STAT 510 Homework 7 Solutions Spring 2019

1. For this problem, let μ_{ijk} denote the mean for filler type $i = 1, 2$, surface treatment $j = 1, 2$, and filler proportion $k = 1, 2, 3$ (corresponding to 25%, 50%, and 75%, respectively).

   This problem is very simple using SAS. For example, the following code generates all the answers.

```
options nocenter nonumber nodate ls=80;

filename fabric url "https://dnett.github.io/S510/FabricLoss.txt";

proc import datafile=fabric
     dbms=TAB replace out=d;
run;

proc print data=d;
run;

proc mixed data=d;
  class filler surface p;
  model y=filler|surface|p;
  lsmeans filler|surface|p;
run;
```

   The R code below fits the cell means model to these data:

```
> # Read in data and set covariates as factors
> dat <- read.table("https://dnett.github.io/S510/FabricLoss.txt",header = T,
+                   col.names=c("S","F","p","y"),
+                   colClasses = c("factor","factor","factor","numeric"))

> # Fit cell means model.
> fit <- lm(y ~ F:S:p + 0, data = dat)
```

   From the output below, notice that R is using the parameterization

$$\boldsymbol{\beta} = [\mu_{111}, \mu_{211}, \mu_{121}, \mu_{221}, \mu_{112}, \mu_{212}, \mu_{122}, \mu_{222}, \mu_{113}, \mu_{213}, \mu_{123}, \mu_{223}]'. \tag{1}$$

```
> # Look at the vector of estimates.
> coef(fit)
F1:S1:p25 F2:S1:p25 F1:S2:p25 F2:S2:p25 F1:S1:p50 F2:S1:p50
    201.0     213.0     164.0     148.5     237.0     233.5
F1:S2:p50 F2:S2:p50 F1:S1:p75 F2:S1:p75 F1:S2:p75 F2:S2:p75
    187.5     113.5     267.0     234.5     232.0     143.5
```

   Depending how you called `lm()`, you may not have the same parameterization of $\boldsymbol{\beta}$ as in (1), and hence may require different $\boldsymbol{C}$ matrices in the following problems.

   (a) From the output below,

$$\hat{\mu}_{221} = 148.50 \quad \text{and} \quad \text{se}(\hat{\mu}_{221}) = 11.59.$$

```
> # 1a: mean and se for F2, S2, 25% filler.
> summary(fit)

Call:
lm(formula = y ~ F:S:p + 0, data = dat)

Residuals:
    Min      1Q  Median      3Q     Max
-26.000  -9.125   0.000   9.125  26.000

Coefficients:
          Estimate Std. Error t value Pr(>|t|)
F1:S1:p25   201.00      11.59  17.340 7.33e-10 ***
F2:S1:p25   213.00      11.59  18.375 3.74e-10 ***
F1:S2:p25   164.00      11.59  14.148 7.57e-09 ***
F2:S2:p25   148.50      11.59  12.811 2.33e-08 ***
F1:S1:p50   237.00      11.59  20.445 1.08e-10 ***
F2:S1:p50   233.50      11.59  20.143 1.28e-10 ***
F1:S2:p50   187.50      11.59  16.175 1.64e-09 ***
F2:S2:p50   113.50      11.59   9.791 4.50e-07 ***
F1:S1:p75   267.00      11.59  23.033 2.67e-11 ***
F2:S1:p75   234.50      11.59  20.229 1.22e-10 ***
F1:S2:p75   232.00      11.59  20.014 1.38e-10 ***
F2:S2:p75   143.50      11.59  12.379 3.42e-08 ***
```

(b) From the code and output below (which uses Dr. Nettleton's `estimate()` function),

$$\hat{\bar{\mu}}_{.1.} = 231.0000 \qquad \text{and} \qquad \hat{\bar{\mu}}_{.2.} = 164.8333.$$

```
> # 1b: LSMEANS for S1 and S2.
> C.b <- c(rep(c(1,1,0,0), 3),   # S1
+          rep(c(0,0,1,1), 3))   # S2
> C.b <- matrix(C.b / 6, nrow = 2, byrow = TRUE)
> estimate(fit, C.b)
            c1        c2        c3        c4        c5        c6        c7        c8
[1,] 0.1666667 0.1666667 0.0000000 0.0000000 0.1666667 0.1666667 0.0000000 0.0000000
[2,] 0.0000000 0.0000000 0.1666667 0.1666667 0.0000000 0.0000000 0.1666667 0.1666667
            c9       c10       c11       c12 estimate       se 95% Conf.   limits
[1,] 0.1666667 0.1666667 0.0000000 0.0000000 231.0000 4.732424  220.6889 241.3111
[2,] 0.0000000 0.0000000 0.1666667 0.1666667 164.8333 4.732424  154.5223 175.1444
```

