3. The F Test for Comparing Reduced vs. Full Models

Assume the Gauss-Markov Model with normal errors:

$$\boldsymbol{y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon}, \qquad \boldsymbol{\epsilon} \sim N(\boldsymbol{0}, \sigma^2 \boldsymbol{I}).$$

Suppose $\mathcal{C}(\boldsymbol{X}_0) \subset \mathcal{C}(\boldsymbol{X})$ and we wish to test

$$H_0 : E(\boldsymbol{y}) \in \mathcal{C}(\boldsymbol{X}_0) \text{ vs. } H_A : E(\boldsymbol{y}) \in \mathcal{C}(\boldsymbol{X}) \setminus \mathcal{C}(\boldsymbol{X}_0).$$

The "reduced" model corresponds to the null hypothesis and says that $E(\boldsymbol{y}) \in \mathcal{C}(\boldsymbol{X}_0)$, a specified subspace of $\mathcal{C}(\boldsymbol{X})$.

The "full" model says that $E(\boldsymbol{y})$ can be anywhere in $\mathcal{C}(\boldsymbol{X})$.

For example, suppose

$$\boldsymbol{X}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \text{ and } \boldsymbol{X} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

In this case, the reduced model says that all 6 observations have the same mean.

The full model says that there are three groups of two observations. Within each group, observations have the same mean. The three group means may be different from one another.

For this example, let μ_1, μ_2, and μ_3 be the elements of $\boldsymbol{\beta}$ in the full model, i.e., $\boldsymbol{\beta} = [\mu_1, \mu_2, \mu_3]'$. Then, for the full model,

$$E(\boldsymbol{y}) = \boldsymbol{X\beta} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_1 \\ \mu_2 \\ \mu_2 \\ \mu_3 \\ \mu_3 \end{bmatrix}, \text{ and}$$

$$H_0 : E(\boldsymbol{y}) \in \mathcal{C}(\boldsymbol{X}_0) \text{ vs. } H_A : E(\boldsymbol{y}) \in \mathcal{C}(\boldsymbol{X}) \setminus \mathcal{C}(\boldsymbol{X}_0)$$

is equivalent to

$$H_0 : \mu_1 = \mu_2 = \mu_3 \text{ vs. } H_A : \mu_i \neq \mu_j, \text{ for some } i \neq j.$$

For the general case, consider the test statistic

$$F = \frac{\boldsymbol{y}'(\boldsymbol{P_X} - \boldsymbol{P_{X_0}})\boldsymbol{y} / \left[\text{rank}(\boldsymbol{X}) - \text{rank}(\boldsymbol{X_0})\right]}{\boldsymbol{y}'(\boldsymbol{I} - \boldsymbol{P_X})\boldsymbol{y} / \left[n - \text{rank}(\boldsymbol{X})\right]}.$$

To show that this statistic has an F distribution, we will use the following fact:

$$\boldsymbol{P_{X_0}} \boldsymbol{P_X} = \boldsymbol{P_X} \boldsymbol{P_{X_0}} = \boldsymbol{P_{X_0}}.$$

There are many ways to see that this fact is true. First,

$$
\begin{aligned}
\mathcal{C}(\boldsymbol{X}_0) \subset \mathcal{C}(\boldsymbol{X}) \quad &\Longrightarrow \quad \text{Each column of } \boldsymbol{X}_0 \in \mathcal{C}(\boldsymbol{X}) \\
&\Longrightarrow \quad \boldsymbol{P}_X \boldsymbol{X}_0 = \boldsymbol{X}_0.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\boldsymbol{P}_X \boldsymbol{P}_{X_0} = \boldsymbol{P}_X \boldsymbol{X}_0 (\boldsymbol{X}_0' \boldsymbol{X}_0)^- \boldsymbol{X}_0' \quad &= \quad \boldsymbol{X}_0 (\boldsymbol{X}_0' \boldsymbol{X}_0)^- \boldsymbol{X}_0' \\
&= \quad \boldsymbol{P}_{X_0}.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
(\boldsymbol{P}_X \boldsymbol{P}_{X_0})' = \boldsymbol{P}_{X_0}' \quad &\Longrightarrow \quad \boldsymbol{P}_{X_0}' \boldsymbol{P}_X' = \boldsymbol{P}_{X_0}' \\
&\Longrightarrow \quad \boldsymbol{P}_{X_0} \boldsymbol{P}_X = \boldsymbol{P}_{X_0}. \quad \square
\end{aligned}
$$

