1. (a) Let

$$\begin{aligned} a_{ik} &= \frac{y_{i1k} + y_{i2k}}{2} = \bar{\mu}_{i.} + p_k + \bar{e}_{i.k} \\ &= \bar{\mu}_{i.} + \varepsilon_{ik}, \end{aligned}$$

where $\varepsilon_{ik} \equiv p_k + \bar{e}_{i.k}$. Note that the ε_{ik} terms are *iid* $N(0, \sigma^2)$, where $\sigma^2 \equiv \sigma_p^2 + \frac{\sigma_e^2}{2}$. Thus, a two sample t-test can be used to test $H_0 : \bar{\mu}_{1.} = \bar{\mu}_{2.}$. From the R output of the analysis of averages, we have

$$t = \frac{84.892 - 80.454}{\sqrt{2.169^2 + 1.534^2}}.$$

(b) Let

$$\begin{aligned} d_{ik} &= y_{i1k} - y_{i2k} = \mu_{i1} - \mu_{i2} + e_{i1k} - e_{i2k} \\ &\equiv \delta_i + \eta_{ik}, \end{aligned}$$

where $\delta_i = \mu_{i1} - \mu_{i2}$ and $\eta_{ik} = e_{i1k} - e_{i2k}$. Note that the η_{ik} terms are *iid* $N(0, \sigma_\eta^2)$, where $\sigma_\eta^2 \equiv 2\sigma_e^2$. The test of infection main effect is a test of $H_0 : \frac{\mu_{11}+\mu_{21}}{2} = \frac{\mu_{12}+\mu_{22}}{2} \Longleftrightarrow H_0 : \mu_{11} - \mu_{21} + \mu_{21} - \mu_{22} = 0 \Longleftrightarrow \delta_1 + \delta_2 = 0$. From the last analysis of the differences in R, we can test $H_0 : \delta_1 + \delta_2 = 0$ with

$$t = \frac{8.250 + 1.492}{\sqrt{2.439^2 + 1.724^2}}.$$

(c) It is straightforward to see that a test for interaction is a test of $H_0 : \delta_1 = \delta_2 \Longleftrightarrow H_0 : \delta_1 - \delta_2 = 0$. Thus,

$$t = \frac{8.250 - 1.492}{\sqrt{2.439^2 + 1.724^2}}$$

is the relevant test statistic.

(d) $\hat{\sigma}_\eta^2 = 2\hat{\sigma}_e^2 = 5.974^2 \Longrightarrow \hat{\sigma}_e^2 = \frac{5.974^2}{2}$.

(e) $\hat{\sigma}^2 = \hat{\sigma}_p^2 + \frac{\hat{\sigma}_e^2}{2} = 5.313^2 \Longrightarrow \hat{\sigma}_p^2 = 5.313^2 - \frac{5.974^2}{4}$.

The answers to parts a) through e) above match tests and estimates obtained by fitting the full linear mixed effects model $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{Zu} + \boldsymbol{e}$.

2. (a) $(k-1)(l-1) = kl - k - l + 1$

$$\sum_{i=1}^{2}\sum_{j=1}^{5}\sum_{k=1}^{2}\sum_{l=1}^{2}(\bar{y}_{\cdot\cdot kl} - \bar{y}_{\cdot\cdot k\cdot} - \bar{y}_{\cdot\cdot\cdot l} + \bar{y}_{\cdot\cdot\cdot\cdot})^2 = 10\sum_{k=1}^{2}\sum_{l=1}^{2}(\bar{y}_{\cdot\cdot kl} - \bar{y}_{\cdot\cdot k\cdot} - \bar{y}_{\cdot\cdot\cdot l} + \bar{y}_{\cdot\cdot\cdot\cdot})^2$$

(b) This is a split-split-plot experiment. $H_0 : \bar{\mu}_{1\cdot\cdot} = \bar{\mu}_{2\cdot\cdot}$ is the null hypothesis that says there is no whole-plot-factor (i.e., Type) main effect. The whole-plot-experimental units correspond to Helmet(Type), so the F statistic for testing H_0 is

$$F = \frac{MS_{Type}}{MS_{Helmet(Type)}} = \frac{226/1}{254/8}.$$

Thus,

$$t = \sqrt{\frac{MS_{Type}}{MS_{Helmet(Type)}}} = \sqrt{\frac{226/1}{254/8}}.$$

(c) Since we have a balanced design, we know the BLUE of $\bar{\mu}_{1\cdot\cdot} - \bar{\mu}_{2\cdot\cdot}$ is $\bar{y}_{1\cdots} - \bar{y}_{2\cdots}$.Thus, the t test statistic in (b) can also be obtained by using $\bar{y}_{1\cdots} - \bar{y}_{2\cdots}$. Since $\bar{y}_{1\cdots} - \bar{y}_{2\cdots}$ is normally distributed with mean

