

STAT 510 Homework 1**Due Date:** 11:00 A.M., Wednesday, January 17

1. The very last expression on slide 8 of slide set 1 shows how to compute a matrix product as a sum of n matrices, each of dimension $m \times k$. Write down the matrices summed together to give the matrix product \mathbf{AB} , where

$$\mathbf{A} = \begin{bmatrix} 1 & 5 \\ 4 & -1 \\ 0 & 1 \end{bmatrix} \text{ and } \mathbf{B} = \begin{bmatrix} 3 & 3 & -2 & 4 \\ 5 & -1 & 2 & 3 \end{bmatrix}.$$

2. Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 5 & 2 \\ 3 & -1 & 7 \\ 5 & -7 & 12 \end{bmatrix}.$$

- Use the algorithm on slide 15 of slide set 1 to find a generalized inverse of \mathbf{A} .
 - Use the R function `ginv` in the MASS package to find a generalized inverse of \mathbf{A} . (To load the MASS package into your R workspace use the command `library(MASS)`. If the MASS package is not already installed, you will need to install it before loading. The command `install.packages('MASS')` should install the MASS package if necessary.)
3. Use the definitions of t and F distributions to explain why the relationship between t and F distributions described on slide 37 of slide set 1 is true.
4. Imagine extending a string from $(0, 0)$, the origin in \mathbb{R}^2 , to a random point (x, y) in \mathbb{R}^2 , where $x \sim N(2, 1)$ independent of $y \sim N(1, 1)$. Use R to find the probability that the string will need to be longer than 6 units to reach from $(0, 0)$ to (x, y) .
5. Suppose $z_1, z_2 \stackrel{iid}{\sim} N(0, 1)$. Find the distribution of the following random variables and prove that your answer is correct.
- $(z_1 - z_2)^2/2$
 - $(z_1 + z_2)/|z_1 - z_2|$
6. Suppose $y_1, \dots, y_n \stackrel{iid}{\sim} N(\mu, \sigma^2)$. Let $\mathbf{y} = [y_1, \dots, y_n]'$, and let $\bar{y} = \frac{1}{n} \sum_{i=1}^n y_i$.
- Show that $s^2 = \frac{1}{n-1} \sum_{i=1}^n (y_i - \bar{y})^2$ can be written as $\mathbf{y}'\mathbf{B}\mathbf{y}$ for some matrix \mathbf{B} .
 - Prove that $(n-1)s^2/\sigma^2 \sim \chi_{n-1}^2$ using the result on slide 31 of slide set 1.
7. The main purpose of this question is to prove several results about orthogonal project matrices stated on slide 5 of slide set 2 (*A Review of Some Key Linear Model Results*).
- Prove that a matrix \mathbf{A} is $\mathbf{0}$ if and only if $\mathbf{A}'\mathbf{A} = \mathbf{0}$. (Hint: What are the diagonal elements of $\mathbf{A}'\mathbf{A}$?)
 - Prove that $\mathbf{X}'\mathbf{X}\mathbf{A} = \mathbf{X}'\mathbf{X}\mathbf{B}$ if and only if $\mathbf{X}\mathbf{A} = \mathbf{X}\mathbf{B}$. (Note that the “if” part of the proof, i.e.,

$$\mathbf{X}\mathbf{A} = \mathbf{X}\mathbf{B} \implies \mathbf{X}'\mathbf{X}\mathbf{A} = \mathbf{X}'\mathbf{X}\mathbf{B},$$

holds trivially. Thus, proving the converse, i.e.,

$$X'XA = X'XB \implies XA = XB,$$

is the challenging part. One proof starts like this

$$\begin{aligned} X'XA = X'XB &\implies X'XA - X'XB = 0 \\ &\implies X'X(A - B) = 0 \end{aligned}$$

Now if you multiply on the left by the appropriate matrix, you can use the result of part (a) to help complete the proof.)

- (c) Use the definition of generalized inverse and the result of part (b) to prove that

$$X(X'X)^-X'X = X$$

for any $(X'X)^-$ a generalized inverse of $X'X$.

- (d) Prove that if A is any symmetric matrix and G is any generalized inverse of A , then it must be true that G' is also a generalized inverse of A .
(e) Use the results of problems (c) and (d) to prove that

$$X'X(X'X)^-X' = X'$$

for any $(X'X)^-$ a generalized inverse of $X'X$.

- (f) Show that idempotency of P_X (i.e., $P_X P_X = P_X$) follows from the result of part (c) and, alternatively, from the result of part (e).
(g) Use parts (c) and (e) to prove that

$$XG_1X' = XG_2X'$$

for any two generalized inverses of $X'X$ denoted by G_1 and G_2 . (This says that P_X is the same matrix no matter which generalized inverse of $X'X$ is used to compute it.)

- (h) Use (d) and (g) to prove that P_X is symmetric (i.e., $P_X' = P_X$).