Alternatively,

$$\forall\, \boldsymbol{a} \in \mathbb{R}^n, \qquad \boldsymbol{P}_{\boldsymbol{X}_0}\boldsymbol{a} \in \mathcal{C}(\boldsymbol{X}_0) \subset \mathcal{C}(\boldsymbol{X}).$$

Thus, $\forall\, \boldsymbol{a} \in \mathbb{R}^n, \quad \boldsymbol{P}_{\boldsymbol{X}}\boldsymbol{P}_{\boldsymbol{X}_0}\boldsymbol{a} = \boldsymbol{P}_{\boldsymbol{X}_0}\boldsymbol{a}$.

This implies $\boldsymbol{P}_{\boldsymbol{X}}\boldsymbol{P}_{\boldsymbol{X}_0} = \boldsymbol{P}_{\boldsymbol{X}_0}$.

Transposing both sides of this equality and using symmetry of projection matrices yields

$$\boldsymbol{P}_{\boldsymbol{X}_0}\boldsymbol{P}_{\boldsymbol{X}} = \boldsymbol{P}_{\boldsymbol{X}_0}. \quad \square$$

Alternatively, $\mathcal{C}(\boldsymbol{X}_0) \subset \mathcal{C}(\boldsymbol{X}) \implies \boldsymbol{X}\boldsymbol{B} = \boldsymbol{X}_0$ for some $\boldsymbol{B}$ because every column of $\boldsymbol{X}_0$ must be in $\mathcal{C}(\boldsymbol{X})$.

Thus,

$$\begin{aligned}
\boldsymbol{P}_{\boldsymbol{X}_0}\boldsymbol{P}_{\boldsymbol{X}} &= \boldsymbol{X}_0(\boldsymbol{X}_0'\boldsymbol{X}_0)^-\boldsymbol{X}_0'\boldsymbol{P}_{\boldsymbol{X}} = \boldsymbol{X}_0(\boldsymbol{X}_0'\boldsymbol{X}_0)^-(\boldsymbol{X}\boldsymbol{B})'\boldsymbol{P}_{\boldsymbol{X}} \\
&= \boldsymbol{X}_0(\boldsymbol{X}_0'\boldsymbol{X}_0)^-\boldsymbol{B}'\boldsymbol{X}'\boldsymbol{P}_{\boldsymbol{X}} = \boldsymbol{X}_0(\boldsymbol{X}_0'\boldsymbol{X}_0)^-\boldsymbol{B}'\boldsymbol{X}' \\
&= \boldsymbol{X}_0(\boldsymbol{X}_0'\boldsymbol{X}_0)^-(\boldsymbol{X}\boldsymbol{B})' = \boldsymbol{X}_0(\boldsymbol{X}_0'\boldsymbol{X}_0)^-\boldsymbol{X}_0' = \boldsymbol{P}_{\boldsymbol{X}_0}. \\
\boldsymbol{P}_{\boldsymbol{X}}\boldsymbol{P}_{\boldsymbol{X}_0} &= \boldsymbol{P}_{\boldsymbol{X}}\boldsymbol{X}_0(\boldsymbol{X}_0'\boldsymbol{X}_0)^-\boldsymbol{X}_0' = \boldsymbol{P}_{\boldsymbol{X}}\boldsymbol{X}\boldsymbol{B}(\boldsymbol{X}_0'\boldsymbol{X}_0)^-\boldsymbol{X}_0' \\
&= \boldsymbol{X}\boldsymbol{B}(\boldsymbol{X}_0'\boldsymbol{X}_0)^-\boldsymbol{X}_0' = \boldsymbol{X}_0(\boldsymbol{X}_0'\boldsymbol{X}_0)^-\boldsymbol{X}_0' = \boldsymbol{P}_{\boldsymbol{X}_0}.
\end{aligned}$$