(c) From the code and output below (which uses Dr. Nettleton's `estimate()` function),

$$\hat{\bar{\mu}}_{1.2} = 212.25.$$

```
> # 1 c-d. LSMEAN and se for F1 & p50.
> C.c <- c(0,0,0,0,1,0,1,0,0,0,0,0)
> C.c <- matrix(C.c / 2, nrow = 1)
> estimate(fit, C.c)
     c1 c2 c3 c4  c5 c6  c7 c8 c9 c10 c11 c12 estimate       se 95% Conf.   limits
[1,]  0  0  0  0 0.5  0 0.5  0  0   0   0   0   212.25 8.196798  194.3907 230.1093
```

(d) From the output in part (c),

$$\text{se}(\hat{\bar{\mu}}_{1.2}) = 8.197.$$

(e) The test for surface treatment main effects is given by

$$H_0 : \bar{\mu}_{.1.} - \bar{\mu}_{.2.} = 0.$$

The code below (which uses Dr. Nettleton's `test()` function) gives $F = 97.7420$ and $p = 4.051053e-07$. These data provide strong evidence that there are surface treatment main effects. That is, averaging over filler type and proportion of filler, there is strong evidence that surface S1 differs from S2 at preventing fabric loss in abrasive tests.

```
> # 1e. surface treatment  main effects.
> C.e <- matrix(rep(c(1,1,-1,-1), 3), nrow = 1)
> test(fit, C.e) # Note d = 0 by default.
$Fstat
[1] 97.74202
$pvalue
[1] 4.051053e-07
```

(f) The test for three-way interactions is given by

$$H_0 : \begin{bmatrix} \mu_{111} - \mu_{121} - \mu_{211} + \mu_{221} - \mu_{112} + \mu_{122} + \mu_{212} - \mu_{222} \\ \mu_{111} - \mu_{121} - \mu_{211} + \mu_{221} - \mu_{113} + \mu_{123} + \mu_{213} - \mu_{223} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The code below (which uses Dr. Nettleton's `test()` function) gives $F = 0.89$ and $p = 0.44$. These data do not provide evidence of three-way interactions among filler type, surface treatment, and filler proportion.

```
> # 1f. Three-factor interaction.
> C.f <- c(1, -1, -1, 1, -1, 1, 1, -1, 0, 0, 0, 0,
+          1, -1, -1, 1, 0, 0, 0, 0, -1, 1, 1, -1)
> C.f <- matrix(C.f, nrow = 2, byrow = TRUE)
> test(fit, C.f)
$Fstat
[1] 0.8903876
$pvalue
[1] 0.4359589
```

<u>Comments</u>: a large p-value (i.e., $p > \alpha$) suggests that it *possible* that there are no three-way interactions, but it does not say there are no three-way interactions. Why? There could be three-way interactions, but they are too small (relative to the variability in the study) to detect.

(g) The test for two-way interactions between filler type and filler proportion is given by

$$H_0 : \begin{bmatrix} \bar{\mu}_{1.1} - \bar{\mu}_{1.2} - \bar{\mu}_{2.1} + \bar{\mu}_{2.2} \\ \bar{\mu}_{1.1} - \bar{\mu}_{1.3} - \bar{\mu}_{2.1} + \bar{\mu}_{2.3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

The code below (which uses Dr. Nettleton's `test()` function) gives $F = 6.57$ and $p = 0.012$. These data provide evidence that there are two-way interactions between filler type and filler proportion.

```
> # 1g. Two-factor interaction between filler type and proportion.
> C.g <- c(1, -1, 1, -1, -1, 1, -1, 1, 0, 0, 0, 0,
+          1, -1, 1, -1, 0, 0, 0, 0, -1, 1, -1, 1)
> C.g <- matrix(C.g, nrow = 2, byrow = TRUE)
> test(fit, C.g)
$Fstat
[1] 6.565736
$pvalue
[1] 0.01185168
```

2. (a) Describe the distribution of these differences.

