

1. (a) Let

$$\begin{aligned} a_{ik} &= \frac{y_{i1k} + y_{i2k}}{2} = \bar{\mu}_i + p_k + \bar{e}_{i.k} \\ &= \bar{\mu}_i + \varepsilon_{ik}, \end{aligned}$$

where $\varepsilon_{ik} \equiv p_k + \bar{e}_{i.k}$. Note that the ε_{ik} terms are *iid* $N(0, \sigma^2)$, where $\sigma^2 \equiv \sigma_p^2 + \frac{\sigma_e^2}{2}$. Thus, a two sample t-test can be used to test $H_0 : \bar{\mu}_1 = \bar{\mu}_2$. From the R output of the analysis of averages, we have

$$t = \frac{84.892 - 80.454}{\sqrt{2.169^2 + 1.534^2}}.$$

- (b) Let

$$\begin{aligned} d_{ik} &= y_{i1k} - y_{i2k} = \mu_{i1} - \mu_{i2} + e_{i1k} - e_{i2k} \\ &\equiv \delta_i + \eta_{ik}, \end{aligned}$$

where $\delta_i = \mu_{i1} - \mu_{i2}$ and $\eta_{ik} = e_{i1k} - e_{i2k}$. Note that the η_{ik} terms are *iid* $N(0, \sigma_\eta^2)$, where $\sigma_\eta^2 \equiv 2\sigma_e^2$. The test of infection main effect is a test of $H_0 : \frac{\mu_{11} + \mu_{21}}{2} = \frac{\mu_{12} + \mu_{22}}{2} \iff H_0 : \mu_{11} - \mu_{21} + \mu_{21} - \mu_{22} = 0 \iff \delta_1 + \delta_2 = 0$. From the last analysis of the differences in R, we can test $H_0 : \delta_1 + \delta_2 = 0$ with

$$t = \frac{8.250 + 1.492}{\sqrt{2.439^2 + 1.724^2}}.$$

- (c) It is straightforward to see that a test for interaction is a test of
- $H_0 : \delta_1 = \delta_2 \iff H_0 : \delta_1 - \delta_2 = 0$
- . Thus,

$$t = \frac{8.250 - 1.492}{\sqrt{2.439^2 + 1.724^2}}$$

is the relevant test statistic.

- (d)
- $\hat{\sigma}_\eta^2 = 2\hat{\sigma}_e^2 = 5.974^2 \implies \hat{\sigma}_e^2 = \frac{5.974^2}{2}$
- .

- (e)
- $\hat{\sigma}^2 = \hat{\sigma}_p^2 + \frac{\hat{\sigma}_e^2}{2} = 5.313^2 \implies \hat{\sigma}_p^2 = 5.313^2 - \frac{5.974^2}{4}$
- .

The answers to parts a) through e) above match tests and estimates obtained by fitting the full linear mixed effects model $\mathbf{y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{Z}\mathbf{u} + \mathbf{e}$.

2. (a)
- $(k-1)(l-1) = kl - k - l + 1$

$$\sum_{i=1}^2 \sum_{j=1}^5 \sum_{k=1}^2 \sum_{l=1}^2 (\bar{y}_{..kl} - \bar{y}_{..k} - \bar{y}_{..l} + \bar{y}_{....})^2 = 10 \sum_{k=1}^2 \sum_{l=1}^2 (\bar{y}_{..kl} - \bar{y}_{..k} - \bar{y}_{..l} + \bar{y}_{....})^2$$

- (b) This is a split-split-plot experiment. $H_0 : \bar{\mu}_{1..} = \bar{\mu}_{2..}$ is the null hypothesis that says there is no whole-plot-factor (i.e., Type) main effect. The whole-plot-experimental units correspond to *Helmet*(Type), so the F statistic for testing H_0 is

$$F = \frac{MS_{Type}}{MS_{Helmet(Type)}} = \frac{226/1}{254/8}.$$

Thus,

$$t = \sqrt{\frac{MS_{Type}}{MS_{Helmet(Type)}}} = \sqrt{\frac{226/1}{254/8}}.$$

- (c) From the provided expected mean squares, it is straightforward to see that

$$\frac{MS_{Helmet(Type)} - MS_{Direction \times Helmet(Type)}}{4}$$

has expectation σ_a^2 .