$\square$

Note that $\boldsymbol{P_X} - \boldsymbol{P_{X_0}}$ is a symmetric and idempotent matrix:

$$(\boldsymbol{P_X} - \boldsymbol{P_{X_0}})' = \boldsymbol{P_X'} - \boldsymbol{P_{X_0}'} = \boldsymbol{P_X} - \boldsymbol{P_{X_0}}.$$

$$\begin{aligned}
(\boldsymbol{P_X} - \boldsymbol{P_{X_0}})(\boldsymbol{P_X} - \boldsymbol{P_{X_0}}) &= \boldsymbol{P_X P_X} - \boldsymbol{P_X P_{X_0}} - \boldsymbol{P_{X_0} P_X} + \boldsymbol{P_{X_0} P_{X_0}} \\
&= \boldsymbol{P_X} - \boldsymbol{P_{X_0}} - \boldsymbol{P_{X_0}} + \boldsymbol{P_{X_0}} \\
&= \boldsymbol{P_X} - \boldsymbol{P_{X_0}}.
\end{aligned}$$

Now back to determining the distribution of

$$F = \frac{\boldsymbol{y}'(\boldsymbol{P_X} - \boldsymbol{P_{X_0}})\boldsymbol{y}/\left[\text{rank}(\boldsymbol{X}) - \text{rank}(\boldsymbol{X}_0)\right]}{\boldsymbol{y}'(\boldsymbol{I} - \boldsymbol{P_X})\boldsymbol{y}/\left[n - \text{rank}(\boldsymbol{X})\right]}.$$

An important first step is to note that

$$F = \frac{\boldsymbol{y}'\left(\frac{\boldsymbol{P_X} - \boldsymbol{P_{X_0}}}{\sigma^2}\right)\boldsymbol{y}/\left[\text{rank}(\boldsymbol{X}) - \text{rank}(\boldsymbol{X}_0)\right]}{\boldsymbol{y}'\left(\frac{\boldsymbol{I} - \boldsymbol{P_X}}{\sigma^2}\right)\boldsymbol{y}/\left[n - \text{rank}(\boldsymbol{X})\right]}.$$

Now we can show that the numerator is a chi-square random variable divided by its degrees of freedom, independent of the denominator, which is a central chi-square random variable divided by its degrees of freedom. Once we show all these things, we will have established that the statistic F has an F distribution (see Slide Set 1 Slide 35).

We will use the following result from Slide Set 1.

- Suppose $\boldsymbol{\Sigma}$ is an $n \times n$ positive definite matrix.
- Suppose $\boldsymbol{A}$ is an $n \times n$ symmetric matrix of rank m such that $\boldsymbol{A\Sigma}$ is idempotent (i.e., $\boldsymbol{A\Sigma A\Sigma} = \boldsymbol{A\Sigma}$).
- Then $\boldsymbol{y} \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \Longrightarrow \boldsymbol{y}'\boldsymbol{A}\boldsymbol{y} \sim \chi^2_m(\boldsymbol{\mu}'\boldsymbol{A}\boldsymbol{\mu}/2)$.