Based on the model assumptions of $e_{ij} \overset{iid}{\sim} N(0, \sigma_e^2)$, for each subject $j = 1, \cdots, 20$,

$$\begin{aligned} d_j &= y_{1j} - y_{2j} \\ &= \mu_1 + u_j + e_{1j} - (\mu_2 + u_j + e_{2j}) \\ &= (\mu_1 - \mu_2) + e_{1j} - e_{2j} \end{aligned}$$

$E(d_j) = \mu_1 - \mu_2$, $Var(d_j) = Var(e_{1j}) + Var(e_{2j}) = 2\sigma_e^2$. Because a linear combination of independent normal distributions is still normal, we have $d_j \sim N(\mu_1 - \mu_2, 2\sigma_e^2)$.
For any $j \neq j'$, $Cov(d_j, d_{j'}) = Cov(e_{1j} - e_{2j}, e_{1j'} - e_{2j'}) = 0$, so all d_j's are independent.
Therefore $d_j \overset{iid}{\sim} N(\mu_1 - \mu_2, 2\sigma_e^2)$, which is a constant mean model. We can write this as a special case of a Gauss-Markov model as follows:

$$\boldsymbol{d} = \mathbf{1}[\mu_1 - \mu_2] + \boldsymbol{\epsilon}, \text{ where } \boldsymbol{d} = (d_1, \ldots, d_{20})' \text{ and } \boldsymbol{\epsilon} \sim N(\mathbf{0}, 2\sigma_e^2 \boldsymbol{I}).$$

(b) Provide a formula for a test statistic (as a function of $d_1, \cdots, d_{20}$) to test $H_0 : \mu_1 = \mu_2$.

Given the Gauss-Markov model above, we can find the formula for a test statistic by considering either a t test or and F test of $H_0 : \boldsymbol{C\beta} = \mathbf{0}$. The general formulas for a Gauss-Markov model can be simplified in this case because the "$\boldsymbol{X}$" matrix is just $\mathbf{1}$, the "$\boldsymbol{\beta}$" vector is just the one-element vector with $\mu_1 - \mu_2$ as the only element, and the "$\boldsymbol{C}$" matrix is just the 1×1 matrix with the element 1. Alternatively, can rewrite the model for differences as $d_1, \ldots, d_{20} \overset{iid}{\sim} N(\mu_d, \sigma_d^2)$, where $\mu_d = \mu_1 - \mu_2, \sigma_d^2 = 2\sigma_e^2$. Now the null hypothesis is equivalent to $H_0 : \mu_1 - \mu_2 = \mu_d = 0$. We can now see this as a STAT 101 type of question that asks us to test whether the mean of a normal distribution is zero based on an i.i.d. sample.

Let $\bar{d}. = \frac{\sum_{j=1}^{20} d_j}{20}$. Then $\bar{d}. \sim N\left(\mu_d, \frac{\sigma_d^2}{20}\right)$, and we can build up a t statistic to test $H_0 = \mu_d = 0$ as follows:

$$\begin{aligned} t &= \frac{\bar{d}. - 0}{\sqrt{\widehat{Var}(\bar{d}.)}} \\ &= \frac{\bar{d}.}{\sqrt{\hat{\sigma}_d^2/20}} \\ &= \frac{\bar{d}.}{\sqrt{\left[\frac{1}{20-1}\sum_{j=1}^{20}(d_j - \bar{d}.)^2\right]/20}} \end{aligned}$$

Or use F test statistic $F = t^2 = \dfrac{380\,\bar{d}.^2}{\sum_{j=1}^{20}(d_j - \bar{d}.)^2}$

(c) Fully state the exact distribution of the test statistic provided in part (b).