Thus, an unbiased estimator of σ_a^2 takes the value

$$\frac{254/8 - 114/8}{4} = \frac{140}{32} = 4.375.$$

- (d) Because we have a balanced design, we know the BLUE of $\bar{\mu}_{12.} - \bar{\mu}_{11.}$ is $\bar{y}_{1.2.} - \bar{y}_{1.1.}$, which has variance

$$\begin{aligned} \text{Var}(\bar{a}_{1.} + \bar{b}_{1.2.} + \bar{e}_{1.2.} - \bar{a}_{1.} - \bar{b}_{1.1.} - \bar{e}_{1.1.}) &= \text{Var}(\bar{b}_{1.2.} - \bar{b}_{1.1.} + \bar{e}_{1.2.} - \bar{e}_{1.1.}) \\ &= \frac{2\sigma_b^2}{5} + \frac{2\sigma_e^2}{5 \times 2} \\ &= \frac{1}{5}(2\sigma_b^2 + \sigma_e^2) \\ &= \frac{1}{5}E(MS_{Direction \times Helmet(Type)}). \end{aligned}$$

Thus,

$$\begin{aligned} \widehat{\text{Var}}(\bar{y}_{1.2.} - \bar{y}_{1.1.}) &= \frac{1}{5} \left(\frac{114}{8} \right) \\ &= \frac{57}{20} \\ &= 2.85 \end{aligned}$$

Thus, the confidence interval is $0.5 \pm 2.306 \sqrt{2.85}$, where $t_{0.975,8} = 2.306$ and 8 is DF for $Direction \times Helmet(Type)$.

(e)

$$\begin{aligned}
\text{Var}(\bar{y}_{1.21} - \bar{y}_{1.11}) &= \text{Var}(\bar{a}_{1.} + \bar{b}_{1.2} + \bar{e}_{1.21} - \bar{a}_{1.} - \bar{b}_{1.1} - \bar{e}_{1.11}) \\
&= \text{Var}(\bar{b}_{1.2} - \bar{b}_{1.1} + \bar{e}_{1.21} - \bar{e}_{1.11}) \\
&= \frac{2\sigma_b^2}{5} + \frac{2\sigma_e^2}{5} = \frac{1}{5}(2\sigma_b^2 + 2\sigma_e^2) \\
&= \frac{1}{5} [E(MS_{\text{Direction} \times \text{Helmet}(\text{Type})}) + E(MS_{\text{Error}})] .
\end{aligned}$$

Thus, SE is $\sqrt{\frac{1}{5} \left(\frac{114}{8} + \frac{59}{16} \right)}$. To see that DF for *Error* is 16, note that

$\text{Error} = \text{Intensity} \times \text{Helmet}(\text{Type}) + \text{Intensity} \times \text{Direction} \times \text{Helmet}(\text{Type}) = 16$.

So, DF is $(2-1) \times (5-1) \times 2 + (2-1) \times (2-1) \times (5-1) \times 2 = 16$.

Alternatively, by subtraction, we have $(40-1) - (1+1+1+1+1+1+1+1+8+8) = 16$.

(f) Because we have a balanced design, we know the BLUE of $\mu_{121} - \mu_{111}$ is $\bar{y}_{1.21} - \bar{y}_{1.11}$, which has variance

$$\begin{aligned}
\text{Var}(\bar{y}_{1.21} - \bar{y}_{1.11}) &= \text{Var}(\bar{a}_{1.} + \bar{b}_{1.2} + \bar{e}_{1.21} - \bar{a}_{1.} - \bar{b}_{1.1} - \bar{e}_{1.11}) \\
&= \text{Var}(\bar{b}_{1.2} - \bar{b}_{1.1}) + \text{Var}(\bar{e}_{1.21} - \bar{e}_{1.11}) \\
&= \frac{2}{5}\sigma_b^2 + \frac{2}{5}\sigma_e^2 \\
&= \frac{2}{5} \left[\frac{1}{2} \{ E(MS_{\text{Direction} \times \text{Helmet}(\text{Type})}) + E(MS_{\text{Error}}) \} \right] .
\end{aligned}$$