For the numerator of our F statistic, we have

$$\boldsymbol{\mu} = \boldsymbol{X\beta}, \quad \boldsymbol{\Sigma} = \sigma^2\boldsymbol{I}, \quad \boldsymbol{A} = \left(\frac{\boldsymbol{P_X} - \boldsymbol{P_{X_0}}}{\sigma^2}\right), \quad \text{and}$$

$$\begin{aligned} m &= \text{rank}(\boldsymbol{A}) = \text{rank}\left(\frac{\boldsymbol{P_X} - \boldsymbol{P_{X_0}}}{\sigma^2}\right) = \text{rank}(\boldsymbol{P_X} - \boldsymbol{P_{X_0}}) \\ &= \text{tr}(\boldsymbol{P_X} - \boldsymbol{P_{X_0}}) = \text{tr}(\boldsymbol{P_X}) - \text{tr}(\boldsymbol{P_{X_0}}) \\ &= \text{rank}(\boldsymbol{P_X}) - \text{rank}(\boldsymbol{P_{X_0}}) = \text{rank}(\boldsymbol{X}) - \text{rank}(\boldsymbol{X}_0). \end{aligned}$$

(Multiplying by a nonzero constant does not affect the rank of a matrix. Rank is the same as trace for idempotent matrices. Trace of a difference is the same as the difference of traces. The rank of a projection matrix is equal to the rank of the matrix whose column space is projected onto.)

To verify that Σ is positive definite, note that for any $\boldsymbol{a} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$,

$$\boldsymbol{a}'\boldsymbol{\Sigma}\boldsymbol{a} = \boldsymbol{a}'\left(\sigma^2\boldsymbol{I}\right)\boldsymbol{a} = \sigma^2\boldsymbol{a}'\boldsymbol{a} = \sigma^2\sum_{i=1}^{n} a_i^2 > 0.$$

To verify that $\boldsymbol{A}\boldsymbol{\Sigma}$ is idempotent, we have

$$\boldsymbol{A}\boldsymbol{\Sigma} = \left(\frac{\boldsymbol{P_X} - \boldsymbol{P_{X_0}}}{\sigma^2}\right)(\sigma^2\boldsymbol{I}) = \boldsymbol{P_X} - \boldsymbol{P_{X_0}}.$$

Thus, we have

$$\boldsymbol{y}'\left(\frac{\boldsymbol{P_X} - \boldsymbol{P_{X_0}}}{\sigma^2}\right)\boldsymbol{y} \sim \chi^2_{\text{rank}(\boldsymbol{X})-\text{rank}(\boldsymbol{X}_0)}\left(\frac{1}{2}\boldsymbol{\beta}'\boldsymbol{X}'\left(\frac{\boldsymbol{P_X} - \boldsymbol{P_{X_0}}}{\sigma^2}\right)\boldsymbol{X}\boldsymbol{\beta}\right).$$

Denominator Distribution

From the result on Slide Set 2 Slide 19, we have

$$\boldsymbol{y}'\left(\frac{\boldsymbol{I}-\boldsymbol{P_X}}{\sigma^2}\right)\boldsymbol{y} \sim \chi^2_{n-\text{rank}(\boldsymbol{X})}.$$

(This fact can be proved using the result on Slide Set 1 Slide 31, following the same type of argument used to show the distribution of the numerator.)

Independence of Numerator and Denominator

By Slide Set 1 Slide 38,

$$\boldsymbol{y}'\left(\frac{\boldsymbol{P_X}-\boldsymbol{P_{X_0}}}{\sigma^2}\right)\boldsymbol{y} \text{ is independent of } \boldsymbol{y}'\left(\frac{\boldsymbol{I}-\boldsymbol{P_X}}{\sigma^2}\right)\boldsymbol{y} \text{ because}$$

$$\begin{aligned}\left(\frac{\boldsymbol{P_X}-\boldsymbol{P_{X_0}}}{\sigma^2}\right)(\sigma^2\boldsymbol{I})\left(\frac{\boldsymbol{I}-\boldsymbol{P_X}}{\sigma^2}\right) &= \frac{1}{\sigma^2}(\boldsymbol{P_X}-\boldsymbol{P_X}\boldsymbol{P_X}-\boldsymbol{P_{X_0}}+\boldsymbol{P_{X_0}}\boldsymbol{P_X}) \\ &= \frac{1}{\sigma^2}(\boldsymbol{P_X}-\boldsymbol{P_X}-\boldsymbol{P_{X_0}}+\boldsymbol{P_{X_0}}) = \boldsymbol{0}.\end{aligned}$$