$$t \sim t_{19}\left(\frac{\mu_d}{\sqrt{\sigma_d^2/20}}\right) \overset{d}{=} t_{19}\left(\frac{\mu_1 - \mu_2}{\sqrt{\sigma_e^2/10}}\right)$$

$$F \sim F_{1,19}\left(\frac{5(\mu_1-\mu_2)^2}{\sigma_e^2}\right)$$

(d) Provide a formula for a 95% confidence interval for $\mu_1 - \mu_2$.
Given only the 40 scores of the subjects who received only drink one type, the model for these scores is simplfied to be a Markov model as

$$\boldsymbol{y} = \underbrace{[\boldsymbol{I}_{2\times 2} \otimes \mathbf{1}_{20\times 1}]}_{\boldsymbol{X}} \begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix} + \boldsymbol{\varepsilon}$$

with $\boldsymbol{y} = [a_1, \cdots, a_{20}, b_1, \cdots, b_{20}]'$ and $\boldsymbol{\varepsilon}$ is a vector of random errors $[\varepsilon_{11}, \ldots, \varepsilon_{1,20}, \varepsilon_{21}, \ldots, \varepsilon_{2,20}]'$ where $\varepsilon_{ik} \overset{iid}{\sim} N(0, \sigma_u^2 + \sigma_e^2)$ for $i = 1, 2; k = 1, \cdots, 20$.
So the BLUE for $\mu_1 - \mu_2$ is $\bar{a}. - \bar{b}.$.

$$\begin{aligned}\widehat{Var}(\bar{a}. - \bar{b}.) &= \widehat{Var(\bar{a}.) + Var}(\bar{b}.)\\ &= 2 \times \frac{1}{20}\widehat{(\sigma_u^2 + \sigma_e^2)} \qquad \text{MSE for the Markov model above}\\ &= \frac{1}{10} \cdot \frac{1}{40-2}\left(\sum_{j=1}^{20}(a_j - \bar{a}.)^2 + \sum_{j=1}^{20}(b_j - \bar{b}.)^2\right)\end{aligned}$$

Therefore the 95% confidence interval for $\mu_1 - \mu_2$ is

$$(\bar{a}. - \bar{b}.) + t_{38,\,0.975}\sqrt{\frac{1}{380}\left(\sum_{j=1}^{20}(a_j - \bar{a}.)^2 + \sum_{j=1}^{20}(b_j - \bar{b}.)^2\right)}$$

with $df = n - rank(\boldsymbol{X}) = 38$

(e) Provide formulas for unbiased estimators of σ_u^2 and σ_e^2

From part (b), we have $\hat{\sigma}_d^2 = 2\hat{\sigma}_e^2 = \frac{1}{20-1}\sum_{j=1}^{20}(d_j - \bar{d}.)^2$.
From part (d) we have $\widehat{\sigma_u^2 + \sigma_e^2} = \frac{1}{40-2}\left(\sum_{j=1}^{20}(a_j - \bar{a}.)^2 + \sum_{j=1}^{20}(b_j - \bar{b}.)^2\right)$.
By solving the equations above, we can obtain

$$\begin{cases}\hat{\sigma}_e^2 = \dfrac{\sum_{j=1}^{20}(d_j - \bar{d}.)^2}{38}\\[2ex] \hat{\sigma}_u^2 = \dfrac{\left(\sum_{j=1}^{20}(a_j - \bar{a}.)^2 + \sum_{j=1}^{20}(b_j - \bar{b}.)^2\right)}{38} - \dfrac{\sum_{j=1}^{20}(d_j - \bar{d}.)^2}{38}\end{cases}$$

(f) Provide a simplified expression for the best linear unbiased estimator of $\mu_1 - \mu_2$.