Thus,

$$\begin{aligned}
\widehat{\text{Var}}(\bar{y}_{1.21} - \bar{y}_{1.11}) &= \frac{1}{5} \left(\frac{114}{8} + \frac{59}{16} \right) \\
&= \frac{1}{5} \frac{287}{16} \\
&= 3.5875
\end{aligned}$$

Thus, a standard error for the BLUE of $\mu_{121} - \mu_{111}$ is $\sqrt{3.5875}$.

3. Let y_{ijk} be the weight gain for drug i ($i = 1, 2$), dose j ($j=1$ for dose 0 and $j=2$ for dose 10), and pig k ($k=1,2,3,4$ for drug 1 and $k=1,2,3$ for drug2). Then, we can suppose

$$y_{ijk} = \mu_{ij} + e_{ijk},$$

where $\mu_{11}, \mu_{12}, \mu_{21}$, and μ_{22} are unknown parameters and $e_{ijk} \stackrel{\text{iid}}{\sim} N(0, \sigma^2)$ for all i, j , and k .

(a) From the slide 60 of set 8,

$$SS(\text{drug}|1, \text{dose}) = SS(\text{drug} \times \text{dose}, \text{drug}, \text{dose}|1) - SS(\text{dose}|1) - SS(\text{drug} \times \text{dose}|1, \text{drug}, \text{dose}).$$

By slide 63 of set 8, $SS(\text{drug} \times \text{dose}|1, \text{drug}, \text{dose})$ is sum of squares which is relevant to the test for the $\text{drug} \times \text{dose}$ interaction, where $H_0 : \mu_{11} - \mu_{12} - \mu_{21} + \mu_{22} = 0$.

Let $\mathbf{c}=(1, -1, -1, 1)'$, $\hat{\boldsymbol{\mu}}=(\bar{y}_{11.}, \bar{y}_{12.}, \bar{y}_{21.}, \bar{y}_{22.})'=(4, 9, 4, 13)'$ and $\mathbf{X}=\begin{pmatrix} \mathbf{1}_{2 \times 1} & 0 & 0 & 0 \\ 0 & \mathbf{1}_{2 \times 1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{2 \times 1} \end{pmatrix}$.

Then,

$$\begin{aligned} SS(drug \times dose|1, drug, dose) &= (\mathbf{c}'\hat{\boldsymbol{\mu}})' [\mathbf{c}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{c}]^{-1} (\mathbf{c}'\hat{\boldsymbol{\mu}}) \\ &= (\mathbf{c}'\hat{\boldsymbol{\mu}})' \left[\mathbf{c}' \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \mathbf{c} \right]^{-1} (\mathbf{c}'\hat{\boldsymbol{\mu}}) \\ &= 4 \times (2.5)^{-1} \times 4 \\ &= 6.4. \end{aligned}$$

Since

$$\begin{aligned} SS(drug \times dose, drug, dose|1) &= 2(\bar{y}_{11.} - \bar{y}_{...})^2 + 2(\bar{y}_{12.} - \bar{y}_{...})^2 + (\bar{y}_{21.} - \bar{y}_{...})^2 + 2(\bar{y}_{22.} - \bar{y}_{...})^2 \\ &= 2(4 - 8)^2 + 2(9 - 8)^2 + (4 - 8)^2 + 2(13 - 8)^2 \\ &= 100 \end{aligned}$$

and

$$\begin{aligned} SS(dose|1) &= 3(\bar{y}_{.1.} - \bar{y}_{...})^2 + 4(\bar{y}_{.2.} - \bar{y}_{...})^2 \\ &= 3\left(\frac{6+2+4}{3} - 8\right)^2 + 4\left(\frac{12+6+16+10}{4} - 8\right)^2 \\ &= 84, \end{aligned}$$

$$\begin{aligned} SS(drug|1, dose) &= SS(drug \times dose, drug, dose|1) - SS(dose|1) - SS(drug \times dose|1, drug, dose) \\ &= 100 - 84 - 6.4 \\ &= 9.6 \end{aligned}$$