Thus, it follows that

$$
\begin{aligned}
F &= \frac{\boldsymbol{y}'(\boldsymbol{P}_X - \boldsymbol{P}_{X_0})\boldsymbol{y} / \left[\text{rank}(\boldsymbol{X}) - \text{rank}(\boldsymbol{X}_0)\right]}{\boldsymbol{y}'(\boldsymbol{I} - \boldsymbol{P}_X)\boldsymbol{y} / \left[n - \text{rank}(\boldsymbol{X})\right]} \\
&\sim F_{\text{rank}(\boldsymbol{X}) - \text{rank}(\boldsymbol{X}_0), n - \text{rank}(\boldsymbol{X})} \left(\frac{\boldsymbol{\beta}'\boldsymbol{X}'(\boldsymbol{P}_X - \boldsymbol{P}_{X_0})\boldsymbol{X}\boldsymbol{\beta}}{2\sigma^2} \right).
\end{aligned}
$$

If H_0 is true, i.e. if $E(\boldsymbol{y}) = \boldsymbol{X}\boldsymbol{\beta} \in \mathcal{C}(\boldsymbol{X}_0)$, then the noncentrality parameter is 0 because

$$
\begin{aligned}
(\boldsymbol{P}_X - \boldsymbol{P}_{X_0})\boldsymbol{X}\boldsymbol{\beta} &= \boldsymbol{P}_X\boldsymbol{X}\boldsymbol{\beta} - \boldsymbol{P}_{X_0}\boldsymbol{X}\boldsymbol{\beta} \\
&= \boldsymbol{X}\boldsymbol{\beta} - \boldsymbol{X}\boldsymbol{\beta} = \boldsymbol{0}.
\end{aligned}
$$

In general, the noncentrality parameter quantifies how far the mean of $\boldsymbol{y}$ is from $\mathcal{C}(\boldsymbol{X}_0)$ because

$$
\begin{aligned}
&\boldsymbol{\beta}'\boldsymbol{X}'(\boldsymbol{P_X} - \boldsymbol{P}_{\boldsymbol{X}_0})\boldsymbol{X\beta} \\
&= \boldsymbol{\beta}'\boldsymbol{X}'(\boldsymbol{P_X} - \boldsymbol{P}_{\boldsymbol{X}_0})'(\boldsymbol{P_X} - \boldsymbol{P}_{\boldsymbol{X}_0})\boldsymbol{X\beta} \\
&= ||\, (\boldsymbol{P_X} - \boldsymbol{P}_{\boldsymbol{X}_0})\boldsymbol{X\beta} \,||^2 = \; ||\, \boldsymbol{P_X X\beta} - \boldsymbol{P}_{\boldsymbol{X}_0}\boldsymbol{X\beta} \,||^2 \\
&= ||\, \boldsymbol{X\beta} - \boldsymbol{P}_{\boldsymbol{X}_0}\boldsymbol{X\beta} \,||^2 = \; ||\, E(\boldsymbol{y}) - \boldsymbol{P}_{\boldsymbol{X}_0}E(\boldsymbol{y}) \,||^2 .
\end{aligned}
$$

Note that

$$
\begin{aligned}
\boldsymbol{y}'(\boldsymbol{P_X} - \boldsymbol{P_{X_0}})\boldsymbol{y} &= \boldsymbol{y}'\left[(\boldsymbol{I} - \boldsymbol{P_{X_0}}) - (\boldsymbol{I} - \boldsymbol{P_X})\right]\boldsymbol{y} \\
&= \boldsymbol{y}'(\boldsymbol{I} - \boldsymbol{P_{X_0}})\boldsymbol{y} - \boldsymbol{y}'(\boldsymbol{I} - \boldsymbol{P_X})\boldsymbol{y} \\
&= SSE_{\text{REDUCED}} - SSE_{\text{FULL}}.
\end{aligned}
$$

Also rank$(\boldsymbol{X})-$rank$(\boldsymbol{X}_0)$

$$
\begin{aligned}
&= [n - \text{rank}(\boldsymbol{X}_0)] - [n - \text{rank}(\boldsymbol{X})] \\
&= DFE_{\text{REDUCED}} - DFE_{\text{FULL}},
\end{aligned}
$$

where $DFE =$ Degrees of Freedom for Error.