Both $\bar{d}.$ and $(\bar{a}. - \bar{b}.)$ are independent unbiased estimators of $\mu_1 - \mu_2$. Thus, the BLUE of $\mu_1 - \mu_2$ is the weighted average of $\bar{d}.$ and $(\bar{a}. - \bar{b}.)$ with weights proportional to the inverse of the variances.

$$\begin{aligned}
\widehat{\mu_1 - \mu_2} &= \frac{Var^{-1}(\bar{d}.)}{Var^{-1}(\bar{d}.) + Var^{-1}(\bar{a}. - \bar{b}.)} \cdot \bar{d}. + \frac{Var^{-1}(\bar{a}. - \bar{b}.)}{Var^{-1}(\bar{d}.) + Var^{-1}(\bar{a}. - \bar{b}.)} \cdot (\bar{a}. - \bar{b}.) \\
&= \frac{\sigma_u^2 + \sigma_e^2}{\sigma_u^2 + 2\sigma_e^2} \cdot \bar{d}. + \frac{\sigma_e^2}{\sigma_u^2 + 2\sigma_e^2} \cdot (\bar{a}. - \bar{b}.)
\end{aligned}$$

3. Suppose the responses in problem 1 were sorted first by subject and then by drink, the response vector $\boldsymbol{y} = [y_{11}, y_{21}, \cdots, y_{1,20}, y_{2,20}, y_{1,21}, \cdots, y_{1,40}, y_{2,41}, \cdots, y_{2,60}]'$.
In model $\boldsymbol{y} = \boldsymbol{X\beta} + \boldsymbol{Zu} + \boldsymbol{e}$,

$$\boldsymbol{X} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \\ \hline 1 & \\ 1 & \\ \vdots & \\ 1 & \\ \hline & 1 \\ & 1 \\ & \vdots \\ & 1 \end{array}\right] \begin{array}{l} \left.\vphantom{\begin{array}{c} 1 \\ 0 \\ 1 \\ 0 \\ \vdots \\ 1 \\ 0 \end{array}}\right\} \text{40 rows} \\ \left.\vphantom{\begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \end{array}}\right\} \text{20 rows} \\ \left.\vphantom{\begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \end{array}}\right\} \text{20 rows} \end{array} \quad \text{and } \boldsymbol{Z} = \left[\begin{array}{cccc|ccc|ccc} 1 & & & & & & & & & \\ 1 & & & & & & & & & \\ & 1 & & & & & & & & \\ & 1 & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & 1 & & & & & & \\ & & & 1 & & & & & & \\ & & & & 1 & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & 1 & & & \\ & & & & & & & 1 & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & 1 \end{array}\right]$$

the Kronecker product notation for $\boldsymbol{X}$ and $\boldsymbol{Z}$ are

$$\boldsymbol{X}_{80\times 2} = \begin{bmatrix} \mathbf{1}_{20\times 1} \otimes \boldsymbol{I}_{2\times 2} \\ \boldsymbol{I}_{2\times 2} \otimes \mathbf{1}_{20\times 1} \end{bmatrix}$$

$$\boldsymbol{Z}_{80\times 60} = \begin{bmatrix} \boldsymbol{I}_{20\times 20} \otimes \mathbf{1}_{2\times 1} & \boldsymbol{O}_{40\times 40} \\ \boldsymbol{O}_{40\times 20} & \boldsymbol{I}_{40\times 40} \end{bmatrix}$$

4. (a)