$$E\left(\bar{y}_{1\cdots} - \bar{y}_{2\cdots}\right) = \bar{\mu}_{1\cdot\cdot} - \bar{\mu}_{2\cdot\cdot}$$

and variance

$$\begin{aligned}
\text{Var}\left(\bar{y}_{1\cdots} - \bar{y}_{2\cdots}\right) &= \text{Var}\left(\bar{a}_{1\cdot} - \bar{a}_{2\cdot} + \bar{b}_{1\cdot\cdot} - \bar{b}_{2\cdot\cdot} + \bar{e}_{1\cdots} - \bar{e}_{2\cdots}\right) \\
&= \text{Var}\left(\bar{a}_{1\cdot} - \bar{a}_{2\cdot}\right) + \text{Var}\left(\bar{b}_{1\cdot\cdot} - \bar{b}_{2\cdot\cdot}\right) + \text{Var}\left(\bar{e}_{1\cdots} - \bar{e}_{2\cdots}\right) \\
&= \frac{2\sigma_a^2}{5} + \frac{2\sigma_b^2}{10} + \frac{2\sigma_e^2}{20} \\
&= \frac{1}{10}\left(4\sigma_a^2 + 2\sigma_b^2 + \sigma_e^2\right) \\
&= \frac{1}{10} E\left\{MS_{Helmet(Type)}\right\},
\end{aligned}$$

the t statistic can be computed as following:

$$t = \frac{\bar{y}_{1\cdots} - \bar{y}_{2\cdots}}{\sqrt{\widehat{\text{Var}}\left(\bar{y}_{1\cdots} - \bar{y}_{2\cdots}\right)}}$$

where

$$\widehat{\text{Var}}\left(\bar{y}_{1\cdots} - \bar{y}_{2\cdots}\right) = \frac{MS_{Helmet(Type)}}{10} = \frac{1}{10}\frac{SS_{Helmet(Type)}}{(5-1)\times 2}.$$

Thus, by slides 23 $\sim$ 24 of set 2, the noncentrality parameter is expressed as

$$\frac{\bar{\mu}_{1\cdot\cdot} - \bar{\mu}_{2\cdot\cdot}}{\left(4\sigma_a^2 + 2\sigma_b^2 + \sigma_e^2\right)/10}$$

(d) From the provided expected mean squares, it is straightforward to see that

$$\frac{MS_{Helmet(Type)} - MS_{Direction\times Helmet(Type)}}{4}$$

has expectation σ_a^2.
Thus, an unbiased estimator of σ_a^2 takes the value

$$\frac{254/8 - 114/8}{4} = \frac{140}{32} = 4.375.$$

(e) Because we have a balanced design, we know the BLUE of $\bar{\mu}_{12\cdot} - \bar{\mu}_{11\cdot}$ is $\bar{y}_{1\cdot2\cdot} - \bar{y}_{1\cdot1\cdot}$, which has variance

$$\begin{aligned}
\operatorname{Var}\left(\bar{a}_{1\cdot} + \bar{b}_{1\cdot2} + \bar{e}_{1\cdot2\cdot} - \bar{a}_{1\cdot} - \bar{b}_{1\cdot1} - \bar{e}_{1\cdot1\cdot}\right) &= \operatorname{Var}\left(\bar{b}_{1\cdot2} - \bar{b}_{1\cdot1} + \bar{e}_{1\cdot2\cdot} - \bar{e}_{1\cdot1\cdot}\right) \\
&= \frac{2\sigma_b^2}{5} + \frac{2\sigma_e^2}{5 \times 2} \\
&= \frac{1}{5}\left(2\sigma_b^2 + \sigma_e^2\right) \\
&= \frac{1}{5} E\left(MS_{Direction \times Helmet(Type)}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
\widehat{\operatorname{Var}}\left(\bar{y}_{1\cdot2\cdot} - \bar{y}_{1\cdot1\cdot}\right) &= \frac{1}{5}\left(\frac{114}{8}\right) \\
&= \frac{57}{20} \\
&= 2.85
\end{aligned}$$

Thus, the confidence interval is $0.5 \pm 2.306\sqrt{2.85}$, where $t_{0.975,8} = 2.306$ and 8 is DF for $Direction \times Helmet(Type)$.