Thus, the F statistic has the familiar form

$$\frac{(SSE_{\text{REDUCED}} - SSE_{\text{FULL}})/(DFE_{\text{REDUCED}} - DFE_{\text{FULL}})}{SSE_{\text{FULL}}/DFE_{\text{FULL}}}.$$

Equivalence of F Tests

It turns out that this reduced vs. full model F test is equivalent to the F test for testing

$$H_0 : \boldsymbol{C\beta} = \boldsymbol{d} \qquad \text{vs.} \qquad H_A : \boldsymbol{C\beta} \neq \boldsymbol{d}$$

with an appropriately chosen $\boldsymbol{C}$ and $\boldsymbol{d}$.

The equivalence of these tests is proved in STAT 611.

Example: F Test for Lack of Linear Fit

Suppose a balanced, completely randomized design is used to assign 1, 2, or 3 units of fertilizer to a total of 9 plots of land.

The yield harvested from each plot is recorded as the response.

Let y_{ij} denote the yield from the jth plot that received i units of fertilizer ($i, j = 1, 2, 3$).

Suppose all yields are independent and $y_{ij} \sim N(\mu_i, \sigma^2)$ for all $i, j = 1, 2, 3$.

$$\text{If } \boldsymbol{y} = \begin{bmatrix} y_{11} \\ y_{12} \\ y_{13} \\ y_{21} \\ y_{22} \\ y_{23} \\ y_{31} \\ y_{32} \\ y_{33} \end{bmatrix}, \text{ then } E(\boldsymbol{y}) = \begin{bmatrix} \mu_1 \\ \mu_1 \\ \mu_1 \\ \mu_2 \\ \mu_2 \\ \mu_2 \\ \mu_3 \\ \mu_3 \\ \mu_3 \end{bmatrix}.$$

Suppose we wish to determine whether there is a linear relationship between the amount of fertilizer applied to a plot and the expected value of a plot's yield.

In other words, suppose we wish to know if there exists real numbers β_1 and β_2 such that

$$\mu_i = \beta_1 + \beta_2(i) \text{ for all } i = 1, 2, 3.$$

Consider testing $H_0 : E(\boldsymbol{y}) \in \mathcal{C}(\boldsymbol{X}_0)$ vs. $H_A : E(\boldsymbol{y}) \in \mathcal{C}(\boldsymbol{X}) \setminus \mathcal{C}(\boldsymbol{X}_0)$, where

$$\boldsymbol{X}_0 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix} \quad \text{and} \quad \boldsymbol{X} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note $H_0 : E(\boldsymbol{y}) \in \mathcal{C}(\boldsymbol{X}_0) \iff \exists\, \boldsymbol{\beta} \in \mathbb{R}^2 \ni E(\boldsymbol{y}) = \boldsymbol{X}_0\boldsymbol{\beta}$

$$\iff \exists \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} \in \mathbb{R}^2 \ni \begin{bmatrix} \mu_1 \\ \mu_1 \\ \mu_1 \\ \mu_2 \\ \mu_2 \\ \mu_2 \\ \mu_3 \\ \mu_3 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 2 \\ 1 & 2 \\ 1 & 2 \\ 1 & 3 \\ 1 & 3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} \beta_1 + \beta_2(1) \\ \beta_1 + \beta_2(1) \\ \beta_1 + \beta_2(1) \\ \beta_1 + \beta_2(2) \\ \beta_1 + \beta_2(2) \\ \beta_1 + \beta_2(2) \\ \beta_1 + \beta_2(3) \\ \beta_1 + \beta_2(3) \\ \beta_1 + \beta_2(3) \end{bmatrix}$$