$$
\begin{aligned}
EMS_{xu(trt)} &= \frac{1}{df_{xu(trt)}} \mathrm{E}\left(SS_{xu(trt)}\right) \\
&= \frac{1}{tn-t} \mathrm{E}\left(m \sum_{i=1}^{t}\sum_{j=1}^{n} (y_{ij\boldsymbol{.}} - \bar{y}_{i\boldsymbol{..}})^2 \right) \\
&= \frac{1}{tn-t} \mathrm{E}\left(m \sum_{i=1}^{t}\sum_{j=1}^{n} ([\mu + \tau_i + u_{ij} + \bar{e}_{ij\boldsymbol{.}}] - [\mu + \tau_i + \bar{u_{i\boldsymbol{.}}} + \bar{e}_{i\boldsymbol{..}}])^2 \right) \\
&= \frac{m}{tn-t} \sum_{i=1}^{t}\sum_{j=1}^{n} \mathrm{E}\left\{(u_{ij} - \bar{u}_{i\boldsymbol{.}}) + (\bar{e}_{ij\boldsymbol{.}} - \bar{e}_{i\boldsymbol{..}})\right\}^2 \\
&= \frac{m}{tn-t} \sum_{i=1}^{t}\sum_{j=1}^{n} \left\{\mathrm{E}(u_{ij} - \bar{u}_{i\boldsymbol{.}})^2 + \mathrm{E}(\bar{e}_{ij\boldsymbol{.}} - \bar{e}_{i\boldsymbol{..}})^2\right\} \qquad \text{see the comment} \\
&= \frac{m}{tn-t} \sum_{i=1}^{t} \left[\mathrm{E}\left\{ \sum_{j=1}^{n} (u_{ij} - \bar{u}_{i\boldsymbol{.}})^2 \right\} + \mathrm{E}\left\{ \sum_{j=1}^{n} (\bar{e}_{ij\boldsymbol{.}} - \bar{e}_{i\boldsymbol{..}})^2 \right\} \right] \\
&= \frac{m}{tn-t} \sum_{i=1}^{t} \left\{ (n-1)\sigma_u^2 + (n-1)\frac{\sigma_e^2}{m} \right\} \quad \text{since } u_{ij} \overset{\text{iid}}{\sim} \mathcal{N}\left(0, \sigma_u^2\right) \text{ and } \bar{e}_{ij\boldsymbol{.}} \overset{\text{iid}}{\sim} \mathcal{N}\left(0, \frac{\sigma_e^2}{m}\right) \\
&= \frac{m}{tn-t} \left\{ t(n-1)\sigma_u^2 + t(n-1)\frac{\sigma_e^2}{m} \right\} \\
&= m\sigma_u^2 + \sigma_e^2.
\end{aligned}
$$

<u>Comment</u>:

$$
\begin{aligned}
\mathrm{E}\left\{(u_{ij} - \bar{u}_{i\boldsymbol{.}}) + (\bar{e}_{ij\boldsymbol{.}} - \bar{e}_{i\boldsymbol{..}})\right\}^2 &= \mathrm{Var}\left((u_{ij} - \bar{u}_{i\boldsymbol{.}}) + (\bar{e}_{ij\boldsymbol{.}} - \bar{e}_{i\boldsymbol{..}})\right) && \text{since } \mathrm{E}(u_{ij} - \bar{u}_{i\boldsymbol{.}}) = \mathrm{E}(\bar{e}_{ij\boldsymbol{.}} - \bar{e}_{i\boldsymbol{..}}) = 0 \\
&= \mathrm{Var}(u_{ij} - \bar{u}_{i\boldsymbol{.}}) + \mathrm{Var}(\bar{e}_{ij\boldsymbol{.}} - \bar{e}_{i\boldsymbol{..}}) && \text{since by assumption in slide 2 of set 12} \\
&= \mathrm{E}(u_{ij} - \bar{u}_{i\boldsymbol{.}})^2 + \mathrm{E}(\bar{e}_{ij\boldsymbol{.}} - \bar{e}_{i\boldsymbol{..}})^2
\end{aligned}
$$

(b) We can show this in a general case for t, n, m first. From slide 6, the sum of squares can be written as $\boldsymbol{y}'(\boldsymbol{P}_3 - \boldsymbol{P}_2)\boldsymbol{y}$, where

$$
\begin{aligned}
\boldsymbol{P}_2 &= [\mathbf{1}_{tnm\times 1},\ \boldsymbol{I}_{t\times t} \otimes \mathbf{1}_{nm\times 1}]\Big([\mathbf{1}_{tnm\times 1},\ \boldsymbol{I}_{t\times t} \otimes \mathbf{1}_{nm\times 1}]'[\mathbf{1}_{tnm\times 1},\ \boldsymbol{I}_{t\times t} \otimes \mathbf{1}_{nm\times 1}]\Big)^{-}[\mathbf{1}_{tnm\times 1},\ \boldsymbol{I}_{t\times t} \otimes \mathbf{1}_{nm\times 1}]' \\
&= [\mathbf{1}_{tnm\times 1},\ \boldsymbol{I}_{t\times t} \otimes \mathbf{1}_{nm\times 1}] \begin{bmatrix} tnm & nm\mathbf{1}'_{t\times 1} \\ nm\mathbf{1}_{t\times 1} & nm\boldsymbol{I}_{t\times t} \end{bmatrix}^{-} [\mathbf{1}_{tnm\times 1},\ \boldsymbol{I}_{t\times t} \otimes \mathbf{1}_{nm\times 1}]' \\
&= [\mathbf{1}_{tnm\times 1},\ \boldsymbol{I}_{t\times t} \otimes \mathbf{1}_{nm\times 1}] \begin{bmatrix} 0 & \mathbf{0}'_{t\times 1} \\ \mathbf{0}_{t\times 1} & \frac{1}{nm}\boldsymbol{I}_{t\times t} \end{bmatrix} [\mathbf{1}_{tnm\times 1},\ \boldsymbol{I}_{t\times t} \otimes \mathbf{1}_{nm\times 1}]' \\
&= \frac{1}{nm}\boldsymbol{I}_{t\times t} \otimes \mathbf{1}\mathbf{1}'_{nm\times nm}
\end{aligned}
$$