(f) Because we have a balanced design, we know the BLUE of $\mu_{121} - \mu_{111}$ is $\bar{y}_{1\cdot21} - \bar{y}_{1\cdot11}$, which has variance

$$\begin{aligned}
\operatorname{Var}\left(\bar{y}_{1\cdot21} - \bar{y}_{1\cdot11}\right) &= \operatorname{Var}\left(\bar{a}_{1\cdot} + \bar{b}_{1\cdot2} + \bar{e}_{1\cdot21} - \bar{a}_{1\cdot} - \bar{b}_{1\cdot1} - \bar{e}_{1\cdot11}\right) \\
&= \operatorname{Var}\left(\bar{b}_{1\cdot2} - \bar{b}_{1\cdot1}\right) + \operatorname{Var}\left(\bar{e}_{1\cdot21} - \bar{e}_{1\cdot11}\right) \\
&= \frac{2}{5}\sigma_b^2 + \frac{2}{5}\sigma_e^2 \\
&= \frac{2}{5}\left[\frac{1}{2}\left\{E\left(MS_{Direction \times Helmet(Type)}\right) + E\left(MS_{Error}\right)\right\}\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
\widehat{\operatorname{Var}}\left(\bar{y}_{1\cdot21} - \bar{y}_{1\cdot11}\right) &= \frac{1}{5}\left(\frac{114}{8} + \frac{59}{16}\right) \\
&= \frac{1}{5}\frac{287}{16} \\
&= 3.5875
\end{aligned}$$

Thus, a standard error for the BLUE of $\mu_{121} - \mu_{111}$ is $\sqrt{3.5875}$.

3. Let y_{ijk} be the weight gain for drug i (i = 1, 2), dose j (j=1 for dose 0 and j=2 for dose 10), and pig k (k=1,2,3,4 for drug 1 and k=1,2,3 for drug2). Then, we can suppose

$$y_{ijk} = \mu_{ij} + e_{ijk},$$

where $\mu_{11}, \mu_{12}, \mu_{21}$, and μ_{22} are unknown parameters and $e_{ijk} \overset{\text{iid}}{\sim} N(0, \sigma^2)$ for all i,j, and k.

(a) From the slide 60 of set 8,

$$SS(drug|1, dose) = SS(drug{\times}dose, drug, dose|1) - SS(dose|1) - SS(drug{\times}dose|1, drug, dose).$$

By slide 63 of set 8, $SS(drug \times dose|1, drug, dose)$ is sum of squares which is relevant to the test for the $drug \times dose$ interaction, where $H_0 : \mu_{11} - \mu_{12} - \mu_{21} + \mu_{22} = 0$.

Let $\boldsymbol{c}$=$(1, -1, -1, 1)'$, $\hat{\boldsymbol{\mu}}$=$(\bar{y}_{11\cdot}, \bar{y}_{12\cdot}, \bar{y}_{21\cdot}, \bar{y}_{22\cdot})'$=$(4, 9, 4, 13)'$ and $\boldsymbol{X}$=$\begin{pmatrix} \mathbf{1}_{2\times1} & 0 & 0 & 0 \\ 0 & \mathbf{1}_{2\times1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{2\times1} \end{pmatrix}$.

Then,

$$\begin{aligned} SS(drug \times dose|1, drug, dose) &= (\boldsymbol{c}'\hat{\boldsymbol{\mu}})' \left[\boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{c}\right]^{-1} (\boldsymbol{c}'\hat{\boldsymbol{\mu}}) \\ &= (\boldsymbol{c}'\hat{\boldsymbol{\mu}})' \left[\boldsymbol{c}' \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \boldsymbol{c}\right]^{-1} (\boldsymbol{c}'\hat{\boldsymbol{\mu}}) \\ &= 4 \times (2.5)^{-1} \times 4 \\ &= 6.4. \end{aligned}$$

Since

$$\begin{aligned} SS(drug \times dose, drug, dose|1) &= 2\left(\bar{y}_{11\cdot} - \bar{y}_{\cdots}\right)^2 + 2\left(\bar{y}_{12\cdot} - \bar{y}_{\cdots}\right)^2 + \left(\bar{y}_{21\cdot} - \bar{y}_{\cdots}\right)^2 + 2\left(\bar{y}_{22\cdot} - \bar{y}_{\cdots}\right)^2 \\ &= 2(4-8)^2 + 2(9-8)^2 + (4-8)^2 + 2(13-8)^2 \\ &= 100 \end{aligned}$$

and

$$\begin{aligned} SS(dose|1) &= 3\left(\bar{y}_{\cdot1\cdot} - \bar{y}_{\cdots}\right)^2 + 4\left(\bar{y}_{\cdot2\cdot} - \bar{y}_{\cdots}\right)^2 \\ &= 3\left(\frac{6+2+4}{3} - 8\right)^2 + 4\left(\frac{12+6+16+10}{4} - 8\right)^2 \\ &= 84, \end{aligned}$$

$$\begin{aligned} SS(drug|1, dose) &= SS(drug{\times}dose, drug, dose|1) - SS(dose|1) - SS(drug{\times}dose|1, drug, dose) \\ &= 100 - 84 - 6.4 \\ &= 9.6 \end{aligned}$$

