\underset{n \times n}{A} \underset{n \times n}{B} = \underset{n \times n}{I} \Longrightarrow \underset{n \times n}{B} \underset{n \times n}{A} = \underset{n \times n}{I}$

- Thus $AA^{-1} = A^{-1}A = I$.

Proof:

($\Longrightarrow$):

- The columns of A form a basis for $\mathbb{R}^n$ by V4. Thus, there exists a LC of columns of $\underset{n \times n}{A}$ that equals e_i for all $i = 1, \ldots, n$, where e_i is the ith column of the identity matrix $\underset{n \times n}{I}$.

- Let b_i denote the coefficients of the LC of the columns of A that yields e_i. Then, with $B = [b_1, \ldots, b_n]$, we have
$$AB = [Ab_1, \ldots, Ab_n] = [e_1, \ldots, e_n] = I.$$

$(\Longleftarrow)$:

- If $\exists \underset{n \times n}{B} \ni \underset{n \times n}{A} \underset{n \times n}{B} = \underset{n \times n}{I}$, then the columns of A generate $\mathbb{R}^n$

  $\because \forall \, x \in \mathbb{R}^n, ABx = Ix = x$.

- If the columns of A were LD, then a subset of the columns of A would be LI and also generate $\mathbb{R}^n$.

- However, such a subset would be a basis for $\mathbb{R}^n$ and thus must have n elements.

- Thus, the columns of A must be LI. Hence, $r(\underset{n \times n}{A}) = n$. $\qquad\square$

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$$\begin{aligned}
\underset{n \times n}{A}\,\underset{n \times n}{B} = \underset{n \times n}{I} \implies & \text{ Columns of } \underset{n \times n}{A} \text{ are LI} \\
\implies & \text{ Rows of } \underset{n \times n}{A} \text{ are LI} \\
\implies & \text{ Rows of } \underset{n \times n}{A} \text{ are a basis for } \mathbb{R}^n \\
\implies & \exists\, \underset{n \times n}{C} \ni \underset{n \times n}{C}\,\underset{n \times n}{A} = \underset{n \times n}{I}.
\end{aligned}$$

- Thus,

$$AB = I \implies CAB = CI$$
$$\implies IB = C$$
$$\implies B = C.$$

- $\therefore AA^{-1} = A^{-1}A = I$ $\qquad \square$

Singular / Nonsingular Matrix

- If $r(\underset{n \times n}{A}) = n$, $\underset{n \times n}{A}$ is said to be <u>nonsingular</u>.

- If $r(\underset{n \times n}{A}) < n$, $\underset{n \times n}{A}$ is said to be <u>singular</u>.

Inverse of a Nonsingular 2×2 Matrix

- $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ if $ad - bc \neq 0$.

- $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is singular if $ad - bc = 0$.

Column Space of a Matrix

The column space of a matrix $\underset{m \times n}{A}$, denoted by $\mathcal{C}(A)$, is the vector space generated by the columns of $\underset{m \times n}{A}$; i.e.,

$$\mathcal{C}(A) = \{Ax : x \in \mathbb{R}^n\}.$$

$dim(\mathcal{C}(\boldsymbol{A})) = r(\underset{m \times n}{\boldsymbol{A}})$ because the largest possible subset of LI columns of $\underset{m \times n}{\boldsymbol{A}}$ is a basis for $\mathcal{C}(\boldsymbol{A})$.

- Let

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & 4 \\ 4 & 2 & 8 \end{bmatrix}.$$

- Find $r(A)$.

- Give a basis for $\mathcal{C}(A)$.

- Characterize $\mathcal{C}(A)$.

-
$$3 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 8 \end{bmatrix}.$$

- Thus, the columns of A are LD and $r(A) < 3$.