and

$$
\begin{aligned}
\boldsymbol{P}_3 &= [\boldsymbol{I}_{tn\times tn} \otimes \mathbf{1}_{m\times 1}]\Big([\boldsymbol{I}_{tn\times tn} \otimes \mathbf{1}_{m\times 1}]'[\boldsymbol{I}_{tn\times tn} \otimes \mathbf{1}_{m\times 1}]\Big)^{-1}[\boldsymbol{I}_{tn\times tn} \otimes \mathbf{1}_{m\times 1}]' \\
&= [\boldsymbol{I}_{tn\times tn} \otimes \mathbf{1}_{m\times 1}]\Big(m\boldsymbol{I}_{tn\times tn}\Big)^{-1}[\boldsymbol{I}_{tn\times tn} \otimes \mathbf{1}_{m\times 1}]'
\end{aligned}
$$

$= \frac{1}{m}\boldsymbol{I}_{tn\times tn} \otimes \boldsymbol{1}\boldsymbol{1}'_{m\times m}.$

Let

$$\boldsymbol{A} = \frac{\boldsymbol{P}_3 - \boldsymbol{P}_2}{t(n-1)} = \frac{\frac{1}{m}\boldsymbol{I}_{tn\times tn} \otimes \boldsymbol{1}\boldsymbol{1}'_{m\times m} - \frac{1}{nm}\boldsymbol{I}_{t\times t} \otimes \boldsymbol{1}\boldsymbol{1}'_{nm\times nm}}{t(n-1)}.$$