(b) By slide 63 of set 8, Type III sum of squares for $drug$ is relevant to the test for $drug$ main effect. Let $\boldsymbol{c}$=$(1, 1, -1, -1)'$. Then,

$$\begin{aligned} SS(drug|1, drug, dose, drug \times dose) &= (\boldsymbol{c}'\hat{\boldsymbol{\mu}})' \left[\boldsymbol{c}'(\boldsymbol{X}'\boldsymbol{X})^{-1}\boldsymbol{c}\right]^{-1} (\boldsymbol{c}'\hat{\boldsymbol{\mu}}) \\ &= (\boldsymbol{c}'\hat{\boldsymbol{\mu}})' \left[\boldsymbol{c}' \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/2 \end{pmatrix} \boldsymbol{c}\right]^{-1} (\boldsymbol{c}'\hat{\boldsymbol{\mu}}) \\ &= (-4) \times (2.5)^{-1} \times (-4) \\ &= 6.4. \end{aligned}$$

4. (a) Because we have a balanced design, we know the BLUE of $\bar{\mu}_{2\cdot} - \bar{\mu}_{3\cdot}$ is $\bar{y}_{2\cdot\cdot} - \bar{y}_{3\cdot\cdot}$. Thus, $\bar{y}_{2\cdot\cdot} - \bar{y}_{3\cdot\cdot}$ will be used to test $H_0 : \bar{\mu}_{2\cdot} - \bar{\mu}_{3\cdot} = 0$ ($\Longleftrightarrow H_0 : \bar{\mu}_{2\cdot} = \bar{\mu}_{3\cdot}$), which is normally distributed with mean

$$E\left(\bar{y}_{2\cdot\cdot} - \bar{y}_{3\cdot\cdot}\right) = \bar{\mu}_{2\cdot} - \bar{\mu}_{3\cdot}$$

and variance

$$\begin{aligned}
\mathrm{Var}\left(\bar{y}_{2\cdot\cdot} - \bar{y}_{3\cdot\cdot}\right) &= \mathrm{Var}\left(\bar{p}_{2\cdot} - \bar{p}_{3\cdot} + \bar{e}_{2\cdot\cdot} - \bar{e}_{3\cdot\cdot}\right) \\
&= \mathrm{Var}\left(\bar{p}_{2\cdot} - \bar{p}_{3\cdot}\right) + \mathrm{Var}\left(\bar{e}_{2\cdot\cdot} - \bar{e}_{3\cdot\cdot}\right) \\
&= \frac{2\sigma_p^2}{8} + \frac{2\sigma_e^2}{16} \\
&= \frac{1}{8}\left(2\sigma_p^2 + \sigma_e^2\right) \\
&= \frac{1}{8} E\left(MS_{Tray \times SoilMoisture}\right).
\end{aligned}$$

Since $\bar{y}_{2\cdot\cdot} - \bar{y}_{3\cdot\cdot} = \frac{6.3+6.1}{2} - \frac{9.4+8.8}{2} = -2.9$ and $\widehat{\mathrm{Var}}\left(\bar{y}_{2\cdot\cdot} - \bar{y}_{3\cdot\cdot}\right) = \frac{5.3}{8}$, the test statistics for the null hypothesis, $H_0 : \bar{\mu}_{2\cdot} - \bar{\mu}_{3\cdot} = 0$ is

$$\begin{aligned}
t &= \frac{\bar{y}_{2\cdot\cdot} - \bar{y}_{3\cdot\cdot}}{\sqrt{\widehat{\mathrm{Var}}\left(\bar{y}_{2\cdot\cdot} - \bar{y}_{3\cdot\cdot}\right)}} \\
&= \frac{-2.9}{\sqrt{5.3/8}}
\end{aligned}$$

(b) From the Exam 2 solutions, the value of an unbiased estimator for $\sigma_p^2 + \sigma_e^2$ is $\frac{1}{2} MS_{T\times SM} + \frac{1}{2} MS_{Error}$. Thus, the degrees of freedom for test statistic could be approximated by the Cochran-Satterthwaite Method which is as following:

$$d = \frac{\left(0.5MS_{T\times SM} + 0.5MS_{Error}\right)^2}{\frac{(0.5)^2[MS_{T\times SM}]^2}{14} + \frac{(0.5)^2[MS_{Error}]^2}{21}} = \frac{\left\{0.5 \times (5.3 + 3.7)\right\}^2}{\frac{(0.5\times 5.3)^2}{14} + \frac{(0.5\times 3.7)^2}{21}} = 30.4702.$$