$\iff \mu_i = \beta_1 + \beta_2(i)$ for all $i = 1, 2, 3.$

Note $E(\boldsymbol{y}) \in \mathcal{C}(\boldsymbol{X}) \iff \exists\, \boldsymbol{\beta} \in \mathbb{R}^3 \ni E(\boldsymbol{y}) = \boldsymbol{X}\boldsymbol{\beta}$

$$\iff \exists \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} \in \mathbb{R}^3 \ni \begin{bmatrix} \mu_1 \\ \mu_1 \\ \mu_1 \\ \mu_2 \\ \mu_2 \\ \mu_2 \\ \mu_3 \\ \mu_3 \\ \mu_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} \beta_1 \\ \beta_1 \\ \beta_1 \\ \beta_2 \\ \beta_2 \\ \beta_2 \\ \beta_3 \\ \beta_3 \\ \beta_3 \end{bmatrix}.$$

This condition clearly holds with $\beta_i = \mu_i$ for all $i = 1, 2, 3$.

The alternative hypothesis

$$H_A : E(\boldsymbol{y}) \in \mathcal{C}(\boldsymbol{X}) \setminus \mathcal{C}(\boldsymbol{X}_0)$$

is equivalent to

$$H_A : \text{There do not exist } \beta_1, \beta_2 \in \mathbb{R} \text{ such that}$$

$$\mu_i = \beta_1 + \beta_2(i) \qquad \forall\, i = 1, 2, 3.$$

Because the lack of fit test is a reduced vs. full model F test, we can also obtain this test by testing

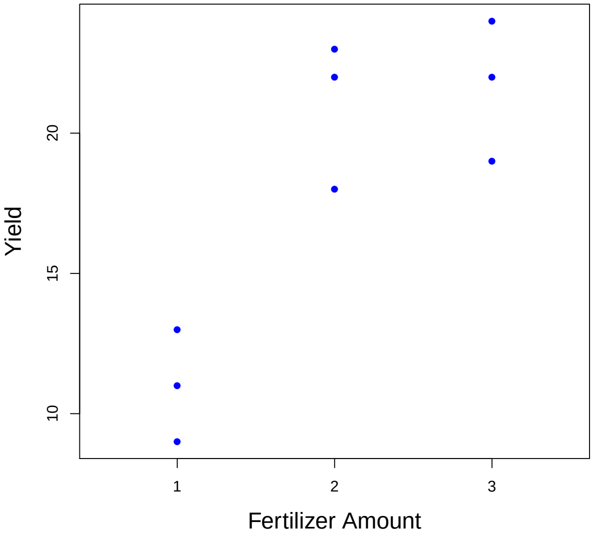
$$H_0 : \boldsymbol{C}\boldsymbol{\beta} = \boldsymbol{d} \quad \text{vs.} \quad H_A : \boldsymbol{C}\boldsymbol{\beta} \neq \boldsymbol{d}$$

for appropriate $\boldsymbol{C}$ and $\boldsymbol{d}$.

$$\boldsymbol{\beta} = \begin{bmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{bmatrix} \qquad \boldsymbol{C} = ? \qquad \boldsymbol{d} = ?$$

R Code and Output

```
> x=rep(1:3,each=3)
> x
[1] 1 1 1 2 2 2 3 3 3
>
> y=c(11,13,9,18,22,23,19,24,22)
>
> plot(x,y,pch=16,col=4,xlim=c(.5,3.5),
+      xlab="Fertilizer Amount",
+      ylab="Yield",axes=F,cex.lab=1.5)
> axis(1,labels=1:3,at=1:3)
> axis(2)
> box()
```