Then, by linearity of trace,

$$\begin{aligned}
\operatorname{tr}(\boldsymbol{A\Sigma}) &= \operatorname{tr}\left(\left[\frac{\frac{1}{m}\boldsymbol{I}_{tn\times tn} \otimes \boldsymbol{1}\boldsymbol{1}'_{m\times m} - \frac{1}{nm}\boldsymbol{I}_{t\times t} \otimes \boldsymbol{1}\boldsymbol{1}'_{nm\times nm}}{t(n-1)}\right]\left[\sigma_u^2\boldsymbol{I}_{tn\times tn} \otimes \boldsymbol{1}\boldsymbol{1}'_{m\times m} + \sigma_e^2\boldsymbol{I}_{tnm\times tnm}\right]\right) \\
&= \frac{1}{t(n-1)nm}\operatorname{tr}\left(\left[n\boldsymbol{I}_{tn\times tn} \otimes \boldsymbol{1}\boldsymbol{1}'_{m\times m} - \boldsymbol{I}_{t\times t} \otimes \boldsymbol{1}\boldsymbol{1}'_{nm\times nm}\right]\left[\sigma_u^2\boldsymbol{I}_{tn\times tn} \otimes \boldsymbol{1}\boldsymbol{1}'_{m\times m} + \sigma_e^2\boldsymbol{I}_{tnm\times tnm}\right]\right) \\
&= \frac{1}{t(n-1)nm}\operatorname{tr}\Big[nm\sigma_u^2\boldsymbol{I}_{tn\times tn} \otimes \boldsymbol{1}\boldsymbol{1}'_{m\times m} + n\sigma_e^2\boldsymbol{I}_{tn\times tn} \otimes \boldsymbol{1}\boldsymbol{1}'_{m\times m} \\
&\qquad - \sigma_u^2(\boldsymbol{I}_{t\times t} \otimes \boldsymbol{1}\boldsymbol{1}'_{nm\times nm})(\boldsymbol{I}_{tn\times tn} \otimes \boldsymbol{1}\boldsymbol{1}'_{m\times m}) - \sigma_e^2\boldsymbol{I}_{t\times t} \otimes \boldsymbol{1}\boldsymbol{1}'_{nm\times nm}\Big] \\
&= \frac{1}{t(n-1)nm}\Big[nm\sigma_u^2\operatorname{tr}(\boldsymbol{I}_{tn\times tn} \otimes \boldsymbol{1}\boldsymbol{1}'_{m\times m}) + n\sigma_e^2\operatorname{tr}(\boldsymbol{I}_{tn\times tn} \otimes \boldsymbol{1}\boldsymbol{1}'_{m\times m}) \\
&\qquad - m\sigma_u^2\operatorname{tr}(\boldsymbol{I}_{t\times t} \otimes \boldsymbol{1}\boldsymbol{1}'_{nm\times nm}) - \sigma_e^2\operatorname{tr}(\boldsymbol{I}_{t\times t} \otimes \boldsymbol{1}\boldsymbol{1}'_{nm\times nm})\Big] \\
&= \frac{1}{t(n-1)nm}(tnm)\Big(nm\sigma_u^2 + n\sigma_e^2 - m\sigma_u^2 - \sigma_e^2\Big) \quad \text{since } \boldsymbol{I}_{tn\times tn} \otimes \boldsymbol{1}\boldsymbol{1}'_{m\times m} \text{ and } \boldsymbol{I}_{t\times t} \otimes \boldsymbol{1}\boldsymbol{1}'_{nm\times nm} \text{ have 1's on the diagonal} \\
&= \frac{1}{n-1}\Big(m\sigma_u^2(n-1) + \sigma_e^2(n-1)\Big) \\
&= m\sigma_u^2 + \sigma_e^2
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{E}(\boldsymbol{y})'\boldsymbol{A}\operatorname{E}(\boldsymbol{y}) &= \operatorname{E}(\boldsymbol{y})'\left(\frac{\frac{1}{m}\boldsymbol{I}_{tn\times tn} \otimes \boldsymbol{1}\boldsymbol{1}'_{m\times m} - \frac{1}{nm}\boldsymbol{I}_{t\times t} \otimes \boldsymbol{1}\boldsymbol{1}'_{nm\times nm}}{t(n-1)}\right)\operatorname{E}(\boldsymbol{y}) \\
&= \frac{1}{tnm(n-1)}\left(n\operatorname{E}(\boldsymbol{y})'\boldsymbol{I}_{tn\times tn} \otimes \boldsymbol{1}\boldsymbol{1}'_{m\times m} - \operatorname{E}(\boldsymbol{y})'\boldsymbol{I}_{t\times t} \otimes \boldsymbol{1}\boldsymbol{1}'_{nm\times nm}\right)\operatorname{E}(\boldsymbol{y}) \\
&= \frac{1}{tnm(n-1)}\left(n(m\operatorname{E}(\boldsymbol{y})') - nm\operatorname{E}(\boldsymbol{y})'\right)\operatorname{E}(\boldsymbol{y}) \\
&= \frac{1}{tnm(n-1)}\boldsymbol{0}'\operatorname{E}(\boldsymbol{y}) \\
&= 0.
\end{aligned}$$

Now,

$$\begin{aligned}
EMS_{ou(xu,trt)} &= \operatorname{E}\left(\boldsymbol{y}'\left(\frac{\boldsymbol{P}_3 - \boldsymbol{P}_2}{t(n-1)}\right)\boldsymbol{y}\right) \\
&= \operatorname{E}\left(\boldsymbol{y}'\boldsymbol{A}\boldsymbol{y}\right) \\
&= \operatorname{tr}(\boldsymbol{A\Sigma}) + \operatorname{E}(\boldsymbol{y})'\boldsymbol{A}\operatorname{E}(\boldsymbol{y}) && \text{by slide 19 of set 12} \\
&= (m\sigma_u^2 + \sigma_e^2) + 0 \\
&= m\sigma_u^2 + \sigma_e^2,
\end{aligned}$$

The result also holds for the special case of $t = 2, n = 2, m = 2$, which is the same result as in part (a).