-
$$c_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} c_1 \\ 2c_1 + c_2 \\ 4c_1 + 2c_2 \end{bmatrix} = \mathbf{0} \implies c_1 = c_2 = 0. \therefore r(A) = 2.$$

- A basis for $\mathcal{C}(A)$ is given by $\left\{ \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$.

- $x \in \mathcal{C}(A) \implies x = c_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ for some $c_1, c_2 \in \mathbb{R}^n$.

- Note

$$\left\{ \boldsymbol{x} = \begin{bmatrix} c_1 \\ 2c_1 + c_2 \\ 4c_1 + 2c_2 \end{bmatrix} : c_1, c_2 \in \mathbb{R} \right\} = \mathcal{C}(\boldsymbol{A})$$

is the set of vectors in $\mathbb{R}^3$ where the first component is arbitrary and the third component is twice the second component, i.e.,

$$\{\boldsymbol{x} \in \mathbb{R}^3 : 2x_2 = x_3\}.$$

Result A.1:

$$rank(\boldsymbol{AB}) \leq \min\{rank(\boldsymbol{A}), rank(\boldsymbol{B})\}$$

Proof of Result A.1:

- Let $b_1, \ldots, b_n$ denote the columns of B so that $B = [b_1, \ldots, b_n]$.

- Then $AB = [Ab_1, \ldots, Ab_n]$. This implies that the columns of AB are in $\mathcal{C}(A)$.

- $dim(\mathcal{C}(\boldsymbol{A}))$ is $rank(\boldsymbol{A})$.

- There does not exist a set of LI vectors in $\mathcal{C}(\boldsymbol{A})$ with more than $rank(\boldsymbol{A})$ vectors by Fact V2.

- It follows that $rank(\boldsymbol{AB}) \leq rank(\boldsymbol{A})$.

- It remains to show that

$$rank(\boldsymbol{AB}) \leq rank(\boldsymbol{B}).$$

- Note that $rank(\boldsymbol{AB})$ is the same as $rank((\boldsymbol{AB})') = rank(\boldsymbol{B}'\boldsymbol{A}')$.

- Our previous argument shows that

$$rank(\boldsymbol{B}'\boldsymbol{A}') \leq rank(\boldsymbol{B}') = rank(\boldsymbol{B}).$$

$\square$

Provide an example where

$$rank(\boldsymbol{AB}) < \min\{rank(\boldsymbol{A}), rank(\boldsymbol{B})\}.$$

-

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

- Then

$$AB = \mathbf{0}.$$

- Therefore,

$$rank(AB) = 0, \quad rank(A) = 1, \quad rank(B) = 1.$$

Result A.2:

(a) If $A = BC$, then $\mathcal{C}(A) \subseteq \mathcal{C}(B)$,

(b) If $\mathcal{C}(A) \subseteq \mathcal{C}(B)$, then there exist C such that $A = BC$.

Proof of A.2(a):

- Suppose $x \in \mathcal{C}(A)$ then $\exists \, y \ni x = Ay$.

- Now $A = BC \implies x = BCy$.

- Thus, $\exists \, z \ni x = Bz$ (namely, $z = Cy$). $\therefore x \in \mathcal{C}(B)$.

- We have shown $x \in \mathcal{C}(A) \implies x \in \mathcal{C}(B)$. $\therefore \mathcal{C}(A) \subseteq \mathcal{C}(B)$. $\square$

Proof of A.2(b):

- Let $a_1, \ldots, a_n$ denote the columns of A.

  $\mathcal{C}(A) \subseteq \mathcal{C}(B) \implies a_1, \ldots, a_n \in \mathcal{C}(B)$.

- Let c_i be such that $Bc_i = a_i \ \forall \ i = 1, \ldots, n$. Then denote

  $C = [c_1, \ldots, c_n]$.

- It follows that

$$
\begin{aligned}
BC &= B[c_1, \ldots, c_n] \\
&= [Bc_1, \ldots, Bc_n] \\
&= [a_1, \ldots, a_n] = A.
\end{aligned}
$$