Yield
Fertilizer Amount
10
15
20
1
2
3

```
> X0=model.matrix(~x)
> X0
  (Intercept) x
1           1 1
2           1 1
3           1 1
4           1 2
5           1 2
6           1 2
7           1 3
8           1 3
9           1 3
```

```
> X=model.matrix(~0+factor(x))
> X
  factor(x)1 factor(x)2 factor(x)3
1          1          0          0
2          1          0          0
3          1          0          0
4          0          1          0
5          0          1          0
6          0          1          0
7          0          0          1
8          0          0          1
9          0          0          1
```

```
> proj=function(x){
+    x%*%ginv(t(x)%*%x)%*%t(x)
+ }
>
> library(MASS)
> PX0=proj(X0)
> PX=proj(X)
```

```
> Fstat=(t(y)%*%(PX-PX0)%*%y/1)/
+        (t(y)%*%(diag(rep(1,9))-PX)%*%y/(9-3))
> Fstat
         [,1]
[1,] 7.538462
>
> pvalue=1-pf(Fstat,1,6)
> pvalue
           [,1]
[1,] 0.03348515
```

```
> reduced=lm(y~x)
> full=lm(y~0+factor(x))
>
> rvsf=function(reduced,full)
+ {
+   sser=deviance(reduced)
+   ssef=deviance(full)
+   dfer=reduced$df
+   dfef=full$df
+   dfn=dfer-dfef
+   Fstat=(sser-ssef)/dfn/
+         (ssef/dfef)
+   pvalue=1-pf(Fstat,dfn,dfef)
+   list(Fstat=Fstat,dfn=dfn,dfd=dfef,
+        pvalue=pvalue)
+ }
```

```
> rvsf(reduced,full)
$Fstat
[1] 7.538462

$dfn
[1] 1

$dfd
[1] 6

$pvalue
[1] 0.03348515
```

```
> anova(reduced,full)
Analysis of Variance Table

Model 1: y ~ x
Model 2: y ~ 0 + factor(x)
  Res.Df    RSS Df Sum of Sq      F  Pr(>F)
1      7 78.222
2      6 34.667  1    43.556 7.5385 0.03349 *
---
Signif. codes:  0 *** 0.001 ** 0.01 * 0.05 . 0.1
```

```
> test=function(lmout,C,d=0){
+   b=coef(lmout)
+   V=vcov(lmout)
+   dfn=nrow(C)
+   dfd=lmout$df
+   Cb.d=C%*%b-d
+   Fstat=drop(
      t(Cb.d)%*%solve(C%*%V%*%t(C))%*%Cb.d/dfn)
+   pvalue=1-pf(Fstat,dfn,dfd)
+   list(Fstat=Fstat,pvalue=pvalue)
+ }
> test(full,matrix(c(1,-2,1),nrow=1))
$Fstat
[1] 7.538462
$pvalue
[1] 0.03348515
```

SAS Code and Output

```
data d;
  input x y;
  cards;
1 11
1 13
1 9
2 18
2 22
2 23
3 19
3 24
3 22
;
run;
```

```
proc glm;
  class x;
  model y=x;
  contrast 'Lack of Linear Fit' x 1 -2 1;
run;
```

```
The SAS System

The GLM Procedure

Dependent Variable: y

                                         Sum of
Source                      DF          Squares     Mean Square    F Value    Pr > F

Model                        2      214.2222222     107.1111111      18.54    0.0027

Error                        6       34.6666667       5.7777778

Corrected Total              8      248.8888889


R-Square     Coeff Var      Root MSE        y Mean

0.860714      13.43684      2.403701      17.88889


Source                      DF        Type I SS     Mean Square    F Value    Pr > F

x                            2      214.2222222     107.1111111      18.54    0.0027


Source                      DF      Type III SS     Mean Square    F Value    Pr > F

x                            2      214.2222222     107.1111111      18.54    0.0027
```

Contrast	DF	Contrast SS	Mean Square	F Value	Pr > F
Lack of Linear Fit	1	43.55555556	43.55555556	7.54	0.0335