- (b) By slide 63 of set 8, Type III sum of squares for *drug* is relevant to the test for *drug* main effect. Let $\mathbf{c}=(1, 1, -1, -1)'$. Then,

$$\begin{aligned} SS(drug|1, drug, dose, drug \times dose) &= (\mathbf{c}'\hat{\boldsymbol{\mu}})' [\mathbf{c}'(\mathbf{X}'\mathbf{X})^{-1}\mathbf{c}]^{-1} (\mathbf{c}'\hat{\boldsymbol{\mu}}) \\ &= (\mathbf{c}'\hat{\boldsymbol{\mu}})' \left[\mathbf{c}' \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \mathbf{c} \right]^{-1} (\mathbf{c}'\hat{\boldsymbol{\mu}}) \\ &= (-4) \times (2.5)^{-1} \times (-4) \\ &= 6.4. \end{aligned}$$

4. (a) Because we have a balanced design, we know the BLUE of $\bar{\mu}_{2.} - \bar{\mu}_{3.}$ is $\bar{y}_{2..} - \bar{y}_{3..}$. Thus, $\bar{y}_{2..} - \bar{y}_{3..}$ will be used to test $H_0 : \bar{\mu}_{2.} - \bar{\mu}_{3.} = 0$ ($\iff H_0 : \bar{\mu}_{2.} = \bar{\mu}_{3.}$), which is normally distributed with mean

$$E(\bar{y}_{2..} - \bar{y}_{3..}) = \bar{\mu}_{2.} - \bar{\mu}_{3.}$$

and variance

$$\begin{aligned}
\text{Var}(\bar{y}_{2..} - \bar{y}_{3..}) &= \text{Var}(\bar{p}_{2..} - \bar{p}_{3..} + \bar{e}_{2..} - \bar{e}_{3..}) \\
&= \text{Var}(\bar{p}_{2..} - \bar{p}_{3..}) + \text{Var}(\bar{e}_{2..} - \bar{e}_{3..}) \\
&= \frac{2\sigma_p^2}{8} + \frac{2\sigma_e^2}{16} \\
&= \frac{1}{8}(2\sigma_p^2 + \sigma_e^2) \\
&= \frac{1}{8}E(MS_{Tray \times SoilMoisture}).
\end{aligned}$$

Since $\bar{y}_{2..} - \bar{y}_{3..} = \frac{6.3+6.1}{2} - \frac{9.4+8.8}{2} = -2.9$ and $\widehat{\text{Var}}(\bar{y}_{2..} - \bar{y}_{3..}) = \frac{5.3}{8}$, the test statistics for the null hypothesis, $H_0 : \bar{\mu}_{2..} - \bar{\mu}_{3..} = 0$ is

$$\begin{aligned}
t &= \frac{\bar{y}_{2..} - \bar{y}_{3..}}{\sqrt{\widehat{\text{Var}}(\bar{y}_{2..} - \bar{y}_{3..})}} \\
&= \frac{-2.9}{\sqrt{5.3/8}}
\end{aligned}$$

- (b) From the Exam 2 solutions, the value of an unbiased estimator for $\sigma_p^2 + \sigma_e^2$ is $\frac{1}{2}MS_{T \times SM} + \frac{1}{2}MS_{Error}$. Thus, the degrees of freedom for test statistic could be approximated by the Cochran-Satterthwaite Method which is as following:

$$d = \frac{(0.5MS_{T \times SM} + 0.5MS_{Error})^2}{\frac{(0.5)^2[MS_{T \times SM}]^2}{14} + \frac{(0.5)^2[MS_{Error}]^2}{21}} = \frac{\{0.5 \times (5.3 + 3.7)\}^2}{\frac{(0.5 \times 5.3)^2}{14} + \frac{(0.5 \times 3.7)^2}{21}} = 30.4702.$$